Gradient

Differential Equations Series

Instructor's Guide

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Introduction

When to Use this Video

- In Math 201, in lecture or recitation, after Unit 3: Multivariable Calculus–Lecture 7: Directional Derivatives and Gradients
- In Phys 201, in recitation–Workshop 3: Electric Potential, during Week 3: Electric Potential
- Prior Knowledge: the definition of the gradient, and the connection to directional derivative

Key Information

Duration: 17:34
Narrator: Prof. Tom Peacock
Materials Needed:
• Paper
• Pencil

Learning Objectives

After watching this video students should be able to:

- Recognize that the gradient vector points in the direction of maximum slope of a scalar function with magnitude equal to that slope.
- Describe the physicality of Fick's first law as it applies to concentration gradients.

Motivation

- Students in a multivariable calculus class are generally aware that the gradient of certain potential functions gives rise to gravity fields and electric fields. They have very limited or no experience using the gradient to describe physical phenomena.
- This video explores and models flux of particles in relation to concentration in order to develop an understanding of how the gradient is involved in describing the relationship between flux and concentration gradient in Fick's first law.
- Students gain exposure to the process of modeling a physical process. In this video, flux is modeled using random walkers. The process of modeling a complex process is explored, beginning with a 1-dimensional model, and then extending the model to higher dimensions.

Student Experience

It is highly recommended that the video is paused when prompted so that students are able to attempt the activities on their own and then check their solutions against the video.

During the video, students will observe the development of a model using random walkers to explore concentration gradients and diffusion.

Video Highlights

This table outlines a collection of activities and important ideas from the video.

Time	Feature	Comments
1:46	Chapter 1: Diffusion equation	The concept of diffusion is introduced through the example of pharmaceuticals diffusing in the body.
1:47	Animation of pill diffusing in stomach	
3:07	Chapter 2: Review of gradient	The properties and meaning of the gradient are described graphically for functions of 1, 2, and 3 variables.
5:35	Chapter 3: Fick's first law	The relationship between flux and the negative gradient of concentration is described.
5:39	Dye drop in water time lapse	This video segment shows a drop of concentrated dye diffusing in a shallow bowl of water.
6:26	Model of random walkers on a line	This video segment models flux and the relationship to concentration through the behavior of random walkers in one dimension.
11:44	Simulation of random walkers in the plane	
13:33	Chapter 4: Examples	A collection of real world examples whose behavior can be described by Fick's first law are shown.
13:35	Solid state diffusion demonstration	This video segment shows a 2-dimensional demonstration of solid state material diffusion using people as random walkers on a grid. You may want to do this activity with the students in your class.
14:58	Glass blowing	This video segment show how heat flux is described by Fick's first law, and explains the need to constantly reheat glass during the glass blowing process.
15:34	Shuttle tile	This video segment shows a piece of shuttle tile designed to have a very small heat flux in comparison to its very large temperature gradient.

Video Summary

This video leads students through an understanding of how the gradient is used in Fick's first law to describe a relationship between the flux of particles and the concentration of particles. This relationship is explored by modeling the process of diffusion using random walkers. Finally, this law is applied to other real-world scenarios involving solid state diffusion and thermodynamics.

Math 201 Materials

Pre-Video Materials

1. Small group discussion reinforcing the meaning of the directional derivatives, and the fundamental theorem of line integrals. Once small groups have discussed the problems posed below, open discussion to the entire class.

Suppose you are walking along a hilly terrain, whose altitude is described by the function $z=f(x,y)$. You walk along some path c. You can compute the total altitude change in two ways. One is that you can find your initial and final positions, and take the difference between the altitude function at those points. Alternatively, you can break up your path into infinitesimal pieces, and compute the change in altitude along each piece. The total altitude change will just be the sum of all of the small changes in altitude along infinitesimal segments of the path.

(a) Explain why these two ways of measuring the change in altitude must be the same.

(b) Use this to explain the fundamental theorem of line integrals.

2. Discussion questions.

(a) What are the units of 1-dimensional flux?

(b) What about the units of flux in 2-dimensions and 3-dimensions?

(c) Why are the units the same or different?

(d) Have students brainstorm the common properties of quantities which depend on dimension.

Post-Video Materials

1. This is the 1-dimensional dye diffusion experiment mentioned in the video. It is intended to be done by individuals or small groups of 2 or 3 students.

The materials that are needed for each group are:

- small test tubes,
- test tube holders,
- dye,
- rulers, and
- time recording devices.

Place water in the narrow test tubes. Because the test tubes are narrow, most measurable diffusion will occur in the vertical direction. Fill with water, and add a small drop of dye. With the ruler, record the height of the dye as it diffuses at different times. Plot the data, and determine the relationship between height and time. This activity may take between 15 and 20 minutes.

2. Consider the examples of Fick's first law that are presented at the end of the video. The first example was a solid-state diffusion example, and Fick's first law was used to describe the flux of material in terms of the concentration gradient. The other two examples connect heat flux to the gradient of the temperature.

(a) Have students write equations that describe these relationships using the following variables:

- J denotes a flux of mass,
- c denotes concentration,
- D the diffusion coefficient,
- q denotes heat flux,
- T denotes the temperature,
- K denotes coefficient (called thermal conductivity)

(b) Determine the units of J and q and compare them.

(c) Determine the units of D and K and compare them.

3. Break students into small groups of 2 or 3 students. Guide students to derive the diffusion equation in 1-dimension using a random walker model. Considering the following steps might be helpful.

(a) Find the concentration of particles $c(x,t)$ at a point x at time t in terms of $x-\Delta x$, $x+\Delta x$, and $t-\Delta t$.

(b) Take a linear approximation in t and a quadratic approximation in x to find $c(x-\Delta x, t-\Delta t)$ in terms of $c(x,t)$ and partial derivatives.

(c) Take a linear in t and a quadratic approximation in x approximation to find $c(x+\Delta x, t-\Delta t)$ in terms of $c(x,t)$ and partial derivatives.

(d) When are the approximations in parts (b) and (c) acceptable?

(e) Plug the approximations found in parts (b) and (c) into the formula found in part (a). Simplify and rearrange to obtain the diffusion equation in one spacial dimension.

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Phys 201 Materials

Pre-Video Materials

One of the most common applications of the gradient in Physics is describing the force (vector) field as the gradient of a potential function. The first two exercises are computations of the gradient of the potential functions for the gravitation force and the electric field created by an electric dipole. They are often represented in spherical coordinates, but are easiest to compute in standard Cartesian coordinates, offering students an opportunity to change coordinate systems.

1. The gravitational potential energy is given in spherical coordinates by the following expression

$$\frac{Gm_1m_2}{r}$$

where G is the gravitational constant, m_1 and m_2 are the masses of two bodies, and r is the distance between them.

(a) Rewrite this expression in cartesian coordinates.

(b) Take the gradient and obtain an expression for the gravitational force.

(c) Rewrite this force field in spherical coordinates.

2. The following problem applies a linear approximation to a superposition of electric charges to find the gradient field of an electric dipole.

Consider the dipole created by placing a postive charge at position $(\Delta, 0)$, and a negative charge placed at $(-\Delta, 0)$. Then the electric potential due to each charge is given by the formulas below.

$$E_+ = \frac{k}{\sqrt{(x-\Delta)^2+y^2}} \qquad E_- = \frac{-k}{\sqrt{(x+\Delta)^2+y^2}}$$

Thus the total electric potential is given by the superposition of these potentials.

$$E = E_+ + E_- = k\left(\frac{1}{\sqrt{(x-\Delta)^2+y^2}} - \frac{1}{\sqrt{(x+\Delta)^2+y^2}}\right)$$

(a) Assume that Δ is very close to zero. Then find a function $f(x,y)$ such that the potential is approximately equal to the partial derivative of $f(x,y)$ with respect to x times a constant.

(b) Using this approximated description of the electric potential to find the electric field due to the dipole.

3. Discussion questions.

(a) What are the units of 1-dimensional flux?

(b) What about the units of flux in 2-dimensions and 3-dimensions?

(c) Why are the units the same or different?

(d) Are there other quantities whose units depend on dimension?
(Students may recognize that density also depends on the dimension.)

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J denotes a flux of mass,	q denotes heat flux,
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D the diffusion coefficient,	K denotes coefficient (called thermal conductivity)

(b) Determine the units of J and q and compare them.

(c) Determine the units of D and K and compare them.

3. Determine the relative magnitudes of the diffusion coefficient and or thermal conductivity constants in the examples from the video. In which example was the constant the smallest?

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Additional Resources

Going Further

The gradient is, perhaps, the most natural generalization of the derivative. But there are further generalizations, allowing the gradient of multivariable functions to obtain tensors. A natural place to recall the gradient is in Soph 301, which explores linear algebra and differential equations. The gradient is a natural tie between functions and matrices. For example, the Jacobian is a key element in changing coordinate systems. Changes of coordinates or reference frame are frequently difficult for students.

Another aspect of this video that has broad applicability beyond Math 201 is modeling a 3-dimensional process by first modeling an analogous 1-dimensional process. This reduction provides a toy model that is simpler to manipulate by hand, and allows one to work through the details of the concepts before moving onto higher dimensions.

References

The use of random processes to model molecular interaction may be unfamiliar to students depending on their background. The first two references are a text about random walks in biology and an online virtual experiment about random walks of molecules. The latter book discusses vector calculus concepts, such as the gradient, through electromagnetism. Both books are very nice and are appropriate for students or instructors.

- Berg, H. C. (1993). *Random Walks in Biology*. Princeton, NJ: Princeton University Press.
- Blauch, D. N. (2009). *Virtual Chemistry Experiments: Kinetic Molecular Theory of Gases.* Retrieved from http://www.chm.davidson.edu/vce/kineticmoleculartheory/index.html
- Schey, H. M. (2005). *Div, Grad, Curl and all that: an informal text on vector calculus.* New York, NY: W. W. Norton & Company, Ltd.

The following articles provide insight into student difficulties in understanding the gradient, as well as activities and problems that may help students overcome these difficulties and gain deeper understanding of the gradient.

- Chorlton, F. (1981). Scalar and Vector Functions Considered Fundamentally. *Int. J. of Math. Ed. in Sci. Technol., 12*(1), 175–191.
- Curjel, C.R. (1990). Understanding Vector Fields. *The American Mathematical Monthly, 97*(6), 524–527.
- Gaze, E. C. (2005). Manipulating the Gradient. *PRIMUS, 15*(2), 109–116.

The final example in the video shows solid material diffusion. References for the description of this process can be found in the following books.

- Abbaschian, R., Abbaschian, L., and Reed-Hill, R. E. (2009). *Physical Metallurgy Principles*. Stamford, CT: Cengage Learning.
- Guy, A. G. (1959). *Elements of Physical Metallurgy.* Boston, MA: Addison–Wesley Publishing Company.
- Porter, D. A., and Easterling, K. E. (1981). *Phase Transformations in Metals and Alloys.* Berkshire, England: Van Nostrand Reinhold (UK) Co. Ltd.

These video references discuss the gradient.

- Auroux, D., *18.02 Multivariable Calculus*, Fall 2007. (Massachusetts Institute of Technology: MIT OpenCouseWare), http://ocw.mit.edu (Accessed March 12, 2010). License: Creative Commons BY-NC-SA
  -Lecture 21: Gradient Fields
- Open University, (2009). *Grad-Grad, Div, and Curl.* (The Open University), http://www.open.edu/openlearn/science-maths-technology/mathematics-and-statistics/grad-div-and-curl?track=37fb6ff4b3 (Accessed March 13, 2010). License: Creative Commons Attribution-NonCommercial-ShareAlike 2.0 Licence