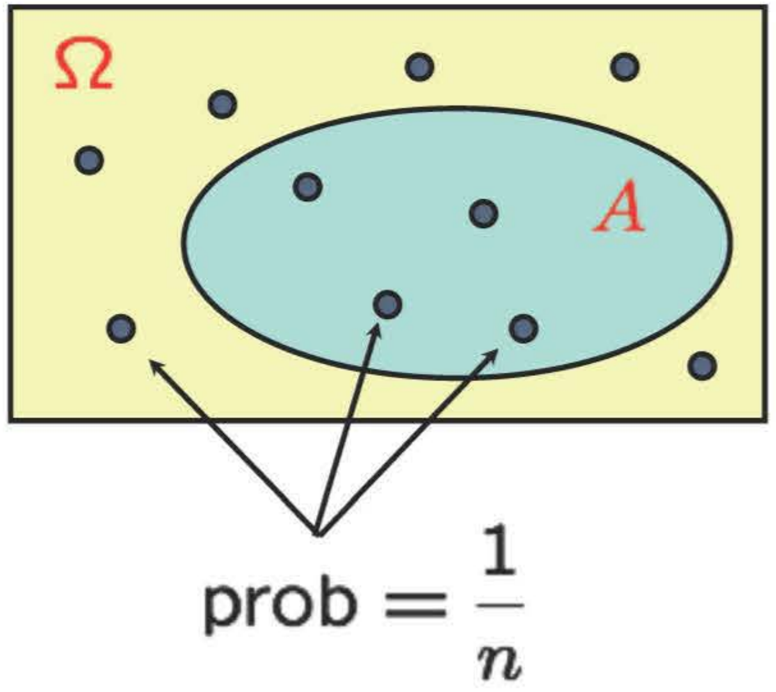
LECTURE 4: Counting



Discrete uniform law

– Assume Ω consists of n equally likely elements
– Assume A consists of k elements

Then : $$\mathbf{P}(A) = \frac{\text{number of elements of } A}{\text{number of elements of } \Omega} = \frac{k}{n}$$

- Basic counting principle

- Applications

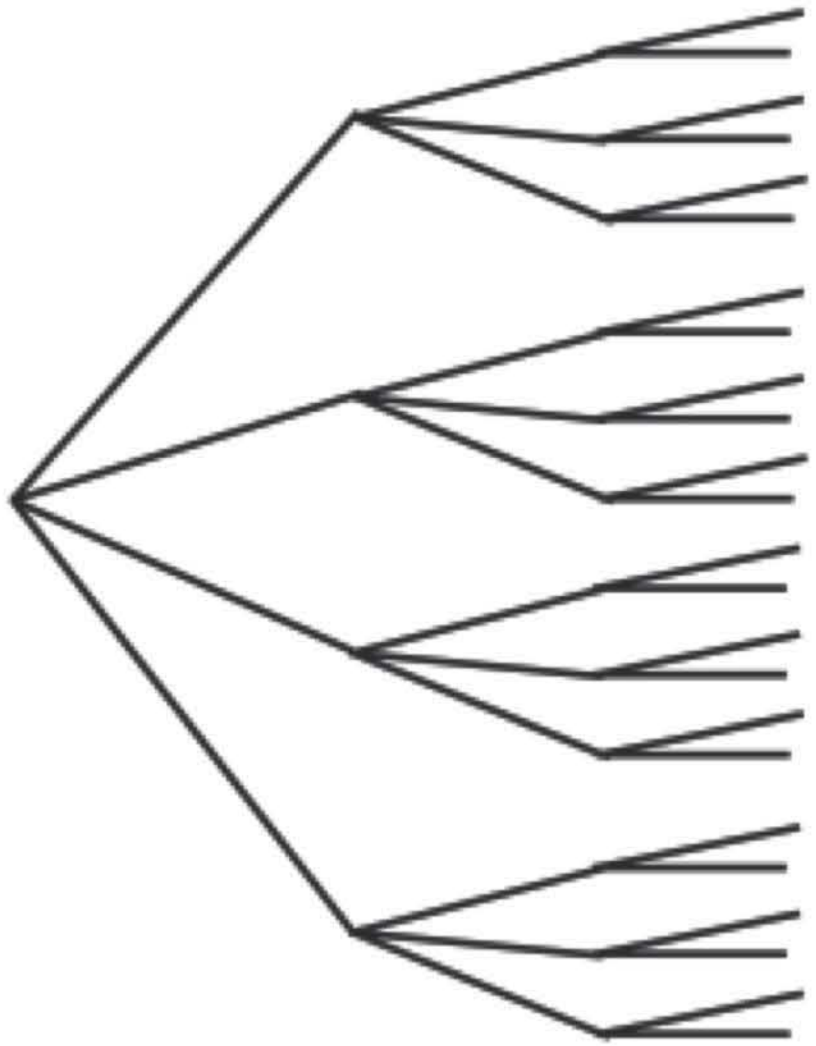
permutations
combinations
partitions

number of subsets
binomial probabilities

Basic counting principle

4 shirts
3 ties
2 jackets

Number of possible attires?



- r stages
- n_i choices at stage i

Number of choices is:

Basic counting principle examples

- Number of license plates with 2 letters followed by 3 digits:

  - ... if repetition is prohibited:

- **Permutations:** Number of ways of ordering n elements:

- Number of subsets of $\{1, \ldots, n\}$:

Example

- Find the probability that:
  six rolls of a (six-sided) die all give different numbers.

(Assume all outcomes equally likely.)

Combinations

- Definition: $\binom{n}{k}$: number of k-element subsets of a given n-element set $\boxed{= \dfrac{n!}{k!(n-k)!}}$

- Two ways of constructing an **ordered** sequence of k **distinct** items:
  - Choose the k items one at a time
  - Choose k items, then order them

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{n}{n} =$$

$$\binom{n}{0} =$$

$$\sum_{k=0}^{n} \binom{n}{k} =$$

Binomial coefficient $\binom{n}{k}$ $\longrightarrow$ Binomial probabilities

$$\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

- $n \geq 1$ independent coin tosses; $\mathbf{P}(H) = p$
- $\mathbf{P}(HTTHHH) =$
- $\mathbf{P}(\text{particular sequence}) =$
- $\mathbf{P}(\text{particular } k\text{–head sequence})$

$\mathbf{P}(k \text{ heads}) =$

A coin tossing problem

- Given that there were 3 heads in 10 tosses, what is the probability that the first two tosses were heads?
  - event A: the first 2 tosses were heads
  - event B: 3 out of 10 tosses were heads

Assumptions:

- independence
- $\mathbf{P}(H) = p$

$$\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

- First solution:

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} =$$

A coin tossing problem

- Given that there were 3 heads in 10 tosses,
  what is the probability that the first two tosses were heads?
  - event A: the first 2 tosses were heads
  - event B: 3 out of 10 tosses were heads

Assumptions:

- independence
- $\mathbf{P}(H) = p$

$$\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

- Second solution: Conditional probability law (on B) is uniform

Partitions

- $n \geq 1$ distinct items; $r \geq 1$ persons
  give n_i items to person i

  - here $n_1, \ldots, n_r$ are given nonnegative integers
  - with $n_1 + \cdots + n_r = n$

- Ordering n items:
  - Deal n_i to each person i, and then order

$$\text{number of partitions} = \frac{n!}{n_1!\, n_2! \cdots n_r!} \qquad \text{(multinomial coefficient)}$$

Example: 52-card deck, dealt (fairly) to four players.
Find $\mathbf{P}$(each player gets an ace)

- Outcomes are:
  - number of outcomes:

- Constructing an outcome with one ace for each person:
  - distribute the aces
  - distribute the remaining 48 cards

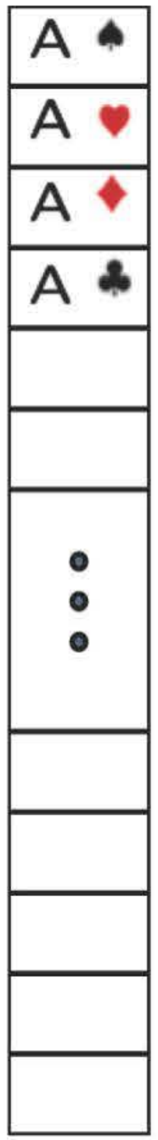
- Answer:

$$\frac{4 \cdot 3 \cdot 2 \cdot \dfrac{48!}{12!\,12!\,12!\,12!}}{\dfrac{52!}{13!\,13!\,13!\,13!}}$$

Example: 52-card deck, dealt (fairly) to four players. Find $\mathbf{P}$(each player gets an ace)

A smart solution

Stack the deck, aces on top



Deal, one at a time, to available "slots"