LECTURE 9: Conditioning on an event; Multiple continuous r.v.'s

- Conditioning a r.v. on an event
  - Conditional PDF
  - Conditional expectation and the expected value rule
  - Exponential PDF: memorylessness
  - Total probability and expectation theorems
  - Mixed distributions
- Jointly continuous r.v.'s and joint PDFs
  - From the joints to the marginals
  - Uniform joint PDF example
  - The expected value rule and linearity of expectations
  - The joint CDF

Conditional PDF, given an event

$P(A) > 0$

$$p_X(x) = \mathbf{P}(X = x)$$

$$f_X(x) \cdot \delta \approx \mathbf{P}(x \le X \le x + \delta)$$

$$p_{X|A}(x) = \mathbf{P}(X = x \mid A)$$

$$\underline{f_{X|A}(x)} \cdot \delta \approx \mathbf{P}(x \le X \le x + \delta \mid A)$$

$$\mathbf{P}(X \in B) = \sum_{x \in B} p_X(x)$$

$$\mathbf{P}(X \in B) = \int_B f_X(x)\, dx$$

$$\mathbf{P}(X \in B \mid A) = \sum_{x \in B} p_{X|A}(x)$$

$$\mathbf{P}(X \in B \mid A) = \int_B f_{X|A}(x)\, dx$$

Def

$$\sum_x p_{X|A}(x) = 1$$

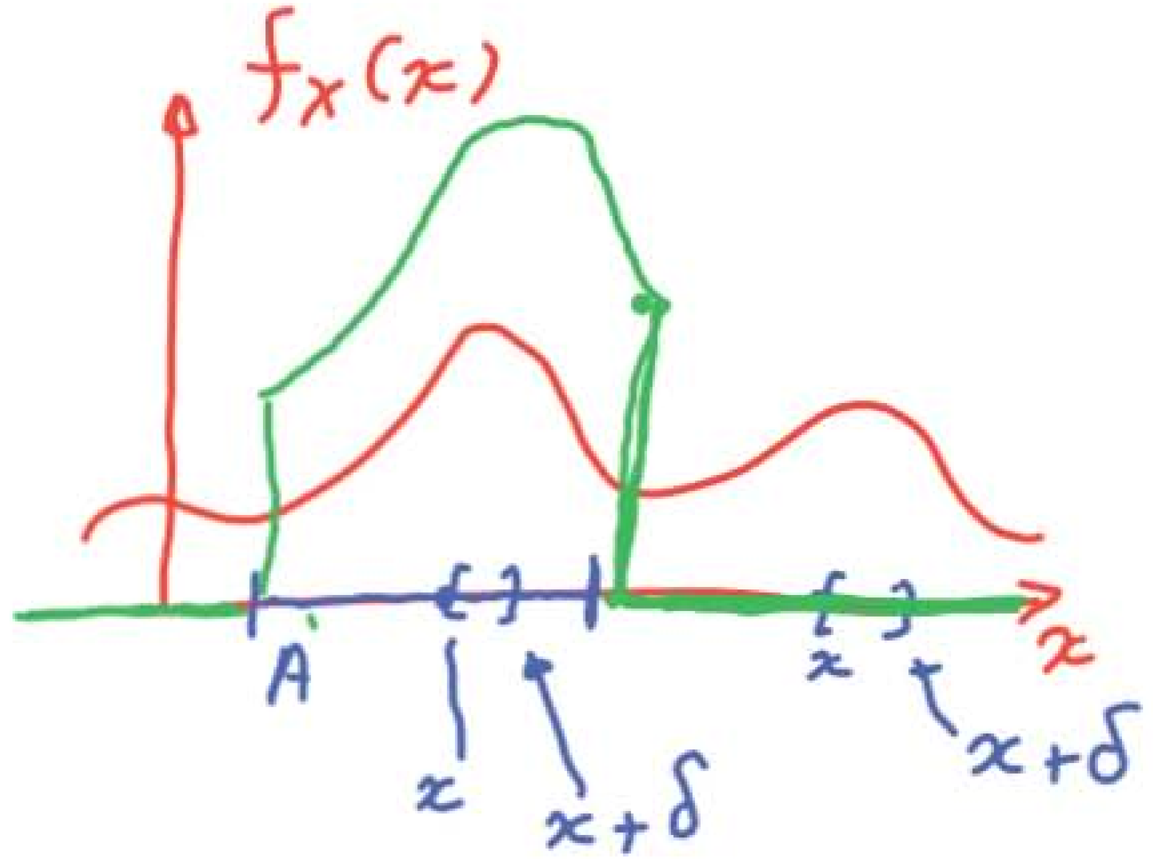
$$\int f_{X|A}(x)\, dx = 1$$

Conditional PDF of X, given that $X \in A$

$$\mathbf{P}(x \le X \le x + \delta \mid X \in A) \approx f_{X|X\in A}(x) \cdot \cancel{\delta}$$

$$= \frac{P(x \le X \le x+\delta,\ X \in A)}{P(A)}$$

$$= \frac{P(x \le X \le x+\delta)}{P(A)} \approx \frac{f_X(x)\,\cancel{\delta}}{P(A)}$$



$$f_{X|X\in A}(x) = \begin{cases} 0, & \text{if } x \notin A \\ \dfrac{f_X(x)}{\mathbf{P}(A)}, & \text{if } x \in A \end{cases}$$

Conditional expectation of X, given an event

$$\mathbf{E}[X] = \sum_x x p_X(x)$$

$$\mathbf{E}[X] = \int x f_X(x)\, dx$$

$$\mathbf{E}[X \mid A] = \sum_x x p_{X|A}(x)$$

$$\mathbf{E}[X \mid A] = \int x f_{X|A}(x)\, dx$$

Def

Expected value rule:

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

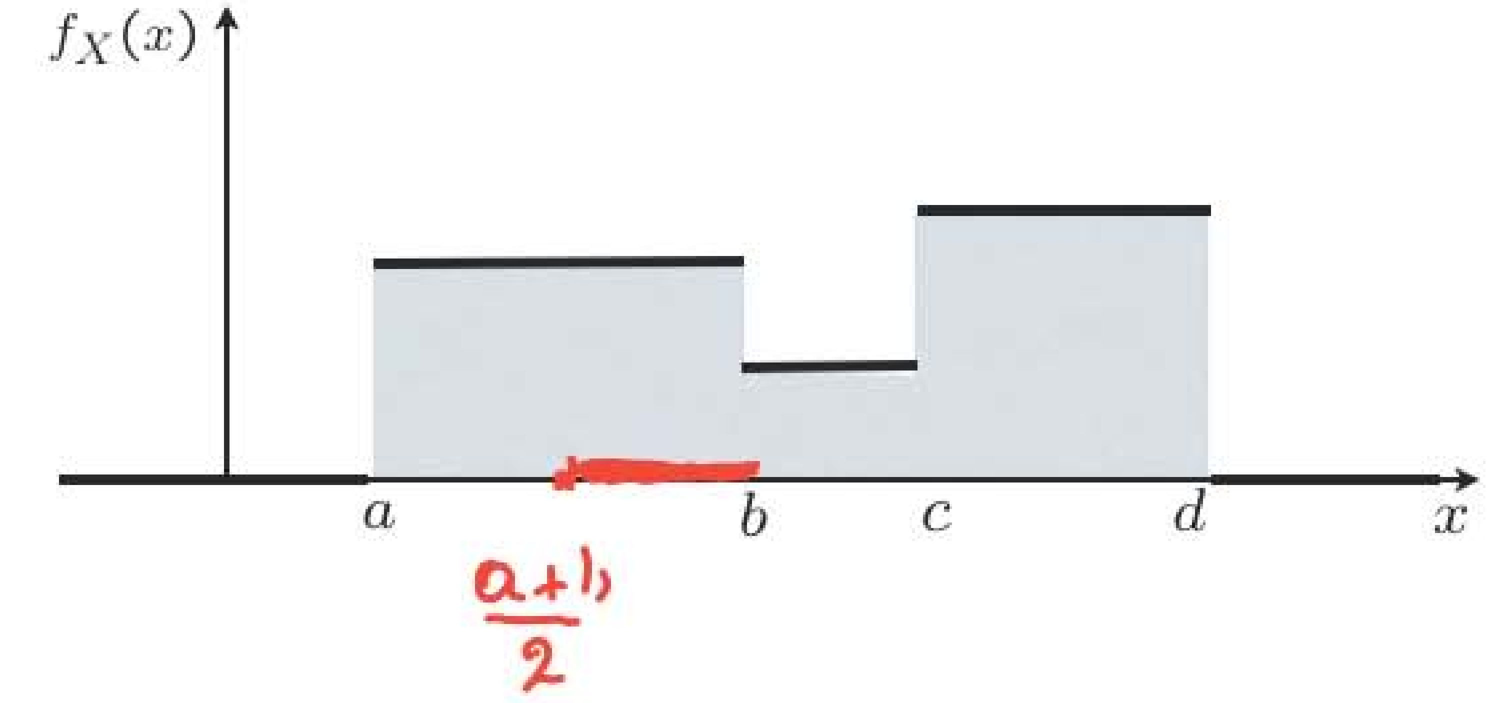
$$\mathbf{E}[g(X)] = \int g(x) f_X(x)\, dx$$

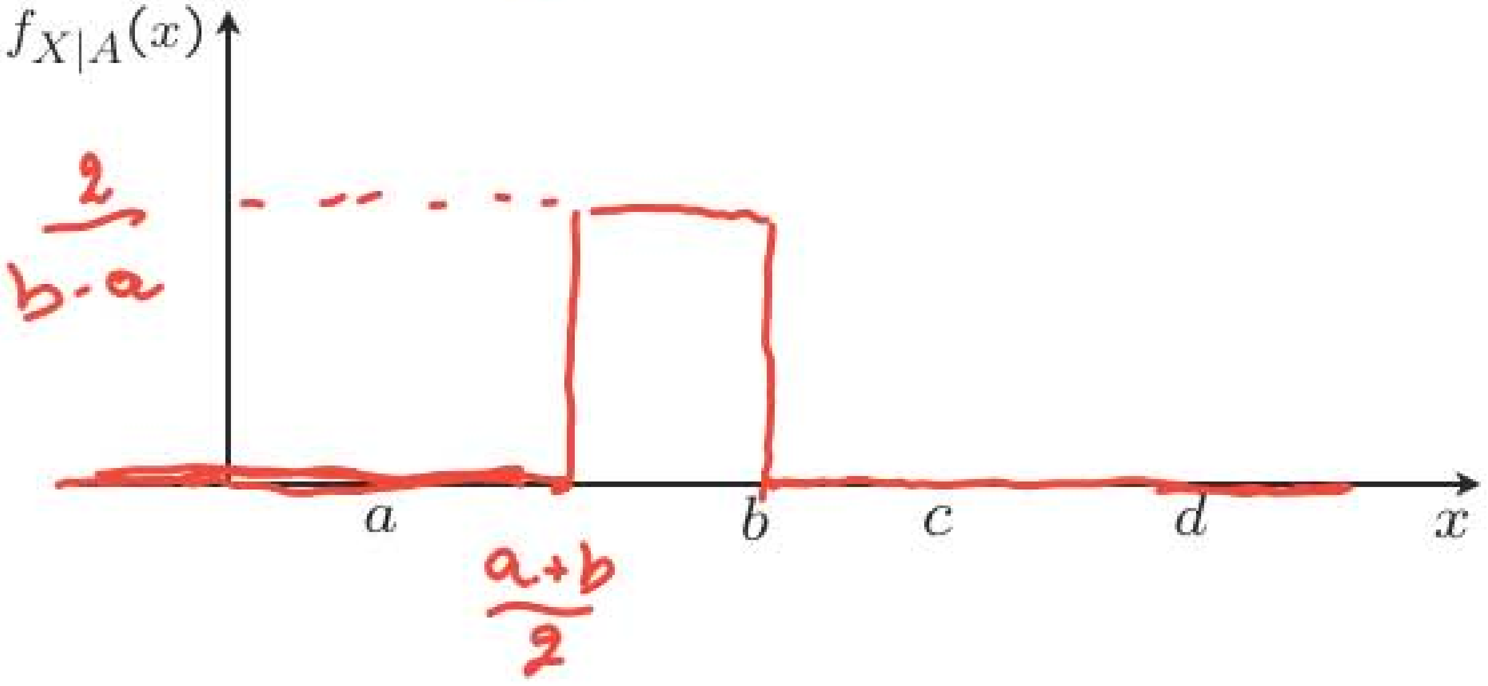
$$\mathbf{E}[g(X) \mid A] = \sum_x g(x) p_{X|A}(x)$$

$$\mathbf{E}[g(X) \mid A] = \int g(x) f_{X|A}(x)\, dx$$

Example

$$A: \quad \frac{a+b}{2} \le X \le b$$





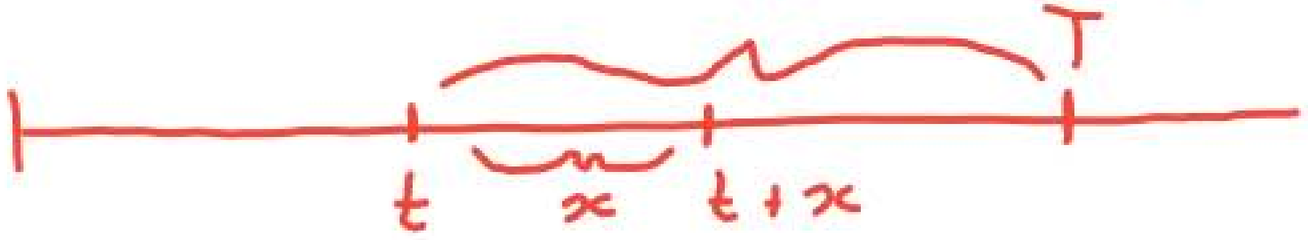
$$\mathbf{E}[X \mid A] = \frac{1}{2} \cdot \frac{a+b}{2} + \frac{1}{2} b$$

$$= \frac{1}{4} a + \frac{3}{4} b$$

$$\mathbf{E}[X^2 \mid A] = \int_{\frac{a+b}{2}}^{b} \frac{2}{b-a} \cdot x^2 \, dx$$

Memorylessness of the exponential PDF

- Do you prefer a used or a new "exponential" light bulb? **Probabilistically identical!**



- Bulb lifetime T: exponential(λ)

  $\mathbf{P}(T > x) = e^{-\lambda x}$, for $x \geq 0$

  – we are told that $T > t$

  – r.v. X: remaining lifetime $= T - t$

  $\mathbf{P}(X > x \,|\, T > t) = e^{-\lambda x}$, for $x \geq 0$

$$= \frac{P(T-t>x,\, T>t)}{P(T>t)} = \frac{P(T>t+x,\, T>t)}{P(T>t)} = \frac{P(T>t+x)}{P(T>t)}$$

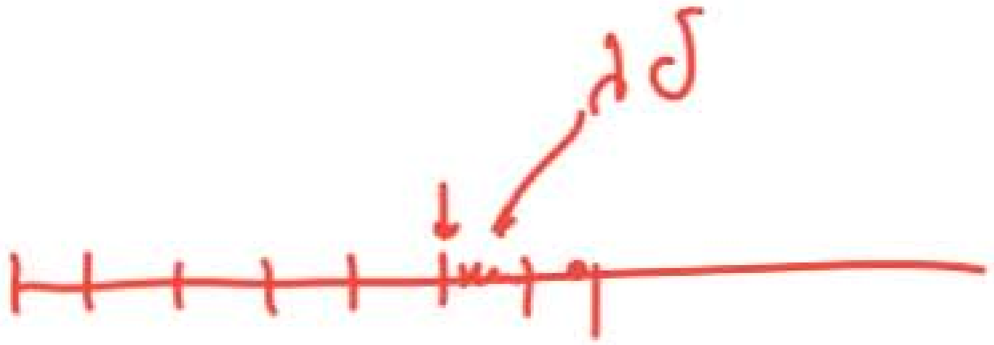
$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}$$

Memorylessness of the exponential PDF

$f_T(x) = \lambda e^{-\lambda x}$, for $x \geq 0$

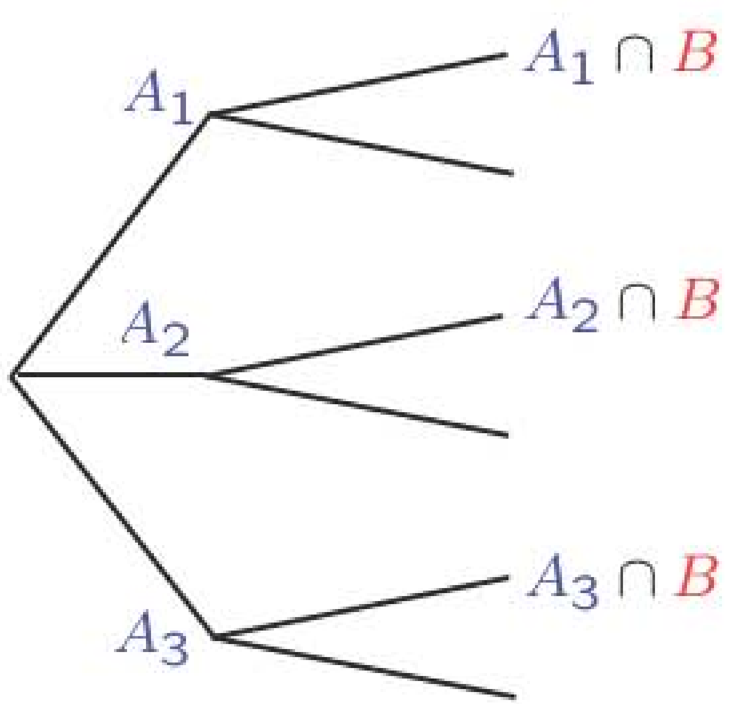
$\mathbf{P}(0 \leq T \leq \delta) \approx f_T(0) \cdot \delta = \lambda\delta$

$\mathbf{P}(t \leq T \leq t + \delta \mid T > t) = \approx \lambda\delta$



similar to an independent coin flip,
every δ time steps,
with $\mathbf{P}(\text{success}) \approx \lambda\delta$

Total probability and expectation theorems



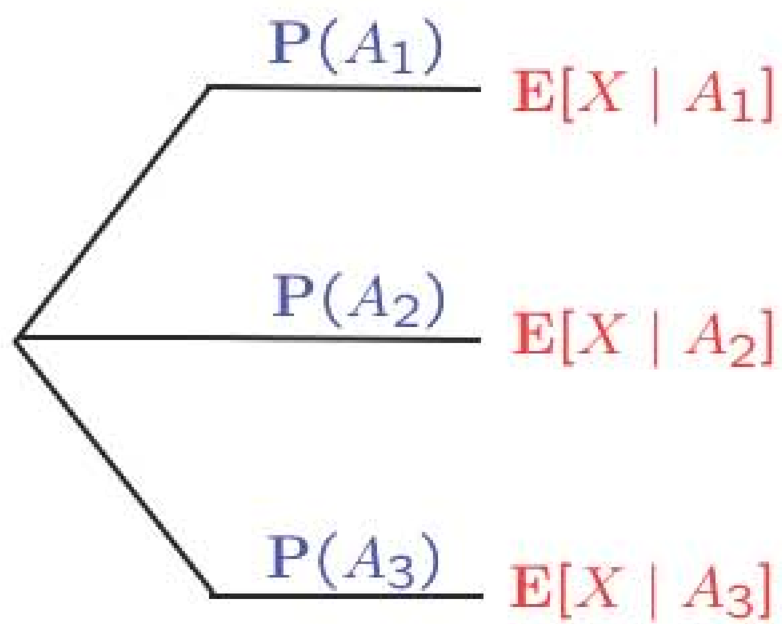
$$\mathbf{P}(B) = \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n)$$

$$p_X(x) = \mathbf{P}(A_1)\, p_{X|A_1}(x) + \cdots + \mathbf{P}(A_n)\, p_{X|A_n}(x)$$

$$F_X(x) = \mathbf{P}(X \le x) = \mathbf{P}(A_1)\,\mathbf{P}(X \le x \mid A_1) + \cdots$$

$$= \mathbf{P}(A_1)\, F_{X|A_1}(x) + \cdots$$

$$\boxed{f_X(x) = \mathbf{P}(A_1) f_{X|A_1}(x) + \cdots + \mathbf{P}(A_n) f_{X|A_n}(x)}$$



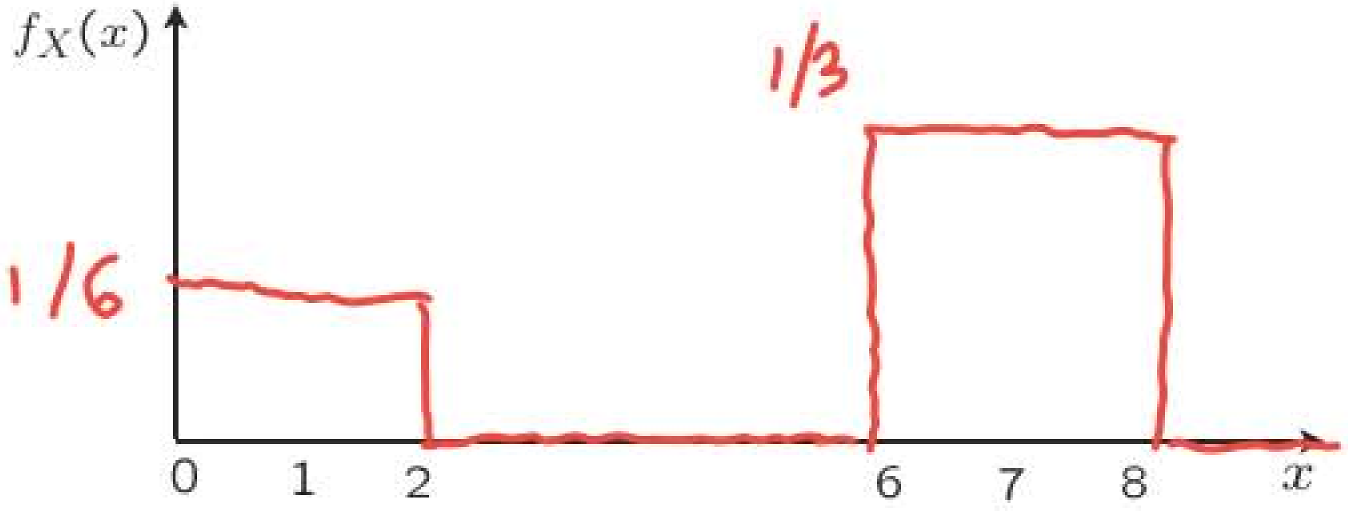
$$\int x f_X(x)\, dx = \mathbf{P}(A_1) \int x f_{X|A_1}(x)\, dx + \cdots$$

$$\boxed{\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X \mid A_n]}$$

Example

- Bill goes to the supermarket shortly, with probability 1/3, at a time uniformly distributed between 0 and 2 hours from now; or with probability 2/3, later in the day at a time uniformly distributed between 6 and 8 hours from now

$P(A_1) = \frac{1}{3}$ $\quad f_{X|A_1} \sim \text{unif}[0,2]$ $\quad P(A_2) = \frac{2}{3}$ $\quad f_{X|A_2} \sim U[6,8]$



$$f_X(x) = \mathbf{P}(A_1) f_{X|A_1}(x) + \cdots + \mathbf{P}(A_n) f_{X|A_n}(x)$$

- $\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X \mid A_n]$

$$\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 7$$

Mixed distributions

$$X = \begin{cases} \text{uniform on } [0,2], & \text{with probability } 1/2 \\ 1, & \text{with probability } 1/2 \end{cases}$$

Is X discrete? No

Is X continuous? No

$P(X=1) = 1/2$

Y discrete

Z continuous

$$X = \begin{cases} Y, & \text{with probability } p \\ Z, & \text{with probability } 1-p \end{cases}$$

X is mixed

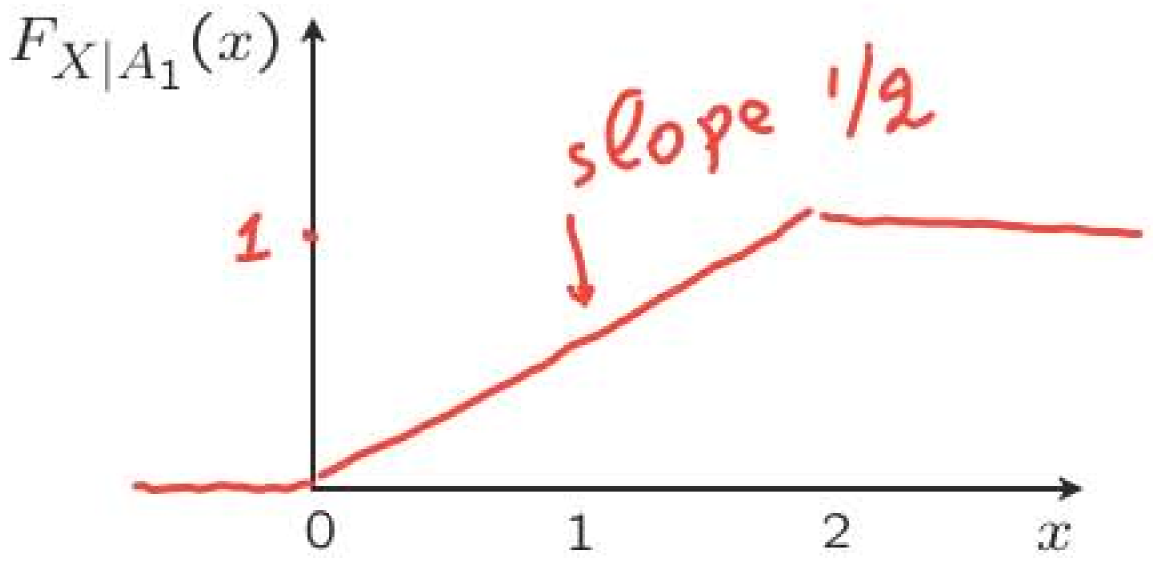
$$F_X(x) = p \cdot P(Y \le x) + (1-p)\, P(Z \le x)$$

$$= p\, F_Y(x) + (1-p)\, F_Z(x)$$

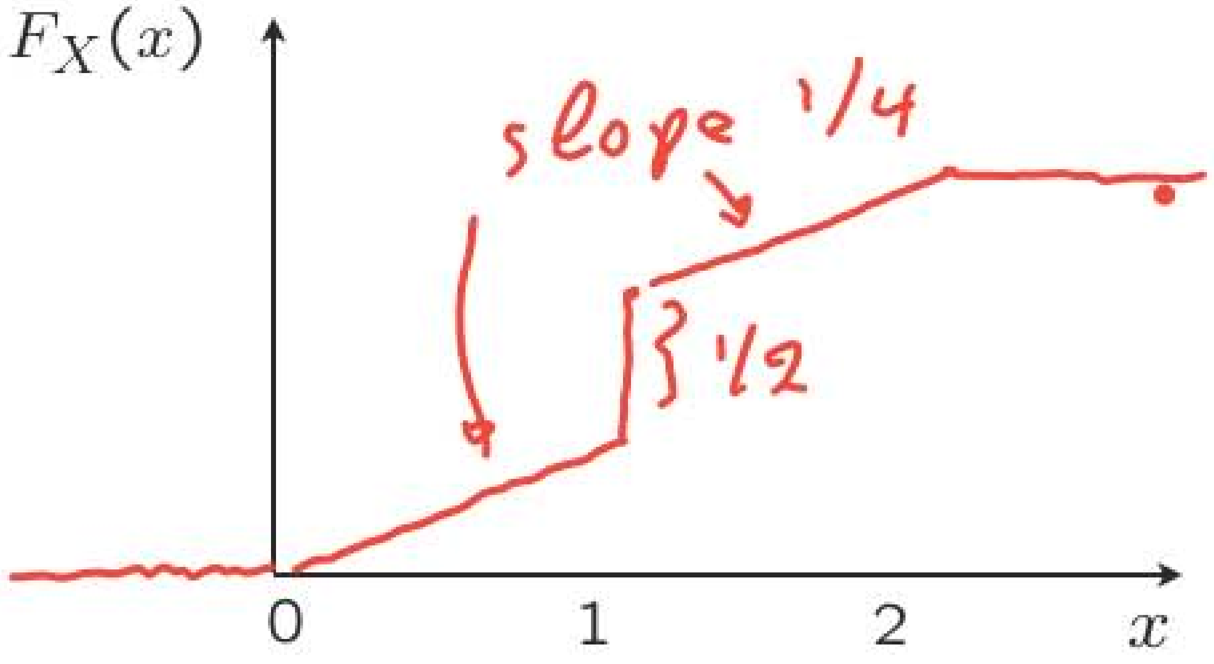
$$E[X] = p\, E[Y] + (1-p)\, E[Z]$$

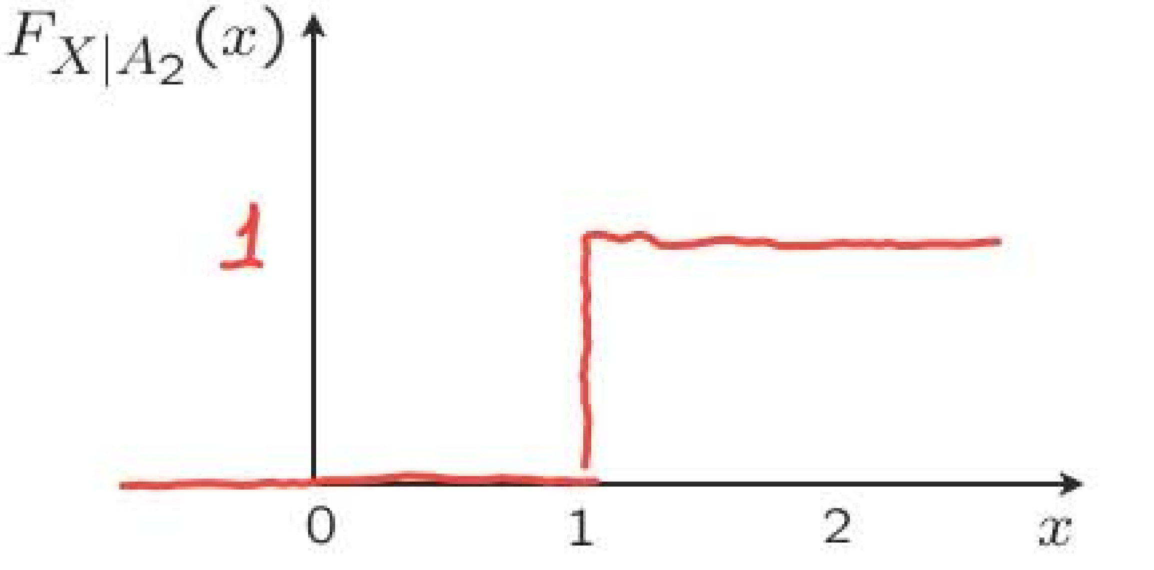
Mixed distributions

$$X = \begin{cases} \text{uniform on } [0,2], & \text{with probability } 1/2 \quad A_1 \\ 1, & \text{with probability } 1/2 \quad A_2 \end{cases}$$



$$F_X(x) = \mathbf{P}(A_1)F_{X|A_1}(x) + \mathbf{P}(A_2)F_{X|A_2}(x)$$





Jointly continuous r.v.'s and joint PDFs

$p_X(x)$	$f_X(x)$
$p_{X,Y}(x,y)$	$f_{X,Y}(x,y)$

$$p_{X,Y}(x,y) = \mathbf{P}(X = x \text{ and } Y = y) \geq 0$$

$$f_{X,Y}(x,y) \geq 0$$

$$\mathbf{P}\big((X,Y) \in B\big) = \sum_{(x,y)\in B}\sum p_{X,Y}(x,y)$$

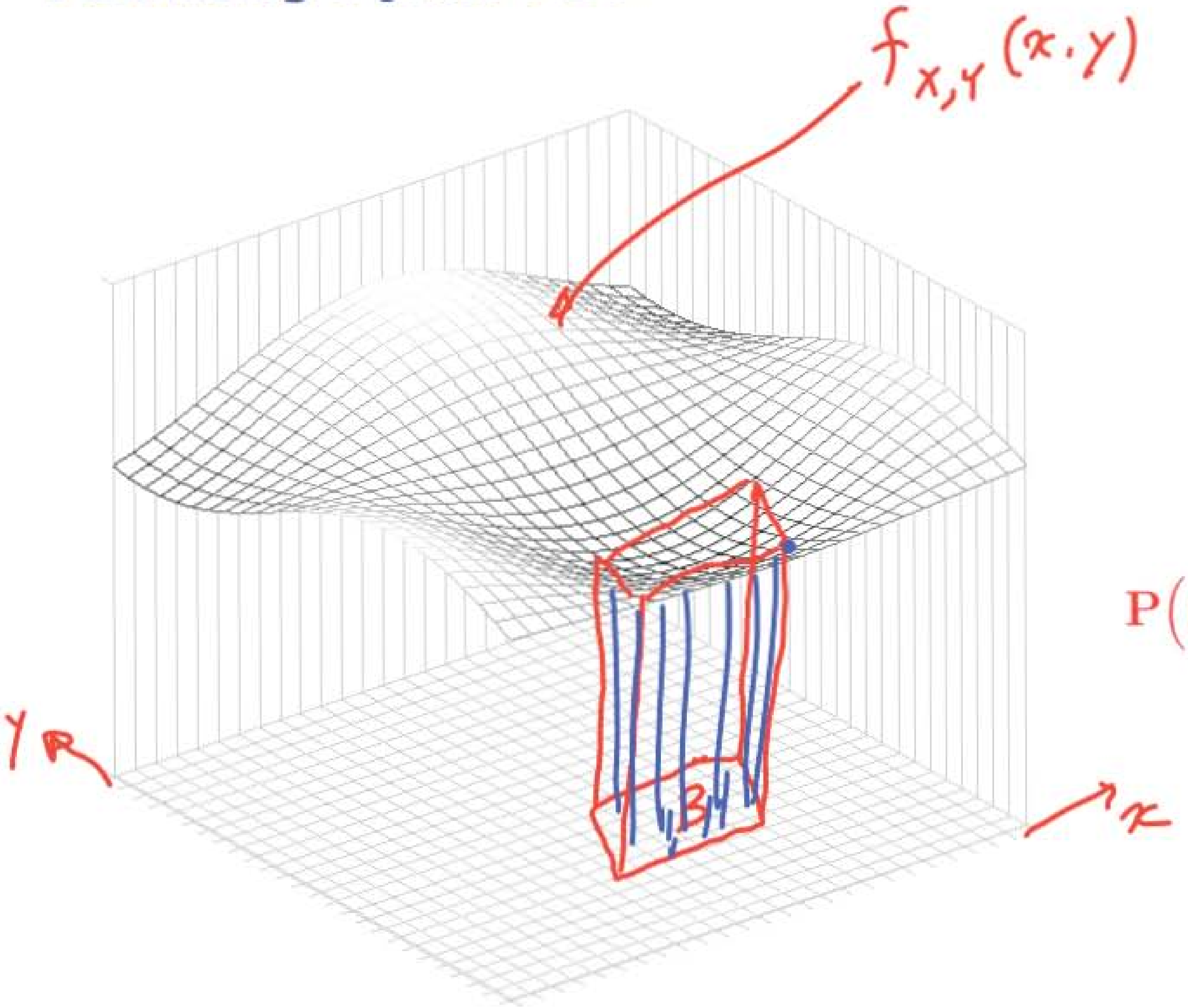
$$\mathbf{P}\big((X,Y) \in B\big) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

$$\sum_x \sum_y p_{X,Y}(x,y) = 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

Definition: Two random variables are **jointly continuous** if they can be described by a joint PDF

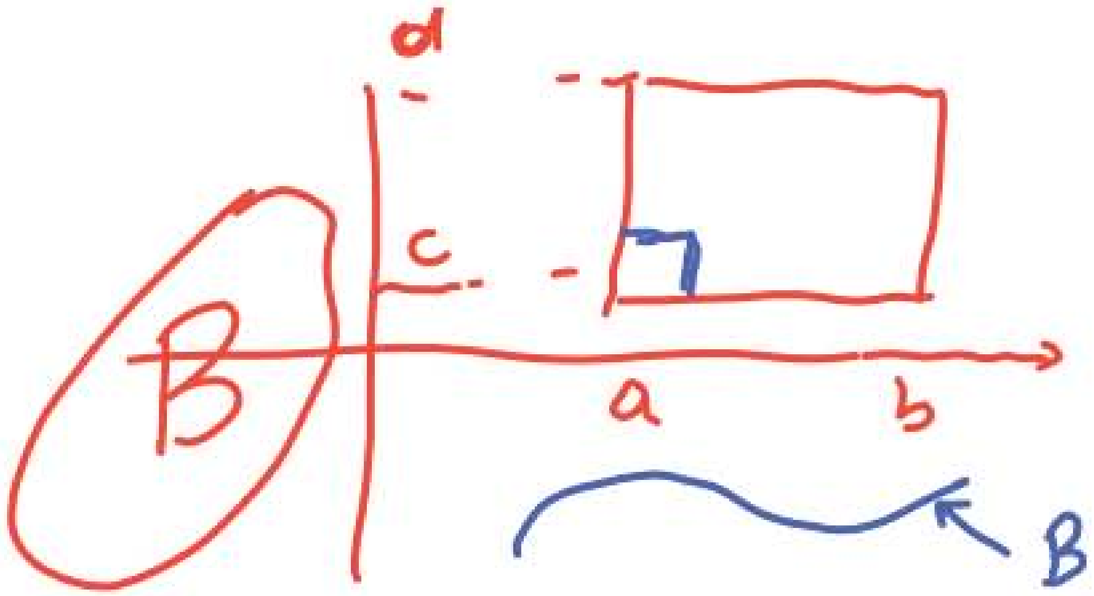
Visualizing a joint PDF



$$\mathbf{P}\big((X,Y) \in B\big) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

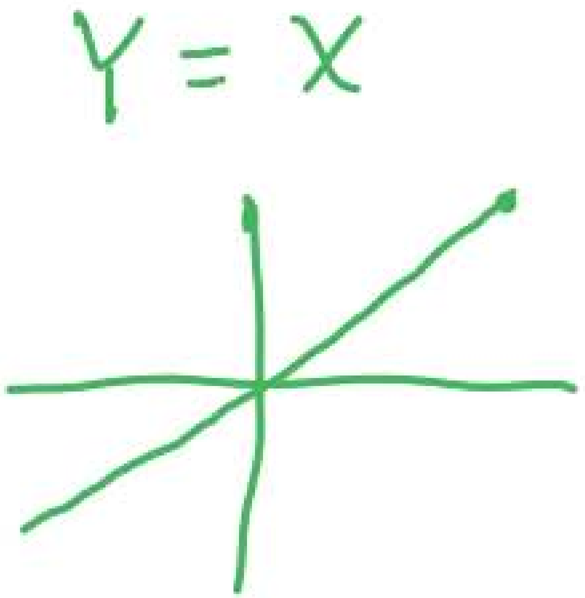
On joint PDFs

$$\mathbf{P}\big((X,Y) \in B\big) = \iint\limits_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$



$$\mathbf{P}(a \le X \le b,\, c \le Y \le d) = \int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy$$

$$\boxed{\mathbf{P}(a \le X \le a+\delta,\, c \le Y \le c+\delta) \approx f_{X,Y}(a,c)\cdot \delta^2}$$



$f_{X,Y}(x,y)$: probability per unit area

$$\text{area}(B) = 0 \quad \Rightarrow \quad \mathbf{P}\big((X,Y) \in B\big) = 0$$

From the joint to the marginals

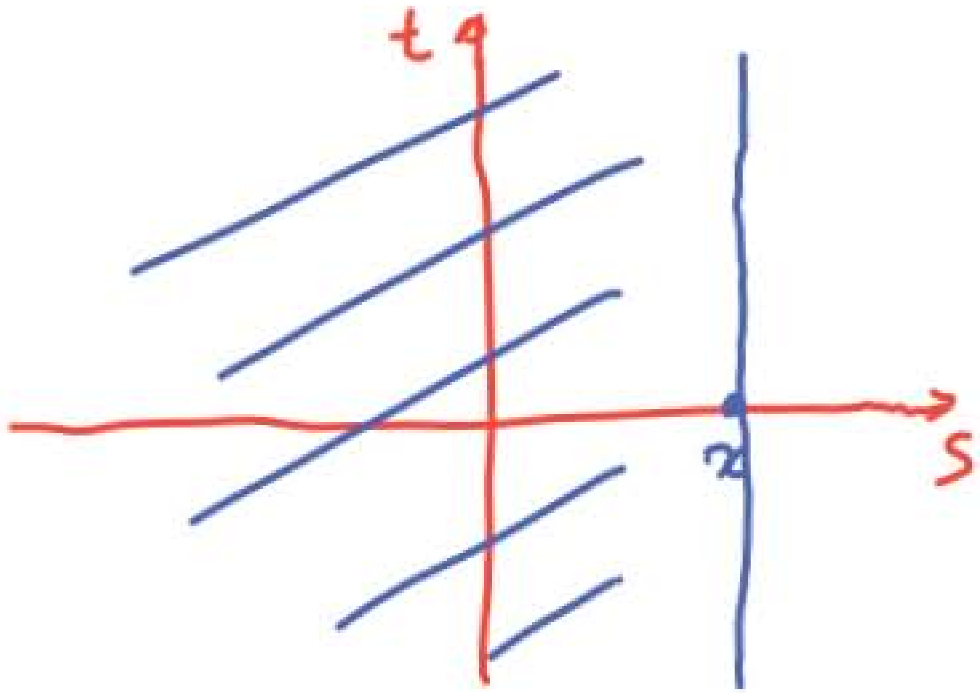
$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

$$f_X(x) = \int f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int f_{X,Y}(x,y)\,dx$$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} \Bigg[\underbrace{\int_{-\infty}^{\infty} f_{X,Y}(s,t)\,dt}_{} \Bigg] ds$$

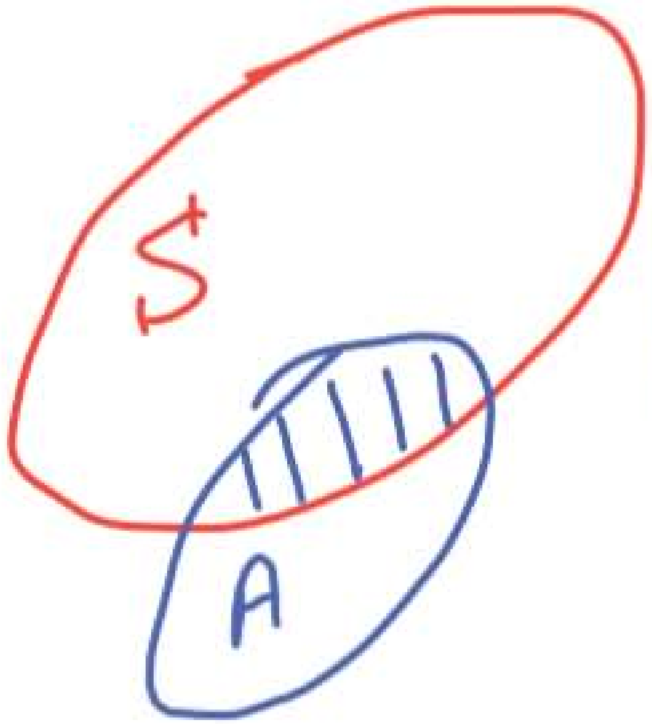


$$f_X(x) = \frac{dF_X}{dx}(x) = [\quad]$$

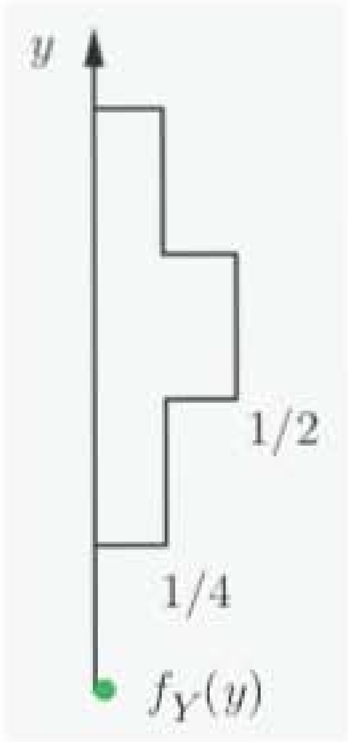
Uniform joint PDF on a set S

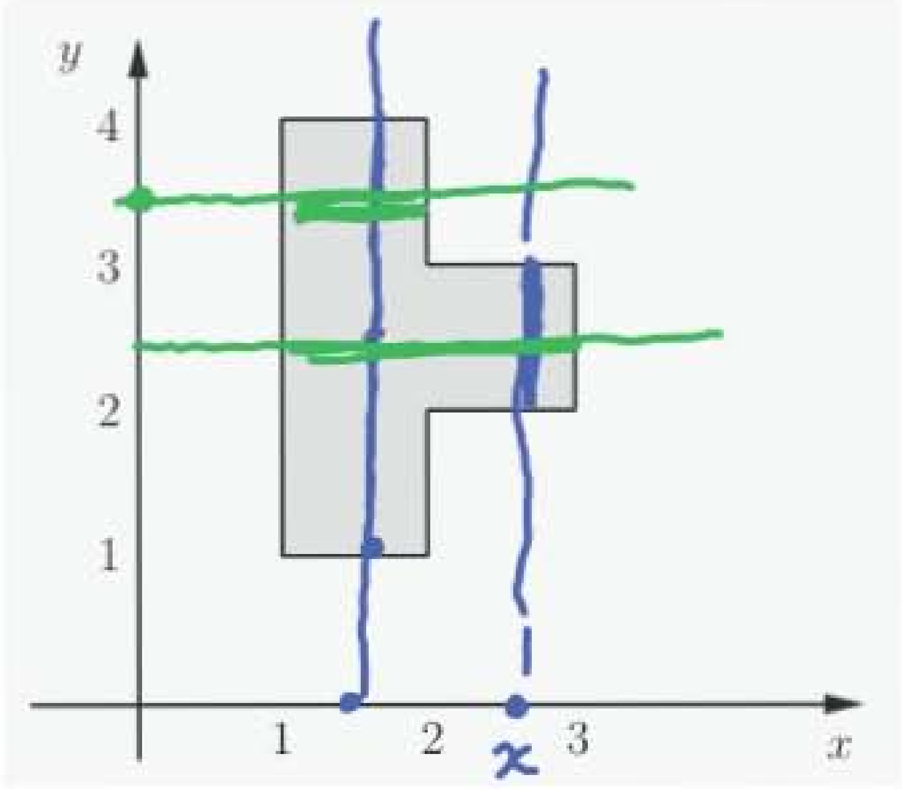
$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{area of } S}, & \text{if } (x,y) \in S, \\ 0, & \text{otherwise.} \end{cases}$$

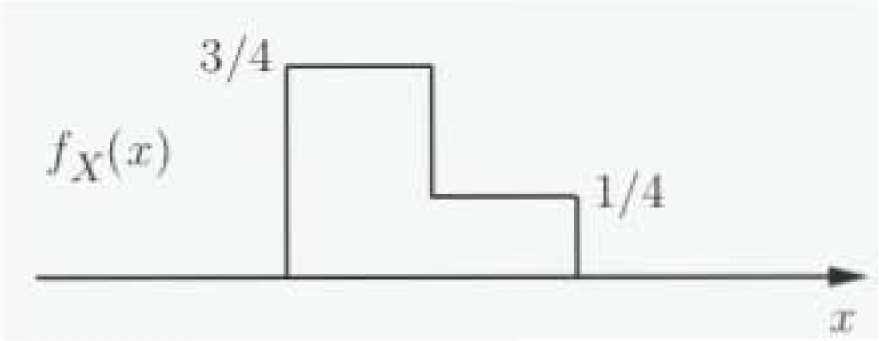
$$f_{X,Y} = \frac{1}{4}$$



$$P(A) = \frac{\text{area}(A \cap S)}{\text{area}(S)}$$







More than two random variables

$p_{X,Y,Z}(x, y, z)$

$f_{X,Y,Z}(x, y, z)$

$$\sum_x \sum_y \sum_z p_{X,Y,Z}(x, y, z) = 1$$

$$p_X(x) = \sum_y \sum_z p_{X,Y,Z}(x, y, z)$$

$$p_{X,Y}(x, y) = \sum_z p_{X,Y,Z}(x, y, z)$$

Functions of multiple random variables

$Z = g(X, Y)$

Expected value rule:

$$\mathbf{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

$$\mathbf{E}[g(X,Y)] = \int \int g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

Linearity of expectations

$$\mathbf{E}[aX + b] = a\mathbf{E}[X] + b$$

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

$$\boxed{\mathbf{E}[X_1 + \cdots + X_n] = \mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n]}$$

The joint CDF

$$F_X(x) = \mathbf{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt \qquad f_X(x) = \frac{dF_X}{dx}(x)$$

$$F_{X,Y}(x,y) = \mathbf{P}(X \le x, Y \le y) = \int_{-\infty}^{y}\left[\int_{-\infty}^{x} \underbrace{f_{X,Y}(s,t)}\,ds\right]dt$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x\,\partial y}(x,y)$$

$$F_{X,Y}(x,y) = xy$$

$$f_{X,Y}(x,y) = 1$$