4 Convolution

Recommended Problems

P4.1

This problem is a simple example of the use of superposition. Suppose that a discrete-time linear system has outputs $y[n]$ for the given inputs $x[n]$ as shown in Figure P4.1-1.

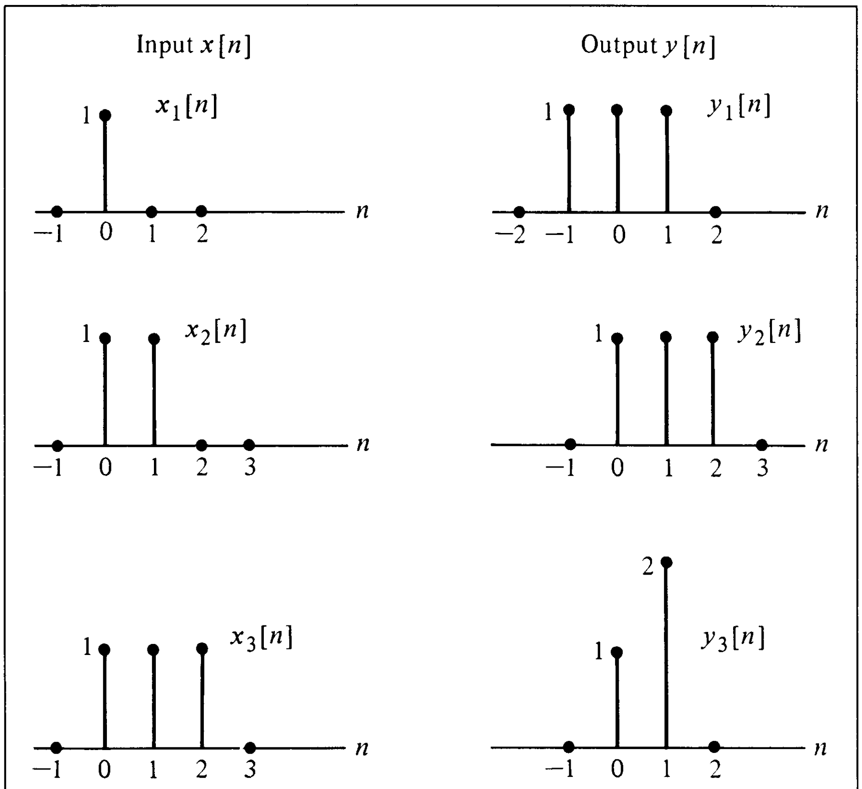

Figure P4.1-1

Determine the response $y_4[n]$ when the input is as shown in Figure P4.1-2.

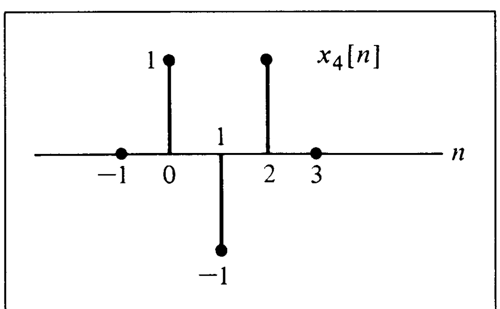

Figure P4.1-2

(a) Express $x_4[n]$ as a linear combination of $x_1[n]$, $x_2[n]$, and $x_3[n]$.

(b) Using the fact that the system is linear, determine $y_4[n]$, the response to $x_4[n]$.

(c) From the input-output pairs in Figure P4.1-1, determine whether the system is time-invariant.

P4.2

Determine the discrete-time convolution of $x[n]$ and $h[n]$ for the following two cases.

(a)

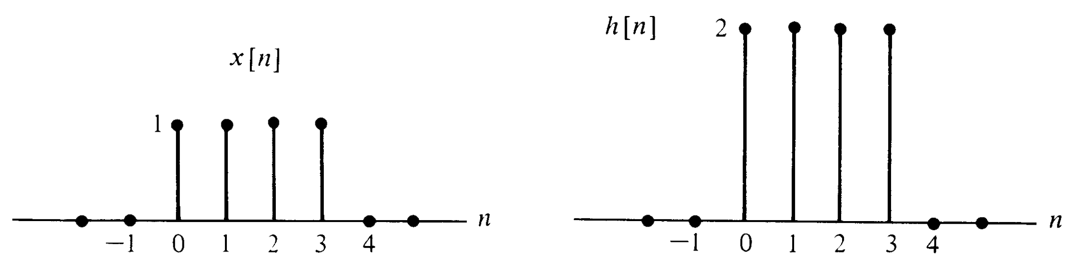

Figure P4.2-1

(b)

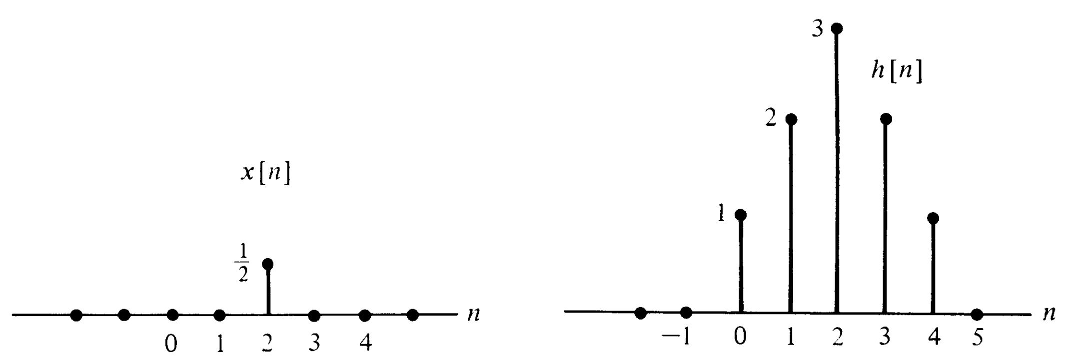

Figure P4.2-2

P4.3

Determine the continuous-time convolution of $x(t)$ and $h(t)$ for the following three cases:

(a)

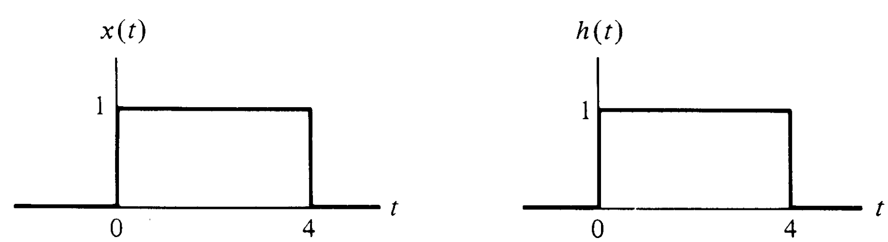

Figure P4.3-1

(b)

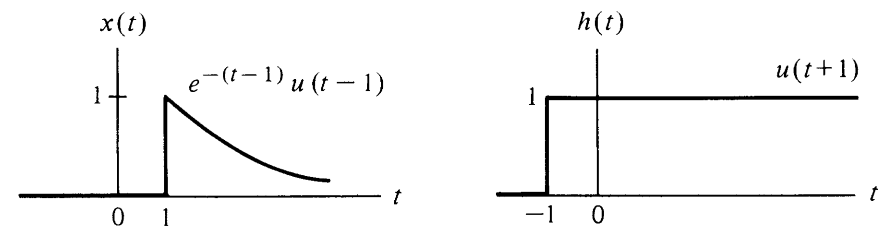

Figure P4.3-2

(c)

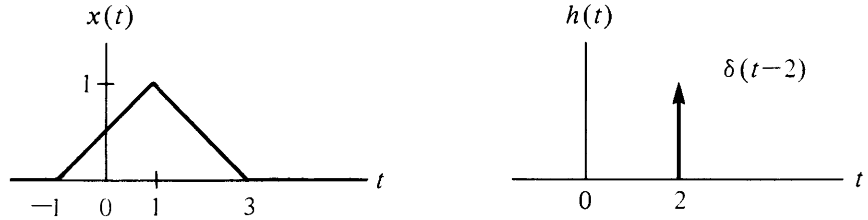

Figure P4.3-3

P4.4

Consider a discrete-time, linear, shift-invariant system that has unit sample response $h[n]$ and input $x[n]$.

(a) Sketch the response of this system if $x[n] = \delta[n - n_0]$, for some $n_0 > 0$, and $h[n] = (\frac{1}{2})^n u[n]$.

(b) Evaluate and sketch the output of the system if $h[n] = (\frac{1}{2})^n u[n]$ and $x[n] = u[n]$.

(c) Consider reversing the role of the input and system response in part (b). That is,

$$h[n] = u[n],$$
$$x[n] = (\tfrac{1}{2})^n u[n]$$

Evaluate the system output $y[n]$ and sketch.

P4.5

(a) Using convolution, determine and sketch the responses of a linear, time-invariant system with impulse response $h(t) = e^{-t/2}\, u(t)$ to each of the two inputs $x_1(t)$, $x_2(t)$ shown in Figures P4.5-1 and P4.5-2. Use $y_1(t)$ to denote the response to $x_1(t)$ and use $y_2(t)$ to denote the response to $x_2(t)$.

(i)

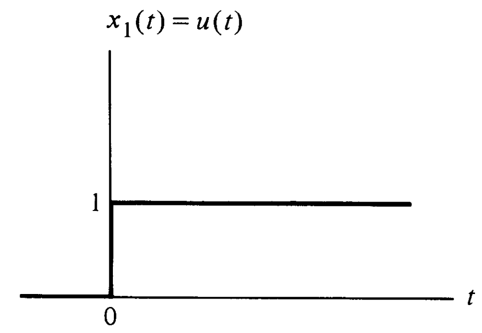

Figure P4.5-1

(ii)

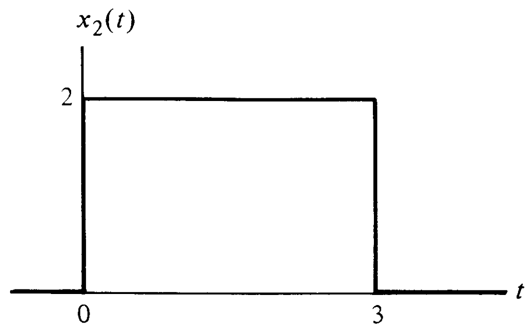

Figure P4.5-2

(b) $x_2(t)$ can be expressed in terms of $x_1(t)$ as

$$x_2(t) = 2[x_1(t) - x_1(t-3)]$$

By taking advantage of the linearity and time-invariance properties, determine how $y_2(t)$ can be expressed in terms of $y_1(t)$. Verify your expression by evaluating it with $y_1(t)$ obtained in part (a) and comparing it with $y_2(t)$ obtained in part (a).

Optional Problems

P4.6

Graphically determine the continuous-time convolution of $h(t)$ and $x(t)$ for the cases shown in Figures P4.6-1 and P4.6-2.

(a)

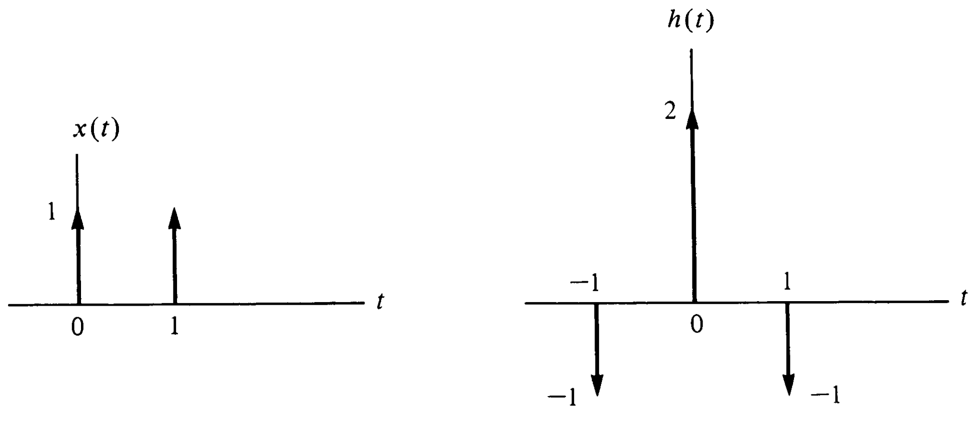

Figure P4.6-1

(b)

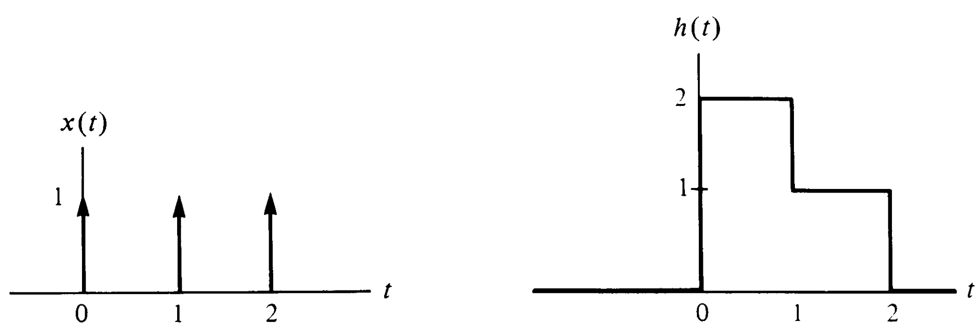

Figure P4.6-2

P4.7

Compute the convolution $y[n] = x[n] * h[n]$ when

$$x[n] = \alpha^n u[n], \qquad 0 < \alpha < 1,$$
$$h[n] = \beta^n u[n], \qquad 0 < \beta < 1$$

Assume that α and β are not equal.

P4.8

Suppose that $h(t)$ is as shown in Figure P4.8 and $x(t)$ is an impulse train, i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

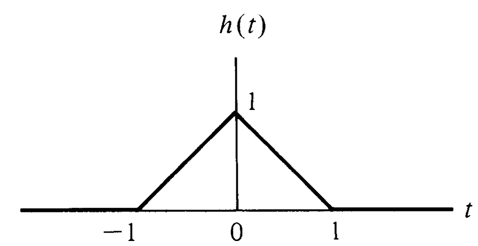

Figure P4.8

(a) Sketch $x(t)$.

(b) Assuming $T = \frac{3}{2}$, determine and sketch $y(t) = x(t) * h(t)$.

P4.9

Determine if each of the following statements is true in general. Provide proofs for those that you think are true and counterexamples for those that you think are false.

(a) $x[n] * \{h[n]g[n]\} = \{x[n] * h[n]\}g[n]$

(b) If $y(t) = x(t) * h(t)$, then $y(2t) = 2x(2t) * h(2t)$.

(c) If $x(t)$ and $h(t)$ are odd signals, then $y(t) = x(t) * h(t)$ is an even signal.

(d) If $y(t) = x(t) * h(t)$, then $Ev\{y(t)\} = x(t) * Ev\{h(t)\} + Ev\{x(t)\} * h(t)$.

P4.10

Let $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ be two periodic signals with a common period T_0. It is not too difficult to check that the convolution of $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ does not converge. However, it is sometimes useful to consider a form of convolution for such signals that is referred to as *periodic convolution*. Specifically, we define the periodic convolution of $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ as

$$\tilde{y}(t) = \int_0^{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t - \tau)\, d\tau = \tilde{x}_1(t) \circledast \tilde{x}_2(t) \qquad \text{(P4.10-1)}$$

Note that we are integrating over exactly one period.

(a) Show that $\tilde{y}(t)$ is periodic with period T_0.

(b) Consider the signal

$$\tilde{y}_a(t) = \int_a^{a + T_0} \tilde{x}_1(\tau)\tilde{x}_2(t - \tau)\, d\tau,$$

where a is an arbitrary real number. Show that

$$\tilde{y}(t) = y_a(t)$$

Hint: Write $a = kT_0 - b$, where $0 \leq b < T_0$.

(c) Compute the periodic convolution of the signals depicted in Figure P4.10-1, where $T_0 = 1$.

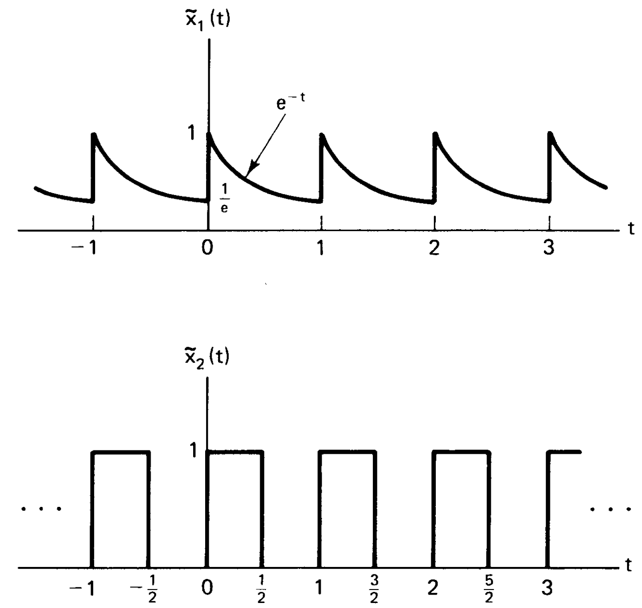

Figure P4.10-1

(d) Consider the signals $x_1[n]$ and $x_2[n]$ depicted in Figure P4.10-2. These signals are periodic with period 6. Compute and sketch their periodic convolution using $N_0 = 6$.

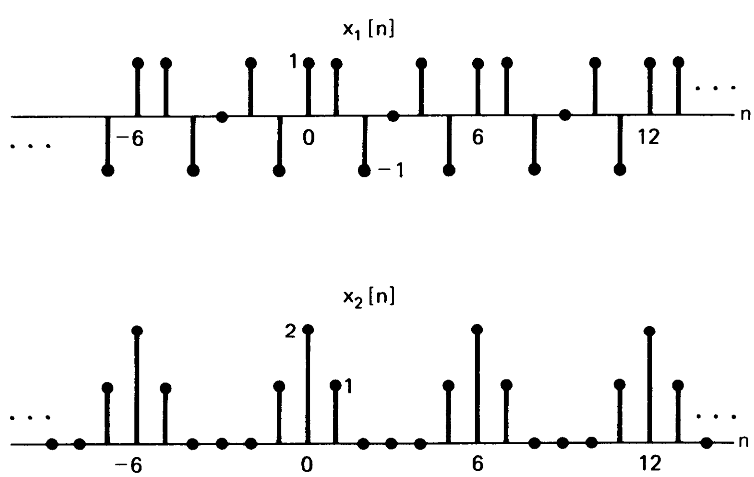

Figure P4.10-2

(e) Since these signals are periodic with period 6, they are also periodic with period 12. Compute the periodic convolution of $x_1[n]$ and $x_2[n]$ using $N_0 = 12$.

P4.11

One important use of the concept of inverse systems is to remove distortions of some type. A good example is the problem of removing echoes from acoustic signals. For example, if an auditorium has a perceptible echo, then an initial acoustic impulse is followed by attenuated versions of the sound at regularly spaced intervals. Consequently, a common model for this phenomenon is a linear, time-invariant system with an impulse response consisting of a train of impulses:

$$h(t) = \sum_{k=0}^{\infty} h_k \delta(t-kT) \tag{P4.11-1}$$

Here the echoes occur T s apart, and h_k represents the gain factor on the kth echo resulting from an initial acoustic impulse.

(a) Suppose that $x(t)$ represents the original acoustic signal (the music produced by an orchestra, for example) and that $y(t) = x(t) * h(t)$ is the actual signal that is heard if no processing is done to remove the echoes. To remove the distortion introduced by the echoes, assume that a microphone is used to sense $y(t)$ and that the resulting signal is transduced into an electrical signal. We will also use $y(t)$ to denote this signal, as it represents the electrical equivalent of the acoustic signal, and we can go from one to the other via acoustic-electrical conversion systems.

The important point to note is that the system with impulse response given in eq. (P4.11-1) is invertible. Therefore, we can find an LTI system with impulse response $g(t)$ such that

$$y(t) * g(t) = x(t)$$

and thus, by processing the electrical signal $y(t)$ in this fashion and then converting back to an acoustic signal, we can remove the troublesome echoes.

The required impulse response $g(t)$ is also an impulse train:

$$g(t) = \sum_{k=0}^{\infty} g_k \delta(t-kT)$$

Determine the algebraic equations that the successive g_k must satisfy and solve for g_1, g_2, and g_3 in terms of the h_k. [*Hint:* You may find part (a) of Problem 3.16 of the text (page 136) useful.]

(b) Suppose that $h_0 = 1$, $h_1 = \frac{1}{2}$, and $h_i = 0$ for all $i \geq 2$. What is $g(t)$ in this case?

(c) A good model for the generation of echoes is illustrated in Figure P4.11. Each successive echo represents a fedback version of $y(t)$, delayed by T s and scaled by α. Typically $0 < \alpha < 1$ because successive echoes are attenuated.

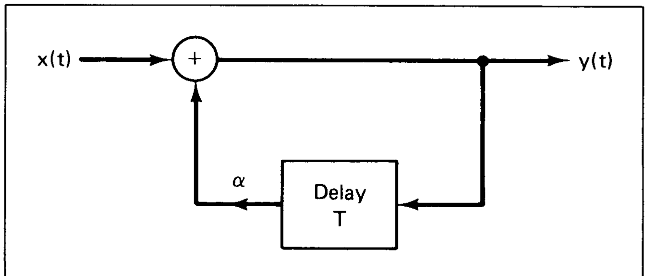

Figure P4.11

(i) What is the impulse response of this system? (Assume initial rest, i.e., $y(t) = 0$ for $t < 0$ if $x(t) = 0$ for $t < 0$.)

(ii) Show that the system is stable if $0 < \alpha < 1$ and unstable if $\alpha > 1$.

(iii) What is $g(t)$ in this case? Construct a realization of this inverse system using adders, coefficient multipliers, and T-s delay elements.

Although we have phrased this discussion in terms of continuous-time systems because of the application we are considering, the same general ideas hold in discrete time. That is, the LTI system with impulse response

$$h[n] = \sum_{k=0}^{\infty} h_k \delta[n-kN]$$

is invertible and has as its inverse an LTI system with impulse response

$$g[n] = \sum_{k=0}^{\infty} g_k \delta[n-kN]$$

It is not difficult to check that the g_i satisfy the same algebraic equations as in part (a).

(d) Consider the discrete-time LTI system with impulse response

$$h[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN]$$

This system is *not* invertible. Find two inputs that produce the same output.

P4.12

Our development of the convolution sum representation for discrete-time LTI systems was based on using the unit sample function as a building block for the representation of arbitrary input signals. This representation, together with knowledge of the response to $\delta[n]$ and the property of superposition, allowed us to represent the system response to an arbitrary input in terms of a convolution. In this problem we consider the use of other signals as building blocks for the construction of arbitrary input signals.

Consider the following set of signals:

$$\phi[n] = (\tfrac{1}{2})^n u[n],$$
$$\phi_k[n] = \phi[n-k], \qquad k = 0, \pm 1, \pm 2, \pm 3, \dots$$

(a) Show that an arbitrary signal can be represented in the form

$$x[n] = \sum_{k=-\infty}^{+\infty} a_k \phi[n-k]$$

by determining an explicit expression for the coefficient a_k in terms of the values of the signal $x[n]$. [*Hint:* What is the representation for $\delta[n]$?]

(b) Let $r[n]$ be the response of an LTI system to the input $x[n] = \phi[n]$. Find an expression for the response $y[n]$ to an arbitrary input $x[n]$ in terms of $r[n]$ and $x[n]$.

(c) Show that $y[n]$ can be written as

$$y[n] = \psi[n] * x[n] * r[n]$$

by finding the signal $\psi[n]$.

(d) Use the result of part (c) to express the impulse response of the system in terms of $r[n]$. Also, show that

$$\psi[n] * \phi[n] = \delta[n]$$