THE Z-TRANSFORM

1. Lecture 5 - 51 minutes

a.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

$$|X(e^{j\omega})| = \left|\sum_{-\infty}^{+\infty} x(n) e^{-j\omega n}\right|$$

$$\leq \sum_{-\infty}^{+\infty} |x(n)| |e^{-j\omega n}|$$

$X(e^{j\omega})$ converges if

$$\sum_{-\infty}^{+\infty} |x(n)| < \infty$$

stable system

$H(e^{j\omega})$ converges

Example

① $x(n) = (\frac{1}{2})^n u(n)$

$$\sum_{-\infty}^{+\infty} |x(n)| = 2$$

② $x(n) = (2)^n u(n)$

$$\sum_{-\infty}^{+\infty} |x(n)| = \infty$$

$$X_r(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} [x(n) r^{-n}] e^{-j\omega n}$$

$$= \sum_{-\infty}^{+\infty} x(n) \underbrace{(re^{j\omega})^{-n}}_{z}$$

The Z-Transform

$$z = r e^{j\omega}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

$$X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}}$$

converges if

$$\sum_{n=-\infty}^{+\infty} |x(n) r^{-n}| < \infty$$

b.

Example

$x(n) = (\frac{1}{2})^n u(n)$

$$X(z) \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} \sum_{n=0}^{\infty} (\frac{1}{2} z^{-1})^n$$

$$\frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$$

$$\sum_{0}^{\infty} |(\frac{1}{2} z^{-1})^n| < \infty \Rightarrow |z| > \frac{1}{2}$$

unit circle $|z|=1$

z-plane

Example

$x(n) = -(\frac{1}{2})^n u(-n-1)$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$$

$$\sum_{-\infty}^{-1} |(\frac{1}{2} z^{-1})^n| < \infty \Rightarrow |z| < \frac{1}{2}$$

unit circle

(1) Region of Convergence bounded by poles or (0/∞)

(2) Finite Length Sequences

$$0 < |z| < \infty$$

(3) Right-sided $x(n)=0$ $n<n_1$

$$R_{x-} < |z| < \infty$$

(4) Left-sided $x(n)=0$ $n>n_1$

$$0 < |z| < R_{x+}$$

(5) Two-sided

$$R_{x-} < |z| < R_{x+}$$

c.

unit circle, z-plane, a, b

$$y(n) = x(n) * h(n)$$

$$Y(z) = X(z) \, H(z)$$

Region of Convergence	Which Sided	Fourier Transform?
$|z|<a$	left	no
$a<|z|<b$	two	yes
$b<|z|$	right	no

$H(z) \triangleq$ System Function

Stable ⟷ unit circle in R. of C.

causal ⟹ h(n) right-sided

⟹ R. of C. outside outermost pole

Example

$$y(n) - \frac{1}{2} y(n-1) = x(n)$$

useful property:

$$y(n) \leftrightarrow Y(z)$$

$$y(n+n_0) \leftrightarrow z^{n_0} Y(z)$$

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z)$$

$$H(z) = Y(z)/X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

z-plane
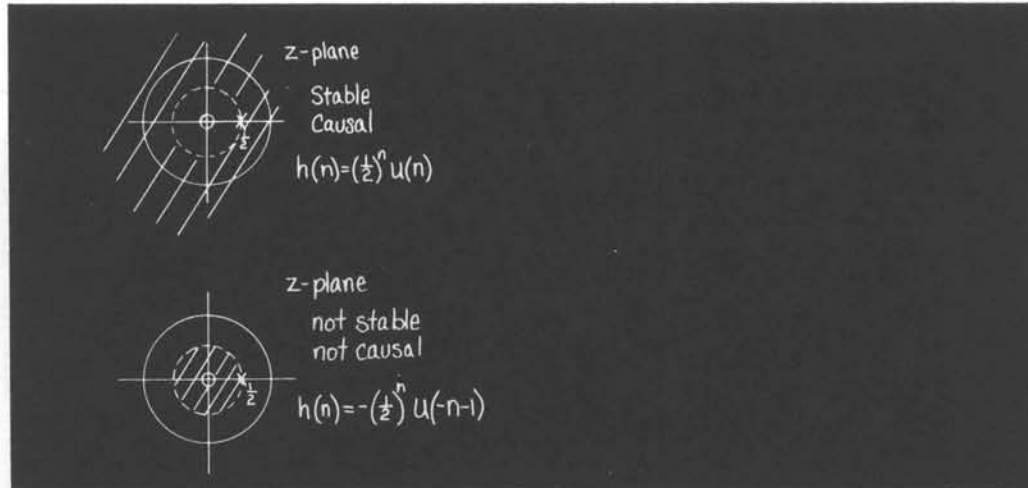Stable
Causal

$h(n) = (\tfrac{1}{2})^n u(n)$

z-plane
not stable
not causal

$h(n) = -(\tfrac{1}{2})^n u(-n-1)$

d.

2. Comments

The Fourier transform does not converge for all sequences. Specifically, depending on the definition of convergence used, the sequence is required to be either absolutely summable or have finite energy. It is useful to note that for an LSI system the condition for stability corresponds to the condition for convergence of the Fourier transform of the unit-sample response (i.e. the frequency response). The z-transform represents a generalization of the Fourier transform to include sequences for which the Fourier transform doesn't converge; and can be interpreted as the Fourier transform of the sequence modified by multiplication with a complex exponential.

Associated with the z-transform is a region of convergence, i.e. a set of values of z for which the transform converges. As illustrated in the lecture two different sequences can have z-transforms of the same functional form and thus for which the z-transforms are distinguished only by their regions of convergence. This emphasizes the importance of the region of convergence in specifying the z-transform. Based on the properties of the region of convergence it is often possible to indicate implicitly the region of convergence for the z-transform. For example an indication that a sequence is right-sided implies that the region of convergence lies outside the outermost pole in the z-plane. Corresponding statements can be made about left-sided and two-sided sequences. Also, for a "stable" sequence, the region of convergence includes the unit circle in the z-plane. Thus if it is indicated that a z-transform corresponds to a stable sequence then its region of convergence is implied.

3. Reading

Text: Read sections 4.0 (page 149) through 4.2.

4. Problems

Problem 5.1

Determine which of the following sequences have Fourier transforms which converge:

(i) $x(n) = 2^n u(n)$

(ii) $x(n) = 2^n u(-n)$

(iii) $x(n) = (\frac{1}{2})^n u(n)$

(iv) $x(n) = u(n)$

Problem 5.2

(a) For each of the following sequences sketch the pole-zero pattern of their z-transforms. Include an indication of the region of convergence.

(i) $\delta(n) + (\frac{1}{2})^n u(n)$

(ii) $(\frac{1}{3})^{-n} u(n)$

(iii) $(\frac{1}{2})^n u(n) + (\frac{1}{3})^n u(n)$

(iv) $\delta(n) - \frac{1}{8} \delta(n - 3)$

(v) $(\frac{1}{2})^{|n|}$

(b) From your results in part (a) determine which of those sequences could correspond to the unit sample response of a stable system.

Problem 5.3

Consider a sequence x(n) with z-transform X(z). The pole zero
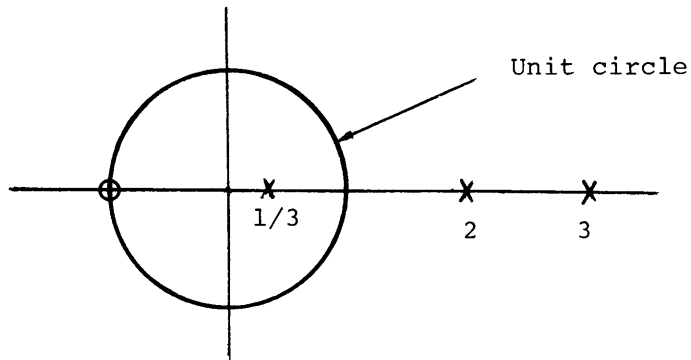pattern for X(z) is shown below:

Figure P5.3-1

(a) Determine the region of convergence of X(z) if it is known that
the Fourier transform of the sequence converges. For this case
determine also whether the sequence is right-sided, left-sided or
two-sided.

(b) If it is not known whether the Fourier transform of x(n) converges
but the sequence is known to be two-sided how many different
possible sequences could the pole-zero pattern of figure P5.3-1
correspond to? For each of these possibilities indicate the associated
region of convergence.

Problem 5.4

Let X(z) denote the z-transform of x(n). Determine in terms of
X(z) the z-transform of $x(n + n_0)$. If you have difficulty deciding
how to proceed you may find it helpful to first consider an example
such as $x(n) = (\frac{1}{2})^n u(n)$ and $n_0 = -1$.

Problem 5.5*

Determine the z-transform of each of the following. Include with your
answer the region of convergence in the z-plane and a sketch of the
pole-zero pattern. Express all sums in closed form. α can be complex.

(a) $x(n) = \alpha^{|n|}$, $0 < |\alpha| < 1$

(b) $x(n) = Ar^n \cos(\omega_0 n + \phi) u(n)$, $0 < r < 1$.

(c) $\quad x(n) = \begin{cases} 1, & 0 \le n < N - 1, \\ 0, & N \le n, \\ 0, & n < 0. \end{cases}$

(d) $\quad x(n) = \begin{cases} n, & 0 \le n \le N, \\ 2N - n, & N + 1 \le n \le 2N, \\ 0, & 2N \le n, \\ 0, & 0 > n. \end{cases}$

[Hint (easy way): First express x(n) in terms of the x(n) in part (c).]

Problem 5.6*

For a sequence x(n) which is zero for n < 0, show that $\lim\limits_{z \to \infty} X(z) = x(0)$. What is the corresponding theorem if the sequence is zero for n > 0?

Problem 5.7*

Let x(n) denote a causal sequence; i.e., x(n) = 0, n < 0. Furthermore, assume that x(0) ≠ 0.

(a) Show that there are no poles or zeros of X(z) at z = ∞.

(b) Show that the number of poles in the finite z-plane equals the number of zeros in the finite z-plane. (The finite z-plane excludes z = ∞.)