LECTURE 1: Probability models and axioms

- Sample space
- Probability laws
  - Axioms
  - Properties that follow from the axioms
- Examples
  - Discrete
  - Continuous
- Discussion
  - Countable additivity
  - Mathematical subtleties
- Interpretations of probabilities

Sample space

- Two steps:
  - Describe possible outcomes
  - Describe beliefs about likelihood of outcomes

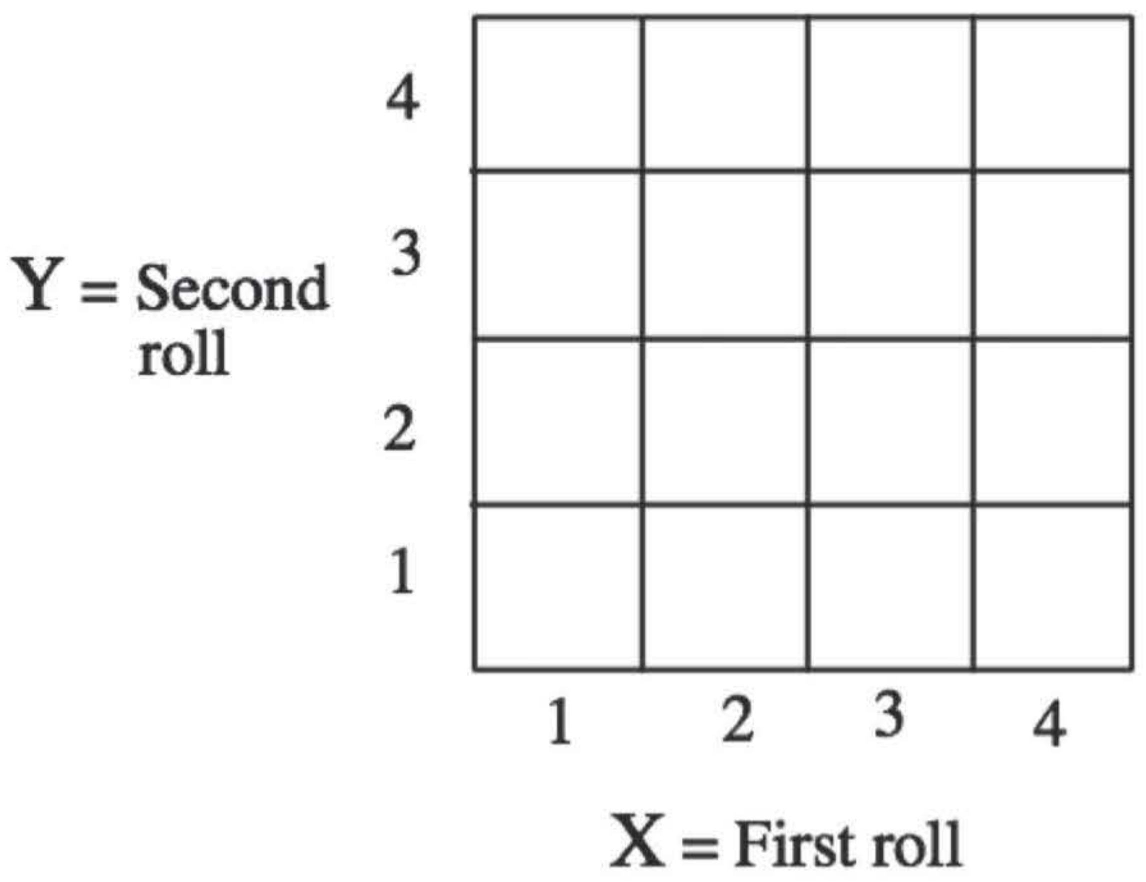
Sample space

- List (set) of possible outcomes, Ω

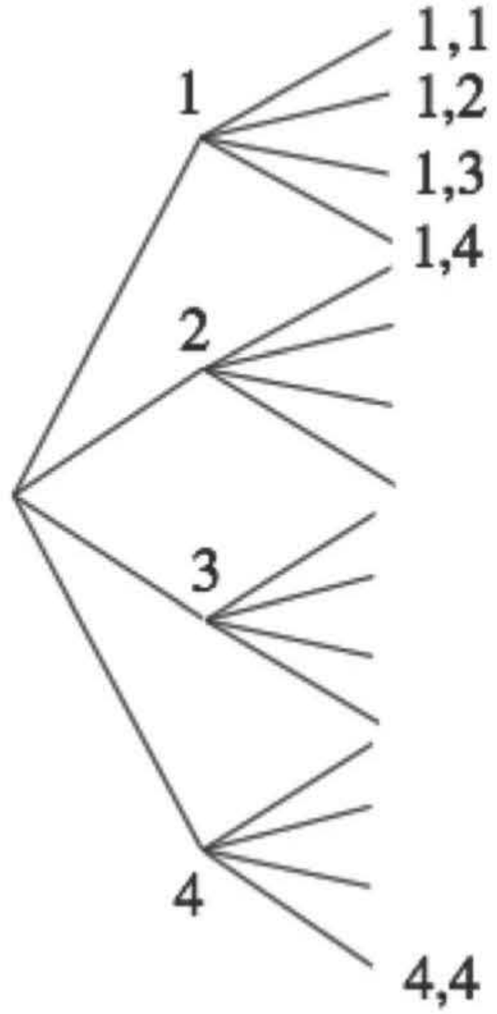
- List must be:
  - Mutually exclusive
  - Collectively exhaustive
  - At the "right" granularity

Sample space: discrete/finite example

- Two rolls of a tetrahedral die

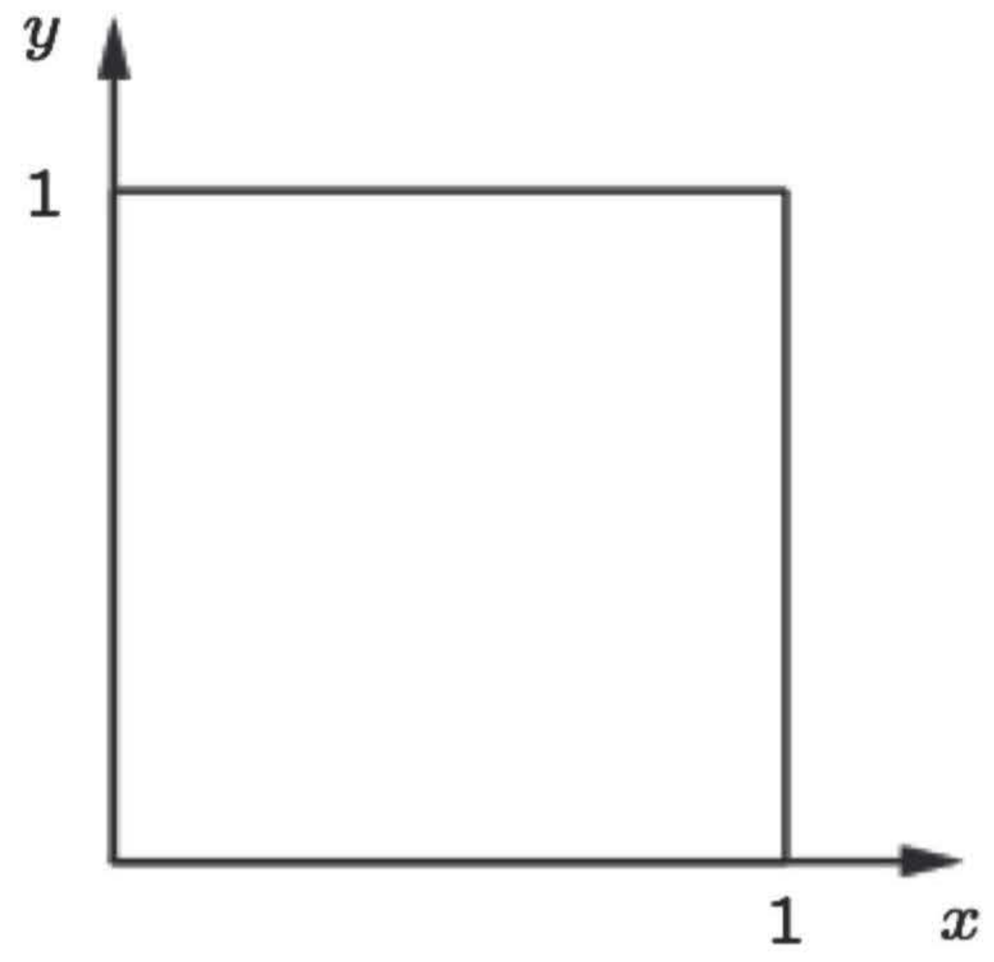


sequential description



Sample space: continuous example

- (x, y) such that $0 \le x, y \le 1$



Probability axioms

- **Event:** a subset of the sample space
  - Probability is assigned to events

- **Axioms:**
  - Nonnegativity: $\mathbf{P}(A) \geq 0$
  - Normalization: $\mathbf{P}(\Omega) = 1$
  - (Finite) additivity: (to be strengthened later)
    If $A \cap B = \varnothing$, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$

Some simple consequences of the axioms

Axioms

$\mathbf{P}(A) \geq 0$

$\mathbf{P}(\Omega) = 1$

For disjoint events:

$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$

Consequences

$\mathbf{P}(A) \leq 1$

$\mathbf{P}(\varnothing) = 0$

$\mathbf{P}(A) + \mathbf{P}(A^c) = 1$

$\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C)$

and similarly for k disjoint events

$$\mathbf{P}(\{s_1, s_2, \ldots, s_k\}) = \mathbf{P}(\{s_1\}) + \cdots + \mathbf{P}(\{s_k\})$$

$$= \mathbf{P}(s_1) + \cdots + \mathbf{P}(s_k)$$

Some simple consequences of the axioms

Axioms

$\mathbf{P}(A) \geq 0$

$\mathbf{P}(\Omega) = 1$

For disjoint events:

$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$

Some simple consequences of the axioms

- A, B, C disjoint: $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C)$

- $\mathbf{P}(\{s_1, s_2, \ldots, s_k\}) =$

More consequences of the axioms

- If $A \subset B$, then $\mathbf{P}(A) \leq \mathbf{P}(B)$

- $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$

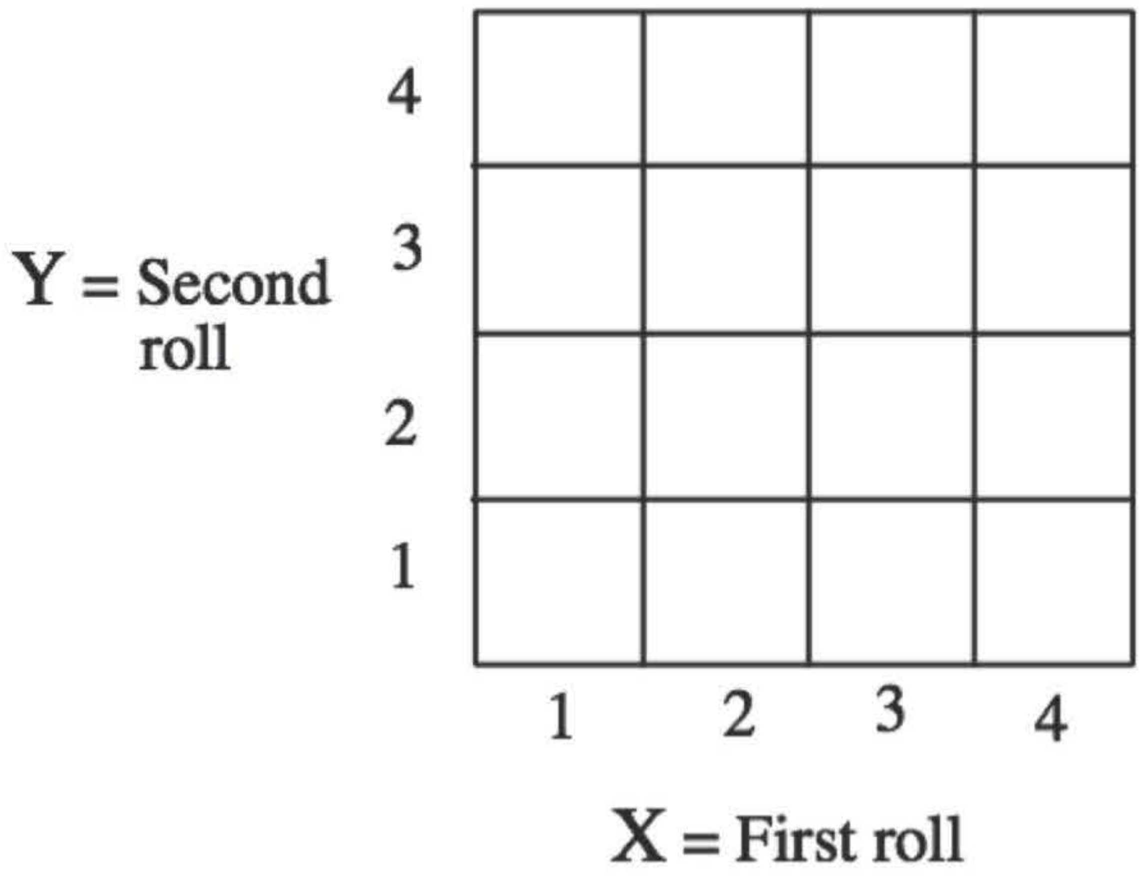
- $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$

More consequences of the axioms

- $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B) + \mathbf{P}(A^c \cap B^c \cap C)$

Probability calculation: discrete/finite example

- Two rolls of a tetrahedral die
- Let every possible outcome have probability 1/16



- $\mathrm{P}(X = 1) =$

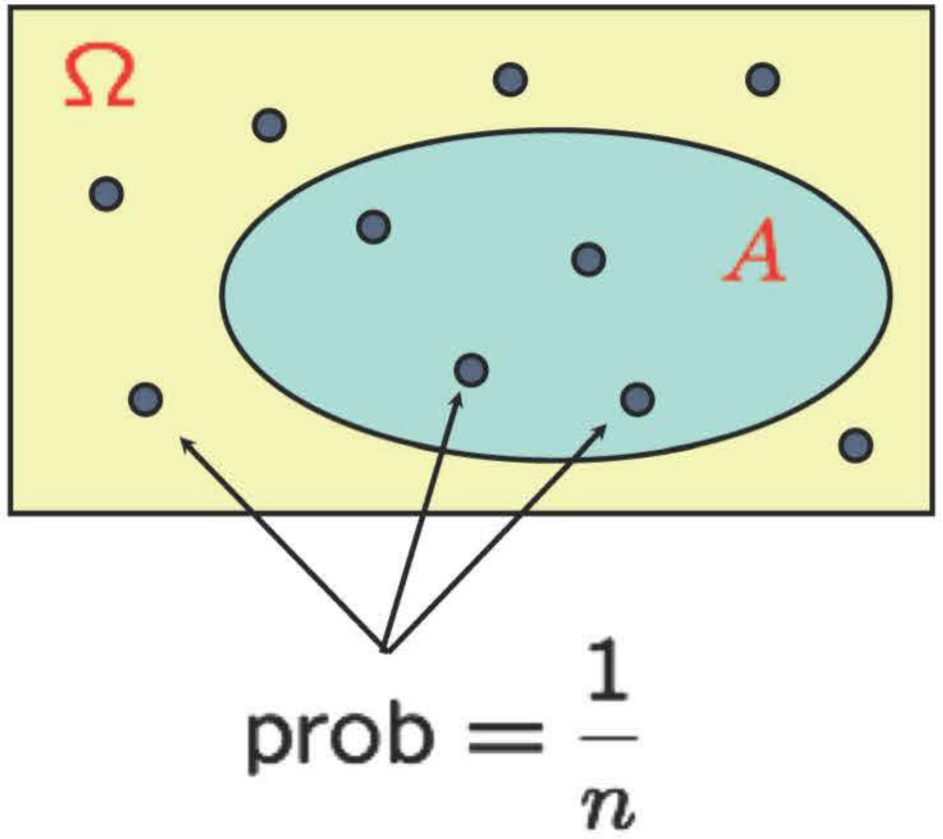
Let $Z = \min(X, Y)$

- $\mathrm{P}(Z = 4) =$
- $\mathrm{P}(Z = 2) =$

Discrete uniform law

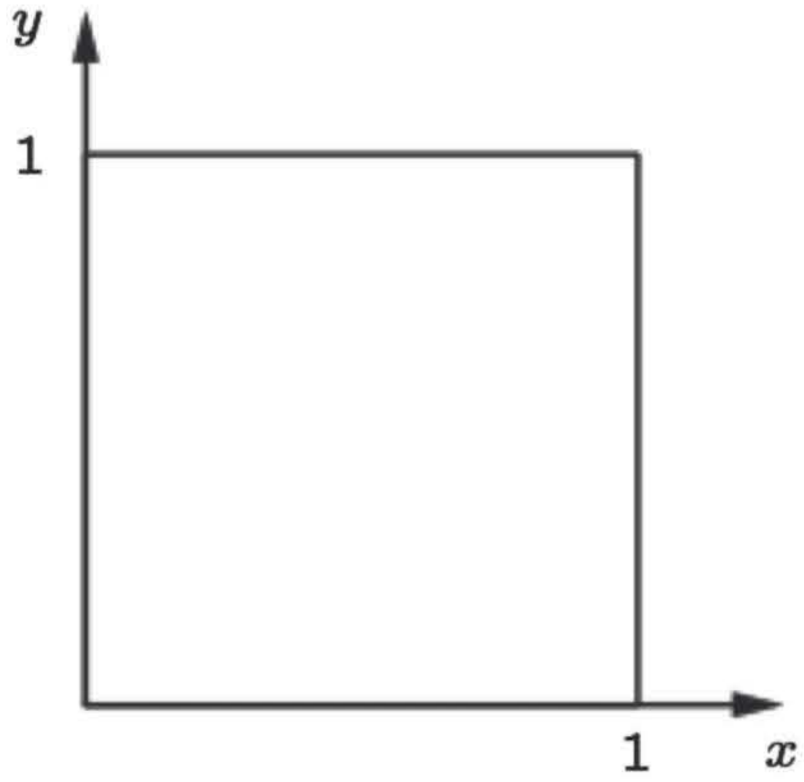
- Assume Ω consists of n equally likely elements
- Assume A consists of k elements

$$\mathbf{P}(A) =$$



Probability calculation: continuous example

- (x, y) such that $0 \leq x, y \leq 1$



- **Uniform** probability law: Probability = Area

$$\mathbf{P}\Big(\{(x,y) \mid x + y \leq 1/2\}\Big) =$$

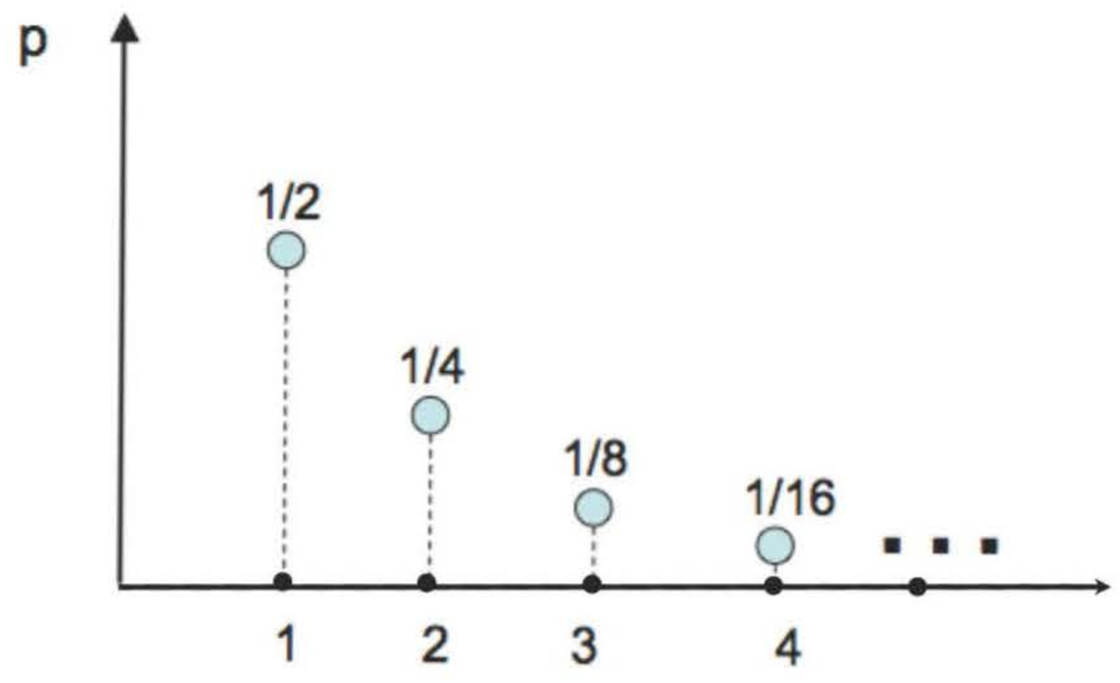
$$\mathbf{P}\Big(\{(0.5, 0.3)\}\Big) =$$

Probability calculation steps

- Specify the sample space
- Specify a probability law
- Identify an event of interest
- Calculate...

Probability calculation: discrete but infinite sample space

- Sample space: $\{1, 2, \ldots\}$
  - We are given $\mathbf{P}(n) = \frac{1}{2^n}$, $n = 1, 2, \ldots$



- $\mathbf{P}(\text{outcome is even}) =$

Countable additivity axiom

- Strengthens the finite additivity axiom

Countable Additivity Axiom:
If A_1, A_2, A_3,... is an infinite **sequence** of disjoint events,
then $\mathbf{P}(A_1 \cup A_2 \cup A_3 \cup \cdots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3) + \cdots$

Mathematical subtleties

Countable Additivity Axiom:
If A_1, A_2, A_3,... is an infinite **sequence** of disjoint events,
then $\mathbf{P}(A_1 \cup A_2 \cup A_3 \cup \cdots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3) + \cdots$

- Additivity holds only for "countable" sequences of events
- The unit square (similarly, the real line, etc.) is **not countable** (its elements cannot be arranged in a sequence)
- "Area" is a legitimate probability law on the unit square, as long as we do not try to assign probabilities/areas to "very strange" sets

Interpretations of probability theory

- A narrow view: a branch of math
  - Axioms $\Rightarrow$ theorems **"Thm:"** "Frequency" of event A "is" $\mathbf{P}(A)$

- Are probabilities frequencies?
  - $\mathbf{P}$(coin toss yields heads) $= 1/2$
  - $\mathbf{P}$(the president of ... will be reelected) $= 0.7$

- Probabilities are often intepreted as:
  - Description of beliefs
  - Betting preferences

The role of probability theory

- A framework for analyzing phenomena with uncertain outcomes
  - Rules for consistent reasoning
  - Used for predictions and decisions