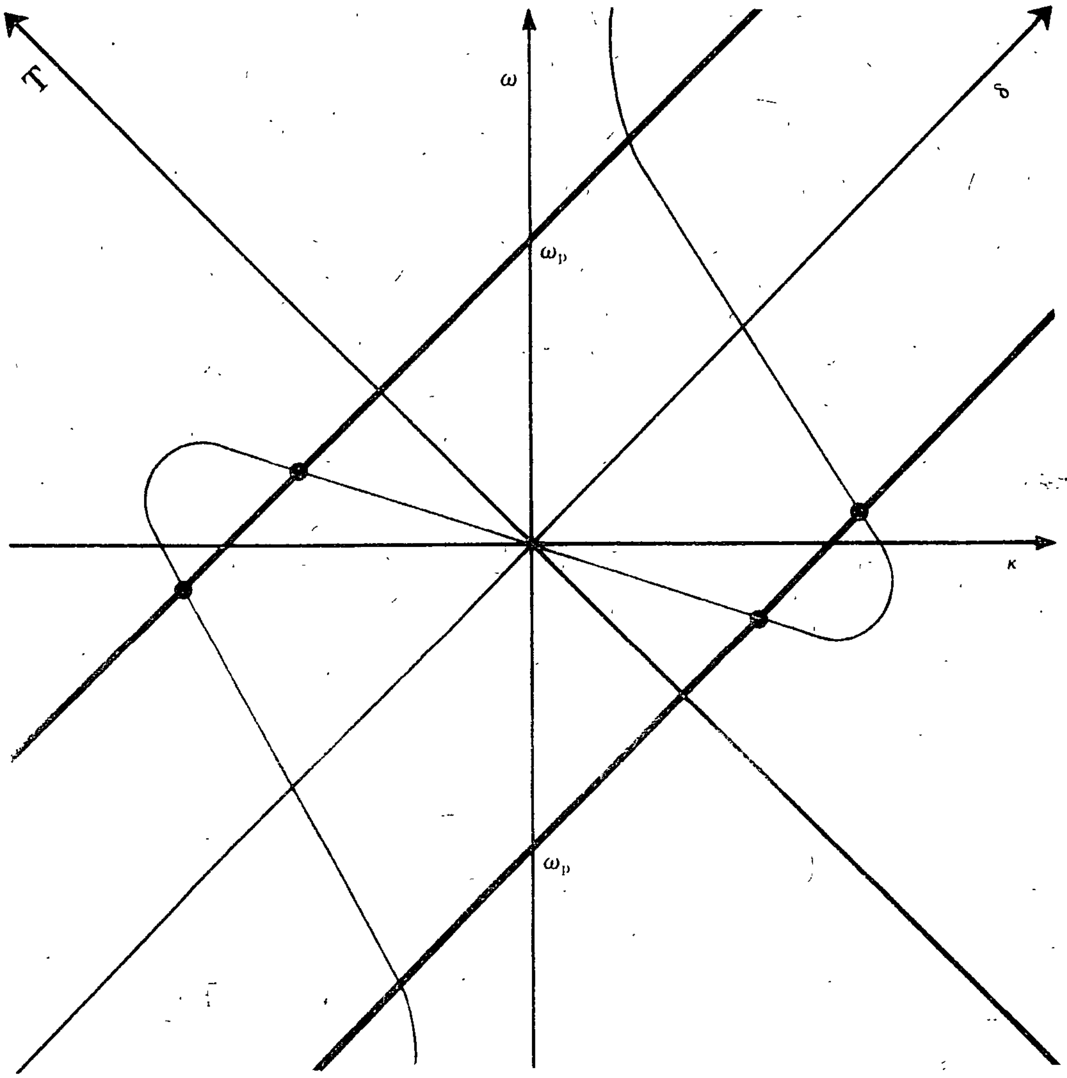
SOLUTIONS MANUAL FOR

ELECTROMECHANICAL DYNAMICS

PART I: Discrete Systems

HERBERT H. WOODSON JAMES R. MELCHER



JOHN WILEY & SONS, INC. NEW YORK • LONDON • SYDNEY

Mark Zahn

SOLUTIONS MANUAL FOR

ELECTROMECHANICAL DYNAMICS

Part I: Discrete Systems

HERBERT H. WOODSON

Philip Sporn Professor of Energy Processing
Departments of Electrical and Mechanical Engineering

JAMES R. MELCHER

Associate Professor of Electrical Engineering
Department of Electrical Engineering

both of Massachusetts Institute of Technology

JOHN WILEY & SONS, INC., NEW YORK · LONDON · SYDNEY

PREFACE TO: SOLUTIONS MANUAL FOR ELECTROMECHANICAL DYNAMICS, PART I: DISCRETE SYSTEMS

This manual presents in an informal format solutions to the problems found at the ends of chapters in Part I of the book, Electromechanical Dynamics. It is intended as an aid for instructors, and in special circumstances for use by students. We have included a sufficient amount of explanatory material that solutions, together with problem statements, are in themselves a teaching aid. They are substantially as found in our records for the course 6.06 as taught at M.I.T. over a period of several years.

Typically, the solutions were originally written up by graduate student tutors whose responsibility it was to conduct one-hour tutorials once a week with students in pairs. These tutorials focused on the homework, with the problem solutions reproduced and given to the students upon receipt of their own homework solutions.

It is difficult to give proper credit to all of those who contributed to these solutions, because the individuals involved range over teaching assistants, instructors, and faculty who have taught the material over a period of more than four years. However, a significant contribution was made by D.S. Guttman who took major responsibility for the solutions in Chapter 6.

The manuscript was typed by Mrs. Barbara Morton, whose patience and expertise were invaluable.

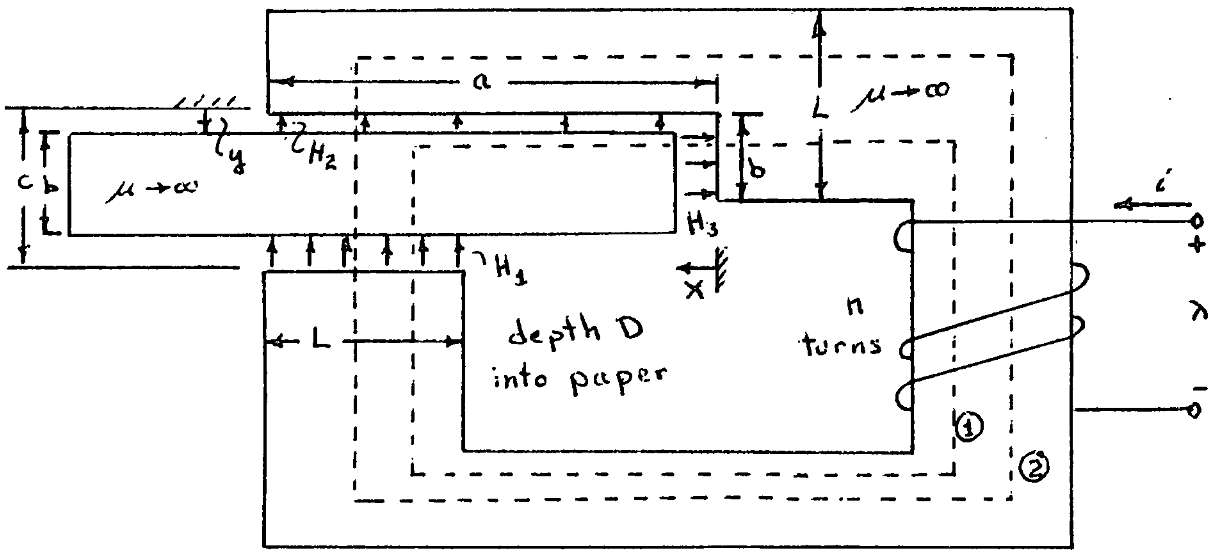
H.H. Woodson

J.R. Melcher

Cambridge, Massachusetts

July, 1968

PROBLEM 2.1



We start with Maxwell's equations for a magnetic system in integral form:

$$\oint_C H\cdot d\bar{\ell} = \int_S \bar{J}\cdot d\bar{a}$$

$$\oint_S B\cdot d\bar{a} = 0$$

Using either path 1 or 2 shown in the figure with the first Maxwell equation we find that

$$\int \bar{J}\cdot d\bar{a} = ni$$

To compute the line integral of H we first note that whenever $\mu\to\infty$ we must have $\bar{H}\to 0$ if $\bar{B}=\mu\bar{H}$ is to remain finite. Thus we will only need to know H in the three gaps (H_1, H_2 and H_3) where the fields are assumed uniform because of the shortness of the gaps. Then

path 1

$$\oint_C H\cdot d\bar{\ell} = H_1(c-b-y) + H_3 x = ni \qquad \text{(a)}$$

path 2

$$\oint_C H\cdot d\bar{\ell} = H_1(c-b-y) + H_2 y = ni \qquad \text{(b)}$$

Using the second Maxwell equation we write that the flux of B into the movable slab equals the flux of B out of the movable slab

$$\mu_o H_1 LD = \mu_o H_2 aD + \mu_o H_3 bD$$

or

$$H_1 L = H_2 a + H_3 b \qquad \text{(c)}$$

Note that in determining the relative strengths of H_1, H_2 and H_3 in this last equation we have let $(a-x) \simeq a$, $(b-y) \simeq b$ to simplify the solution. This means that we are assuming that

$$x/a \ll 1, \; y/b \ll 1 \qquad \text{(d)}$$

Solving for H_1 using (a), (b), and (c)

$$H_1 = \frac{ni(y/a + x/b)}{(c-b-y)(y/a + x/b) + L(y/a \cdot x/b)}$$

The flux of B through the n turns of the coil is then

$$\begin{aligned}(x,y,i) &= nB_1 LD = n\mu_o H_1 LD \\ &= \frac{\mu_o n^2 (y/a + x/b) LD\, i}{(c-b-y)(y/a+x/b) + L(y/a \cdot x/b)}\end{aligned}$$

Because we have assumed that the air gaps are short compared to their cross-sectional dimensions we must have

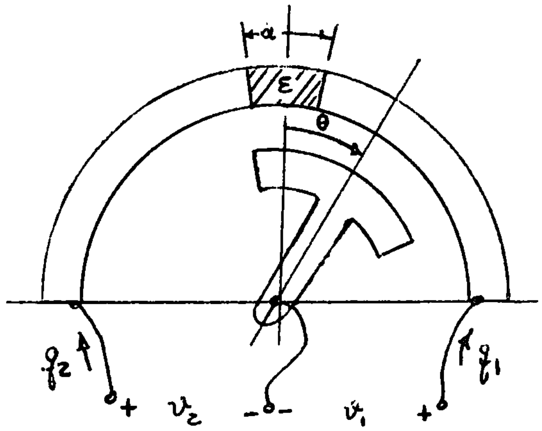
$$\frac{(c-b-y)}{L} \ll 1, \; y/a \ll 1 \text{ and } x/b \ll 1$$

in addition to the constraints of (d) for our expression for λ to be valid. If we assume that $a>L>c>b>(c-b)$ as shown in the diagram, these conditions become

$$x \ll b$$

$$y \ll b$$

PROBLEM 2.2



Because the charge is linearly related to the applied voltages we know that $q_1(V_1,V_2,\theta) = q_1(v_1,0,\theta) + q_1(0,v_2,\theta)$

$$q_1(V_1,0,\theta) = \frac{\varepsilon V_1}{R\alpha} w\ell + \varepsilon_o \frac{V_1}{g}(\frac{\pi}{4} + \theta)R\ell$$

$$q_1(0,V_2,\theta) = -\frac{\varepsilon V_2}{R\alpha} w\ell$$

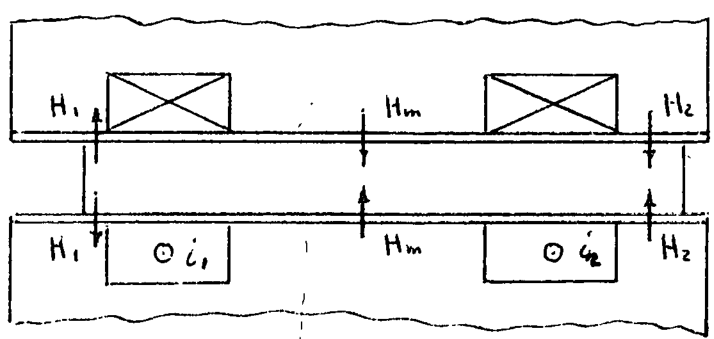
Hence

$$q_1(V_1,V_2,\theta) = V_1\ell(\frac{\varepsilon w}{R\alpha} + \frac{\varepsilon_o(\pi/4+\theta)R}{g}) - V_2\ell\frac{\varepsilon w}{R\alpha}$$

$$q_2(V_1,V_2,\theta) = -V_1\ell\,\frac{\varepsilon w}{R\alpha} + V_2\ell(\frac{\varepsilon w}{R} + \frac{\varepsilon_o(\pi/4-\theta)R}{g})$$

PROBLEM 2.3

The device has cylindrical symmetry so that we assume that the fields in the gaps are essentially radial and denoted as shown in the figure.



Ampere's law can be integrated around each of the current loops to obtain the relations

PROBLEM 2.3 (Continued)

$$gH_1 + gH_m = Ni_1 \tag{a}$$

$$gH_2 - gH_m = Ni_2 \tag{b}$$

In addition, the net flux into the plunger must be zero, and so

$$\mu_o(d-x)2\pi rH_1 - 2d(2\pi r)\mu_o H_m - (d+x)(2\pi r)\mu_o H_2 \tag{c}$$

These three equations can be solved for any one of the intensities. In particular we are interested in H_1 and H_2, because the terminal fluxes can be written simply in terms of these quantities. For example, the flux linking the (1) winding is N times the flux through the air gap to the left

$$\lambda_1 = \mu_o N(d-x)(2\pi r)H_1 \tag{d}$$

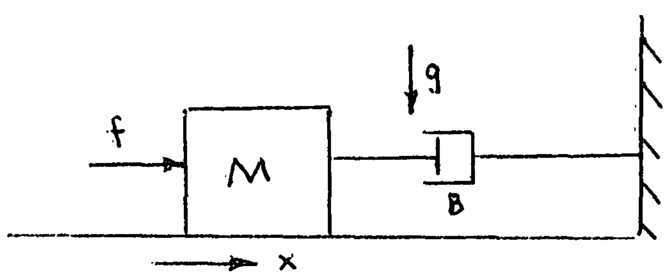
Similarly, to the right,

$$\lambda_2 = \mu_o N(d+x)(2\pi r)H_2 \tag{e}$$

Now, if we use the values of H_1 and H_2 found from (a) - (c), we obtain the terminal relations of Prob. 2.3 with

$$L_o = \frac{\mu_o \pi r N^2 d}{2g}$$

PROBLEM 2.4



PROBLEM 2.4 (Continued)

Part a

$$\sum_i f_i = Ma = M\frac{dx^2}{dt^2}$$

$$f_{DAMPER} = -B\frac{dx}{dt}\ ;\ f_{coul} = -\mu_d Mg\frac{\frac{dx_1}{dt}}{\left|\frac{dx_1}{dt}\right|}$$

$$M\frac{d^2x}{dt^2} = f(t) - B\frac{dx}{dt} + f_{coul}$$

or

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} = f(t) - \mu_d Mg\frac{\frac{dx_1}{dt}}{\left|\frac{dx_1}{dt}\right|}$$

Part b

First we recognize that the block will move so that $\frac{dx_1}{dt} > 0$, hence

$$f_{coul} = -\mu_d Mg;\frac{dx_1}{dt} > 0$$

Then for $t > 0$

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} = -\mu_d Mg$$

which has a solution

$$x(t) = -\frac{\mu_d Mg}{B}t + c_1 + c_2 e^{-(B/M)t}$$

Equating singularities at $t = 0$

$$M\frac{d^2x}{dt^2}(0) = I_o\mu_o(t) \quad \text{or} \quad \frac{d^2x}{dt^2}(0) = \frac{I_o}{M}\mu_o(t)$$

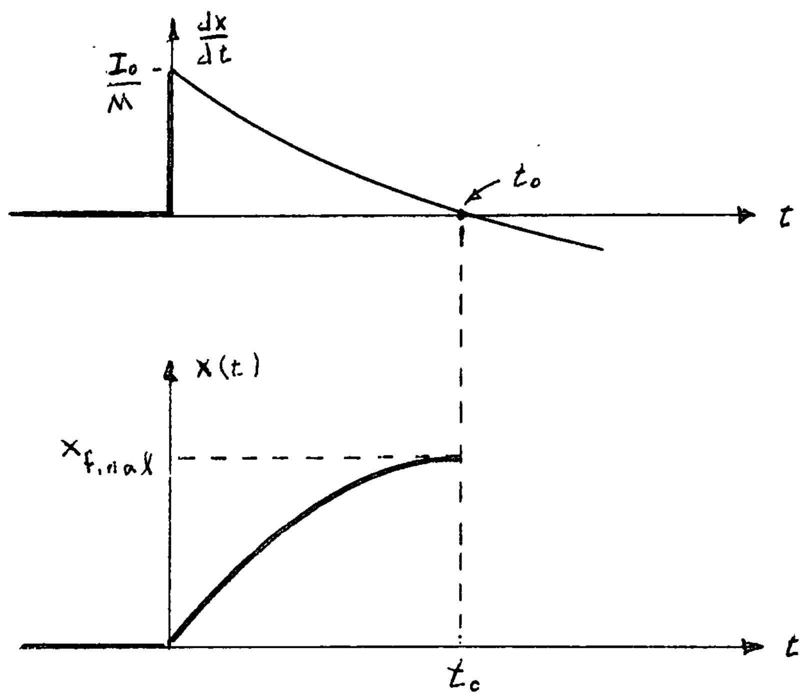
Then since $x(0^-) = \frac{dx}{dt}(0^-) = \frac{d^2x}{dt^2}(0^-) = 0$

PROBLEM 2.4 (Continued)

$$\frac{dx}{dt}(0^+) = \frac{I_o}{M} \; ; \; x(0^+) = 0$$

Hence $x(t) = u_{-1}(t)[-\frac{\mu_d Mg}{B} t + \left(\frac{I_o}{B} + \mu_d g(\frac{M}{B})^2\right)\left(1-e^{-(B/M)t}\right)]$

Actually, this solution will only hold until t_o, where $\frac{dx}{dt}(t_o) = 0$, at which point the mass will stop.



PROBLEM 2.5

Part a

Equation of motion

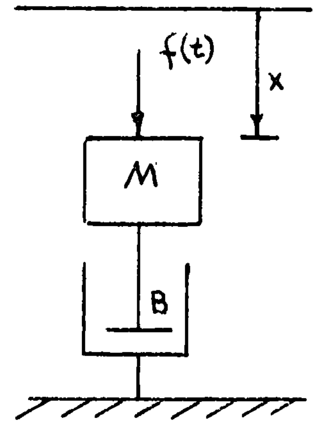
$$M \frac{d^2x}{dt^2} + B \frac{dx}{dt} = f(t)$$

(1) $f(t) = I_o u_o(t)$

$$x(t) = u_{-1}(t) \frac{I_o}{B} (1-e^{-(B/M)t})$$

PROBLEM 2.5 (Continued)

as shown in Prob. 2.4 with $\mu_d = 0$.



(2) $f(t) = F_o u_{-1}(t)$

Integrating the answer in (1)

$$x(t) = \frac{F_o}{B}\left[t + \frac{M}{B}\left(e^{-(B/M)t} - 1\right)\right]u_{-1}(t)$$

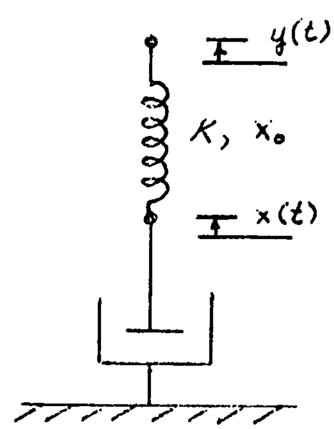
Part b

Consider the node connecting the damper and the spring; there must be no net force on this node or it will suffer infinite acceleration.

$$-B\frac{dx}{dt} + K(y-x) = 0$$

or

$$B/K\frac{dx}{dt} + x = y(t)$$



1. Let $y(t) = Au_o(t)$

$$\frac{B}{K}\frac{dx}{dt} + x = 0 \qquad t > 0$$

$$x(t) = C\,e^{-K/Bt} \qquad t > 0$$

But at $t = 0$

$$\frac{B}{K}\frac{dx}{dt}(0) = Au_o(0)$$

Now since $x(t)$ and $\frac{dx}{dt}(t)$ are zero for $t < 0$

$$x(0^+) = \frac{AK}{B} = C$$

$$x(t) = u_{-1}(t)\,\frac{AK}{B}\,e^{-(K/B)t} \quad \text{all } t$$

2. Let $y(t) = Au_{-1}(t)$

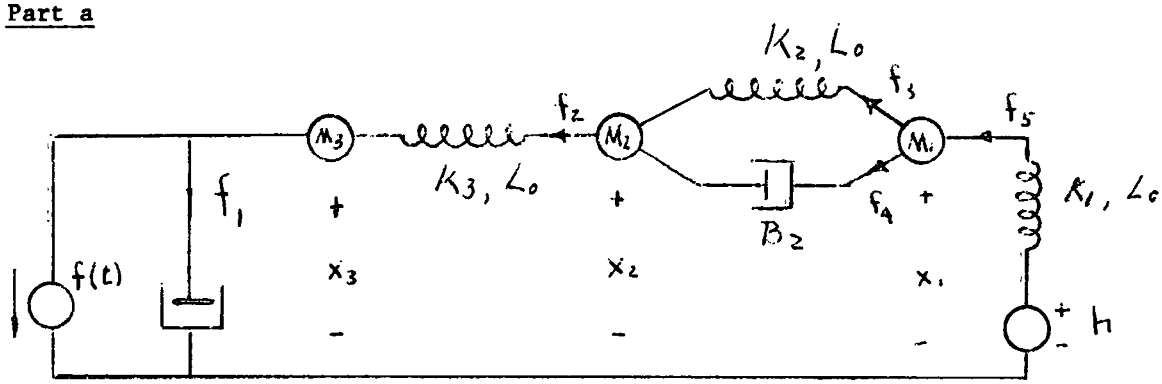
PROBLEM 2.5 (Continued)

Integrating the answer in (1)

$$x(t) = u_{-1}(t)\, Y_o(1-e^{-(K/B)t} \qquad \text{all } t$$

PROBLEM 2.6

Part a



$$f_1 = B_3 \frac{dx_3}{dt} \; ; \; f_2 = K_3(x_2-x_3-t-L_o)$$

$$f_3 = K_2(x_1-x_2-t-L_o); \; f_4 = B_2 \frac{d}{dt}(x_1-x_2)$$

$$f_5 = K_1(h-x_1-L_o)$$

Part b

Summing forces at the nodes and using Newton's law

$$K_1(h-x_1-L_o) = K_2(x_1-x_2-t-L_o) + B_2 \frac{d\,(x_1-x_2)}{dt} + M_1 \frac{d^2x_1}{dt^2}$$

$$K_2(x_1-x_2-t-L_o) + B_2 \frac{d\,(x_1-x_2)}{dt} = K_3(x_2-x_3-t-L_o) + M_2 \frac{d^2x_2}{dt^2}$$

$$K_3(x_2-x_3-t-L_o) = f(t) + B_3 \frac{dx_3}{dt} + M \frac{d^2x_3}{dt^2}$$

PROBLEM 2.6 (Continued)

Let's solve these equations for the special case

$$M_1 = M_2 = M_3 = B_2 = B_3 = L_o = 0$$

Now nothing is left except three springs pulled by force f(t). The three equations are now

$$K_1(h-x_1) = K_2(x_1-x_2) \tag{a}$$

$$K_2(x_1-x_2) = K_3(x_2-x_3) \tag{b}$$

$$K_3(x_2-x_3) = f(t) \tag{c}$$

We write the equation of geometric constraint

$$x_3 + (x_2-x_3) + (x_1-x_2) + (h-x_1)-h = 0$$

or

$$(h-x_3) = (x_2-x_3) + (x_1-x_2) + (h-x_1) \tag{d}$$

which is really a useful identity rather than a new independent equation. Substituting in (a) and (b) into (d)

$$(h-x_3) = \frac{K_3(x_2-x_3)}{K_3} + \frac{K_3(x_2-x_3)}{K_2} + \frac{K_3(x_2-x_3)}{K_1}$$

$$= K_3(x_2-x_3)\ (\frac{1}{K_3} + \frac{1}{K_2} + \frac{1}{K_1})$$

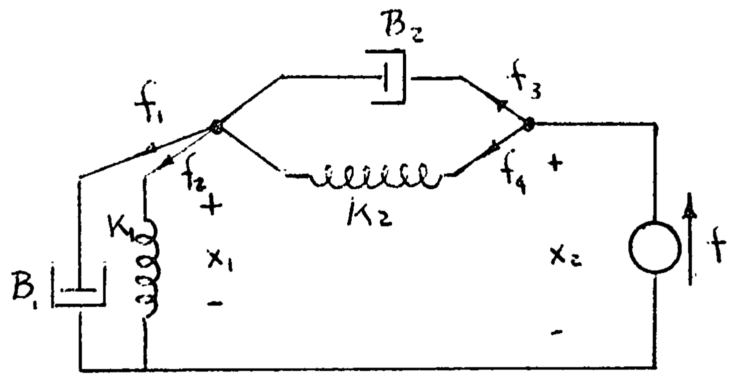
which can be plugged into (c)

$$(\frac{1}{K_3} + \frac{1}{K_2} + \frac{1}{K_1})^{-1}(h-x_3) = f(t)$$

which tells us that three springs in series act like a spring with

$$K' = (\frac{1}{K_3} + \frac{1}{K_2} + \frac{1}{K_1})^{-1}$$

PROBLEM 2.7



$$f_1 = B_1 \frac{dx_1}{dt} \qquad f_2 = K_1 x_1$$

$$f_3 = B_2 \frac{d(x_2-x_1)}{dt} \quad f_4 = K_2(x_2-x_1)$$

Node equations:

Node 1 $$B_1 \frac{dx_1}{dt} + K_1 x_1 = B_2 \frac{d(x_2-x_1)}{dt} + K_2(x_2-x_1)$$

Node 2 $$B_2 \frac{d(x_2-x_1)}{dt} + K_2(x_2-x_1) = f$$

To find natural frequencies let $f = 0$

$$B_1 \frac{dx_1}{dt} + K_1 x_1 = 0 \quad \text{Let} \quad x_1 = e^{st}$$

$$B_1 s + K_1 = 0 \qquad\qquad s_1 = - K_1/B_1$$

$$B_2 \frac{d(x_2-x_1)}{dt} + K_2(x_2-x_1) = 0 \quad \text{Let} \ (x_2-x_1) = e^{st}$$

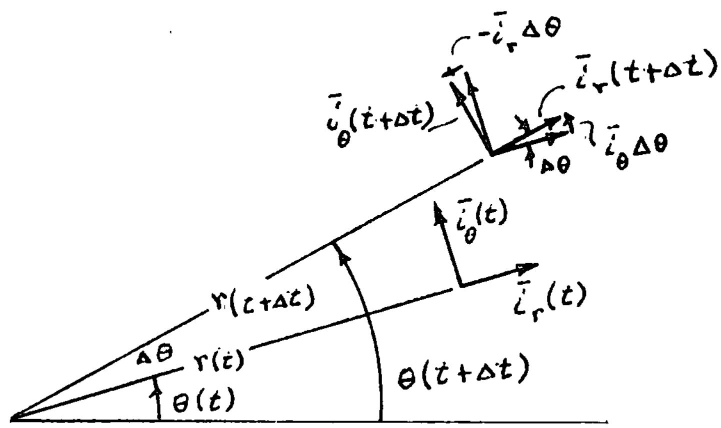
$$B_2 s + K_2 = 0 \qquad\qquad s_2 = - K_2/B$$

The general solution when $f = 0$ is then

$$x_1 = c_1 \, e^{-(K_1/B_1)t}$$

$$x_2 = (x_2-x_1) + x_1 = c_1 e^{-(K_1/B_1)t} + c_2 e^{-(K_2/B_2)t}$$

PROBLEM 2.8



From the diagram, the change in $\bar{i}_r$ in the time Δt is $\bar{i}_\theta\Delta\theta$. Hence

$$\frac{d\bar{i}_r}{dt} = \lim_{\Delta t\to 0} \bar{i}_\theta \frac{\Delta\theta}{\Delta t} = \bar{i}_\theta \frac{d\theta}{dt} \qquad \text{(a)}$$

Similarly,

$$\frac{d\bar{i}_\theta}{dt} = \lim_{\Delta t\to 0} - \bar{i}_r \frac{\Delta\theta}{\Delta t} = - \bar{i}_r \frac{d\theta}{dt} \qquad \text{(b)}$$

Then, the product rule of differentiation on $\bar{v}$ gives

$$\frac{d\bar{v}}{dt} = \frac{d\bar{i}_r}{dt}\frac{dr}{dt} + \bar{i}_r \frac{d^2r}{dt^2} + \frac{di_\theta}{dt}\left(r\frac{d\theta}{dt}\right) + i_\theta \frac{d}{dt}\left(r\frac{d\theta}{dt}\right) \qquad \text{(c)}$$

and the required acceleration follows by combining these equations.

PROBLEM 3.1

This problem is a simple extension of that considered in Sec. 3.2, having the purpose of emphasizing how the geometric dependence of the electrical force depends intimately on the electrical constraints.

Part a

The system is electrically linear. Hence, $W_m' = \frac{1}{2} L i^2$ and the force f that must be applied to the plunger is

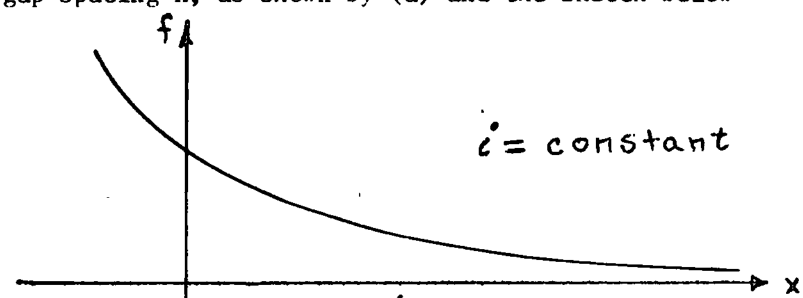
$$f = -f^e = \frac{1}{2a} \frac{L_o i^2}{(1 + \frac{x}{a})^2} \tag{a}$$

The terminal equation can be used to write this force in terms of λ

$$f = -f^e = \lambda^2/2aL_o \tag{b}$$

Part b

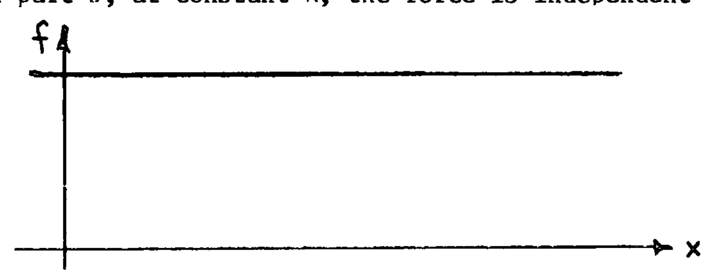
With the current constant, the force decreases rapidly as a function of the plunger gap spacing x, as shown by (a) and the sketch below



With the current constant, the drop in $\int \bar{H} \cdot d\bar{\ell}$ across the gap increases with x, and hence the field in the gap is reduced by increasing x.

Part c

By contrast with part b, at constant λ, the force is independent of x



PROBLEM 3.1 (Continued)

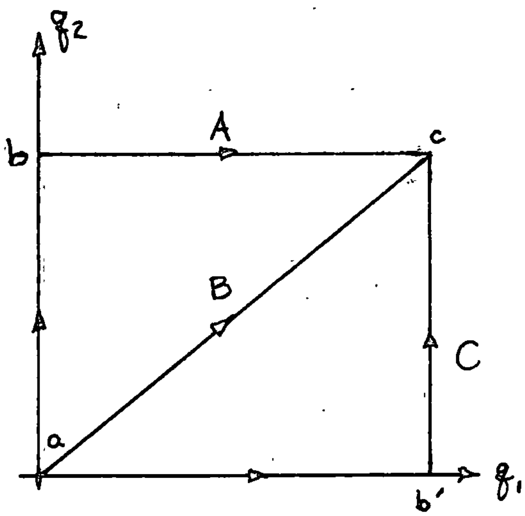
With this constraint, the field in the gap must remain constant, independent of the position x.

PROBLEM 3.2

Part a

The terminal relations are

$$v_1 = S_{11}q_1 + S_{12}q_2$$
$$v_2 = S_{21}q_1 + S_{22}q_2 \qquad \text{(a)}$$



Energy input can result only through the electrical terminal pairs, because the mechanical terminal pairs are constrained to constant position. Thus,

$$W_e = \int v_1 dq_1 + v_2 dq_2 \qquad \text{(b)}$$

First carry out this line integral along the contour A: from a→b, $q_1 = 0$, while from b→c, $dq_2 = 0$. Hence,

$$W_e = \int_o^{Q_2} v_2(0,q_2)dq_2 + \int_o^{Q_1} v_1(q_1,Q_2)dq_1 \qquad \text{(c)}$$

and using (a),

$$W_e = \int_o^{Q_2} S_{22}q_2dq_2 + \int_o^{Q_1}(S_{11}q_1 + S_{12}Q_2)dq_1 \qquad \text{(d)}$$

and for path A,

$$W_e = \frac{1}{2}\, S_{22}Q_2^2 + S_{12}Q_1Q_2 + S_{11}Q_1^2 \qquad \text{(e)}$$

If instead of path A, we use C, the roles of q_1 and q_2 are simply reversed. Mathematically this means 1→2 and 2→1 in the above. Hence, for path C

$$W_e = \frac{1}{2}\, S_{11}Q_1^2 + S_{21}Q_2Q_1 + S_{22}Q_2^2 \qquad \text{(f)}$$

To use path B in carrying out the integration of (b), we relate q_2 and q_1

PROBLEM 3.2 (Continued)

$$q_2 = \frac{Q_2}{Q_1} q_1 \qquad \text{(g)}$$

Then, (a) becomes,

$$v_1 = [S_{11} + \frac{S_{12}Q_2}{Q_1}]q_1; \quad v_2 = [S_{21} + \frac{S_{22}Q_2}{Q_1}]q_1 \qquad \text{(h)}$$

and, from (b), where dq_2 and Q_2dq_1/Q_1

$$W_e = \int_0^{Q_1}[S_{11} + \frac{S_{12}Q_2}{Q_1}]q_1 dq_1 + \int_0^{Q_1}[S_{21} + \frac{S_{22}Q_2}{Q_1}]q_1 \frac{Q_2}{Q_1} dq_1 \qquad \text{(i)}$$

or

$$W_e = \frac{1}{2} S_{11}Q_1^2 + \frac{1}{2} S_{12}Q_2Q_1 + \frac{1}{2} S_{21}Q_1Q_2 + \frac{1}{2} S_{22}Q_2^2 \qquad \text{(j)}$$

Part b

The integrations along paths A, B and C are the same only if $S_{21} = S_{12}$ as can be seen by comparing (e), (f) and (j).

Part c

Conservation of energy requires

$$dW_e(q_1,q_2) = v_1dq_1 + v_2dq_2 = \frac{\partial W_e}{\partial q_1} dq_1 + \frac{\partial W_e}{\partial q_2} dq_2 \qquad \text{(k)}$$

Since q_1 and q_2 are independent variables

$$v_1 = \frac{\partial W_e}{\partial q_1}; \quad v_2 = \frac{\partial W_e}{\partial q_2} \qquad (\ell)$$

Taking cross derivatives of these two expressions and combining gives

$$\frac{\partial v_1}{\partial q_2} = \frac{\partial v_2}{\partial q_1} \qquad \text{(m)}$$

or, from (a), $S_{12} = S_{21}$.

PROBLEM 3.3

The electric field intensity between the plates is

$$E = v/a \qquad \text{(a)}$$

PROBLEM 3.3 (Continued)

Hence, the surface charge adjacent to the free space region on the upper plate is

$$\sigma_f = \varepsilon_o v/a \tag{b}$$

while that next to the nonlinear dielectric slab is

$$\sigma_f = \alpha \frac{v^3}{a^3} + \varepsilon_o \frac{v}{a} \tag{c}$$

It follows that the total charge on the upper plate is

$$q = \frac{dx\varepsilon_o v}{a} + d(\ell-x)[\frac{\alpha v^3}{a^3} + \frac{\varepsilon_o v}{a}] \tag{d}$$

The electric co-energy is

$$W_e' = \int q dv = \frac{d\ell\varepsilon_o v^2}{2a} + \frac{d(\ell-x)\alpha v^4}{4a^3} \tag{e}$$

Then, the force of electrical origin is

$$f^e = \frac{\partial W_e'}{\partial x} = -\frac{d\alpha v^4}{4a^3} \tag{f}$$

PROBLEM 3.4

Part a

The magnetic field intensity in the gap must first be related to the excitation current. From Ampere's law,

$$Ni = dH_d + xH_x \tag{a}$$

where the fields H_d and H_x are directed counterclockwise around the magnetic circuit when they are positive. These fields are further related because the magnetic flux into the movable member must equal that out of it

$$\mu_o wbH_d = \mu_o waH_x \tag{b}$$

From these two expressions

$$H_x = Ni/(\frac{da}{b} + x) \tag{c}$$

PROBLEM 3.4 (Continued)

The flux linked by the electrical terminals is $\lambda = N\mu_o awH_x$ which in view of (c) is

$$\lambda = Li;\ L = \frac{N^2\mu_o aw}{(\frac{da}{b} + x)} \tag{d}$$

Part b

The system is electrically linear. Hence, $W_m = \frac{1}{2}\lambda^2/L$ (See Sec. 3.1.2b) and from (d),

$$W_m = \frac{1}{2}\lambda^2 \frac{(\frac{da}{b} + x)}{N^2\mu_o aw} \tag{e}$$

Part c

From conservation of energy $f^e = -\partial W_m/\partial x$, $W_m = W_m(\lambda, x)$. Hence,

$$f^e = -\frac{1}{2}\lambda^2/(N^2\mu_o aw) \tag{f}$$

Part d

In view of (d) the current node equation can be written as (remember that the terminal voltage is $d\lambda/dt$)

$$I(t) = \frac{1}{R}\frac{d\lambda}{dt} + \frac{\lambda(\frac{da}{b} + x)}{N^2\mu_o aw} \tag{g}$$

Part e

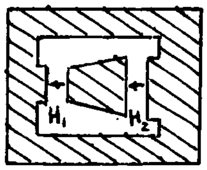
The inertial force due to the mass M must be equal to two other forces, one due to gravity and the other f^e. Hence,

$$M\frac{d^2x}{dt^2} = Mg - \frac{1}{2}\frac{\lambda^2}{N^2\mu_o aw} \tag{h}$$

(g) and (h) are the required equations of motion, where (λ, x) are the dependent variables.

PROBLEM 3.5

Part a



From Ampere's Law

$$H_1(a+x) + H_2(a-x) = N_1 i_1 + N_2 i_2$$

Because $\oint_S \bar{B}\cdot\bar{n}da = 0$

$$\mu_o H_1 A_1 = \mu_o H_2 A_2$$

solving for H_1

$$H_1 = \frac{N_1 i_1 + N_2 i_2}{a(1 + \frac{A_1}{A_2}) + x(1 - \frac{A_1}{A_2})}$$

Now the flux ϕ in each air gap must be the same because

$$\phi = \mu_o H_1 A_1 = \mu_o H_2 A_2$$

and the flux linkages are determined to be $\lambda_1 = N_1\phi$ and $\lambda_2 = N_2\phi$. Using these ideas

$$\lambda_1 = N_1^2 L(x) i_1 + N_1 N_2 L(x) i_2$$

$$\lambda_2 = N_2 N_1 L(x) i_1 + N_2^2 L(x) i_2$$

where $$L(x) = \frac{\mu_o A_1}{a(1 + \frac{A_1}{A_2}) + x(1 - \frac{A_1}{A_2})}$$

Part b

From part a the system is electrically linear, hence

$$W_m' = L(x)[\frac{1}{2} N_1^2 i_1^2 + N_1 N_2 i_1 i_2 + \frac{1}{2} N_2^2 i_2^2]$$

where $$L(x) = \frac{\mu_o A_1}{a(1 + \frac{A_1}{A_2}) + x(1 - \frac{A_1}{A_2})}$$

PROBLEM 3.6

Part a

Conservation of energy requires that

$$dW = i d\lambda - f^e dx \tag{a}$$

In addition,

$$dW = \frac{\partial W}{\partial \lambda} d\lambda + \frac{\partial W}{\partial x} dx \tag{b}$$

so that

$$i = \frac{\partial W}{\partial \lambda} \; ; \; f^e = - \frac{\partial W}{\partial x} \tag{c}$$

Now if we take cross-derivatives of these last relations and combine,

$$\frac{\partial i}{\partial x} = - \frac{\partial f^e}{\partial \lambda} \tag{d}$$

This condition of reciprocity between the electrical and mechanical terminal pairs must be satisfied if the system is to be conservative. For the given terminal relations,

$$\frac{\partial i}{\partial x} = - \frac{I_o}{a}\left[\frac{\lambda}{\lambda_o} + \left(\frac{\lambda}{\lambda_o}\right)^3\right]/\left(1 + \frac{x}{a}\right)^2 \tag{e}$$

$$-\frac{\partial f^e}{\partial \lambda} = - \frac{I_o}{a}\left[\frac{\lambda}{\lambda_o} + \frac{\lambda}{\lambda_o^3}\right]/\left(1 + \frac{x}{a}\right)^2 \tag{f}$$

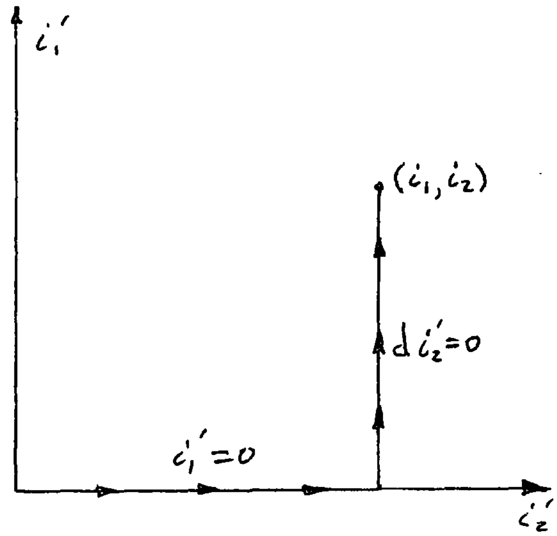
and the system is conservative.

Part b

The stored energy is

$$W_e = \int i d\lambda = \frac{I_o}{[1 + \frac{x}{a}]} \left[\frac{1}{2}\frac{\lambda^2}{\lambda_o} + \frac{1}{4}\frac{\lambda^4}{\lambda_o^3}\right] \tag{g}$$

PROBLEM 3.7



To find the co-energy from the electrical terminal relations alone, we must assume that in the absence of electrical excitations there is no force of electrical origin. Then, the system can be assembled mechanically, with the currents constrained to zero, and there will be no contribution of co-energy in the process (see Sec. 3.1.1). The co-energy input through the electrical terminal pairs with the mechanical system held fixed is

$$W_m' = \int \lambda_1 di_1 + \lambda_2 di_2 \tag{a}$$

For the path shown in the (i_1,i_2) plane of the figure, this becomes

$$W_m' = \int_o^{i_2} \lambda_2(0,i_2')di_2' + \int_o^{i_1} \lambda_1(i_1',i_2)di_1' \tag{b}$$

amd in view of the given terminal relations, the required co-energy is

$$W_m' = \frac{c}{4} x_2 i_2^4 + b x_1 x_2 i_2 i_1 + \frac{a}{4} x_1 i_1^4 \tag{c}$$

PROBLEM 3.8

Steps (a) and (b) establish the flux in the rotor winding.

$$\lambda_2 = I_o L_m \tag{a}$$

With the current constrained on the stator coil, as in step (c), the current i_1 is known, and since the flux λ_2 is also known, we can use the second terminal equations to solve for the current in the rotor winding as a function of the angular position

$$i_2 = \frac{L_m}{L_2}[I_o - I(t)\cos\theta] \tag{b}$$

This is the electrical equation of motion for the system. To complete the picture, the torque equation must be found. From the terminal relations, the co-energy is

PROBLEM 3.8 (Continued)

$$W'_m = \int \lambda_1 di_1 + \lambda_2 di_2 = \frac{1}{2} L_1 i_1^2 + i_1 i_2 L_m \cos\theta + \frac{1}{2} i_2^2 L_2 \qquad \text{(c)}$$

and hence, the electrical torque is

$$T^e = \frac{\partial W'_m}{\partial \theta} = -i_1 i_2 L_m \sin\theta \qquad \text{(d)}$$

Now, we use this expression in the torque equation, with i_2 given by (b) and $i_1 = I(t)$

$$\frac{Jd^2\theta}{dt^2} = -\frac{IL_m^2}{L_2}(I_o - I(t)\cos\theta)\sin\theta \qquad \text{(e)}$$

This is the required equation of motion. Note that we did not substitute i_2 from (b) into the co-energy expression and then take the derivative with respect to θ. This gives the wrong answer because we have assumed in using the basic energy method to find the torque that i_1, i_2 and θ are thermodynamically independent variables.

PROBLEM 3.9

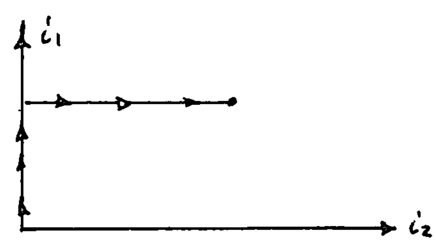
Part a

From the terminal relations, the electrical co-energy is (Table 3.1.1)

$$W'_m = \int \lambda_1 di_1 + \lambda_2 di_2 \qquad \text{(a)}$$

or

$$W'_m = \frac{1}{4} a x_1^2 i_1^4 + b x_2^2 x_1 i_1 i_2 + \frac{1}{4} c x_2^2 i_2^4 \qquad \text{(b)}$$



Part b

It follows that the required forces are

$$f_1^e = \frac{\partial W'_m}{\partial x_1} = \frac{1}{2} a x_1 i_1^4 + b x_2^2 i_1 i_2 \qquad \text{(c)}$$

$$f_2^e = \frac{\partial W'_m}{\partial x_2} = 2b x_2 x_1 i_1 i_2 + \frac{1}{2} c x_2 i_2^4 \qquad \text{(d)}$$

PROBLEM 3.9 (Continued)

Part c

There are four equations of motion in the dependent variables i_1, i_2, x_1 and x_2: two of these are the electrical voltage equations, which in view of the terminal equations for the λ's, are

$$-i_1R_1 = \frac{d}{dt}(ax_1^2 i_1^3 + bx_2^2 x_1 i_2) \tag{e}$$

$$v_2(t) - i_2R_2 = \frac{d}{dt}(bx_2^2 x_1 i_1 + cx_2^2 i_2^3) \tag{f}$$

and two are the mechanical force equations

$$0 = \frac{1}{2}\,ax_1 i_1^4 + bx_2^2 i_1 i_2 - Kx_1 \tag{g}$$

$$0 = 2bx_2 x_1 i_1 i_2 + \frac{1}{2}\,cx_2 i_2^4 - B\,\frac{dx_2}{dt} \tag{h}$$

PROBLEM 3.10

Part a

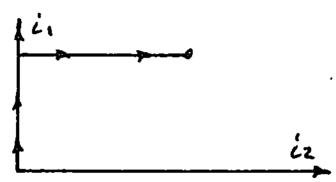
Because the terminal relations are expressed as functions of the current and x, it is most appropriate to use the co-energy to find the force. Hence,

$$W_m' = \int \lambda_1 di_1 + \lambda_2 di_2 \tag{a}$$

which becomes,

$$W_m' = \frac{1}{2}\,L_o i_1^2 + \frac{1}{2}\,Ai_1^2 i_2^2 x + \frac{1}{2}L_o i_2^2 \tag{b}$$

From this it follows that the force is,



$$f^e = \frac{1}{2}\,Ai_1^2 i_2^2 \tag{c}$$

Part b

The currents i_1 and i_2 and x will be used as the dependent variables. Then, the voltage equations for the two electrical circuits can be written, using the electrical terminal equations, as

$$e_1(t) = i_1R_1 + \frac{d}{dt}(L_o i_1 + Ai_1 i_2^2 x) \tag{d}$$

$$e_2(t) = i_2R_2 + \frac{d}{dt}(Ai_1^2 i_2 x + L_o i_2) \tag{e}$$

PROBLEM 3.10 (Continued)

The equation for mechanical equilibrium of the mass M is the third equation of motion

$$M\frac{d^2x}{dt^2} = -K(x-x_o) + \frac{1}{2}Ai_1^2i_2^2 \tag{f}$$

PROBLEM 3.11

Part a

The electrical torques are simply found by taking the appropriate derivatives of the co-energy (see Table 3.1.1)

$$T_1^e = \frac{\partial W_m'}{\partial\theta} = -M\sin\theta\cos\psi\ i_1i_2 \tag{a}$$

$$T_2^e = \frac{\partial W_m'}{\partial\psi} = -M\cos\theta\sin\psi\ i_1i_2 \tag{b}$$

Part b

The only torques acting on the rotors are due to the fields. In view of the above expressions the mechanical equations of motion, written using θ,ψ, i_1 and i_2 as dependent variables, are

$$J\frac{d^2\theta}{dt^2} = -M\sin\theta\cos\psi\ i_1i_2 \tag{c}$$

$$J\frac{d^2\psi}{dt^2} = -M\cos\theta\sin\psi\ i_1i_2 \tag{d}$$

Remember that the terminal voltages are the time rates of change of the respective fluxes. Hence, we can make use of the terminal equations to write the current node equations for each of the circuits as

$$I_1(t) = C\frac{d^2}{dt^2}(L_1i_1 + Mi_2\cos\theta\cos\psi) + i_1 \tag{e}$$

$$I_2(t) = G\frac{d}{dt}(Mi_1\cos\theta\cos\psi + L_2i_2) + i_2 \tag{f}$$

Thus, we have four equations, two mechanical and two electrical, which involve the dependent variables θ,ψ, i_1 and i_2 and the known driving functions I_1 and I_2.

PROBLEM 3.12

We can approach this problem in two ways. First from conservation of energy,

$$dW_m' = \lambda_1 di_1 + \lambda_2 di_2 + \lambda_3 di_3 \tag{a}$$

and

$$dW_m' = \frac{\partial W_m'}{\partial i_1} di_1 + \frac{\partial W_m'}{\partial i_2} di_2 + \frac{\partial W_m'}{\partial i_3} di_3 \tag{b}$$

Hence,

$$\lambda_1 = \frac{\partial W_m'}{\partial i_1} ; \lambda_2 = \frac{\partial W_m'}{\partial i_2} ; \lambda_3 = \frac{\partial W_m'}{\partial i_3} \tag{c}$$

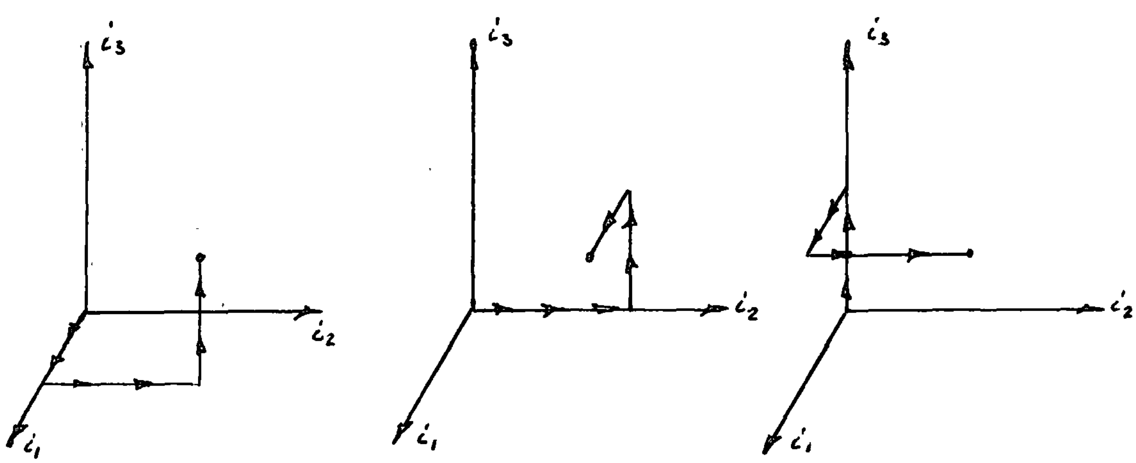
Taking combinations of cross-derivatives, this gives

$$\frac{\partial \lambda_1}{\partial i_2} = \frac{\partial \lambda_2}{\partial i_1} ; \frac{\partial \lambda_2}{\partial i_3} = \frac{\partial \lambda_3}{\partial i_2} ; \frac{\partial \lambda_3}{\partial i_1} = \frac{\partial \lambda_1}{\partial i_3} \tag{d}$$

or

$$L_{12} = L_{21}; L_{23} = L_{32}; L_{31} = L_{13} \tag{e}$$

Another way to show the same thing is to carry out the integrations along the three different paths shown



Since

$$W_m = \int \lambda_1 di_1 + \lambda_2 di_2 + \lambda_3 di_3 \tag{f}$$

PROBLEM 3.12 (Continued)

these paths of integration lead to differing results. For path (a), we have

$$W_m = \frac{1}{2}L_{11}i_1^2 + L_{21}i_1i_2 + \frac{1}{2}L_{22}i_2^2 + L_{31}i_1i_3 + L_{32}i_2i_3 + \frac{1}{2}L_{33}i_3^2 \tag{g}$$

while for path (b)

$$W_m = \frac{1}{2}L_{22}i_2^2 + L_{32}i_2i_3 + \frac{1}{2}L_{33}i_3^2 + \frac{1}{2}L_{11}i_1^2 + L_{12}i_2i_1 + L_{13}i_3i_1 \tag{h}$$

and path (c)

$$W_m = \frac{1}{2}L_{33}i_3^2 + \frac{1}{2}L_{11}i_1^2 + L_{13}i_3i_1 + L_{21}i_1i_2 + \frac{1}{2}L_{22}i_2^2 + L_{23}i_3i_2 \tag{i}$$

These equations will be identical only if (e) holds.

PROBLEM 3.13

Part a

When $\theta = 0$, there is no overlap between the stator and rotor plates, as compared to complete overlap when $\theta = \pi/2$. Because the total exposed area between one pair of stator and rotor plates is $\pi R^2/2$, at an angle θ the area is

$$A = \frac{\pi R^2}{2}\frac{\theta}{(\frac{\pi}{2})} = \theta R^2 \tag{a}$$

There are 2N-1 pairs of such surfaces, and hence the total capacitance is

$$C = (2N-1)\theta R^2\epsilon_o/g \tag{b}$$

The required terminal relation is then $q = Cv$.

Part b

The system is electrically linear. Hence, $W_e' = \frac{1}{2}Cv^2$ and

$$T^e = \frac{\partial W_e'}{\partial\theta} = \frac{(2N-1)R^2\epsilon_o v^2}{2g} \tag{c}$$

Part c

There are three torques acting on the shaft, one due to the torsional spring, the second from viscous damping and the third the electrical torque.

PROBLEM 3.13 (Continued)

$$J \frac{d^2\theta}{dt^2} = -K(\theta-\alpha) - B \frac{d\theta}{dt} + \frac{1}{2} \frac{v^2(2N-1)R^2\varepsilon_o}{g} \qquad \text{(d)}$$

Part d

The voltage circuit equation, in view of the electrical terminal equation is simply

$$V_o(t) = R \frac{d}{dt} [\frac{(2N-1)R^2\theta\varepsilon_o v}{g}] + v \qquad \text{(e)}$$

Part e

When the rotor is in static equilibrium, the derivatives in (d) vanish and we can solve for $\theta-\alpha$,

$$\theta-\alpha = \frac{V_o^2(2N-1)R^2\varepsilon_o}{2gK} \qquad \text{(f)}$$

This equation would comprise a theoretical calibration for the voltmeter if effects of fringing fields could be ignored. In practice, the plates are shaped so as to somewhat offset the square law dependence of the deflections.

PROBLEM 3.14

Part a

Fringing fields are ignored near the ends of the metal coaxial cylinders. In the region between the cylinders, the electric field has the form $\bar{E} = A\bar{i}_r/r$, where r is the radial distance from the axis and A is a constant determined by the voltage. This solution is both divergence and curl free, and hence satisfies the basic electric field equations (See Table 1.2) everywhere between the cylinders. The boundary conditions on the surfaces of the dielectric slab are also satisfied because there is no normal electric field at a dielectric interface and the tangential electric fields are continuous. To determine the constant A, note that

$$\int_a^b E_r dr = -v = A\ln(\frac{b}{a}); \; A = -v/\ln(\frac{b}{a}) \qquad \text{(a)}$$

The surface charge on the inner surface of the outer cylinder in the regions adjacent to free space is then

PROBLEM 3.14 (Continued)

$$\sigma_f = \frac{v\varepsilon_o}{\ln(\frac{b}{a})b} \tag{b}$$

while that adjacent to regions occupied by the dielectric is

$$\sigma_f = \frac{v\varepsilon}{\ln(\frac{b}{a})b} \tag{c}$$

It follows that the total charge on the outer cylinder is

$$q = v\,\frac{\pi}{\ln(\frac{b}{a})}\,[L(\varepsilon_o+\varepsilon)-x(\varepsilon-\varepsilon_o)] \tag{d}$$

Part b

Conservation of power requires

$$v\,\frac{dq}{dt} = \frac{dW_e}{dt} + f_e\,\frac{dx}{dt} \tag{e}$$

Parts c and d

It follows from integration of (c) that

$$W_e = \frac{1}{2}\,\frac{q^2}{C} \quad \text{or} \quad W_e' = \frac{1}{2}\,Cv^2 \tag{f}$$

where

$$C = \frac{\pi}{\ln(\frac{b}{a})}\,[L(\varepsilon_o+\varepsilon)-x(\varepsilon-\varepsilon_o)]$$

Part e

The force of electrical origin is therefore

$$f^e = \frac{\partial W_e'}{\partial x} = -\,\frac{1}{2}\,v^2\,\frac{\pi}{\ln(\frac{b}{a})}\,(\varepsilon-\varepsilon_o) \tag{g}$$

Part f

The electrical constraints of the system have been left unspecified. The mechanical equation of motion, in terms of the terminal voltage v, is

$$M\,\frac{d^2x}{dt^2} = -K(x-\ell)-\,\frac{1}{2}\,\frac{v^2\pi}{\ln(\frac{b}{a})}\,(\varepsilon-\varepsilon_o) \tag{h}$$

PROBLEM 3.14 (Continued)

Part g

In static equilibrium, the inertial term makes no contribution, and (h) can be simply solved for the equilibrium position x.

$$x=\ell-\frac{1}{2}\frac{V_o^2\pi(\varepsilon-\varepsilon_o)}{K\ln(\frac{b}{a})} \tag{i}$$

PROBLEM 3.15

Part a

Call r the radial distance from the origin 0. Then, the field in the gap to the right is, (from Ampere's law integrated across the gaps at a radius r

$$H_\theta^r = Ni/(\beta-\alpha-\theta)r \qquad \text{(directed to the right)} \tag{a}$$

and to the left

$$H_\theta^\ell = Ni/(\beta-\alpha+\theta)r \qquad \text{(directed to the left)} \tag{b}$$

These fields satisfy the conditions that $\nabla\times\bar{H}=0$ and $\nabla\cdot\bar{B}=0$ in the gaps. The flux is computed by integrating the flux density over the two gaps and multiplying by N.

$$\lambda = \mu_o DN\int_a^b (H_\theta^\ell + H_\theta^r)dr \tag{c}$$

which, in view of (a) and (b) becomes,

$$\lambda = Li,\ L = \mu_o DN^2\ln(\frac{b}{a})[\frac{1}{\beta-\alpha+\theta}+\frac{1}{\beta-\alpha-\theta}] \tag{d}$$

Part b

The system is electrically linear, and hence the co-energy is simply (See Sec. 3.1.2b)

$$W_m' = \frac{1}{2}Li^2 \tag{e}$$

Part c

The torque follows from (e) as

PROBLEM 3.15 (Continued)

$$T^e = -\frac{1}{2}\mu_o DN^2 \ln(\frac{b}{a})[\frac{1}{(\beta-\alpha+\theta)^2} - \frac{1}{(\beta-\alpha-\theta)^2}]i^2 \tag{f}$$

Part d

The torque equation is then

$$J\frac{d^2\theta}{dt^2} = -K\theta + T^e \tag{g}$$

Part e

This equation is satisfied if $\theta=0$, and hence it is possible for the wedge to be in static equilibrium at this position.

PROBLEM 3.16

We ignore fringing fields. Then the electric field is completely between the center plate and the outer plates, where it has the value $E = v/b$. The constraints on the electrical terminals further require that $v = V_o - Ax$.

The surface charge on the outer plates is $\varepsilon_o v/b$ and hence the total charge q on these plates is

$$q = 2(a-x)\frac{d\varepsilon_o}{b}v \tag{a}$$

It follows that the co-energy is

$$W_e' = (a-x)\frac{d\varepsilon_o}{b}v^2 \tag{b}$$

and the electrical force is

$$f^e = \frac{\partial W_e'}{\partial x} = -\frac{d\varepsilon_o}{b}v^2 \tag{c}$$

Finally, we use the electrical circuit conditions to write

$$f^e = -\frac{d\varepsilon_o}{b}(V_o - Ax)^2 \tag{d}$$

The major point to be made in this situation is this. One might substitute the voltage, as it depends on x, into (b) before taking the derivative. This clearly gives an answer not in agreement with (d). We have assumed in writing (c) that the variables (v,x) remain thermodynamically independent until after the force has been found. Of course, in the actual situation, external constraints

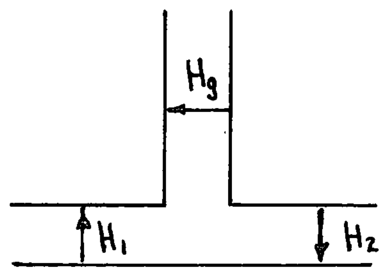
PROBLEM 3.16 (Continued)

relate these variables, but these constraints can only be introduced with care in the energy functions. To be safe they should not be introduced until after the force has been found.

PROBLEM 3.17

Part a

The magnetic field intensities in the gaps can be found by using Ampere's law integrated around closed contours passing through the gaps. These give



$$H_g = N(i_1 + i_2)/g \tag{a}$$

$$H_1 = Ni_1/d \tag{b}$$

$$H_2 = Ni_2/d \tag{c}$$

In the magnetic material, the flux densities are

$$B_1 = \frac{\alpha N^3 i_1^3}{d^3} + \frac{\mu_o N i_1}{d} \tag{d}$$

$$B_2 = \frac{\alpha N^3 i_2^3}{d^3} + \frac{\mu_o N i_2}{d} \tag{e}$$

The flux linking the individual coils can now be computed as simply the flux through the appropriate gaps. For example, the flux λ_1 is

$$\lambda_1 = ND[\ell\mu_o H_g + x\mu_o H_1 + (\ell - x)B_1] \tag{f}$$

which upon substitution from the above equations becomes the first terminal relation. The second is obtained in a similar manner.

Part b

The co-energy is found by integrating, first on i_1 with $i_2 = 0$ and then on i_2 with i_1 fixed at its final value. Hence,

PROBLEM 3.17 (Continued)

$$W_m' = \int \lambda_1 di_1 + \lambda_2 di_2 \tag{g}$$

$$= \frac{1}{2} L_o(1 + \frac{d}{g})i_1^2 + \frac{1}{4} L_o\beta(1 - \frac{x}{\ell})i_1^4 + L_o \frac{d}{g} i_1 i_2$$

$$+ \frac{1}{4} L_o\beta \left(\frac{x}{\ell}\right) i_2^4 + \frac{1}{2} L_o(1 + \frac{d}{g})i_2^2$$

$\left(\frac{x-g}{\ell}\right)$

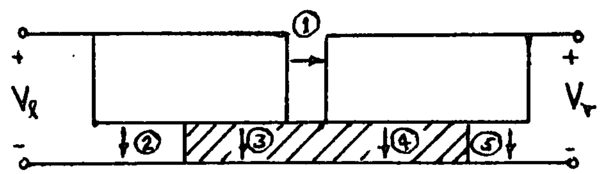
Part c

The force of electrical origin follows from the co-energy functions as,

$$f^e = -\frac{1}{4} L_o \frac{\beta}{\ell} i_1^4 + \frac{1}{4} \frac{L_o\beta}{\ell} i_2^4 \tag{h}$$

PROBLEM 3.18

Part a



Assuming simple uniform E fields in the gaps

$E_1 = (V_\ell - V_r)/g;\ E_2 = V_\ell/d = E_3$

$E_4 = E_5 = V_r/d$

These fields leave surface charge densities on the top electrodes

$$\sigma_1 = \varepsilon_o(V_\ell - V_r)/g,\ \sigma_2 = \varepsilon_o V_\ell/d$$

$$\sigma_3 = [\alpha(V_\ell/d)^2 + \varepsilon_o](V_\ell/d)$$

$$\sigma_4 = [\alpha(V_r/d)^2 + \varepsilon_o](V_r/d)$$

$$\sigma_5 = \varepsilon_o(V_r/d)$$

These surface charge densities cause net charges on the electrodes of

$$q_\ell = \frac{\varepsilon_o wb}{g}(V_\ell - V_r) + \frac{\varepsilon_o wLV_\ell}{d} + \alpha w(L-x)\left(\frac{V_\ell}{d}\right)^3$$

$$q_r = \frac{\varepsilon_o wb}{g}(V_r - V_\ell) + \frac{\varepsilon_o wL}{d} V_r + \alpha w(x-g)\left(\frac{V_r}{d}\right)^3$$

PROBLEM 3.18 (Continued)

Part b

$$W' = \int_{\substack{o\\ q_2=0}}^{q_1} q_\ell dV_\ell + \int_{\substack{o\\ q_1=q_1}}^{q_2} q_r dV_r$$

$$= \varepsilon_o w(\frac{b}{g} + \frac{L}{d}) \frac{V_\ell^2}{2} + \frac{\alpha w(L-x)d}{4} (\frac{V_\ell}{d})^4$$

$$+ \varepsilon_o w(\frac{b}{g} + \frac{L}{d}) \frac{V_r^2}{2} - \frac{\varepsilon_o w b}{g} V_\ell V_r + \frac{\alpha w(x-g)d}{4} (\frac{V_r}{d})^4$$

$$f_e = \frac{\partial W'}{\partial x} = \frac{\alpha wd}{4} [(\frac{V_r}{d})^4 - (\frac{V_\ell}{d})^4] \quad \text{(pulled to side with more voltage)}$$

PROBLEM 3.19

Part a

The rotating plate forms a simple capacitor plate with respect to the other two curved plates. There is no mutual capacitance if the fringing fields are ignored. For example, the terminal relations over the first half cycle of the rotor are

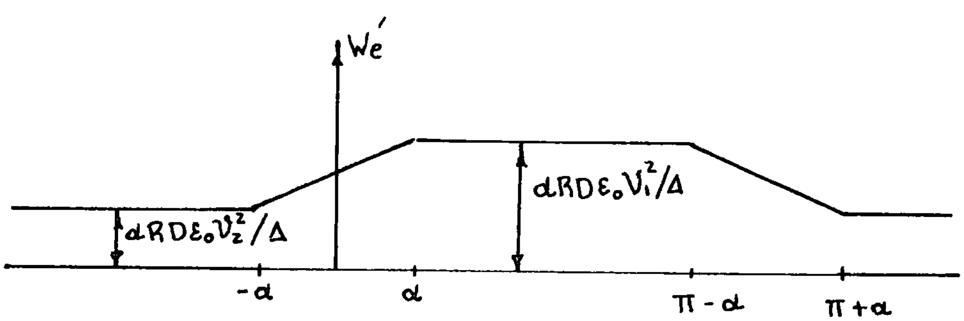
$$-\alpha<\theta<\alpha; \quad q_1 = \frac{(\alpha+\theta)RD\varepsilon_o v_1}{\Delta}; \quad q_2 = \frac{(\alpha-\theta)RD\varepsilon_o v_2}{\Delta} \qquad \text{(a)}$$

$$\alpha<\theta<\pi-\alpha; \quad q_1 = \frac{2\alpha RD\varepsilon_o v_1}{\Delta}; \quad q_2 = 0 \qquad \text{(b)}$$

So that the co-energy can be simply written as the sum of the capacitances for the two outer electrodes relative to the rotor.

$$W_e' = \frac{1}{2} C_1 v_1^2 + \frac{1}{2} C_2 v_2^2 \qquad \text{(c)}$$

The dependence of this quantity on θ is as shown below

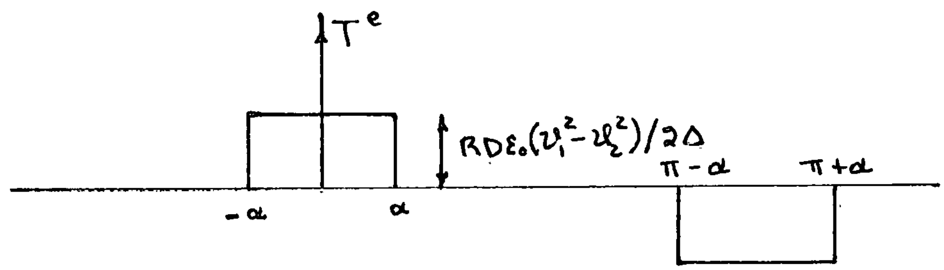


(d)

PROBLEM 3.19 (Continued)

Part b

The torque is the spatial derivative of the above function



Part c

The torque equation is then

$$J\frac{d^2\theta}{dt^2} = T^e$$

where T^e is graphically as above.

PROBLEM 3.20

Part a

The electric energy is

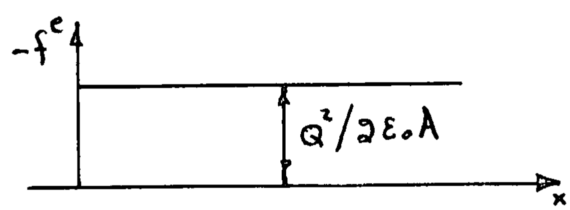
$$W_e = \frac{1}{2}q^2/C \qquad \text{(a)}$$

where

$$C = \varepsilon A/d(1+\frac{\varepsilon x}{\varepsilon_o d}) \qquad \text{(b)}$$

It follows that the force on the upper plate due to the electric field is,

$$-f=f^e=-\frac{\partial W_e}{\partial x}=-\frac{1}{2}\frac{Q^2}{\varepsilon_o A}$$



So long as the charge on the plate is constant, so also is the force.

PROBLEM 3.20 (Continued)

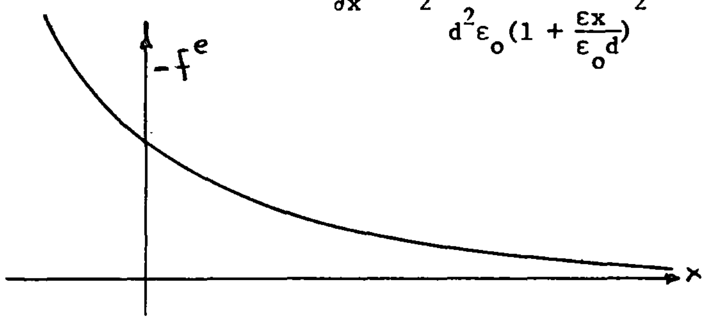
Part b

The electric co-energy is

$$W'_e = \frac{1}{2} Cv^2 \qquad \text{(c)}$$

and hence the force, in terms of the voltage is

$$-f=f^e = \frac{\partial W'_e}{\partial x} = \frac{1}{2} \frac{V^2\varepsilon^2 A}{d^2\varepsilon_o(1 + \frac{\varepsilon x}{\varepsilon_o d})^2} \qquad \text{(d)}$$



The energy converted to mechanical form is $\int f^e\, dx$. The contribution to this integral from d→c and b→a in the figure is zero. Hence,

$$\text{Energy converted to mechanical form} = \int_{\varepsilon_o d/\varepsilon}^{2\varepsilon_o d/\varepsilon} f^e(2Q_o,x)dx$$

$$+ \int_{2\varepsilon_o d/\varepsilon}^{\varepsilon_o d/\varepsilon} f^e(Q_o,x)\,dx = -\frac{3}{2}\frac{dQ_o^2}{A\varepsilon} \qquad \text{(e)}$$

That is, the energy $3dQ_o^2/2A\varepsilon$ is converted from mechanical to electrical form.

PROBLEM 3.21

Part a

The magnetic energy stored in the coupling is

$$W_m = \frac{1}{2}\lambda^2/L \qquad \text{(a)}$$

where $L = L_o/(1 + \frac{x}{a})$

Hence, in terms of λ, the force of electrical origin is

$$-f=f^e= -\frac{\partial W_m}{\partial x} = -\lambda^2/2aL_o \qquad \text{(b)}$$

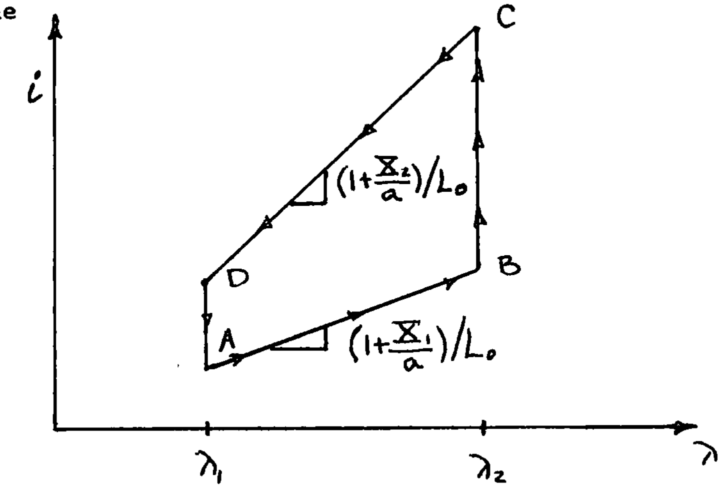
PROBLEM 3.21 (Continued)

Part b

According to the terminal equation, i depends on (λ,x) according to

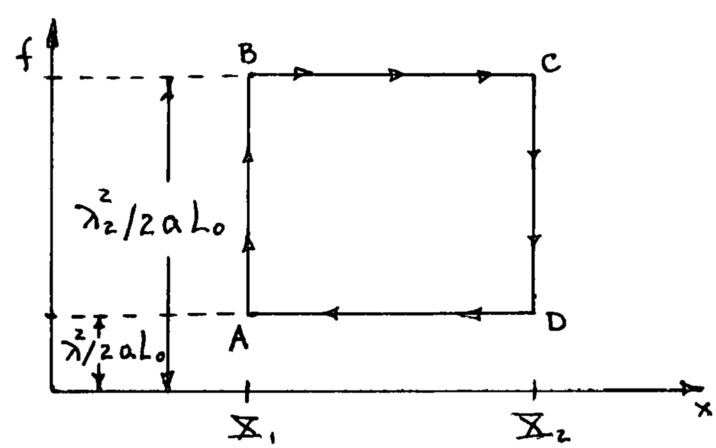
$$i = \frac{\lambda}{L_o}\left(1 + \frac{x}{a}\right) \qquad (c)$$

Thus, the process represented in the λ-x plane has the corresponding path in the i-λ plane



Path c

At the same time, the force traverses a loop in the f-x plane which, from (b) is,



PROBLEM 3.21 (Continued)

Part d

The energy converted per cycle to mechanical form is $\int f^e dx$. Hence,

$$\text{Energy converted to mechanical form} = \int_B^C f^e dx + \int_D^A f^e dx \tag{d}$$

$$= -(\lambda_2^2-\lambda_1^2)(X_2-X_1)/2aL_o \tag{e}$$

That is, the energy converted to electrical form per cycle is $(\lambda_2^2-\lambda_1^2)(X_2-X_1)/2aL_o$. (Note that the energy stored in the coupling, summed around the closed path, is zero because the coupling is conservative.)

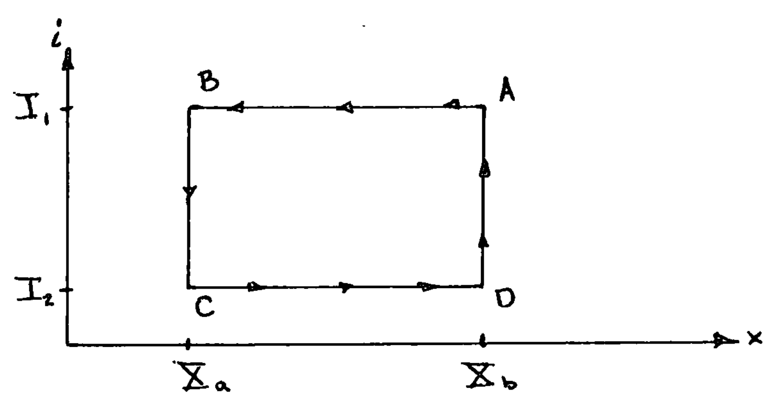
PROBLEM 3.22

Part a

The plates are pushed apart by the fields. Therefore energy is converted from mechanical form to either electrical form or energy storage in the coupling as the plate is moved from X_b to X_a. To make the net conversion from mechanical to electrical form, we therefore make the current the largest during this phase of the cycle or, $I_1>I_2$.

Part b

With the currents related as in part a, the cycle appears in the i-x plane as shown



PROBLEM 3.22 (Continued)

Quantitatively, the magnetic field intensity into the paper is $H = I/D$ so that $\lambda = \mu_o Ixh/D$. Hence,

$$W_m' = \frac{1}{2}\left(\frac{\mu_o xh}{D}\right)I^2 \tag{a}$$

and

$$f^e = \frac{\partial W_m'}{\partial x} = \frac{1}{2}\left(\frac{\mu_o h}{D}\right)I^2 \tag{b}$$

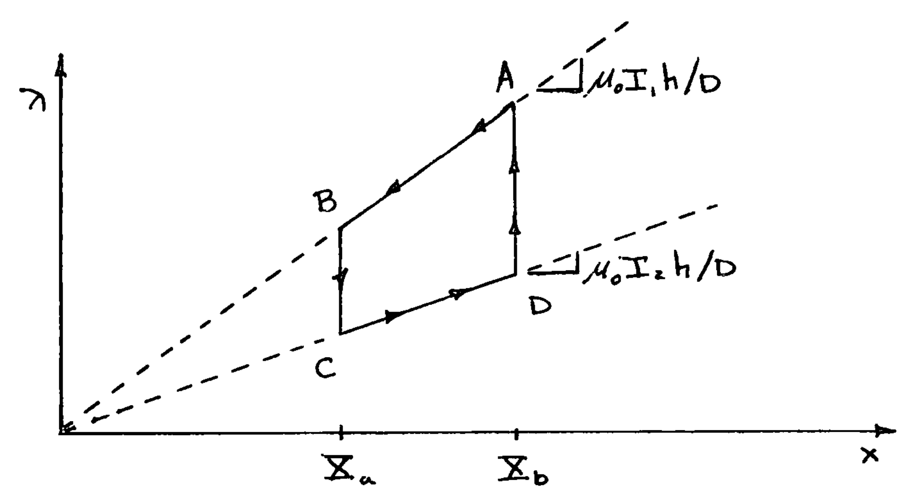
Because the cycle is closed, there is no net energy stored in the coupling, and the energy converted to electrical form is simply that put in in mechanical form:

$$\text{Mechanical to electrical energy per cycle} = -\int_A^B f^e dx - \int_C^D f^e dx \tag{c}$$

$$= \frac{1}{2}\frac{\mu_o h}{D}(X_b - X_a)(I_1^2 - I_2^2) \tag{d}$$

Part c

From the terminal equation and the defined cycle conditions, the cycle in the λ-x plane can be pictured as



The energy converted to electrical form on each of the legs is

PROBLEM 3.22 (Continued)

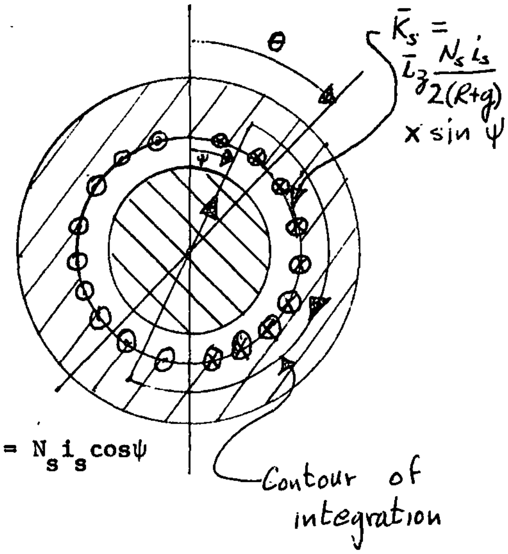
$$(A\rightarrow B) \qquad -\int I_1 d\lambda = -\int_{\mu_o I_1 X_b h/D}^{\mu_o I_1 X_a h/D} I_1 d\lambda = -\frac{\mu_o I_1^2 h}{D}(X_a - X_b) \tag{e}$$

$$(B\rightarrow C) \qquad -\int i d\lambda = -\int_{\mu_o I_1 X_a h/D}^{\mu_o I_2 X_a h/D} \frac{\lambda D d\lambda}{\mu_o X_a h} = -\frac{\mu_o X_a h}{D}(I_2^2 - I_1^2) \tag{f}$$

$$(C\rightarrow D) \qquad -\int \dot{I}_2 d\lambda = \frac{\mu_o I_2^2 h}{D}(X_a - X_b) \tag{g}$$

$$(D\rightarrow A) \qquad -\int i d\lambda = \frac{\mu_o X_b h}{2D}(I_2^2 - I_1^2) \tag{h}$$

The sum of these is equal to (c). Note however that the mechanical energy input on each leg is not necessarily converted to electrical form, but can be stored in the coupling.



PROBLEM 4.1

Part a

With stator current acting alone the situation is as depicted at the right. Recognizing by symmetry that $H_{rs}(\psi+\pi) = -H_{rs}(\psi)$ we use the contour shown and Ampere's law to get

$$2H_{rs}(\psi)g = \int_{\psi}^{\psi+\pi} [\frac{N_s i_s}{2(R+g)} \sin \psi'](R+g)d\psi' = N_s i_s \cos\psi$$

from which

$$H_{rs}(\psi) = \frac{N_s i_s \cos\psi}{2g}$$

and

$$B_{rs}(\psi) = \frac{\mu_o N_s i_s \cos\psi}{2g}$$

Part b

Following the same procedure for rotor excitation alone we obtain

$$B_{rr}(\psi) = \frac{\mu_o N_r i_r \cos(\psi-\theta)}{2g}$$

Note that this result is obtained from part (a) by making the replacements

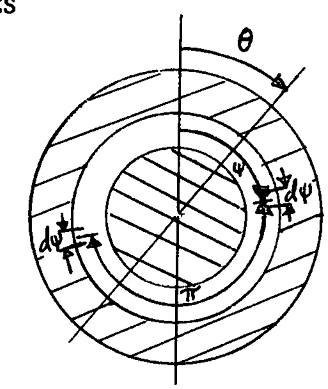
$$N_s \rightarrow N_r$$
$$i_s \rightarrow i_r$$
$$\psi \rightarrow (\psi-\theta)$$

Part c

The flux density varies around the periphery and the windings are distributed, thus a double integration is required to find inductances, whether they are found from stored energy or from flux linkages. We will use flux linkages.

The total radial flux density is

$$B_r = B_{rs} + B_{rr} = \frac{\mu_o}{2g} [N_s i_s \cos\psi + N_r i_r \cos(\psi-\theta)]$$



PROBLEM 4.1 (Continued)

Taking first the elemental coil on the stator having sides of angular span $d\psi$ at positions ψ and $\psi+\pi$ as illustrated. This coil links an amount of flux

$$d\lambda_s = \underbrace{\left[\frac{N_s}{2(R+g)}\sin\psi\right](R+g)\,d\psi}_{\text{number of turns in elemental coil}}\ \underbrace{\left[-\int_{\psi}^{\psi+\pi} B_r(\psi')(R+g)\ell\,d\psi'\right]}_{\text{flux linking one turn of elemental coil}}$$

$$d\lambda_s = -\frac{\mu_o N_s (R+g)\ell}{4\cancel{2}g}\sin\psi d\psi \int_{\psi}^{\psi+\pi}[N_s i_s \cos\psi' + N_r i_r \cos(\psi'-\theta)]d\psi'$$

$$d\lambda_s = \frac{\mu_o N_s (R+g)\ell}{2g}\sin\psi[N_s i_s \sin\psi + N_r i_r \sin(\psi-\theta)]d\psi$$

To find the total flux linkage with the stator coil we add up all of the contributions

$$\lambda_s = \frac{\mu_o N_s (R+g)\ell}{2g}\int_0^{\pi}\sin\psi[N_s i_s \sin\psi + N_r i_r \sin(\psi-\theta)]d\psi$$

$$\lambda_s = \frac{\mu_o N_s (R+g)\ell}{2g}\left[\frac{\pi}{2} N_s i_s + \frac{\pi}{2} N_r i_r \cos\theta\right]$$

This can be written as

$$\lambda_s = L_s i_s + M i_r \cos\theta$$

where

$$L_s = \frac{\pi\mu_o N_s^2 R\ell}{4\cancel{2}g}$$

$$M = \frac{\pi\mu_o N_s N_r R\ell}{4\cancel{2}g}$$

and we have written $R+g \approx R$ because $g \ll R$.

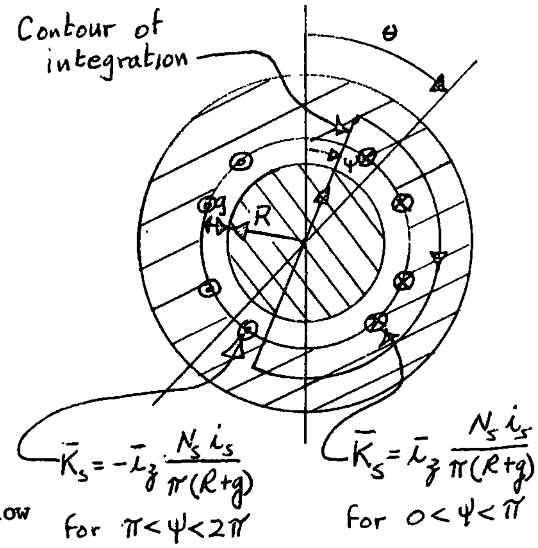
When a similar process is carried out for the rotor winding, it yields

$$\lambda_r = L_r i_r + M i_s \cos\theta$$

where

$$L_r = \frac{\pi\mu_o N_r^2 R\ell}{4\cancel{2}g}$$

and M is the same as calculated before.



PROBLEM 4.2

Part a

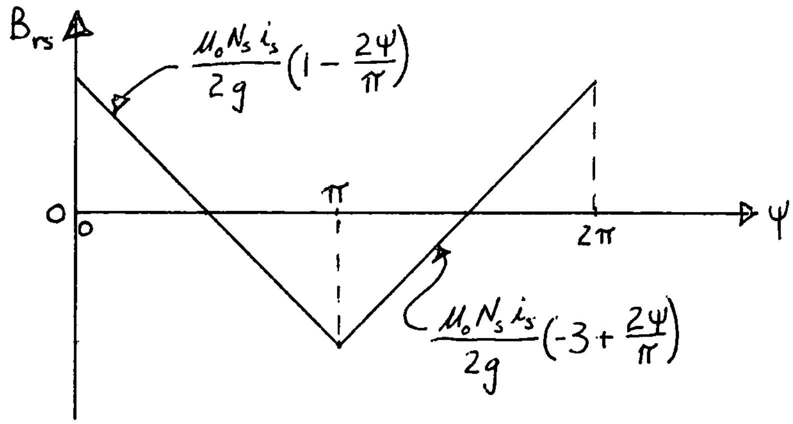
Application of Ampere's law with the contour shown and use of the symmetry condition

$H_{rs}(\psi+\pi) = -H_{rs}(\psi)$ yields

$2H_{rs}(\psi)g = N_s i_s(1-\frac{2\psi}{\pi})$; for $0 < \psi < \pi$

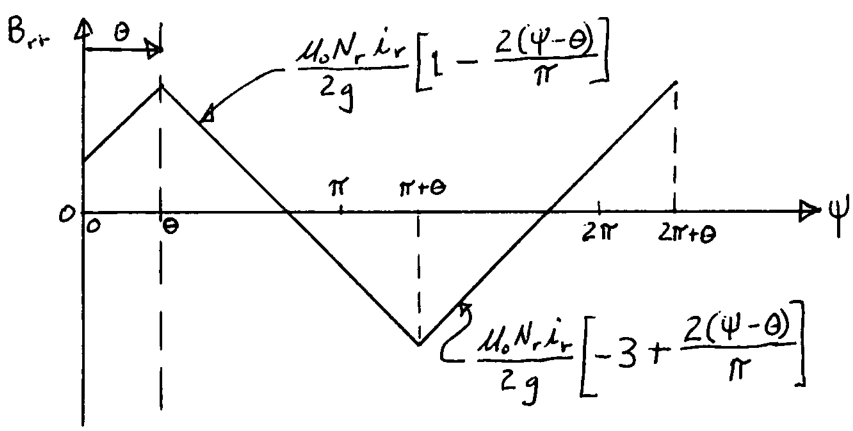
$2H_{rs}(\psi)g = N_s i_s(-3+\frac{2\psi}{\pi})$; for $\pi < \psi < 2\pi$

The resulting flux density is sketched below



Part b

The same process applied to excitation of the rotor winding yields



PROBLEM 4.2 (Continued)

Part c

For calculating inductances it will be helpful to have both flux densities and turn densities in terms of Fourier series. The turn density on the stator is expressible as

$$n_s = \frac{4N_s}{\pi^2(R+g)} \sum_{n \text{ odd}} \frac{1}{n} \sin n\psi$$

and the turn density on the rotor is

$$n_r = \frac{4N_r}{\pi^2 R} \sum_{n \text{ odd}} \frac{1}{n} \sin(\psi-\theta)$$

and the flux densities are expressible as

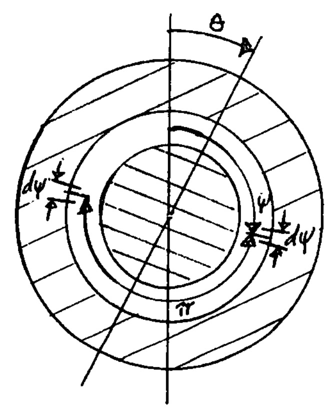
$$B_{rs} = \sum_{n \text{ odd}} \frac{4\mu_o N_s i_s}{\pi^2 g n^2} \cos n\psi$$

$$B_{rr} = \sum_{n \text{ odd}} \frac{4\mu_o N_r i_r}{\pi^2 g n^2} \cos n(\psi-\theta)$$

The total radial flux density is

$$B_r = B_{rs} + B_{rr}$$

First calculating stator flux linkages, we first consider the elemental coil having sides $d\psi$ long and π radians apart



$$d\lambda_s = \underbrace{n_s(R+g)d\psi}_{\text{number of turns}} \underbrace{\left[-\int_{\psi}^{\psi+\pi} B_r(\psi')(R+g)\ell\, d\psi'\right]}_{\text{flux linking one turn of elemental coil}}$$

Substitution of series for B_r yields

$$d\lambda_s = n_s(R+g)^2\ell d\psi \left[\sum_{n \text{ odd}} \frac{8\mu_o N_s i_s}{\pi^2 g n^3} \sin n\psi + \sum_{n \text{ odd}} \frac{8\mu_o N_r i_r}{\pi^2 g n^3} \sin n(\psi-\theta)\right]$$

The total flux linkage with the stator coil is

$$\lambda_s = \frac{32\mu_o N_s(R+g)\ell}{\pi^4 g} \int_0^{\pi} \left[\sum_{n \text{ odd}} \frac{1}{n} \sin n\psi\right]\left[\sum_{n \text{ odd}} \frac{N_s i_s}{n^3} \sin n\psi + \sum_{n \text{ odd}} \frac{N_r i_r}{n^3} \sin n(\psi-\theta)\right] d\psi$$

PROBLEM 4.2 (Continued)

Recognition that

$$\int_0^{\pi} \sin n\psi \sin m(\psi-\theta)d\psi = 0 \text{ when } m \neq n$$

simplifies the work in finding the solution

$$\lambda_s = \frac{32\mu_o N_s(R+g)\ell}{\pi^4 g} \sum_{nodd} \left(\frac{\pi N_s i_s}{2n^4} + \frac{\pi N_r i_r}{2n^4} \cos n\theta\right)$$

This can be written in the form

$$\lambda_s = L_s i_s + \sum_{nodd} M_n \cos n\theta \; i_r$$

where

$$L_s = \frac{16\mu_o N_s^2 R\ell}{\pi^3 g} \sum_{nodd} \frac{1}{n^4}$$

$$M_n = \frac{16\mu_o N_s N_r R\ell}{\pi^3 g n^4}$$

In these expressions we have used the fact that $g << R$ to write $R+g \approx R$.

A similar process with the rotor winding yields

$$\lambda_r = L_r i_r + \sum_{nodd} M_n \cos n\theta \; i_s$$

where

$$L_r = \frac{16\mu_o N_s^2 R\ell}{\pi^3 g} \sum_{nodd} \frac{1}{n^4}$$

and M_n is as given above.

PROBLEM 4.3

With reference to the solution of Prob. 4.2, if the stator winding is sinusoidally distributed, λ_s becomes

$$\lambda_s = \frac{32\mu_o N_s(R+g)\ell}{\pi^4 g} \int_o^{\pi} \sin\psi \left[N_s i_s \sin\psi + \sum_{nodd} \frac{N_r i_r}{n^3} \sin n(\psi-\theta)\right] d\psi$$

Because $\int_o^{\pi} \sin\psi \sin n(\psi-\theta) = 0$ when $n \neq 1$

$$\lambda_s = \frac{32\mu_o N_s(R+g)\ell}{\pi^4 g} \int_o^{\pi} \sin\psi \left[N_s i_s \sin\psi + N_r i_r \sin(\psi-\theta)\right] d\psi$$

and the mutual inductance will contain no harmonic terms.

Similarly, if the rotor winding is sinusoidally distributed,

PROBLEM 4.3 (Continued)

$$\lambda_s = \frac{32\mu_o N_s (R+g)\ell}{\pi^4 g} \int_o^{\pi} \left[\sum_{nodd} \frac{1}{n} \sin n\psi\right]\left[\sum_{nodd} \frac{N_s i_s}{n^3} \sin n\psi + N_r i_r \sin(\psi-\theta)\right] d\psi$$

Using the orthogonality condition

$$\int_o^{\pi} \sin n\psi \sin(\psi-\theta) d\psi = 0 \text{ when } n \neq 1$$

$$\lambda_s = \frac{32\mu_o N_s (R+g)\ell}{\pi^4 g} \int_o^{\pi} \left[\sum_{nodd} \frac{N_s i_s}{n^4} \sin^2 n\psi + N_r i_r \sin\psi \sin(\psi-\theta)\right] d\psi$$

and the mutual inductance once again contains only a space fundamental term.

PROBLEM 4.4

Part a

The open-circuit stator voltage is

$$v_s = \frac{d\lambda_s}{dt} = \frac{d}{dt}\left[I \sum_{nodd} \frac{M_o}{n^4} \cos n\omega t\right]$$

$$v_s(t) = - \sum_{nodd} \frac{\omega M_o I}{n^3} \sin n\omega t$$

Part b

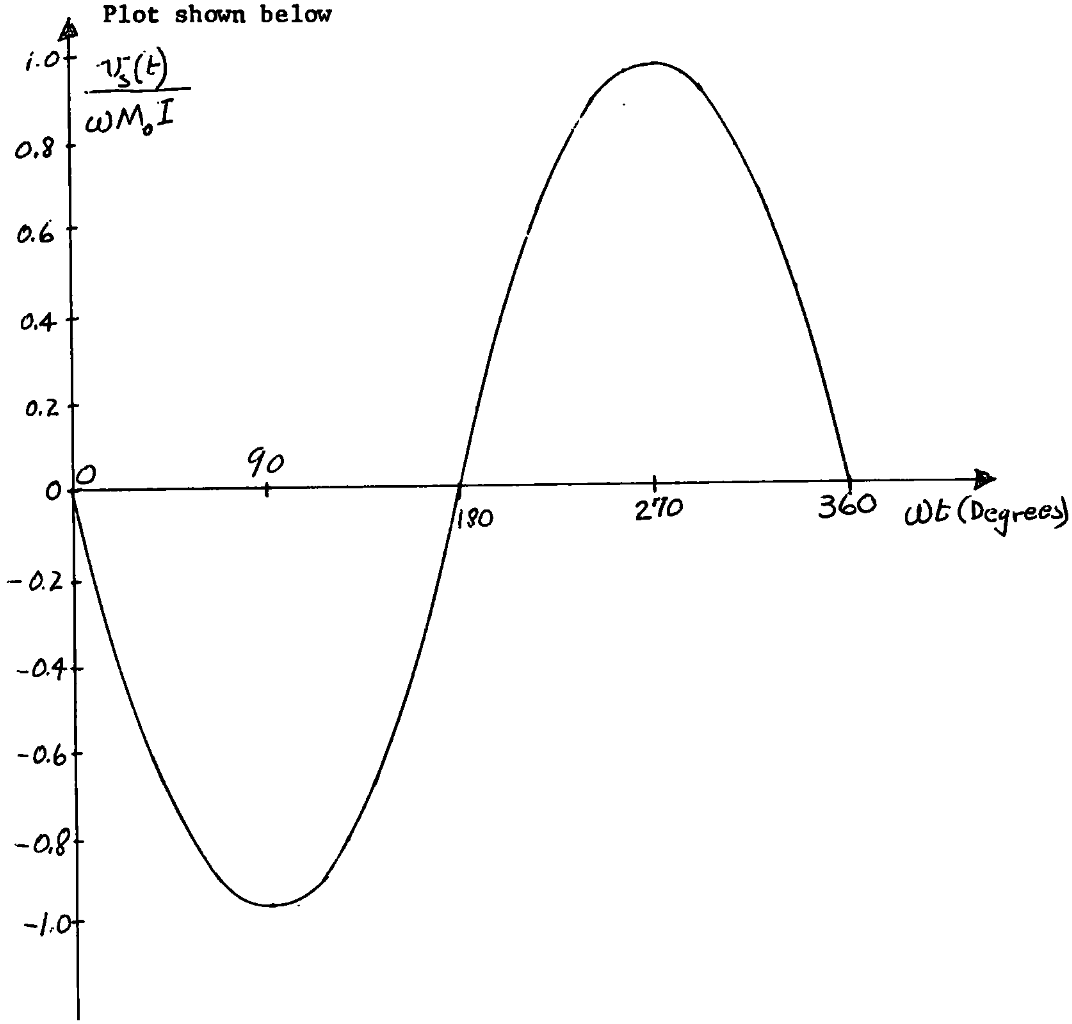
$$\frac{V_{sn}}{V_{s1}} = \frac{1}{n^3} \; ; \; \frac{V_{s3}}{V_{s1}} = \frac{1}{27} \approx 4 \text{ percent}$$

This indicates that uniform turn density does not yield unreasonably high values of harmonics.

PROBLEM 4.4 (Continued)

Part c

Plot shown below



PROBLEM 4.5

Given electrical terminal relations are

$$\lambda_s = L_s i_s + M i_r \cos\theta$$

$$\lambda_r = M i_s \cos\theta + L_r i_r$$

System is conservative so energy or coenergy is independent of path. Select currents and θ as independent variables and use coenergy (see Table 3.1). Assemble system first mechanically, then electrically so torque is not needed in calculation of coenergy. Selecting one of many possible paths of integration for i_s and i_r we have

PROBLEM 4.5 (Continued)

$$W_m'(i_s,i_r,\theta) = \int_o^{i_s} \lambda_s(i_s',0,\theta)di_s' + \int_o^{i_r} \lambda_r(i_s,i_r',\theta)di_r'$$

$$W_m'(i_s,i_r,\theta) = \frac{1}{2} L_s i_s^2 + Mi_r i_s \cos\theta + \frac{1}{2} L_r i_r^2$$

$$T^e = \frac{\partial W_m'(i_s,i_r,\theta)}{\partial\theta} = - Mi_r i_s \cos\theta$$

PROBLEM 4.6

The conditions existing at the time the rotor winding terminals are short-circuited lead to the constant rotor winding flux linkages

$$\lambda_r = MI_o$$

This constraint leads to a relation between i_r and $i_s = i(t)$

$$MI_o = Mi_s \cos\theta + L_r i_r$$

$$i_r = \frac{M}{L_r} [I_o - i(t)\cos\theta]$$

The torque equation (4.1.8) is valid for any terminal constraint, thus

$$T^e = -Mi_r i_s \cos\theta = - \frac{M^2}{L_r} i(t)[I_o - i(t)\cos\theta]\sin\theta$$

The equation of motion for the shaft is then

$$J \frac{d^2\theta}{dt^2} = - \frac{M^2}{L_r} i(t)[I_o - i(t)\cos\theta]\sin\theta$$

PROBLEM 4.7

Part a

Coenergy is

$$W_m'(i_s,i_r,\theta) = \frac{1}{2} L_s i_s^2 + \frac{1}{2} L_r i_r^2 + L_{sr}(\theta) i_s i_r$$

$$T^e = \frac{\partial W_m'(i_s,i_r,\theta)}{\partial\theta} = i_s i_r \frac{dL_{sr}(\theta)}{d\theta}$$

$$T^e = -i_s i_r [M_1 \sin\theta + 3M_3 \sin 3\theta]$$

Part b

With the given constraints

$$T^e = -I_s I_r \sin\omega_s t \sin\omega_r t[M_1 \sin(\omega_m t+\gamma) + 3M_3 \sin 3(\omega_m t+\gamma)]$$

PROBLEM 4.7 (Continued)

Repeated application of trigonometric identities leads to:

$$T^e = -\frac{M_1 I_s I_r}{4}\Big\{\sin[(\omega_m+\omega_s-\omega_r)t+\gamma] + \sin[(\omega_m-\omega_s+\omega_r)t+\gamma]$$

$$-\sin[(\omega_m+\omega_s+\omega_r)t+\gamma] - \sin[(\omega_m-\omega_s-\omega_r)t+\gamma]\Big\}$$

$$-\frac{3M_3 I_s I_r}{4}\Big\{\sin[(3\omega_m+\omega_s-\omega_r)t+3\gamma] + \sin[(3\omega_m-\omega_s+\omega_r)t+3\gamma]$$

$$-\sin[(3\omega_m+\omega_s+\omega_r)t+3\gamma] - \sin[(3\omega_m-\omega_s-\omega_r)t+3\gamma]\Big\}$$

To have a time-average torque, one of the coefficients of time must equal zero. This leads to the eight possible mechanical speeds

$$\omega_m = \pm\,\omega_s \pm \omega_r \quad \text{and} \quad \omega_m = \pm\frac{\omega_s \pm \omega_r}{3}$$

For

$$\omega_m = \pm(\omega_s - \omega_r)$$

$$T^e_{avg} = -\frac{M_1 I_s I_r}{4}\sin\gamma$$

For

$$\omega_m = \pm(\omega_s + \omega_r)$$

$$T^e_{avg} = \frac{M_1 I_s I_r}{4}\sin\gamma$$

For

$$\omega_m = \pm\frac{(\omega_s-\omega_r)}{3}$$

$$T^e = -\frac{3M_3 I_s I_r}{4}\sin 3\gamma$$

For

$$\omega_m = \pm\frac{(\omega_s+\omega_r)}{3}$$

$$T^e_{avg} = \frac{3M_3 I_s I_r}{4}\sin 3\gamma$$

PROBLEM 4.8

From 4.1.8 and the given constraints the instantaneous torque is

$$T^e = -I_r M \sin\omega_r t\,\cos(\omega_m t+\gamma)(I_{s1}\sin\omega_s t + I_{s3}\sin 3\omega_s t)$$

Repeated use of trigonometric identities leads to:

PROBLEM 4.8 (Continued)

$$T^e = -\frac{I_r I_{s1} M}{4}\Big\{\cos[(\omega_r+\omega_m-\omega_s)t+\gamma]-\cos[(\omega_r+\omega_m+\omega_s)t+\gamma]$$

$$+\cos[(\omega_r-\omega_m-\omega_s)t-\gamma]-\cos[(\omega_r-\omega_m+\omega_s)t-\gamma]\Big\}$$

$$-\frac{I_r I_{s3} M}{4}\Big\{\cos[(\omega_r+\omega_m-3\omega_s)t+\gamma]-\cos[(\omega_r+\omega_m+3\omega_s)t+\gamma]$$

$$+\cos[(\omega_r-\omega_m-3\omega_s)t-\gamma]-\cos[(\omega_r-\omega_m+3\omega_s)t-\gamma]\Big\}$$

For a time-average torque one of the coefficients of t must be zero. This leads to eight values of ω_m:

$$\omega_m = \pm\,\omega_r \pm \omega_s \quad \text{and} \quad \omega_m = \pm\,\omega_r \pm 3\omega_s$$

For

$$\omega_m = \pm(\omega_r - \omega_s)$$

$$T^e_{avg} = -\frac{I_r I_{s1} M}{4}\cos\gamma$$

For

$$\omega_m = \pm(\omega_r + \omega_s)$$

$$T^e_{avg} = \frac{I_r I_{s1} M}{4}\cos\gamma$$

For

$$\omega_m = \pm(\omega_r - 3\omega_s)$$

$$T^e_{avg} = -\frac{I_r I_{s3} M}{4}\cos\gamma$$

For

$$\omega_m = \pm(\omega_r + 3\omega_s)$$

$$T^e_{avg} = \frac{I_r I_{s3} M}{4}\cos\gamma$$

PROBLEM 4.9

Electrical terminal relations are 4.1.19-4.1.22. For conservative system, coenergy is independent of path and if we bring system to its final mechanical configuration before exciting it electrically there is no contribution to the coenergy from the torque term. Thus, of the many possible paths of integration we choose one

PROBLEM 4.9 (Continued)

$$W_m'(i_{as},i_{bs},i_{ar},i_{br},\theta) = \int_o^{i_{as}} \lambda_{as}(i_{as}',0,0,0,\theta)di_{as}'$$

$$+ \int_o^{i_{bs}} \lambda_{bs}(i_{as},i_{bs}',0,0,\theta)di_{bs}'$$

$$+ \int_o^{i_{ar}} \lambda_{ar}(i_{as},i_{bs},i_{ar}',0,\theta)di_{ar}'$$

$$+ \int_o^{i_{br}} \lambda_{br}(i_{as},i_{bs},i_{ar},i_{br}',\theta)di_{br}'$$

The use of 4.1.19-4.1.22 in this expression yields

$$W_m' = \int_o^{i_{as}} L_s i_{as}' \; i_{as}' + \int_o^{i_{bs}} L_s i_{bs}' di_{bs}'$$

$$+ \int_o^{i_{ar}} (L_r i_{ar}' + Mi_{as} \cos\theta + Mi_{bs}\sin\theta)di_{ar}'$$

$$+ \int_o^{i_{br}} (L_r i_{br}' - Mi_{as}\sin\theta + Mi_{bs}\cos\theta)di_{br}'$$

Evaluation of these integrals yields

$$W_m' = \frac{1}{2} L_s i_{as}^2 + \frac{1}{2} L_s i_{bs}^2 + \frac{1}{2} L_r i_{ar}^2 + \frac{1}{2} L_r i_{br}^2$$

$$+ Mi_{as}i_{ar} \cos\theta + Mi_{bs}i_{ar}\sin\theta$$

$$- Mi_{as}i_{br}\sin \theta + Mi_{bs}i_{br}\cos\theta$$

The torque of electric origin is then (see Table 3.1)

$$T^e = \frac{\partial W_m'(i_{as},i_{bs},i_{ar},i_{br},\theta)}{\partial\theta}$$

$$T^e = -M[i_{as}i_{ar}\sin\theta - i_{bs}i_{ar}\cos\theta + i_{as}i_{br}\cos\theta + i_{bs}i_{br}\sin\theta]$$

PROBLEM 4.10

Part a

Substitution of currents into given expressions for flux density

$$B_r = B_{ra} + B_{rb}$$

$$B_r = \frac{\mu_o N}{2g} [I_a \cos \omega t \cos \psi + I_b \sin \omega t \sin \psi]$$

PROBLEM 4.10 (Continued)

Part b

Application of trigonometric identities and simplification yield.

$$B_r = \frac{\mu_o N}{2g}\left[\frac{I_a}{2}\cos(\omega t-\psi) + \frac{I_a}{2}\cos(\omega t + \psi)\right.$$

$$\left. + \frac{I_b}{2}\cos(\omega t-\psi) - \frac{I_b}{2}\cos(\omega t + \psi)\right]$$

$$B_r = \frac{\mu_o N}{4g}\left[(I_a+I_b)\cos(\omega t- \psi)+(I_a-I_b)\cos(\omega t+\psi)\right]$$

The forward wave is

$$B_{rf} = \frac{\mu_o N(I_a + I_b)}{4g}\cos(\omega t-\psi)$$

For constant phase on the forward wave

$$\omega t - \psi = \text{constant}$$

$$\omega_f = \frac{d\psi}{dt} = \omega$$

The backward wave is

$$B_{rb} = \frac{\mu_o N(I_a - I_b)}{4g}\cos(\omega t + \psi)$$

For

$$\omega t + \psi = \text{constant}$$

$$\omega_b = \frac{d\psi}{dt} = - \omega$$

Part c

The ratio of amplitudes is

$$\frac{B_{rbm}}{B_{rfm}} = \frac{I_a-I_b}{I_a+I_b}$$

$$\frac{B_{rbm}}{B_{rfm}} \rightarrow 0 \quad \text{as} \quad I_a \rightarrow I_b$$

Part d

When $I_b = - I_a$

$$B_{rf} = 0$$

This has simply reversed the phase sequence.

PROBLEM 4.11

Part a

$$B_r = B_{ra} + B_{rb}$$

$$B_r = \frac{\mu_o NI}{2g}[\cos \omega t \cos \psi + \sin(\omega t + \beta)\sin\psi]$$

Part b

Using trigonometric identities

$$B_r = \frac{\mu_o NI}{2g}[\cos \omega t \cos \psi + \cos \beta \sin \omega t \sin \psi + \sin \beta \cos \omega t \sin \psi]$$

$$B_r = \frac{\mu_o NI}{2g} [\frac{1}{2} \cos(\omega t-\psi)+ \frac{1}{2} \cos(\omega t+\psi)$$

$$+ \frac{\cos\beta}{2} \cos(\omega t-\psi)- \frac{\cos\beta}{2} \cos(\omega t+\beta)$$

$$+ \frac{\sin\beta}{2} \sin(\omega t+\psi)- \frac{\sin\beta}{2} \sin(\omega t-\psi)]$$

$$B_r = \frac{\mu_o NI}{4g} [(1+\cos\beta)\cos(\omega t-\psi)-\sin\beta\sin(\omega t-\psi)$$

$$+ (1-\cos\beta)\cos(\omega t+\psi)+\sin\beta\sin(\omega t+\psi)]$$

Forward wave is

$$B_{rf} = \frac{\mu_o NI}{4g} [(1+\cos\beta)\cos(\omega t-\psi)-\sin\beta\sin(\omega t-\psi)]$$

For constant phase

$$\omega t - \psi = \text{constant}$$

and

$$\omega_f = \frac{d\psi}{dt} = \omega$$

Backward wave is

$$B_{rb} = \frac{\mu_o NI}{4g}[(1-\cos\beta)\cos(\omega t+\psi)+\sin\beta\sin(\omega t+\psi)]$$

For constant phase

$$\omega t + \psi = \text{constant}$$

and

$$\omega_b = \frac{d\psi}{dt} = - \omega$$

PROBLEM 4.11 (Continued)

Part c

The ratio of amplitudes is

$$\frac{B_{rbm}}{B_{rfm}} = \frac{\sqrt{(1-\cos\beta)^2 + \sin^2\beta}}{\sqrt{(1+\cos\beta)^2 + \sin^2\beta}} = \sqrt{\frac{1-\cos\beta}{1+\cos\beta}}$$

as $\beta \to 0$, $\frac{B_{rbm}}{B_{rfm}} \to 0$.

Part d

The forward wave amplitude will go to zero when $\beta = \pi$. The phase sequence has been reversed by reversing the phase of the current in the b-winding.

PROBLEM 4.12

Equation 4.1.53 is

$$p_e = v_{as}i_{as} + v_{bs}i_{bs}$$

For steady state balanced conditions we can write

$$i_{as} = I\cos\omega t; \quad i_{bs} = I\sin\omega t$$

$$v_{as} = V\cos(\omega t+\phi); \quad v_{bs} = V\sin(\omega t+\phi)$$

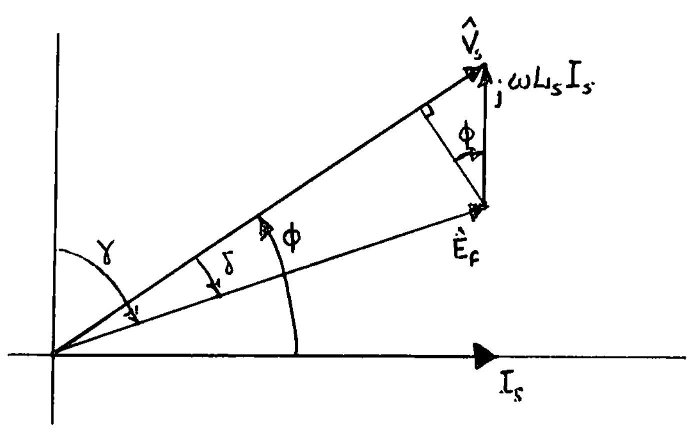
then

$$p_e = VI[\cos\omega t\cos(\omega t+\phi) + \sin\omega t\sin(\omega t+\phi)]$$

Using trigonometric identities

$$p_e = VI\cos\phi$$

Referring to Fig. 4.1.13(b) we have the vector diagram



PROBLEM 4.12 (Continued)

From this figure it is clear that

$$\omega L_s I_s \cos\phi = -E_f \sin\delta$$

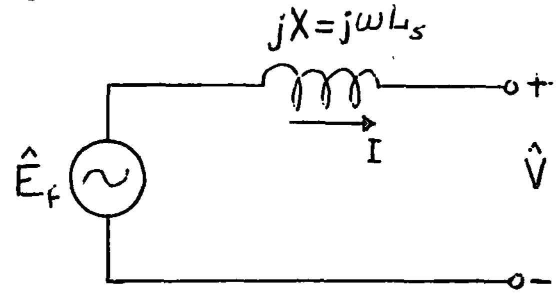
(remember that $\delta < 0$)

Then $p_e = -\dfrac{VE_f}{\omega L_s} \sin\delta$

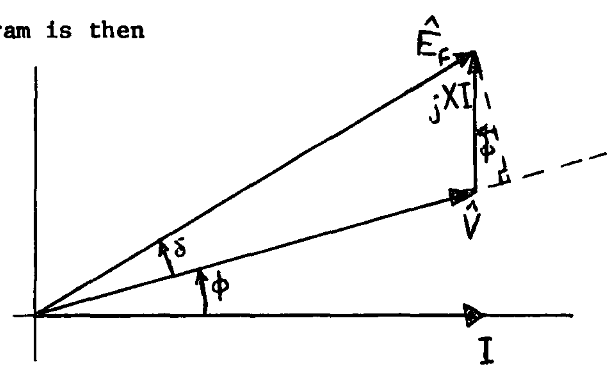
which was to be shown.

PROBLEM 4.13

For the generator we adopt the notation for one phase of the armature circuit (see Fig. 4.1.12 with current convention reversed)



The vector diagram is then



From the vector diagram

$$XI \sin\phi = E_f \cos\delta - V$$

$$XI \cos\phi = E_f \sin\delta$$

Also, the mechanical power input is

$$P = \frac{E_f V}{X} \sin\delta$$

Eliminating ϕ and δ from these equations and solving for I yields

PROBLEM 4.13 (Continued)

$$I = \frac{V}{X}\sqrt{\left(\frac{E_f}{V}\right)^2 - 2\sqrt{\left(\frac{E_f}{V}\right)^2 - \left(\frac{PX}{V^2}\right)^2} + 1}$$

Normalizing as indicated in the problem statement we define

I_o = rated armature current

I_{fo} = field current to give rated voltage on open circuit.

P_o = rated power

$$\frac{I}{I_o} = \frac{V}{I_o X}\sqrt{\left(\frac{I_f}{I_{fo}}\right)^2 + 1 - 2\sqrt{\left(\frac{I_f}{I_{fo}}\right)^2 - \left(\frac{P}{P_o}\right)^2\left(\frac{P_o X}{V^2}\right)^2}}$$

Injecting given numbers and being careful about rms and peak quantities we have

$$\frac{I}{I_o} = 0.431\sqrt{\left(\frac{I_f}{I_{fo}}\right)^2 + 1 - 2\sqrt{\left(\frac{I_f}{I_{fo}}\right)^2 - 3.92\left(\frac{P}{P_o}\right)^2}}$$

$$I_{fo} = 2{,}030 \text{ amps}$$

and

$$\left(\frac{I_f}{I_{fo}}\right)_{max} = 3.00$$

The condition that $\delta = \frac{\pi}{2}$ is

$$E_f = \frac{PX}{V}$$

$$\left(\frac{I_f}{I_{fo}}\right)_{min} = \frac{PX}{\omega M I_{fo} V} = \frac{PX}{V^2} = 1.98\,\frac{P}{P_o}$$

For unity p.f., $\cos\phi = 1$, $\sin\phi = 0$

$$E_f \cos\delta = V \quad \text{and} \quad E_f \sin\delta = IX$$

eliminating δ we have

$$\frac{I}{I_o} = \frac{V}{XI_o}\sqrt{\left(\frac{E_f}{V}\right)^2 - 1}$$

$$\frac{I}{I_o} = 0.431\sqrt{\left(\frac{I_f}{I_{fo}}\right)^2 - 1}$$

PROBLEM 4.13 (Continued)

for 0.85 p.f.

$$E_f \sin \delta = 0.85\ IX$$

$$E_f \cos \delta - V = \sqrt{1-(0.85)^2}\ \ IX$$

eliminating δ, solving for I, and normalizing yields

$$\frac{I}{I_o} = 0.431 \left| [-0.527 \pm \sqrt{(\frac{I_f}{I_{fo}})^2 - 0.722}] \right|$$

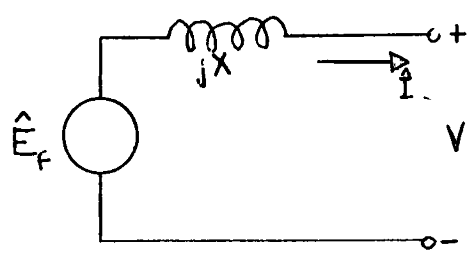
This is double-valued and the magnitude of the bracketed term is used.

The required curves are shown on the next page.

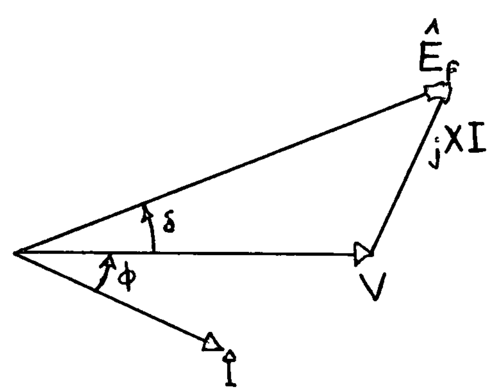
PROBLEM 4.14

The armature current limit is defined by a circle of radius VI_o, where I_o is the amplitude of rated armature current.

To find the effect of the field current limit we must express the complex power in terms of field current. Defining quantities in terms of this circuit

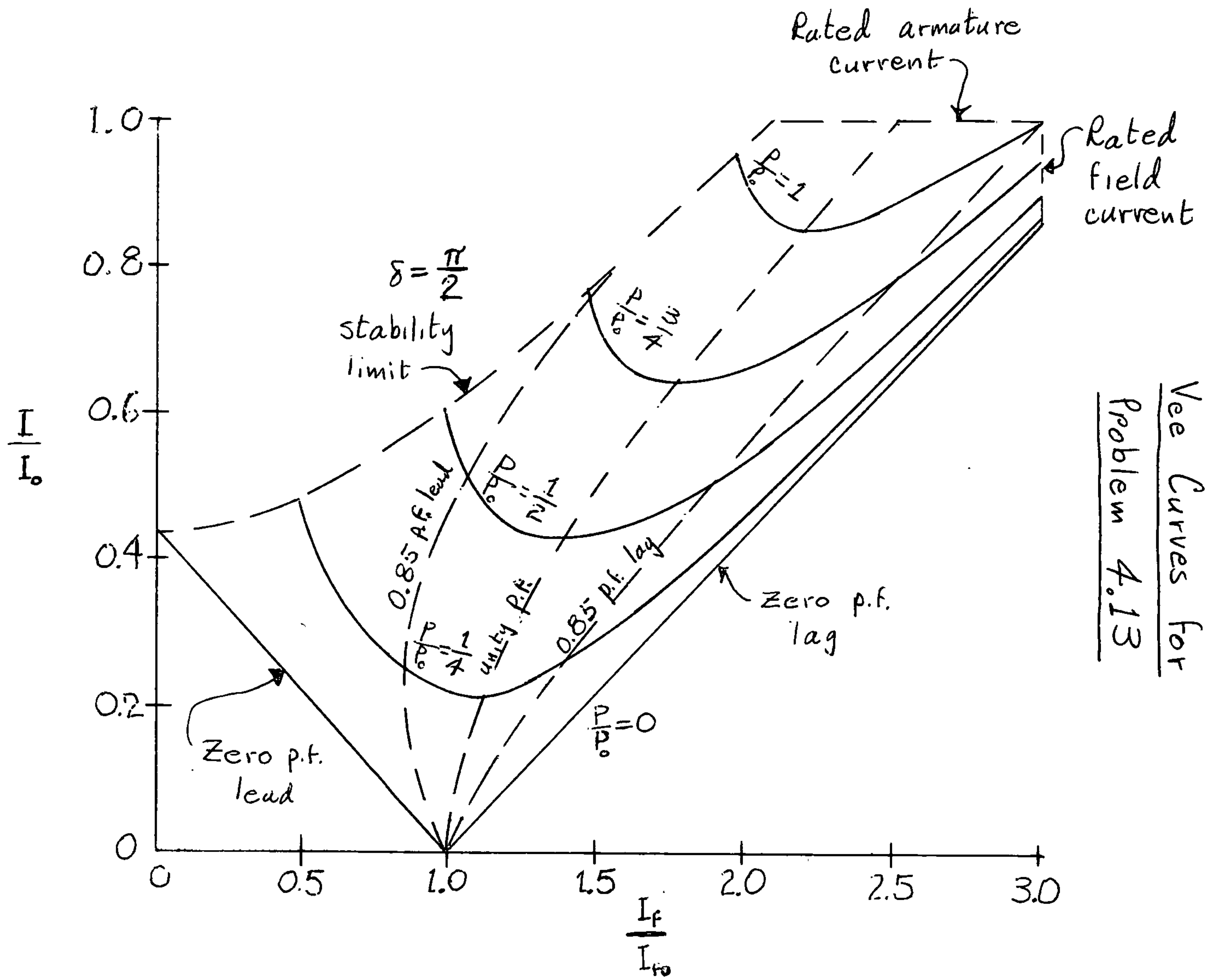


The vector diagram is



$$\hat{I} = \frac{\hat{E}_f - V}{jX}$$

$$P + jQ = V\hat{I}^* = \frac{V\hat{E}_f^* - V^2}{-jX} = j\,\frac{VE_f e^{-j\delta}}{X} - j\,\frac{V^2}{X}$$



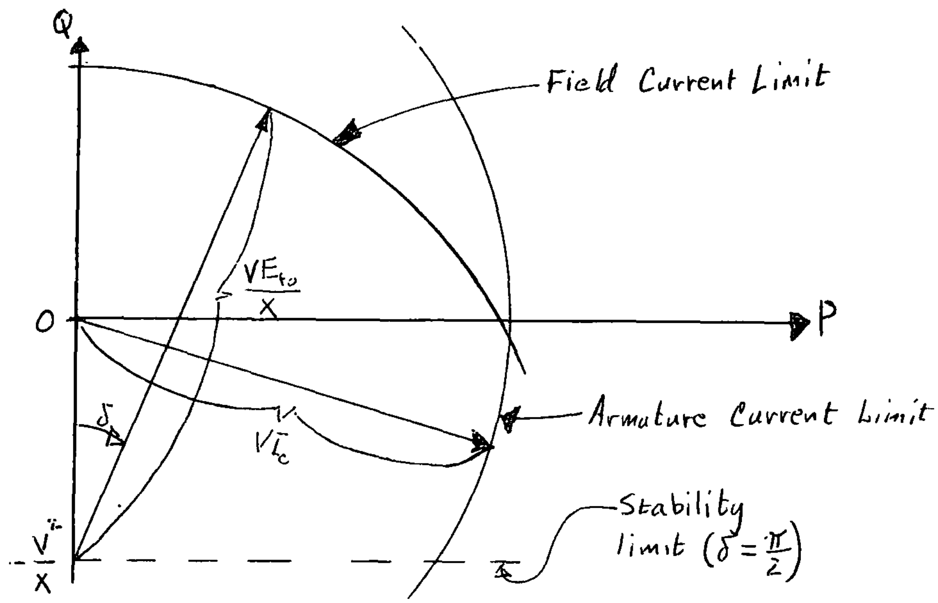
Rated armature current
Rated field current
$\delta = \frac{\pi}{2}$
stability limit
$\frac{P}{P_o} = 1$
$\frac{P}{P_o} = \frac{3}{4}$
$\frac{P}{P_o} = \frac{1}{2}$
$\frac{P}{P_o} = \frac{1}{4}$
$\frac{P}{P_o} = 0$
0.85 p.f. lead
unity p.f.
0.85 p.f. lag
Zero p.f. lag
Zero p.f. lead
$\frac{I}{I_o}$
$\frac{I_f}{I_{fo}}$
0
0.2
0.4
0.6
0.8
1.0
0.5
1.5
2.0
2.5
3.0
Vee Curves for Problem 4.13

PROBLEM 4.14 (Continued)

If we denote the voltage for maximum field current as E_{fo}, this expression becomes

$$P+jQ = -j\,\frac{V^2}{X} + \frac{VE_{fo}}{X}\sin\delta + j\,\frac{VE_{fo}}{X}\cos\delta$$

On a P+jQ plane this trajectory is as sketched below



The stability limit ($\delta = \frac{\pi}{2}$) is also shown in the sketch, along with the armature current limit.

The capability curve for the generator of Prob. 4.13 is shown on the next page.

P and Q are normalized to 724 MVA.

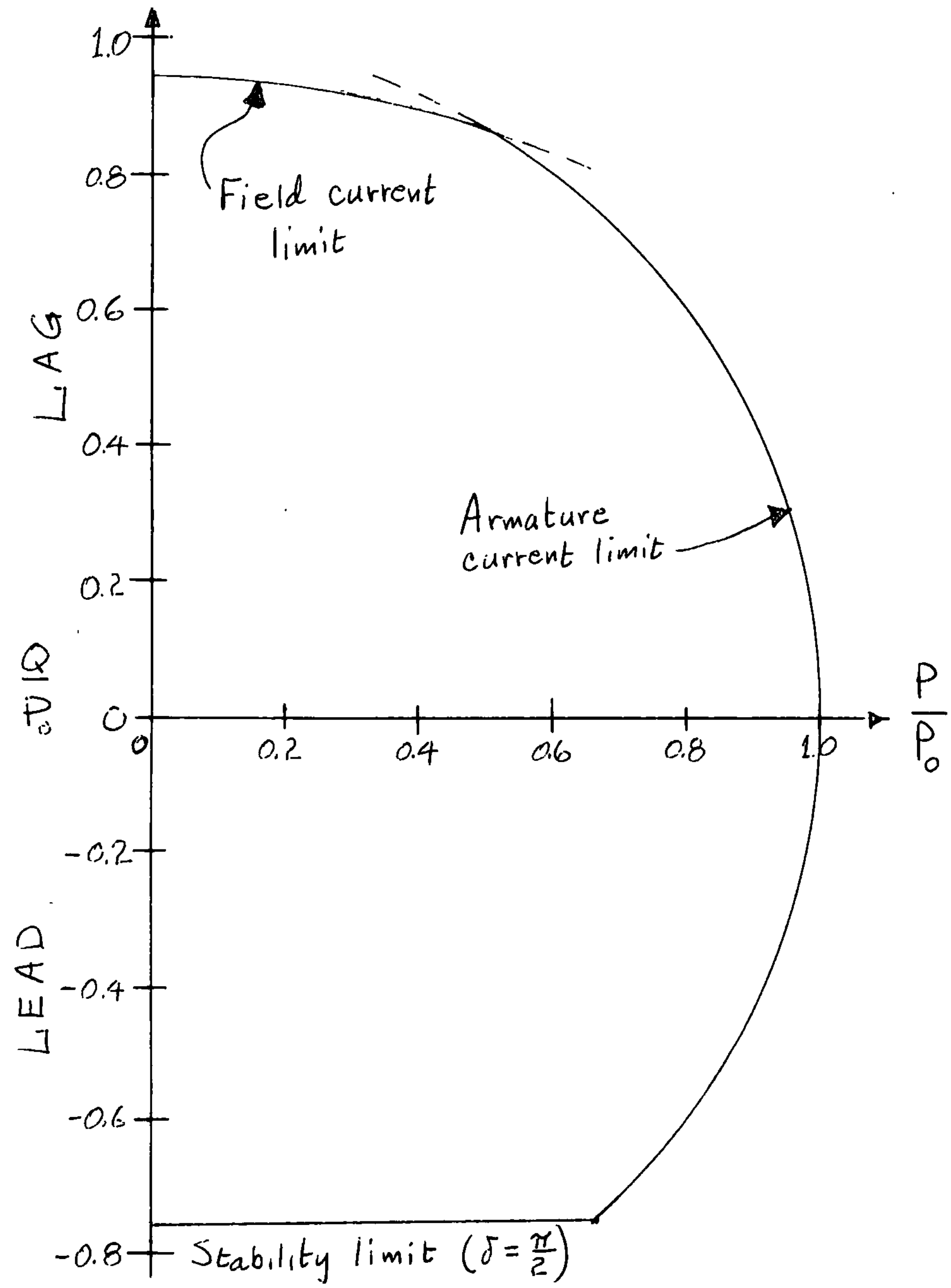
PROBLEM 4.15

The steady state deflection ψ of the rotatable frame is found by setting sum of torques to zero

$$T^e + T_s = 0 = T^e - K\psi \qquad (1)$$

where T^e is electromagnetic torque. This equation is solved for ψ.

Torque T^e is found from



Capability curve of Problem 4.14

PROBLEM 4.15 (Continued)

$$T^e = \frac{\partial W_m'(i_1,i_2,i_3,\phi,\psi)}{\partial\psi}$$

and the magnetic coenergy for this electrically linear system is

$$W_m' = \frac{1}{2}Li_1^2 + \frac{1}{2}Li_2^2 + \frac{1}{2}L_3i_3^2$$
$$+ Mi_1i_3\cos(\phi-\psi)+Mi_2i_3\sin(\phi-\psi)$$

from which

$$T^e = Mi_1i_3\sin(\phi-\psi) - Mi_2i_3\cos(\phi-\psi)$$

For constant shaft speed ω, the shaft position is

$$\phi = \omega t.$$

Then, with $i_3 = I_o$ as given

$$\frac{d\lambda_1}{dt} = -\omega MI_o\sin(\omega t-\psi)+L\frac{di_1}{dt} = -i_1R$$

and

$$\frac{d\lambda_2}{dt} = \omega MI_o\cos(\omega t-\psi)+L\frac{di_2}{dt} = -i_2R$$

Using the given assumptions that

$$\left|L\frac{di_1}{dt}\right| << \left|Ri_1\right| \quad \text{and} \quad \left|L\frac{di_2}{dt}\right| << \left|Ri_2\right|$$

we have

$$i_1 = \frac{\omega MI_o}{R}\sin(\omega t-\psi)$$
$$i_2 = -\frac{\omega MI_o}{R}\cos(\omega t-\psi)$$

and the torque T^e is

$$T^e = MI_o\left(\frac{\omega MI_o}{R}\right)[\sin^2(\omega t-\psi)+\cos^2(\omega t-\psi)]$$

Hence, from (1)

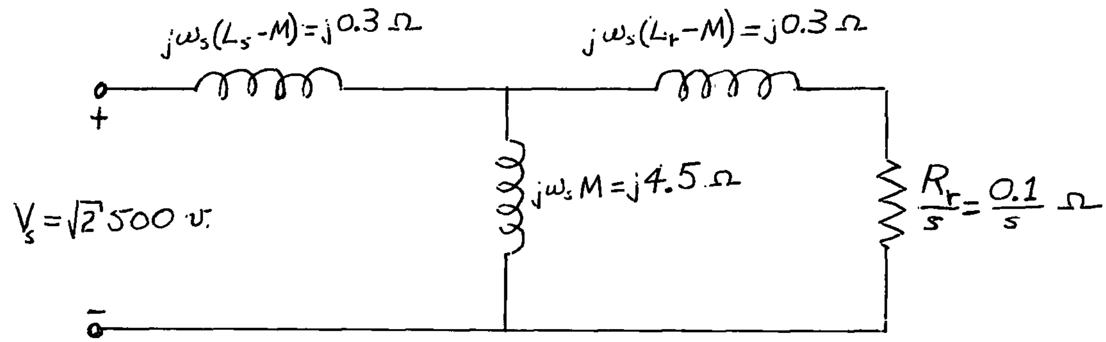
$$\psi = \frac{(MI_o)^2}{KR}\omega$$

which shows that pointer displacement ψ is a linear function of shaft speed ω which is in turn proportional to car speed.

Suppose we had not neglected the voltage drops due to self inductance. Would the final result still be the same?

PROBLEM 4.16

The equivalent circuit with parameter values as given is



From (4.1.82) the torque is

$$T^e = \frac{(\frac{k^2}{\omega_s})(\frac{L_r}{L_s})(\frac{R_r}{s})V_s^2}{[\omega_s(1-k^2)L_r]^2+(R_r/s)^2}$$

where $k^2 = \frac{M^2}{L_r L_s}$ and $s = \frac{\omega_s-\omega_m}{\omega_s}$

Solution of (4.1.81) for I_s yields

$$I_s = \sqrt{\frac{(\frac{R_r}{s})^2 + (\omega_s L_r)^2}{(\frac{R_r}{s})^2 + [\omega_s L_r(1-k^2)]^2}}\quad(\frac{V_s}{\omega_s L_s})$$

volt-ampere input is simply (for two phases)

$$(VA)_{in} = V_s I_s$$

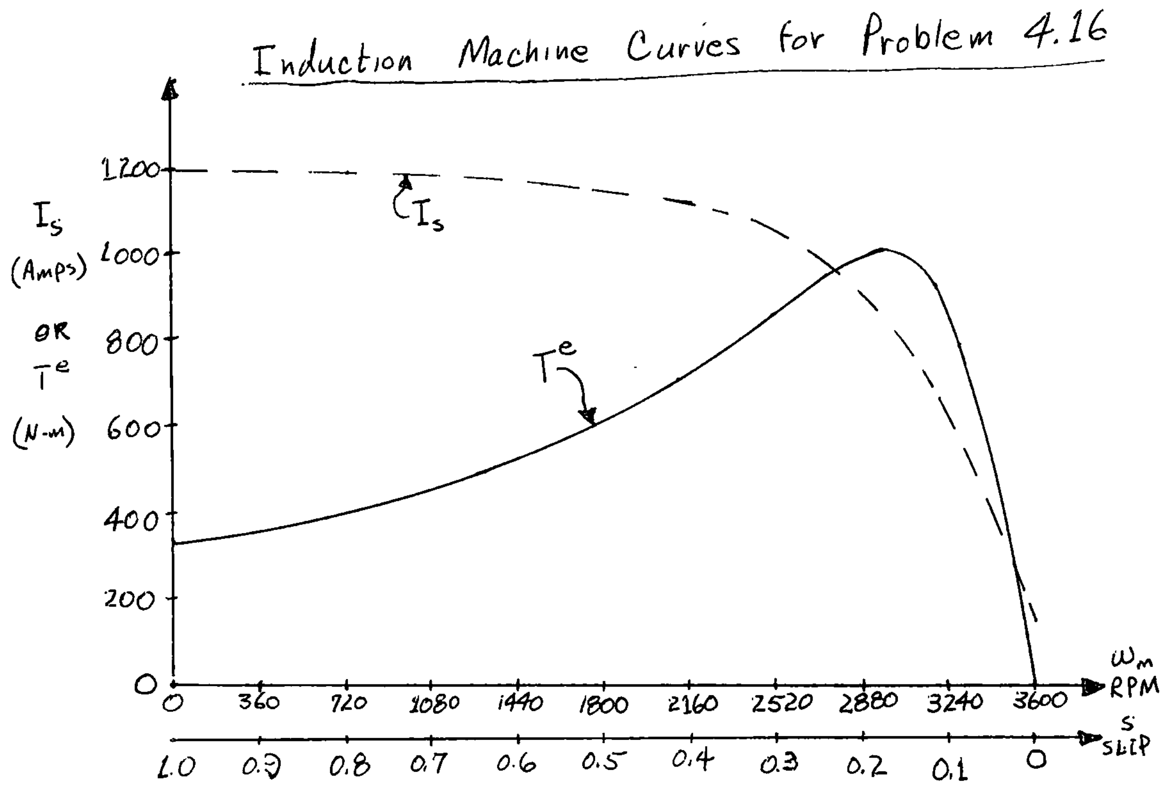
The electrical input power can be calculated in a variety of ways, the simplest being to recognize that in the equivalent circuit the power dissipated in R_r/s (for two phases) is just ω_s times the electromagnetic torque, hence

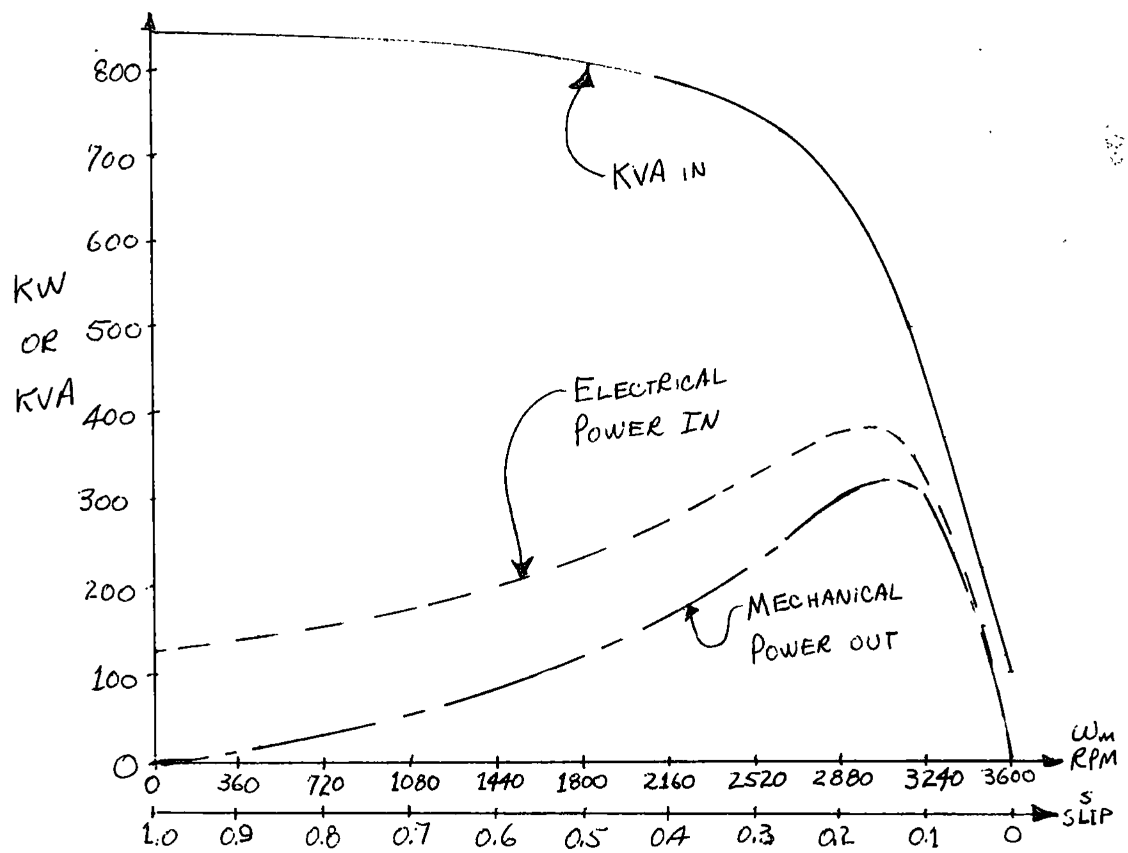
$$P_{in} = T^e\omega_s$$

Finally, the mechanical power output is

$$P_{mech} = T^e\omega_m$$

These five quantities are shown plotted in the attached graphs. Numerical constants used in the computations are


Induction Machine Curves for Problem 4.16
I_s (Amps) OR T^e (N-m)
1200
1000
800
600
400
200
0
I_s
T^e
0 360 720 1080 1440 1800 2160 2520 2880 3240 3600
ω_m RPM
s SLIP
1.0 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0


KW OR KVA
800
700
600
500
400
300
200
100
0
KVA IN
ELECTRICAL POWER IN
MECHANICAL POWER OUT
0 360 720 1080 1440 1800 2160 2520 2880 3240 3600
ω_m RPM
s SLIP
1.0 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0

PROBLEM 4.16 (Continued)

$$\omega_s L_s = \omega_s L_r = \omega_s M + 0.3 = 4.8\Omega$$

$$k^2 = \left(\frac{4.5}{4.8}\right)^2 = 0.878$$

$$T^e = \frac{\frac{117}{s}}{0.342 + \frac{0.01}{s^2}} \text{ newton-meters}$$

$$I_s = \sqrt{\frac{23.0 + \frac{0.01}{s^2}}{0.342 + \frac{0.01}{s^2}}}\ 147 \text{ amps pk.}$$

$$s_{mT} = 0.188$$

PROBLEM 4.17

Part a

For ease in calculation it is useful to write the mechanical speed as

$$\omega_m = (1-s)\omega_s$$

and the fan characteristic as

$$T_m = -B\omega_s^3(1-s)^3$$

With $\omega_s = 120\pi$ rad/sec

$$B\omega_s^3 = 400 \text{ newton-meters}$$

The results of Prob. 4.16 for torque yields

$$400(1-s)^3 = \frac{\frac{117}{s}}{0.342 + \frac{0.01}{s^2}}$$

Solution of this equation by cut-and-try for s yields:

$$s = 0.032$$

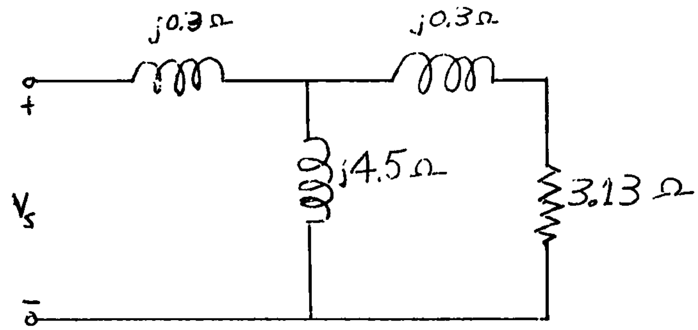
Then $P_{mech} = (400)\ (1-s)^3\omega_m = (400)(\omega_s)(1-s)^4$

$$P_{mech} = 133 \text{ kilowatts into fan}$$

$$P_{input} = \frac{P_{mech}}{1-s} = 138 \text{ kilowatts}$$

Circuit seen by electrical source is

PROBLEM 4.17 (Continued)



Input impedance is

$$Z_{in} = j0.3 + \frac{(j4.5)(3.13+j0.3)}{3.13 + j4.8} = \frac{-2.79+j15.0}{3.13+j4.8}$$

$$\angle Z_{in} = 100.6° - 56.8° = 43.8°$$

Hence,

$$\text{p.f.} = \cos\angle Z_{in} = 0.72 \text{ lagging}$$

Part b

Electromagnetic torque scales as the square of the terminal voltage, thus

$$T^e = \frac{\frac{117}{s}}{0.342 + \frac{0.01}{s^2}} \left(\frac{V_s}{V_{so}}\right)^2$$

where $V_{so} = \sqrt{2}\ 500$ volts peak. The slip for any terminal voltage is now found from

$$400(1-s)^3 = \frac{\frac{117}{s}}{0.342 + \frac{0.01}{s^2}} \left(\frac{V_s}{V_{so}}\right)^2$$

The mechanical power into the fan is

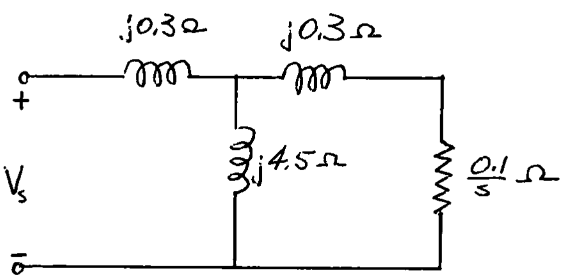
$$P_{mech} = 400\ \omega_s^4 (1-s)^4$$

electrical power input is

$$P_{in} = \frac{P_{mech}}{1-s}$$

PROBLEM 4.17 (Continued)

and the power factor is found as the cosine of the angle of the input impedance of the circuit



These quantities are plotted as required on the attached graph.

PROBLEM 4.18

Part a

The solution to Prob. 4.1 can be used to find the flux densities here. For the stator a-winding, the solution of Prob. 4.1 applies directly, thus, the radial component of flux density due to current in stator winding a is

$$B_{ra}(\psi) = \frac{\mu_o N_s i_a}{2g} \cos\psi$$

Windings b and c on the stator are identical with the a winding except for the indicated angular displacements, thus,

$$B_{rb}(\psi) = \frac{\mu_o N_s i_b}{2g} \cos(\psi - \frac{2\pi}{3})$$

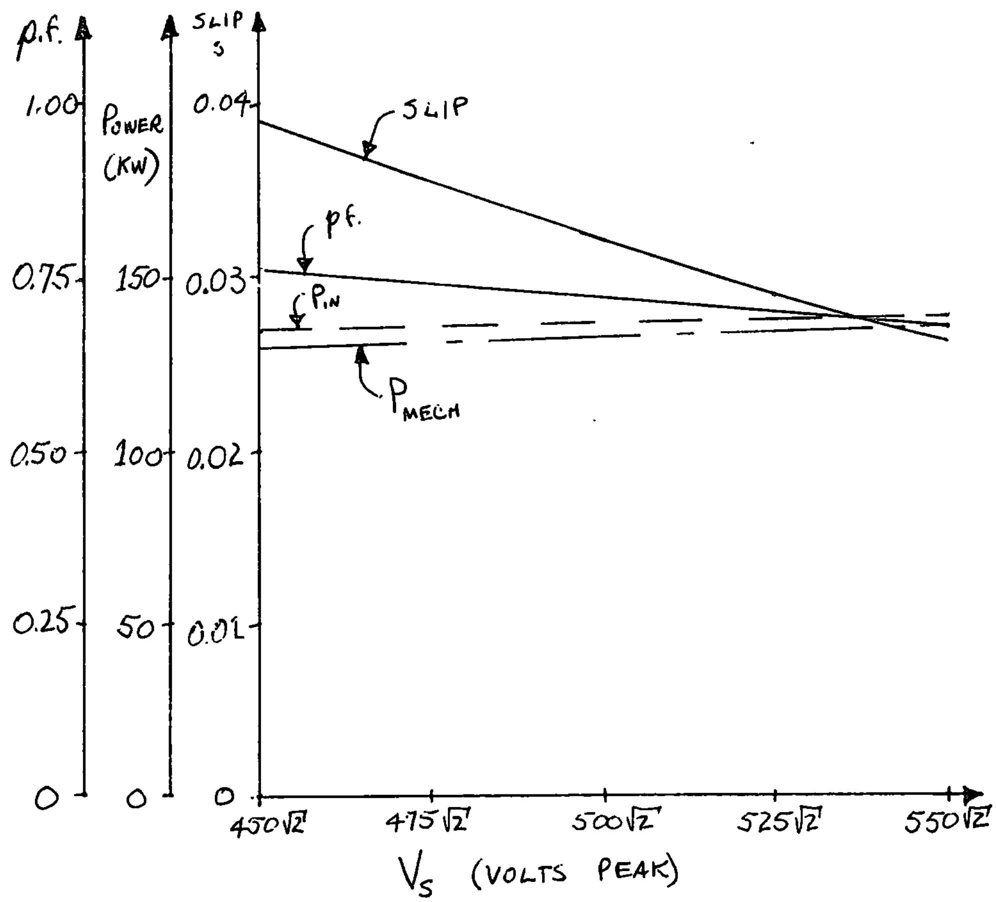
$$B_{rc}(\psi) = \frac{\mu_o N_s i_c}{2g} \cos(\psi - \frac{4\pi}{3})$$

The solution in Prob. 4.1 for the flux density due to rotor winding current applies directly here, thus,

$$B_{rr}(\psi) = \frac{\mu_o N_r i_r}{2g} \cos(\psi - \theta)$$

Part b

The method of part (c) of Prob. 4.1 can be used and the results of that analysis applied directly by replacing rotor quantities by stator b-winding quantities and θ by $2\pi/3$. The resulting mutual inductance is (assuming $g \ll R$)



Induction Machine Curves for Problem 4.17

PROBLEM 4.18 (Continued)

$$L_{ab} = \frac{\pi\mu_o N_s^2 R\ell}{2g} \cos\frac{2\pi}{3}$$

$$L_{ab} = -\frac{\mu_o N_s^2 R\ell}{4g} = -\frac{L_s}{2}$$

where L_s is the self inductance of one stator winding alone. Note that $L_{ac} = L_{ab}$ because of relative geometry.

Part c

The λ-i relations are thus

$$\lambda_a = L_s i_a - \frac{L_s}{2} i_b - \frac{L_s}{2} i_c + M\cos\theta i_r$$

$$\lambda_b = -\frac{L_s}{2} i_a + L_s i_b - \frac{L_s}{2} i_c + M\cos(\theta - \frac{2\pi}{3}) i_r$$

$$\lambda_c = -\frac{L_s}{2} i_a - \frac{L_s}{2} i_b + L_s i_c + M\cos(\theta - \frac{4\pi}{3}) i_r$$

$$\lambda_r = M\cos\theta i_a + M\cos(\theta - \frac{2\pi}{3}) i_b$$

$$+ M\cos(\theta - \frac{4\pi}{3}) i_c + L_r i_r$$

where from Prob. 4.1,

$$L_s = \frac{\pi\mu_o N_s^2 R\ell}{2g}$$

$$M = \frac{\pi\mu_o N_s N_r R\ell}{2g}$$

$$L_r = \frac{\pi\mu_o N_r^2 R\ell}{2g}$$

Part d

The torque of electric origin is found most easily by using magnetic coenergy which for this electrically linear system is

$$W_m'(i_a, i_b, i_c, i_r, \theta) = \frac{1}{2} L_s (i_a^2 + i_b^2 + i_c^2)$$

$$+ \frac{1}{2} L_s (i_a i_b + i_a i_c + i_b i_c) + M\cos\theta i_r i_a$$

$$+ M\cos(\theta - \frac{2\pi}{3}) i_r i_b + M\cos(\theta - \frac{4\pi}{3}) i_r i_c$$

The torque of electric origin is

PROBLEM 4.18 (Continued)

$$T^e = \frac{\partial W'_m(i_a,i_b,i_c,i_r,\theta)}{\partial\theta}$$

$$T^e = -Mi_r[i_a\sin\theta + i_b\sin(\theta-\frac{2\pi}{3}) + i_c\sin(\theta-\frac{4\pi}{3})]$$

PROBLEM 4.19

Part a

Superimposing the three component stator flux densities from Part a of Prob. 4.18, we have

$$B_{rs} = \frac{\mu_o N_s}{2g}[i_a\cos\psi + i_b\cos(\psi-\frac{2\pi}{3}) + i_c\cos(\psi-\frac{4\pi}{3})]$$

Substituting the given currents

$$B_{rs} = \frac{\mu_o N_s}{2g}[I_a\cos\omega t\cos\psi + I_b\cos(\omega t-\frac{2\pi}{3})\cos(\psi-\frac{2\pi}{3})$$

$$+ I_c\cos(\omega t-\frac{4\pi}{3})\cos(\psi-\frac{4\pi}{3})$$

Using trigonometric identities and simplifying yields

$$B_{rs} = \frac{\mu_o N_s}{2g}\Bigg[(\frac{I_a + I_b + I_c}{2})\cos(\omega t-\psi)$$

$$+ \frac{1}{2}(I_a + I_b\cos\frac{4\pi}{3} + I_c\cos\frac{2\pi}{3})\cos(\omega t+\psi)$$

$$+ \frac{1}{2}(I_b\sin\frac{4\pi}{3} + I_c\sin\frac{2\pi}{3})\sin(\omega t+\psi)\Bigg]$$

Positive traveling wave has point of constant phase defined by

$$\omega t - \psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = \omega$$

This is positive traveling wave with amplitude

$$B_{rfm} = \frac{\mu_o N_s}{4g}(I_a + I_b + I_c)$$

Negative traveling wave has point of constant phase

$$\omega t + \psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = -\omega$$

This defines negative traveling wave with amplitude

PROBLEM 4.19 (Continued)

$$B_{rbm} = \frac{\mu_o N_s}{4g} \sqrt{\left(I_a - \frac{I_b}{2} - \frac{I_c}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2} I_b + \frac{\sqrt{3}}{2} I_c\right)^2}$$

Part b

When three phase currents are balanced

$$I_a = I_b = I_c$$

and $B_{rbm} = 0$ leaving only a forward (positive) traveling wave.

PROBLEM 4.20

Part a

Total radial flux density due to stator excitation is

$$B_{rs} = \frac{\mu_o N}{2g} (i_a \cos 2\psi + i_b \sin 2\psi)$$

Substituting given values for currents

$$B_{rs} = \frac{\mu_o N}{2g} (I_a \cos \omega t \cos 2\psi + I_b \sin \omega t \sin 2\psi)$$

Part b

$$B_{rs} = \frac{\mu_o N}{2g} \left| \left(\frac{I_a + I_b}{2}\right) \cos(\omega t - 2\psi) + \left(\frac{I_a - I_b}{2}\right) \cos(\omega t + 2\psi) \right|$$

The forward (positive-traveling) component has constant phase defined by

$$\omega t - 2\psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = \frac{\omega}{2}$$

The backward (negative-traveling) component has constant phase defined by

$$\omega t + 2\psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = -\frac{\omega}{2}$$

Part c

From part b, when $I_a = I_b$, $I_a - I_b = 0$ and the backward-wave amplitude goes to zero. When $I_b = - I_a$, $I_a + I_b = 0$ and the forward-wave amplitude goes to zero.

PROBLEM 4.21

Referring to the solution for Prob. 4.20,

Part a

$$B_{rs} = \frac{\mu_o N}{2g} (i_a \cos p\psi + i_b \sin p\psi)$$

$$B_{rs} = \frac{\mu_o N}{2g} (I_a \cos \omega t \cos p\psi + I_b \sin \omega t \sin p\psi)$$

Part b

Using trigonometric identities yields

$$B_{rs} = \frac{\mu_o N}{2g} \left[\left(\frac{I_a + I_b}{2}\right)\cos(\omega t - p\psi) + \left(\frac{I_a - I_b}{2}\right)\cos(\omega t + p\psi)\right]$$

Forward wave has constant phase

$$\omega t - p\psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = \frac{\omega}{p}$$

Backward wave has constant phase

$$\omega t + p\psi = \text{constant}$$

from which

$$\frac{d\psi}{dt} = -\frac{\omega}{p}$$

Part c

From part b, when $I_b = I_a$, $I_a - I_b = 0$, and backward-wave amplitude goes to zero. When $I_b = -I_a$, $I_a + I_b = 0$, and forward-wave amplitude goes to zero.

PROBLEM 4.22

This is an electrically linear system, so the magnetic coenergy is

$$W_m'(i_s, i_r, \theta) = \frac{1}{2}(L_o + L_2 \cos 2\theta) i_s^2 + \frac{1}{2} L_r i_r^2 + M i_r i_s \cos \theta$$

Then the torque is

$$T^e = \frac{\partial W_m'(i_s, i_r, \theta)}{\partial \theta} = -M i_r i_s \sin \theta - L_2 i_s^2 \sin 2\theta$$

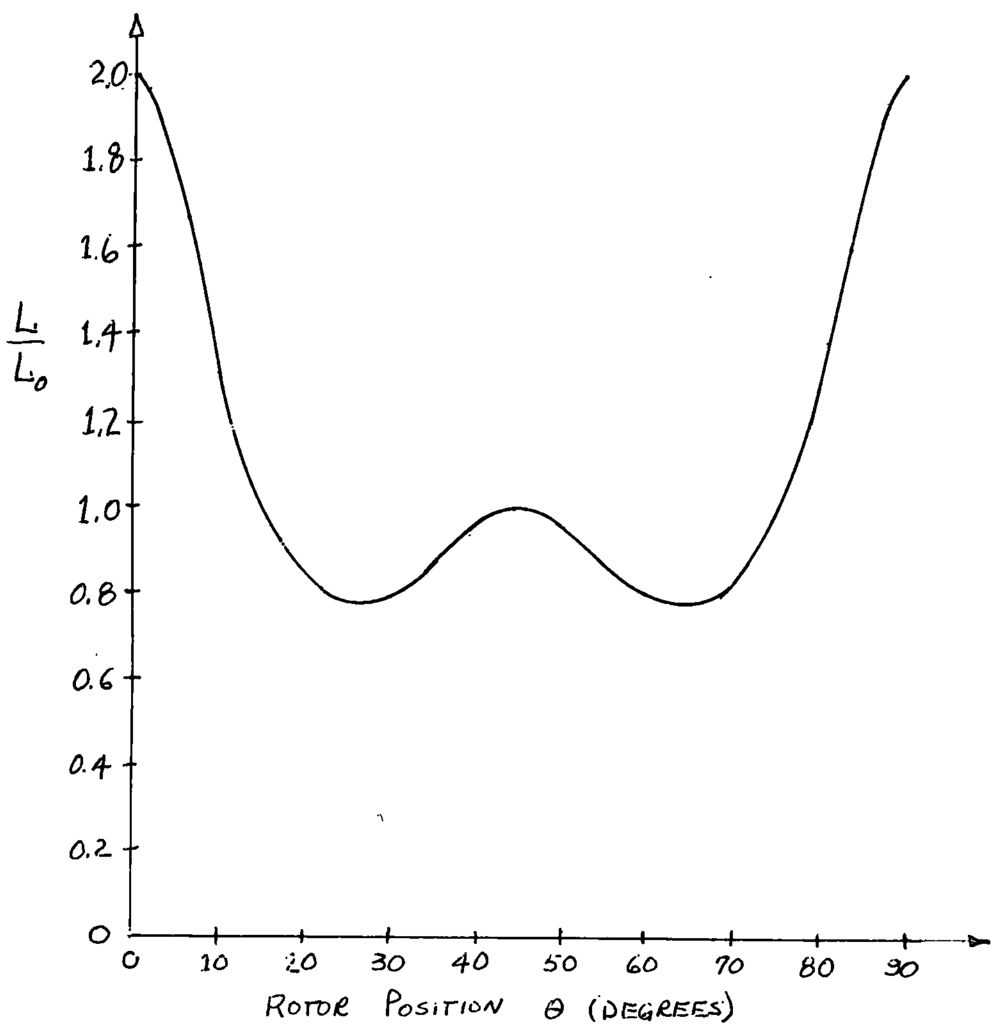
PROBLEM 4.23

Part a

$$L = \frac{L_o}{(1 - 0.25 \cos 4\theta - 0.25 \cos 8\theta)}$$

PROBLEM 4.23 (Continued)

The variation of this inductance with θ is shown plotted below.



PROBLEM 4.23 (Continued)

From this plot and the configuration of Fig. 4P.23, it is evident that minimum reluctance and maximum inductance occur when $\theta = 0, \pi/2, \pi, \ldots \frac{n}{2}\pi, \ldots$. The inductance is symmetrical about $\theta = 0, \frac{\pi}{2}, \ldots$ and about $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots \frac{\pi}{4} + \frac{n\pi}{2}, \ldots$ as it should be. Minimum inductance occurs on both sides of $\theta = \frac{\pi}{4}$ which ought to be maximum reluctance.

The general trend of the inductance is correct for the geometry of Fig. 4P.23 but the equation would probably be a better representation if the sign of the 8θ term were reversed.

Part b

For this electrically linear system, the magnetic stored energy is

$$W_m(\lambda,\theta) = \frac{1}{2}\frac{\lambda^2}{L}$$

$$W_m(\lambda,\theta) = \frac{\lambda^2(1-0.25\cos 4\theta - 0.25\cos 8\theta)}{2L_o}$$

The torque is then

$$T^e = -\frac{\partial W_m(\lambda,\theta)}{\partial\theta}$$

$$T^e = -\frac{\lambda^2}{2L_o}(\sin 4\theta + 2\sin 8\theta)$$

Part c

With $\lambda = \Lambda_o \cos\omega t$ and $\theta = \Omega t + \delta$

$$T^e = -\frac{\Lambda_o^2 \cos^2\omega t}{2L_o}[\sin(4\Omega t+4\delta)+2\sin(8\Omega t+8\delta)]$$

Repeated use of trig identities yields for the instantaneous converted power

$$\Omega T^e = -\frac{\Omega\Lambda_o^2}{4L_o}[\sin(4\Omega t+4\delta) + 2\sin(8\Omega t+8\delta)$$
$$+ \frac{1}{2}\sin(2\omega t + 4\Omega t + 4\delta) + \frac{1}{2}\sin(4\Omega t - 2\omega t + 4\delta)$$
$$+ \sin(2\omega t + 8\Omega t + 8\delta) + \sin(8\Omega t - 2\omega t + 8\delta)]$$

This can only have a non-zero average value when $\Omega \neq 0$ and a coefficient of t in one argument is zero. This gives 4 conditions

$$\Omega = \pm\frac{\omega}{2}, \pm\frac{\omega}{4}$$

When $\Omega = \pm\frac{\omega}{2}$

$$[\Omega T^e]_{avg} = -\frac{\Omega\Lambda_o^2}{8L_o}\sin 4\delta$$

PROBLEM 4.23 (Continued)

and when $\Omega = \pm \frac{\omega}{4}$

$$[\Omega T^e]_{avg} = - \frac{\Omega \Lambda_o^2}{4L_o} \sin 8\delta$$

PROBLEM 4.24

It will be helpful to express the given ratings in alternative ways. Rated output power = 6000 HP = 4480 KW at 0.8 p.f. this is

$$\frac{4480}{0.8} = 5600 \text{ KVA total}$$

or

2800 KVA per phase

The rated phase current is then

$$I_s = \frac{2800 \times 10^3}{3 \times 10^3} = 933 \text{ amps rms} = 1320 \text{ amps pk.}$$

Given:

Direct axis reactance $\omega(L_o+L_2) = 4.0$ ohms

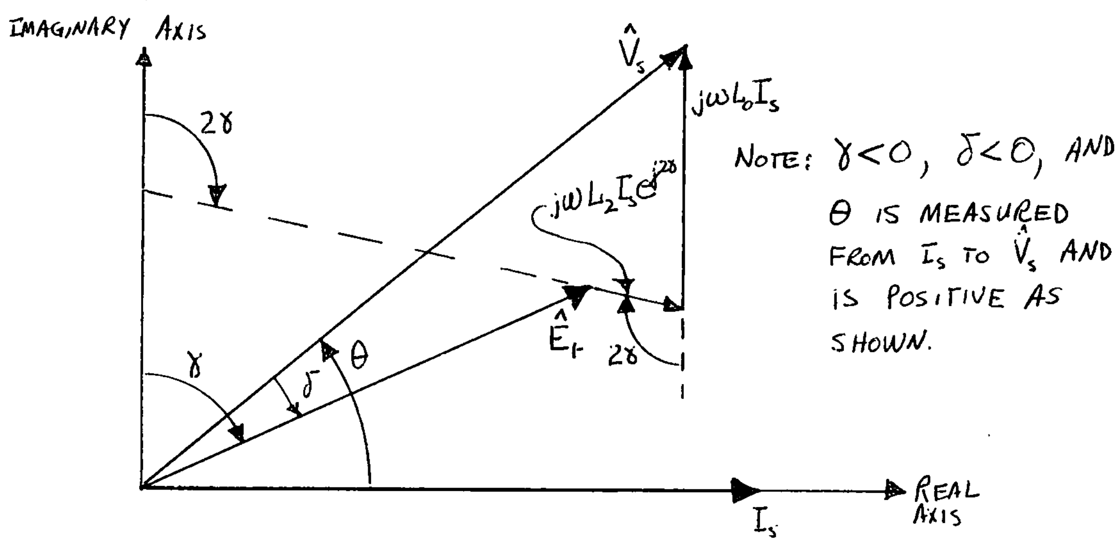
Quadrature axis reactance $\omega(L_o-L_2) = 2.2$ ohms

$\omega L_o = 3.1$ ohms $\qquad\qquad \omega L_2 = 0.9$ ohms

The number of poles is not given in the problem statement. We assume 2 poles.

Part a

Rated field current can be found in several ways, all involving cut-and-try procedures. Our method will be based on a vector diagram like that of Fig. 4.2.5(a), thus



PROBLEM 4.24 (Continued)

Evaluating the horizontal and vertical components of $\hat{V}_s$ we have (remember that $\gamma < 0$)

$$V_s \cos\theta = E_f \cos(\tfrac{\pi}{2} + \gamma) + \omega L_2 I_s \cos(\tfrac{\pi}{2} + 2\gamma)$$

$$V_s \sin\theta = E_f \sin(\tfrac{\pi}{2} + \gamma) + \omega L_2 I_s \sin(\tfrac{\pi}{2} + 2\gamma) + \omega L_o I_s$$

Using trigonometric identities we rewrite these as

$$V_s \cos\theta = -E_f \sin\gamma - \omega L_2 I_s \sin 2\gamma$$

$$V_s \sin\theta = E_f \cos\gamma + \omega L_2 I_s \cos 2\gamma + \omega L_o I_s$$

Next, it will be convenient to normalize these equations to V_s,

$$\cos\theta = -e_f \sin\gamma - \frac{\omega L_2 I_s}{V_s} \sin 2\gamma$$

$$\sin\theta = e_f \cos\gamma + \frac{\omega L_2 I_s}{V_s} \cos 2\gamma + \frac{\omega L_o I_s}{V_s}$$

where

$$e_f = \frac{E_f}{V_s}$$

Solution of these two equations for e_f yields

$$e_f = \frac{\sin\theta - \dfrac{\omega L_2 I_s}{V_s}\cos 2\gamma - \dfrac{\omega L_o I_s}{V_s}}{\cos\gamma}$$

$$e_f = \frac{-\cos\theta - \dfrac{\omega L_2 I_s}{V_s}\sin 2\gamma}{\sin\gamma}$$

For rated conditions as given the constants are:

$$\cos\theta = \text{p.f.} = 0.8$$

$$\sin\theta = -\,0.6 \text{ (negative sign for leading p.f.)}$$

$$\frac{\omega L_2 I_s}{V_s} = 0.280; \quad \frac{\omega L_o I_s}{V_s} = 0.964$$

Solution by trial and error for a value of γ that satisfies both equations simultaneously yields

$$\gamma = -\,148°$$

PROBLEM 4.24 (Continued)

and the resulting value for e_f is

$$e_f = 1.99$$

yielding for the rated field current

$$I_r = \frac{V_s e_f}{\omega M} = 24.1 \text{ amps.}$$

where V_s is in volts peak.

Part b

The V-curves can be calculated in several ways. Our choice here is to first relate power converted to terminal voltage and field generated voltage by multiplying (4.2.46) by ω, thus

$$P = \omega T^e = -\frac{E_f V_s}{X_d} \sin \delta - \frac{(X_d - X_q) V_s^2}{2 X_d X_q} \sin 2\delta$$

where
$$X_d = \omega(L_o + L_2)$$
$$X_q = \omega(L_o - L_2)$$

We normalize this expression with respect to V_s^2/X_d, then

$$\frac{P X_d}{V_s^2} = -e_f \sin \delta - \frac{(X_d - X_q)}{2 X_q} \sin 2\delta$$

Pull-out torque occurs when the derivative of this power with respect to δ goes to zero. Thus pull-out torque angle is defined by

$$\frac{\partial}{\partial \delta}\left(\frac{P X_d}{V_s^2}\right) = -e_f \cos \delta - \frac{(X_d - X_q)}{X_q} \cos 2\delta = 0$$

The use of (4.2.44) and (4.2.45) then yield the armature (stator) current amplitude as

$$I_s = \sqrt{\left(\frac{V_s}{X_q} \sin \delta\right)^2 + \left(\frac{V_s}{X_d} \cos \delta - \frac{E_f}{X_d}\right)^2}$$

A more useful form is

$$I_s = \frac{V_s}{X_d} \sqrt{\left(\frac{X_d}{X_q} \sin \delta\right)^2 + (\cos \delta - e_f)^2}$$

The computation procedure used here was to fix the power and assume values of δ over a range going from either rated armature current or rated field current to pull-out. For each value of δ, the necessary value of e_f is calculated

PROBLEM 4.24 (Continued)

from the expression for power as

$$e_f = \frac{\frac{PX_d}{V_s^2} + \frac{(X_d - X_q)}{2X_q}\sin 2\delta}{-\sin\delta}$$

and then the armature current magnitude is calculated from

$$I_s = \frac{V_s}{X_d}\sqrt{\left(\frac{X_d}{X_q}\sin\delta\right)^2 + (\cos\delta - e_f)^2}$$

For zero load power, $\gamma = 0$ and $\delta = 0$ and, from the vector diagram given earlier, the armature current amplitude is

$$I_s = \frac{|V_s - E_f|}{\omega(L_o + L_2)}$$

with pull-out still defined as before. The required V-curves are shown in the followinggraph. Note that pull-out conditions are never reached because range of operation is limited by rated field current and rated armature current.

PROBLEM 4.25

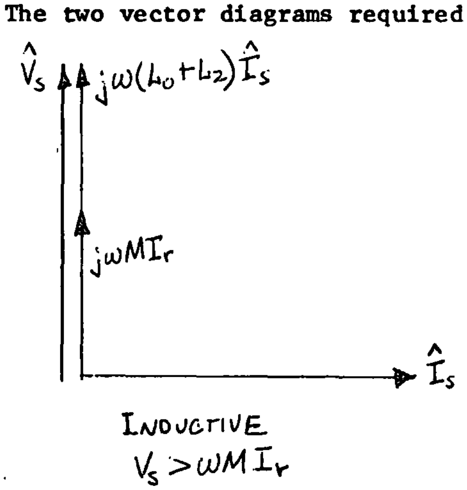
Equation (4.2.41) is (assuming arbitrary phase for I_s)

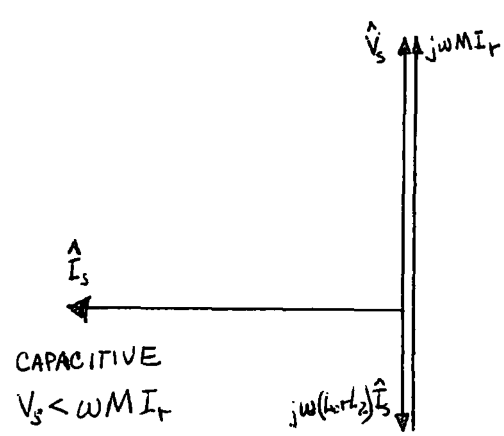
$$\hat{V}_s = j\omega L_o\,\hat{I}_s + j\omega L_2\,\hat{I}_s e^{j2\gamma} + j\omega M I_r\,e^{j2\gamma}$$

With $\gamma = 0$ as specified

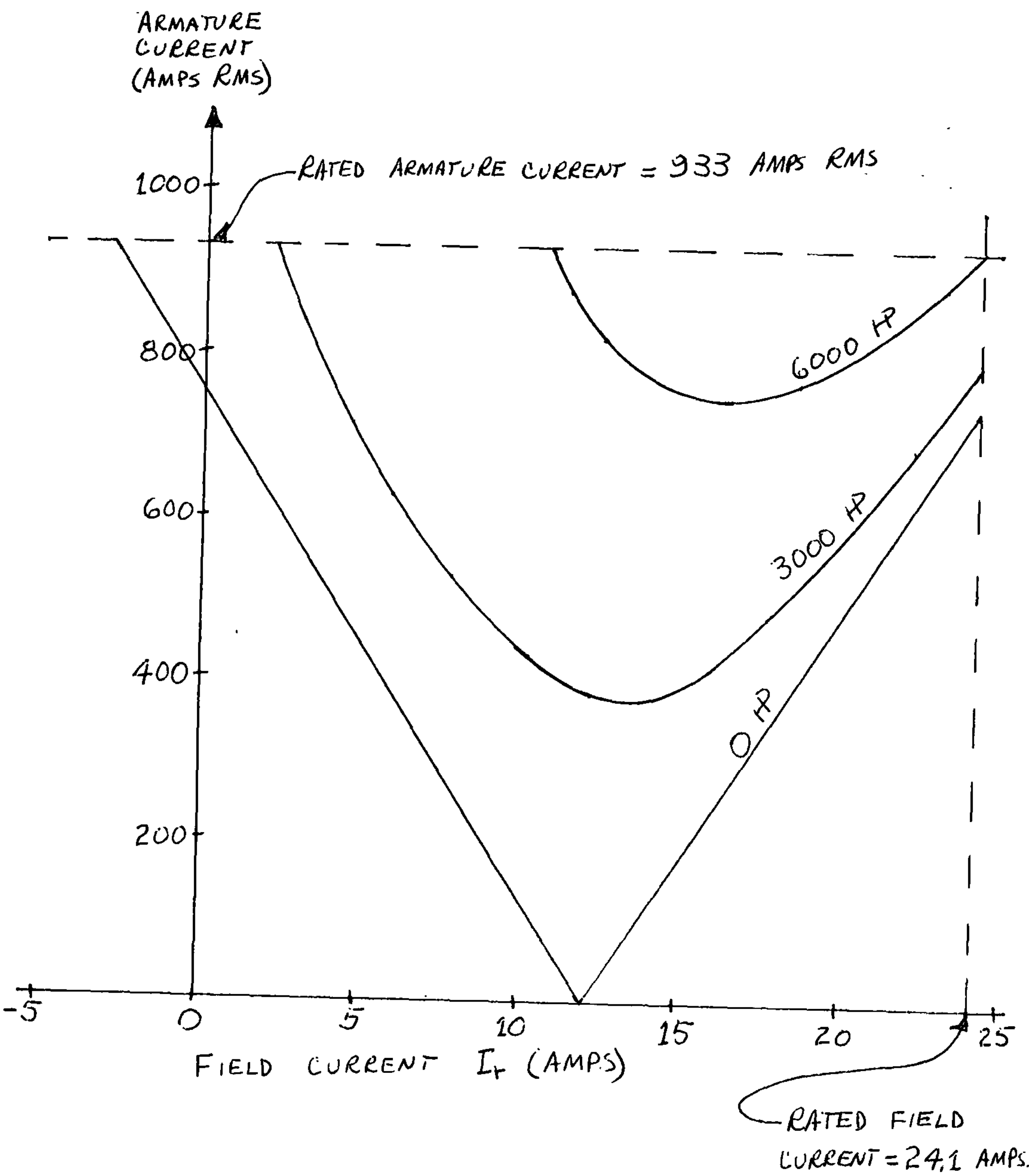
$$\hat{V}_s = j\omega(L_o + L_2)\hat{I}_s + j\omega M I_r$$

The two vector diagrams required are





V- CURVES FOR PROBLEM 4.24



PROBLEM 4.26

Part a

From Fig. 4P.26(a)

$$\frac{\hat{V}}{\hat{V}_s} = \frac{\frac{1}{Y} e^{j\phi}}{jX_s + \frac{1}{Y} e^{j\phi}}$$

from which the ratio of the magnitudes is

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{\frac{1}{Y}}{\sqrt{(\frac{1}{Y}\cos\phi)^2 + (\frac{1}{Y}\sin\phi + X_s)^2}}$$

For the values Y = 0.01 mho, X_s = 10 ohms

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{100}{\sqrt{(100\cos\phi)^2 + (100\sin\phi + 10)^2}}$$

Then, for $\phi = 0$

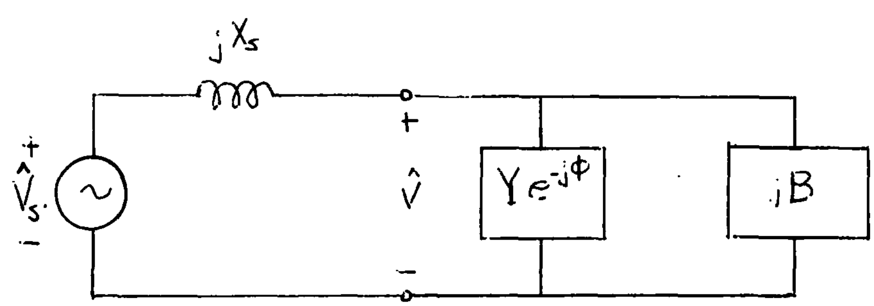
$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{100}{\sqrt{10{,}000 + 100}} = 0.995$$

and, for $\phi = 45°$

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{100}{\sqrt{(\frac{100}{\sqrt{2}})^2 + (\frac{100}{\sqrt{2}} + 10)^2}} = 0.932$$

Part b

It is instructive to represent the synchronous condenser as a susceptance jB, then when B is positive the synchronous condenser appears capacitive. Now the circuit is



PROBLEM 4.26 (Continued)

Now the voltage ratio is

$$\frac{\hat{V}}{\hat{V}_s} = \frac{\dfrac{1}{Ye^{-j\phi}+jB}}{\dfrac{1}{Ye^{-j\phi}+jB}+jX_s} = \frac{1}{1+jX_sYe^{-j\phi}-BX_s}$$

$$\frac{\hat{V}}{\hat{V}_s} = \frac{1}{1-BX_s+X_sY\sin\phi+jX_sY\cos\phi}$$

Then

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{1}{\sqrt{(1-BX_s+X_sY\sin\phi)^2+(X_sY\cos\phi)^2}}$$

For $\phi = 0$

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{1}{\sqrt{(1-BX_s)^2+(X_sY)^2}}$$

If this is to be unity

$$(1-BX_s)^2+(X_sY)^2 = 1$$

$$1-BX_s = \sqrt{1-(X_sY)^2}$$

$$B = \frac{1-\sqrt{1-(X_sY)^2}}{X_s}$$

for the constants given

$$B = \frac{1-\sqrt{1-0.01}}{10} = \frac{0.005}{10} = \underline{0.0005\ \text{mho}}$$

Volt-amperes required from synchronous condenser

$$(VA)_{sc} = |\hat{V}|^2B = (2)(10^{10})(5)(10^{-4}) = 10{,}000\ \text{KVA}$$

Real power supplied to load

$$P_L = |\hat{V}|^2Y\cos\phi = |\hat{V}|^2Y \quad \text{for } \phi = 0$$

Then

$$\frac{(VA)_{sc}}{P_L} = \frac{B}{Y} = \frac{0.0005}{0.01} = 0.05$$

For $\phi = 0$ the synchronous condenser needs to supply reactive volt amperes equal to 5 percent of the load power to regulate the voltage perfectly.

PROBLEM 4.26 (Continued)

For $\phi = 45°$

$$\frac{|\hat{V}|}{|\hat{V}_s|} = \frac{1}{\sqrt{\left(1-BX_s + \frac{X_sY}{\sqrt{2}}\right)^2 + \left(\frac{X_sY}{\sqrt{2}}\right)^2}}$$

In order for this to be unity

$$\left(1-BX_s + \frac{X_sY}{\sqrt{2}}\right)^2 + \left(\frac{X_sY}{\sqrt{2}}\right)^2 = 1$$

$$B = \frac{1+\frac{X_sY}{\sqrt{2}} - \sqrt{1-\left(\frac{X_sY}{\sqrt{2}}\right)^2}}{X_s}$$

For the constants given

$$B = \frac{1 + 0.0707 - \sqrt{1-0.005}}{10} = 0.00732 \text{ mho}$$

Volt-amperes required from synchronous condenser

$$(VA)_{sc} = |\hat{V}|^2B = (2)(10^{10})(7.32)(10^{-3}) = 146{,}400 \text{ KVA}$$

Real power supplied to load

$$P_L = |\hat{V}|^2Y \cos\phi = \frac{|\hat{V}|^2Y}{\sqrt{2}} \text{ for } \phi = 45°$$

Then

$$\frac{(VA)_{sc}}{P_L} = \frac{B\sqrt{2}}{Y} = \frac{(\sqrt{2})(0.00732)}{0.01} = 1.04$$

Thus for a load having power factor of 0.707 lagging a synchronous condenser needs to supply reactive volt-amperes equal to 1.04 times the power supplied to the load to regulate the voltage perfectly.

These results, of course, depend on the internal impedance of the source. That given is typical of large power systems.

PROBLEM 4.27

Part a

This part of this problem is very much like part a of Prob. 4.24. Using results from that problem we define

PROBLEM 4.27 (Continued)

$$e_f = \frac{E_f}{V_s} = \frac{\omega M I_r}{V_s}$$

where V_s is in volts peak. Then

$$e_f = \frac{\sin\theta - \frac{\omega L_2 I_s}{V_s}\cos 2\gamma - \frac{\omega L_o I_s}{V_s}}{\cos\gamma}$$

$$e_f = \frac{-\cos\theta - \frac{\omega L_2 I_s}{V_s}\sin 2\gamma}{\sin\gamma}$$

From the constants given

$$\cos\theta = 1.0; \quad \sin\theta = 0$$

$$\omega L_o = 2.5 \text{ ohms} \quad \omega L_2 = 0.5 \text{ ohm}$$

Rated power

$$P_L = 1000 \text{ HP} = 746 \text{ KW}$$

Armature current at rated load is

$$I_s = \frac{746{,}000}{\sqrt{2}\ 1000} = 527 \text{ amps peak} = 373 \text{ amps RMS}$$

Then

$$\frac{\omega L_2 I_s}{V_s} = 0.186; \quad \frac{\omega L_o I_s}{V_s} = 0.932$$

Using the constants

$$e_f = \frac{-0.186\cos 2\gamma - 0.932}{\cos\gamma}$$

$$e_f = \frac{-1 - 0.186\sin 2\gamma}{\sin\gamma}$$

The use of trial-and-error to find a value of γ that satisfies these two equations simultaneously yields

$$\gamma = -127° \text{ and } e_f = 1.48$$

Using the given constants we obtain

$$I_r = \frac{e_f V_s}{\omega M} = \frac{(1.48)(\sqrt{2})(1000)}{150} = 14.0 \text{ amps}$$

For L_f/R_f very large compared to a half period of the supply voltage the field

PROBLEM 4.27 (Continued)

current will essentially be equal to the peak of the supply voltage divided by the field current; thus, the required value of R_f is

$$R_f = \frac{V_s}{I_r} = \frac{\sqrt{2}\ (1000)}{14.0} \approx 100 \text{ ohms}$$

Part b

We can use (4.2.46) multiplied by the rotational speed ω to write the output power as

$$P_L = \omega T^e = -\frac{E_f V_s}{X_d} \sin\delta - \frac{(X_d - X_q)V_s^2}{2X_d X_q} \sin 2\delta$$

where

$$X_d = \omega(L_o + L_2) = \text{direct axis reactance}$$

$$X_q = \omega(L_o - L_2) = \text{quadrature axis reactance}$$

With the full-wave rectifier supplying the field winding we can express

$$E_f = \omega M I_r = \frac{\omega M}{R_f} V_s$$

Then

$$P_L = -\frac{\omega M\ V_s^2}{R_f X_d} \sin\delta - \frac{(X_d - X_q)V_s^2}{2X_d X_q} \sin 2\delta$$

Factoring out V_s^2 yields

$$P_L = V_s^2 \left[-\frac{\omega M}{R_f X_d} \sin\delta - \frac{(X_d - X_q)}{2X_d X_q} \sin 2\delta \right]$$

Substitution of given constants yields

$$746 \times 10^3 = V_s^2\ [-0.500 \sin\delta - 0.083 \sin 2\delta]$$

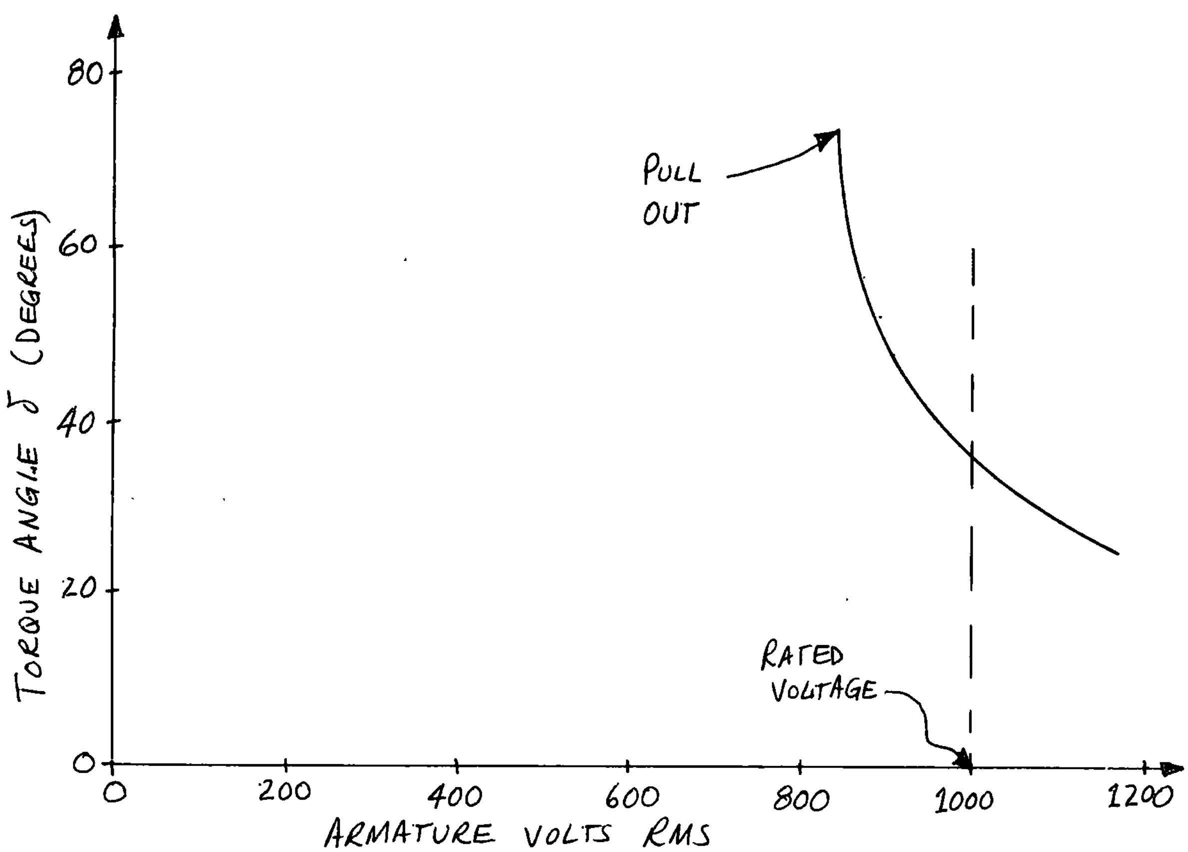
To find the required curve it is easiest to assume δ and calculate the required V_s, the range of δ being limited by pull-out which occurs when

$$\frac{\partial P_L}{\partial \delta} = 0 = -\ 0.500 \cos\delta\ -\ 0.166 \cos 2\delta$$

The resulting curve of δ as a function of V_s is shown in the attached graph.

Note that the voltage can only drop 15.5% before the motor pulls out of step.

PROBLEM 4.27 (Continued)



Although it was not required for this problem, calculations will show that operation at reduced voltage will lead to excessive armature current, thus, operation in this range must be limited to transient conditions.

PROBLEM 4.28

Part a

This is similar to part a of Prob. 4.24 except that now we are considering a number of pole pairs greater than two and we are treating a generator. Considering first the problem of pole pairs, reference to Sec. 4.1.8 and 4.2.4 shows that when we define electrical angles γ_e and δ_e as

$$\gamma_e = p\gamma \text{ and } \delta_e = p\delta$$

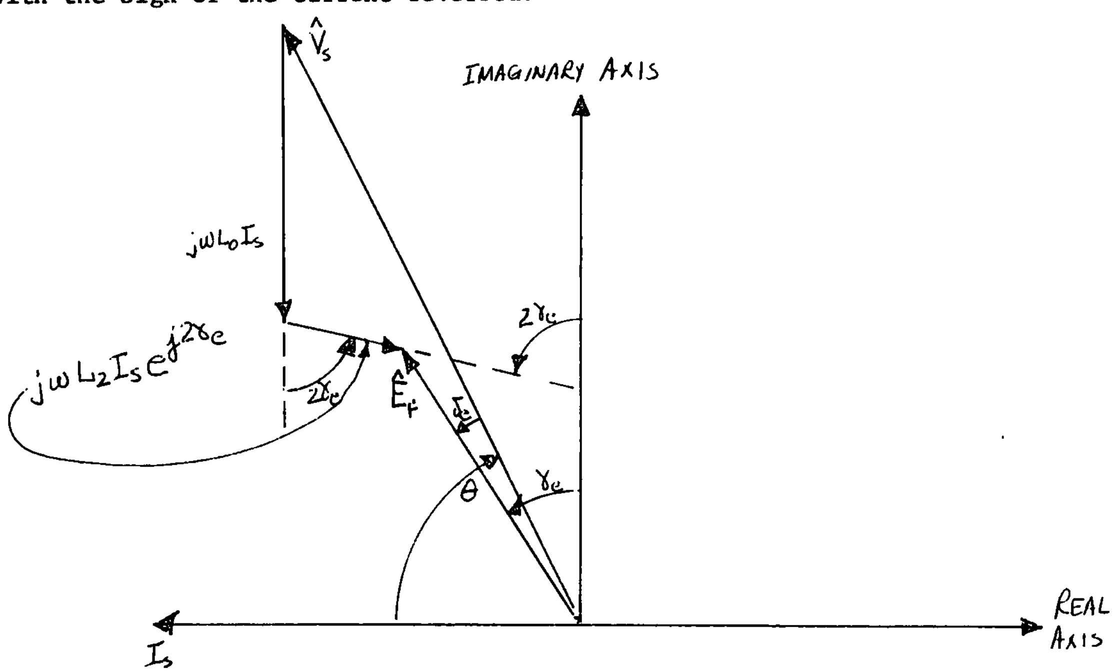
where p is number of pole pairs (36 in this problem) and when we realize that the electromagnetic torque was obtained as a derivative of inductances with respect to angle we get the results

$$T^e = -\frac{p}{\omega}\frac{V_sE_f}{X_d}\sin\delta_e - \frac{p(X_d-X_q)\,V_s^2}{\omega 2X_dX_q}\sin 2\delta_e$$

where $X_d = \omega(L_o+L_2)$ and $X_q = \omega(L_o-L_2)$, and, because the synchronous speed is ω/p (see 4.1.95) the electrical power output from the generator is

$$P = -\frac{\omega}{p}T^e = \frac{V_sE_f}{X_d}\sin\delta_e + \frac{(X_d-X_q)V_s^2}{2X_dX_q}\sin 2\delta_e$$

Next, we are dealing with a generator so it is convenient to replace I_s by $-I_s$ in the equations. To make clear what is involved we redraw Fig. 4.2.5(a) with the sign of the current reversed.



PROBLEM 4.28 (Continued)

Now, evaluating horizontal and vertical components of V_s we have

$$V_s \cos\theta - \omega L_2 I_s \sin 2\gamma_e = E_f \sin\gamma_e$$

$$-V_s \sin\theta = \omega L_o I_s + \omega L_2 I_s \cos 2\gamma_e + E_f \cos\gamma_e$$

From these equations we obtain

$$e_f = \frac{\cos\theta - \frac{\omega L_2 I_s}{V_s}\sin 2\gamma_e}{\sin\gamma_e}$$

$$e_f = \frac{-\sin\theta - \frac{\omega L_o I_s}{V_s} - \frac{\omega L_2 I_s}{V_s}\cos 2\gamma_e}{\cos\gamma_e}$$

where

$$e_f = \frac{E_f}{V_s} = \frac{\omega M I_r}{V_s}$$

with V_s in volts peak

I_s in amps peak

ω is the electrical frequency

For the given constants

$$\cos\theta = \text{p.f.} = 0.850 \qquad \sin\theta = 0.528$$

$$\frac{\omega L_2 I_s}{V_s} = 0.200 \qquad\qquad \frac{\omega L_o I_s}{V_s} = 1.00$$

and

$$e_f = \frac{0.850 - 0.200 \sin 2\gamma_e}{\sin\gamma_e}$$

$$e_f = \frac{-1.528 - 0.200 \cos 2\gamma_e}{\cos\gamma_e}$$

Trial-and-error solution of these two equations to find a positive value of γ_e that satisfies both equations simultaneously yields

$$\gamma_e = 147.5^\circ \quad \text{and} \quad e_f = 1.92$$

From the definition of e_f we have

$$I_r = \frac{e_f V_s}{\omega M} = \frac{(1.92)(\sqrt{2})(10{,}000)}{(120)(\pi)(0.125)} = 576 \text{ amps}$$

PROBLEM 4.28 (Continued)

Part b

From Prob. 4.14 the definition of complex power is

$$\hat{V}_s\hat{I}_s^* = P + jQ$$

where $\hat{V}_s$ and $\hat{I}_s$ are complex amplitudes.

The capability curve is not as easy to calculate for a salient-pole machine as it was for a smooth-air-gap machine in Prob. 4.14. It will be easiest to calculate the curve using the power output expression of part a

$$P = \frac{V_s E_f}{X_d} \sin \delta_e + \frac{(X_d - X_q)V_s^2}{2X_d X_q} \sin 2\delta_e$$

the facts that

$$P = V_s I_s \cos \theta$$

$$Q = V_s I_s \sin \theta$$

and that I_s is given from (4.2.44) and (4.2.45) as

$$I_s = \sqrt{\left(\frac{V_s}{X_q} \sin \delta_e\right)^2 + \left(\frac{V_s}{X_d} \cos \delta_e - \frac{E_f}{X_d}\right)^2}$$

First, assuming operation at rated field current the power is

$$P = 320 \times 10^6 \sin \delta_e + 41.7 \times 10^6 \sin 2\delta_e \text{ watts.}$$

We assume values of δ_e starting from zero and calculate P; then we calculate I_s for the same values of δ_e from

$$I_s = 11{,}800 \sqrt{(1.50 \sin \delta_e)^2 + (\cos \delta_e - 1.92)^2} \text{ amps peak}$$

Next, because we know P, V_s, and I_s we find θ from

$$\cos \theta = \frac{P}{V_s I_s}$$

From θ we then find Q from

$$Q = V_s I_s \sin \theta .$$

This process is continued until rated armature current

$$I_s = \sqrt{2}\ 10{,}000 \text{ amps peak}$$

is reached.

The next part of the capability curve is limited by rated armature current which defines the trajectory

PROBLEM 4.28 (Continued)

$$\sqrt{P^2 + Q^2} = V_s I_s$$

where V_s and I_s are rated values.

For $Q < 0$, the capability curve is limited by pull-out conditions defined by the condition

$$\frac{dP}{d\delta_e} = 0 = \frac{V_s E_f}{X_d} \cos \delta_e + \frac{(X_d - X_q) V_s^2}{X_d X_q} \cos 2\delta_e$$

To evaluate this part of the curve we evaluate e_f in terms of δ_e from the power and current expressions

$$e_f = \frac{\frac{PX_d}{V_s^2} - \frac{(X_d - X_q)}{2X_q} \sin 2\delta_e}{\sin \delta_e}$$

$$e_f = \cos \delta_e - \sqrt{\left(\frac{I_s X_d}{V_s}\right)^2 - \left(\frac{X_d}{X_q} \sin \delta_e\right)^2}$$

For each level of power at a given power factor we find the value of δ_e that simultaneously satisfies both equations. The resulting values of e_f and δ_e are used in the stability criterion

$$\frac{dP}{d\delta_e} = \frac{V_s^2 e_f}{X_d} \cos \delta_e + \frac{(X_d - X_q) V_s^2}{X_d\, X_q} \cos 2\delta_e \geq 0$$

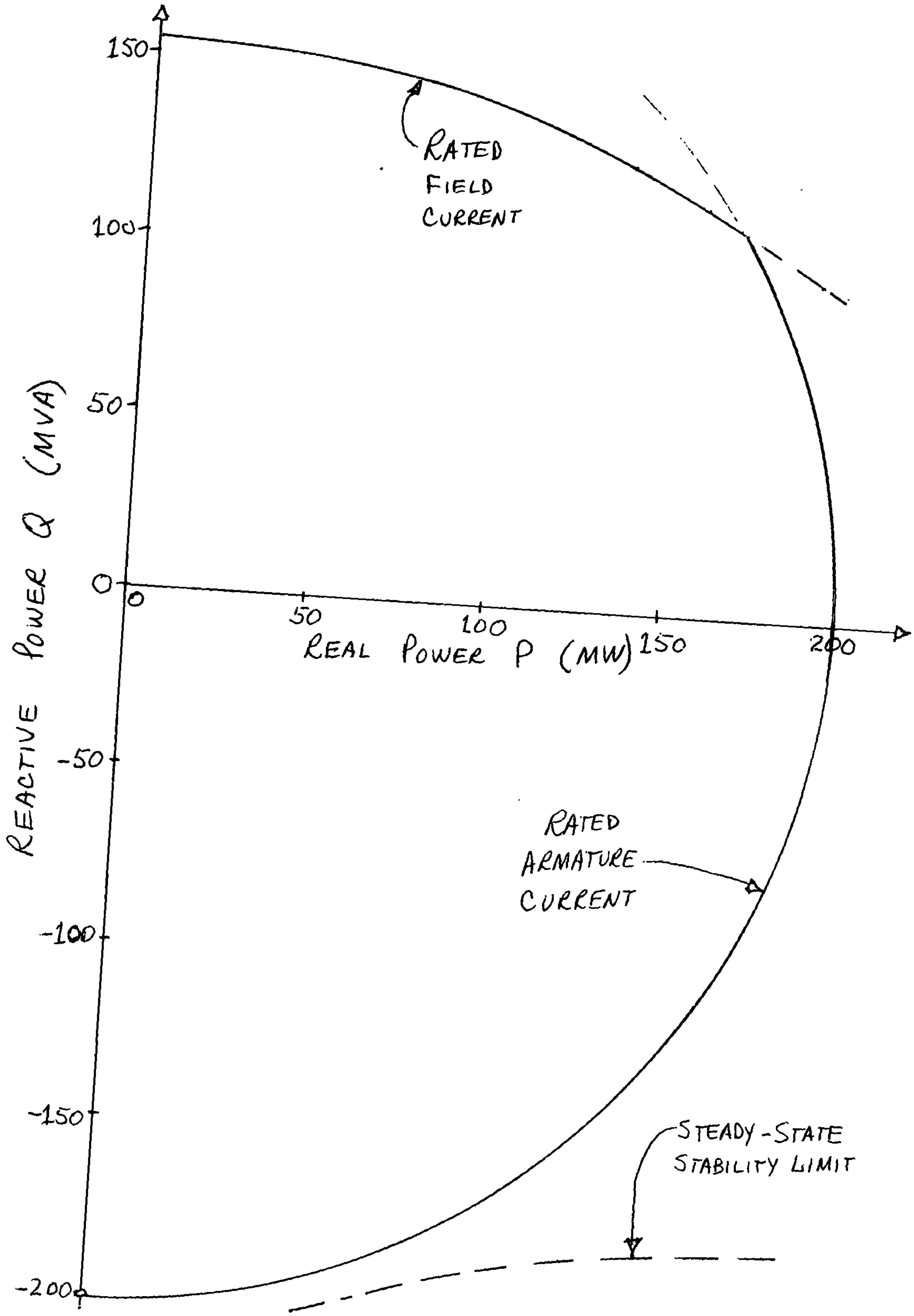
When this condition is no longer met (equal sign holds) the stability limit is reached. For the given constants

$$e_f = \frac{\frac{P}{167 \times 10^6} - 0.25 \sin 2\delta_e}{\sin \delta_e}$$

$$e_f = \cos \delta_e - \sqrt{\left(\frac{I_s}{11{,}800}\right)^2 - (1.5 \sin \delta_e)^2}$$

$$\frac{dP}{d\delta_e} = e_f \cos \delta_e + 0.5 \cos 2\delta_e \geq 0$$

The results of this calculation along with the preceding two are shown on the attached graph. Note that the steady-state stability never limits the capability. In practice, however, more margin of stability is required and the capability in the fourth quadrant is limited accordingly.



150
100
50
0
-50
-100
-150
-200
REACTIVE POWER Q (MVA)
0
50
100
150
200
REAL POWER P (MW)
RATED FIELD CURRENT
RATED ARMATURE CURRENT
STEADY-STATE STABILITY LIMIT

PROBLEM 4.29

Part a

For this electrically linear system the electric coenergy is

$$W_e'(v_1,v_2,\theta) = \frac{1}{2} C_o(1 + \cos 2\theta)v_1^2 + \frac{1}{2} C_o(1 + \sin 2\theta)v_2^2$$

The torque of electric origin is

$$T^e = \frac{\partial W_e'(v_1,v_2,\theta)}{\partial \theta} = C_o(v_2^2 \cos 2\theta - v_1^2 \sin 2\theta)$$

Part b

With $v_1 = V_o \cos \omega t$; $v_2 = V_o \sin \omega t$

$$T^e = C_o V_o^2(\sin^2 \omega t \cos 2\theta - \cos^2 \omega t \sin 2\theta)$$

Using trig identities

$$T^e = \frac{C_o V_o^2}{2}[\cos 2\theta - \cos 2\omega t \cos 2\theta - \sin 2\theta - \cos 2\omega t \cos 2\theta]$$

$$T^e = \frac{C_o V_o^2}{2} (\cos 2\theta - \sin 2\theta) - \frac{C_o V_o^2}{2}[\cos(2\omega t-2\theta) + \cos(2\omega t + 2\theta)]$$

Three possibilities for time-average torque:

Case I:

Shaft sitting still at fixed angle θ

Case II:

Shaft turning in positive θ direction

$$\theta = \omega t + \gamma$$

where γ is a constant

Case III:

Shaft turning in negative θ direction

$$\theta = - \omega t + \delta$$

where δ is a constant.

Part c

The time average torques are:

Case I: θ = const.

$$<T^e> = \frac{C_o V_o^2}{2}(\cos 2\theta - \sin 2\theta)$$

PROBLEM 4.29 (Continued)

Case II: $\theta = \omega t + \gamma$

$$\langle T^e \rangle = -\frac{C_o V_o^2}{2} \cos 2\gamma$$

Case III: $\theta = -\omega t + \delta$

$$\langle T^e \rangle = -\frac{C_o V_o^2}{2} \cos 2\delta$$

PROBLEM 4.30

For an applied voltage v(t) the electric coenergy for this electrically linear system is

$$W_e'(v,\theta) = \frac{1}{2}(C_o + C_1 \cos 2\theta)v^2$$

The torque of electric origin is then

$$T^e = \frac{\partial W_e'(v,\theta)}{\partial \theta} = -C_1 \sin 2\theta\, v^2$$

For $v = V_o \sin \omega t$

$$T^e = -C_1 V_o^2 \sin^2 \omega t \sin 2\theta$$

$$T^e = -\frac{C_1 V_o^2}{2} (\sin 2\theta - \cos 2\omega t \cos 2\theta)$$

$$T^e = -\frac{C_1 V_o^2}{2} \sin 2\theta + \frac{C_1 V_o^2}{4} [\cos(2\omega t - 2\theta) + \cos(2\omega t + 2\theta)]$$

For rotational velocity ω_m we write

$$\theta = \omega_m t + \gamma$$

and then

$$T^e = -\frac{C_1 V_o^2}{2} \sin 2(\omega_m t + \gamma)$$

$$+ \frac{C_1 V_o^2}{4} \{\cos[2(\omega - \omega_m)t - 2\gamma] + \cos[2(\omega + \omega_m)t + 2\gamma]\}$$

This device can behave as a motor if it can produce a time-average torque for ω_m = constant. This can occur when

$$\omega_m = \pm \omega$$

PROBLEM 5.1

Part a

The capacitance of the system of plane parallel electrodes is

$$C = (L+x)d\varepsilon_o/s \tag{a}$$

and since the co-energy W' of an electrically linear system is simply $\frac{1}{2}Cv^2$ (remember v is the terminal voltage of the capacitor, not the voltage of the driving source)

$$f^e = \frac{\partial W'}{\partial x} = \frac{1}{2}\frac{d\varepsilon_o}{s}v^2 \tag{b}$$

The plates tend to increase their area of overlap.

Part b

The force equation is

$$M\frac{d^2x}{dt^2} = -Kx + \frac{1}{2}\frac{d\varepsilon_o}{s}v^2 \tag{c}$$

while the electrical loop equation, written using the fact that the current dq/dt through the resistance can be written as Cv, is

$$V(t) = R\frac{d}{dt}[(L+x)\frac{d\varepsilon_o}{s}v] + v \tag{d}$$

These are two equations in the dependent variables (x,v).

Part c

This problem illustrates the important point that unless a system involving electromechanical components is either intrinsically or externally biased, its response will not in general be a linear reproduction of the input. The force is proportional to the square of the terminal voltage, which in the limit of small R is simply $V^2(t)$. Hence, the equation of motion is (c) with

$$v^2 = V^2(t) = u_{-1}(t)\frac{V_o^2}{2}(1-\cos 2\omega t) \tag{e}$$

where we have used the identity $\sin^2\omega t = \frac{1}{2}(1-\cos 2\omega t)$. For convenience the equation of motion is normalized

PROBLEM 5.1 (Continued)

$$\frac{d^2x}{dt^2} + \omega_o^2 x = \alpha u_{-1}(t)(1-\cos 2\omega t) \tag{f}$$

where

$$\omega_o^2 = K/M \; ; \; \alpha = V_o^2 d \varepsilon_o/4sM$$

To solve this equation, we note that there are two parts to the particular solution, one a constant

$$x = \frac{\alpha}{\omega_o^2} \tag{g}$$

and the other a cosinusoid having the frequency 2ω. To find this second part solve the equation

$$\frac{d^2x}{dt^2} + \omega_o^2 x = - \operatorname{Re} \alpha e^{2j\omega t} \tag{h}$$

for the particular solution

$$x = \frac{-\alpha \cos 2\omega t}{\omega_o^2 - 4\omega^2} \tag{i}$$

The general solution is then the sum of these two particular solutions and the homogeneous solution $t > 0$

$$x(t) = \frac{\alpha}{\omega_o^2} - \frac{\alpha \cos 2\omega t}{\omega_o^2 - 4\omega^2} + A \sin\omega_o t + B\cos\omega_o t \tag{j}$$

The constants A and B are determined by the initial conditions. At $t=0$, $dx/dt = 0$, and this requires that $A = 0$. The spring determines that the initial position is $x = 0$, from which it follows that

$$B = \alpha 4\omega^2/\omega_o^2(\omega_o^2 - 4\omega^2) \tag{k}$$

Finally, the required response is $(t > 0)$

$$x(t) = \frac{\alpha}{\omega_o^2}\left[1 - \frac{\cos 2\omega t}{[1-(\frac{2\omega}{\omega_o})^2]} + \frac{(\frac{2\omega}{\omega_o})^2 \cos \omega_o t}{[1-(\frac{2\omega}{\omega_o})^2]}\right] \tag{l}$$

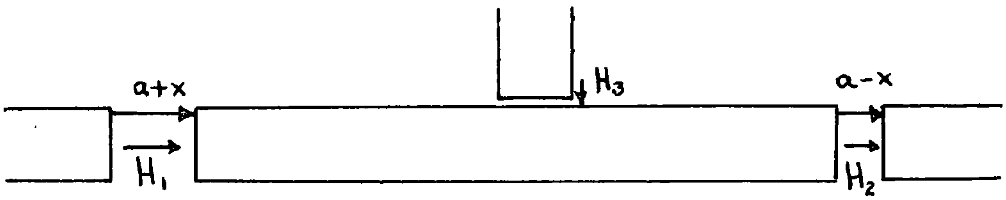
PROBLEM 5.1 (Continued)

Note that there are constant and double frequency components in this response, reflecting the effect of the drive. In addition, there is the response frequency ω_o reflecting the natural response of the spring mass system. No part of the response has the same frequency as the driving voltage.

PROBLEM 5.2

Part a

The field intensities are defined as in the figure



Ampere's law, integrated around the outside magnetic circuit gives

$$2N_1i_1 = H_1(a+x) + H_2(a-x) \tag{a}$$

and integrated around the left inner circuit gives

$$N_1i_1 - N_2i_2 = H_1(a+x) - H_3a \tag{b}$$

In addition, the net flux into the movable plunger must be zero

$$0 = H_1 - H_2 + H_3 \tag{c}$$

These three equations can be solved for H_1, H_2 and H_3 as functions of i_1 and i_2. Then, the required terminal fluxes are

$$\lambda_1 = N_1\mu_o dW(H_1+H_2) \tag{d}$$

$$\lambda_2 = N_2\mu_o dWH_3 \tag{e}$$

Hence, we have

$$\lambda_1 = \frac{N_1\mu_o dW}{3a^2-x^2}[i_1 6aN_1 + i_2 2N_2 x] \tag{f}$$

$$\lambda_2 = \frac{N_2\mu_o dW}{3a^2-x^2}[i_1 2N_1 x + i_2 2aN_2] \tag{g}$$

Part b

To use the device as a differential transformer, it would be excited at a frequency such that

PROBLEM 5.2 (Continued)

$$\frac{2\pi}{\omega} << T \tag{h}$$

where T is a period characterizing the movement of the plunger. This means that in so far as the signal induced at the output terminals is concerned, the effect of the motion can be ignored and the problem treated as though x is a constant (a quasi-static situation, but not in the sense of Chap. 1). Put another way, because the excitation is at a frequency such that (h) is satisfied, we can ignore idL/dt compared to Ldi/dt and write

$$v_2 = \frac{d\lambda_2}{dt} = - \frac{\omega 2N_1N_2\mu_o dWxI_o}{(3a^2-x^2)} \sin \omega t$$

At any instant, the amplitude is determined by x(t), but the phase remains independent of x(t), with the voltage leading the current by 90°. By design, the output signal is zero at x=0 and tends to be proportional to x over a range of x << a.

PROBLEM 5.3

Part a

The potential function which satisfies the boundary conditions along constant θ planes is

$$\phi = \frac{V\theta}{\psi} \tag{a}$$

where differentiation shows that Laplaces equation is satisfied. The constant has been set so that the potential is V on the upper electrode where $\theta = \psi$, and zero on the lower electrode where $\theta = 0$. Then, the electric field is

$$\bar{E} = -\nabla\phi = -\bar{i}_\theta \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\bar{i}_\theta \frac{V}{r\psi} \tag{b}$$

Part b

The charge on the upper electrode can be written as a function of (V,ψ) by writing

$$q = D\varepsilon_o \int_a^b \frac{V}{r\psi} dr = \frac{D\varepsilon_o V}{\psi} \ln(\frac{b}{d}) \tag{c}$$

PROBLEM 5.3 (Continued)

Part c

Then, the energy stored in the electromechanical coupling follows as

$$W = \int V dq = \int \frac{q\psi dq}{D\varepsilon_o \ln(\frac{b}{a})} = \frac{1}{2} q^2 \frac{\psi}{D\varepsilon_o \ln(\frac{b}{a})} \quad \text{(d)}$$

and hence

$$T^e = -\frac{\partial W}{\partial \psi} = -\frac{1}{2} \frac{q^2}{D\varepsilon_o \ln(\frac{b}{a})} \quad \text{(e)}$$

Part d

The mechanical torque equation for the movable plate requires that the inertial torque be balanced by that due to the torsion spring and the electric field

$$\frac{Jd^2\psi}{dt^2} = \alpha(\psi_o - \psi) - \frac{1}{2} \frac{q^2}{D\varepsilon_o \ln(\frac{b}{a})} \quad \text{(f)}$$

The electrical equation requires that currents sum to zero at the current node, and makes use of the terminal equation (c).

$$\frac{dQ}{dt} = \frac{dq}{dt} + \frac{Gd}{dt}\left[\frac{q\psi}{D\varepsilon_o \ln(\frac{b}{a})}\right] \quad \text{(g)}$$

Part e

With $G = 0$, $Q(t) = q(t)$. (This is true to within a constant, corresponding to charge placed on the upper plate initially. We will assume that this constant is zero.) Then, (f) reduces to

$$\frac{d^2\psi}{dt^2} + \frac{\alpha}{J} = \frac{\alpha}{J}\psi_o - \frac{1}{4} \frac{Q_o^2}{J D\varepsilon_o \ln(\frac{b}{a})} (1+\cos 2\omega t) \quad \text{(h)}$$

where we have used the identity $\cos^2\omega t = \frac{1}{2}(1 + \cos 2\omega t)$. This equation has a solution with a constant part

$$\psi_1 = \psi_o - \frac{1}{4} \frac{Q_o^2}{\alpha D\varepsilon_o \ln(\frac{b}{a})} \quad \text{(i)}$$

and a sinusoidal steady state part

$$\psi' = -\frac{Q_o^2 \cos 2\omega t}{J 4 D\varepsilon_o \ln(\frac{b}{a})\left[\frac{\alpha}{J} - (2\omega)^2\right]} \quad \text{(j)}$$

PROBLEM 5.3 (Continued)

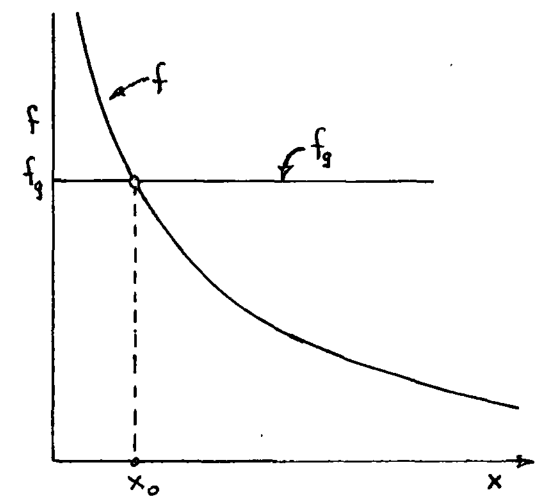
as can be seen by direct substitution. The plate responds with a d-c part and a part which has twice the frequency of the drive. As can be seen from the mathematical description itself, this is because regardless of whether the upper plate is positive or negative, it will be attracted toward the opposite plate where the image charges reside. The plates always attract. Hence, if we wish to obtain a mechanical response that is proportional to the driving signal, we must bias the system with an additional source and used the drive to simply increase and decrease the amount of this force.

PROBLEM 5.4

Part a

The equation of motion is found from (d) and (h) with $i=I_o$, as given in the solution to Prob. 3.4.

$$M\frac{d^2x}{dt^2} = Mg - \frac{1}{2}I_o^2\frac{(N^2\mu_o aw)}{(\frac{da}{b}+x)^2} \qquad \text{(a)}$$



Part b

The mass M can be in static equilibrium if the forces due to the field and gravity just balance,

$$f_g = f$$

or

$$Mg = \frac{1}{2}I_o^2\frac{(N^2\mu_o aw)}{(\frac{da}{b}+x)^2} \qquad \text{(b)}$$

A solution to this equation is shown graphically in the figure. The equilibrium is statically unstable because if the mass moves in the positive x direction from x_o, the gravitational force exceeds the magnetic force and tends to carry it further from equilibrium.

Part c

Because small perturbations from equilibrium are being considered it is appropriate to linearize. We assume $x = x_o + x'(t)$ and expand the last term in (a) to obtain

PROBLEM 5.4 (Continued)

$$-\frac{1}{2} I_o^2 \frac{(N^2\mu_o aw)}{(\frac{da}{b} + x_o)^2} + I_o^2 \frac{(N^2\mu_o aw)}{(\frac{da}{b} + x_o)^3} x' + \dots \tag{c}$$

(see Sec. 5.1.2a). The constant terms in the equation of motion cancel out by virtue of (b) and the equation of motion is

$$\frac{d^2x'}{dt^2} - \alpha^2 x' = 0; \quad \alpha = \sqrt{\frac{I_o^2(N^2\mu_o aw)}{(\frac{da}{b} + x_o)^3 M}} \tag{d}$$

Solutions are exp $\pm\, \alpha t$, and the linear combination which satisfies the given initial conditions is

$$x' = \frac{v_o}{\alpha}[e^{\alpha t} - e^{-\alpha t}] \tag{e}$$

PROBLEM 5.5

Part a

For small values of x relative to d, the equation of motion is

$$M\frac{d^2x}{dt^2} = \frac{Q_oQ}{4\pi\varepsilon}[\frac{1}{d^2} - \frac{2x}{d^3} - \frac{1}{d^2} - \frac{2x}{d^3}] \tag{a}$$

which reduces to

$$\frac{d^2x}{dt^2} + \omega_o^2 x = 0 \text{ where } \omega_o^2 = \frac{Q_oQ_1}{M\pi\varepsilon d^3} \tag{b}$$

The equivalent spring constant will be positive if

$$\frac{Q_oQ_1}{\pi\varepsilon d^3} > 0 \tag{c}$$

and hence this is the condition for stability. The system is stable if the charges have like signs.

Part b

The solution to (b) has the form

$$x = A\cos\omega_o t + B\sin\omega_o t \tag{d}$$

and in view of the initial conditions, $B = 0$ and $A = x_o$.

PROBLEM 5.6

Part a

Questions of equilibrium and stability are of interest. Therefore, the equation of motion is written in the standard form

$$M \frac{d^2x}{dt^2} = - \frac{\partial V}{\partial x} \tag{a}$$

where

$$V = Mgx - W' \tag{b}$$

Here the contribution of W' to the potential is negative because $F^e = \partial W'/\partial x$. The separate potentials are shown in the figure, together with the total potential. From this plot it is clear that there will be one point of static equilibrium as indicated.

Part b

An analytical expression for the point of equilibrium follows by setting the force equal to zero

$$\frac{\partial V}{\partial x} = Mg + \frac{2L_o X^3}{b^4} I^2 \tag{c}$$

Solving for X, we have

$$X = - [\frac{Mgb^4}{2L_o I^2}]^{1/3} \tag{d}$$

Part c

It is clear from the potential plot that the equilibrium is stable.

PROBLEM 5.7

From Prob. 3.15 the equation of motion is, for small θ

$$J \frac{d^2\theta}{dt^2} = - K\theta + \frac{1}{2} \mu_o DN^2 \ln(\frac{b}{a}) I_o^2 [\frac{4\theta}{(\beta-\alpha)^3}] \tag{a}$$

Thus, the system will have a stable static equilibrium at $\theta = 0$ if the effective spring constant is positive, or if

$$K > \frac{2\mu_o DN^2}{(\beta-\alpha)^3} \ln(\frac{b}{a}) I_o^2 \tag{b}$$

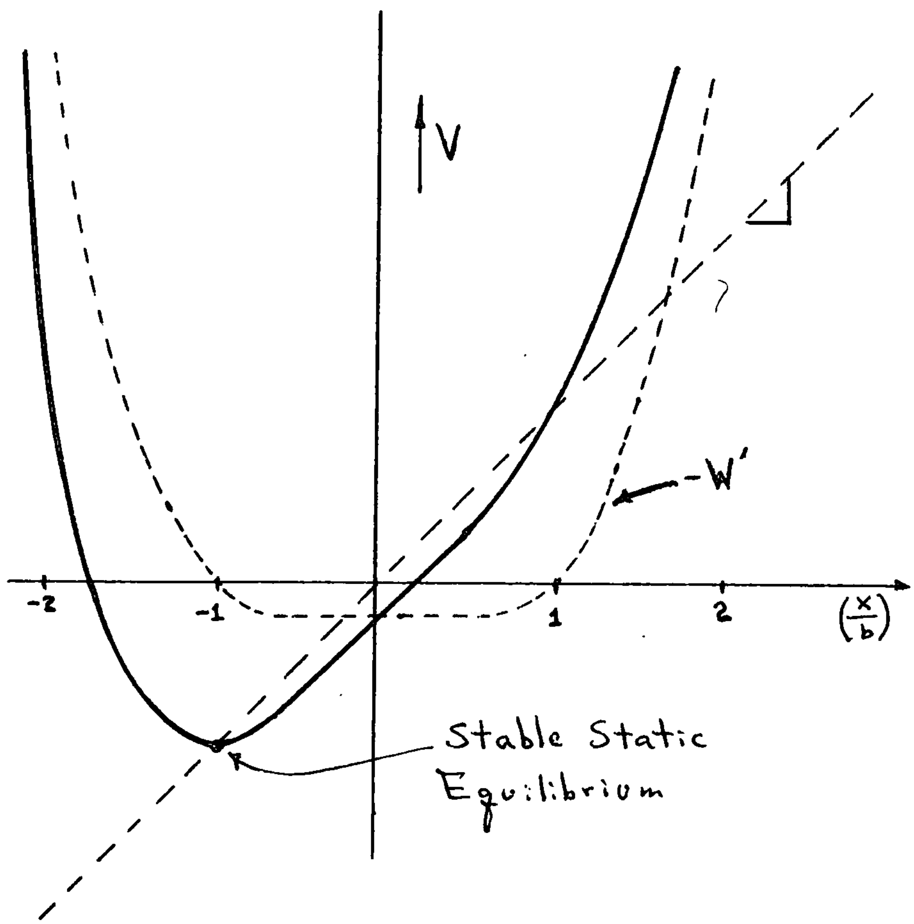


Figure for Prob. 5.6

PROBLEM 5.8

Part a

The coenergy is

$$W' = \int_o^{i_1} \lambda_1(i_1',0,x)di_1' + \int_o^{i_2} \lambda_2(i_1,i_2',x)di_2' \tag{a}$$

which can be evaluated using the given terminal relations

$$W' = [\tfrac{1}{2}L_1\, i_1^2 + Mi_1i_2 + \tfrac{1}{2}\, L_2i_2^2]/(1+\tfrac{x}{a})^3 \tag{b}$$

If follows that the force of electrical origin is

$$f^e = \frac{\partial W'}{\partial x} = -\frac{3}{2a}[L_1i_1^2 + 2Mi_1i_2 + L_2i_2^2]/(1+\tfrac{x}{a})^4 \tag{c}$$

Part b

The static force equation takes the form

$$-f^e = Mg \tag{d}$$

or, with $i_2=0$ and $i_1=I$,

$$\frac{3}{2a}\frac{L_1\; I^2}{[1+\frac{X_o}{a}]^4} = Mg \tag{e}$$

Solution of this equation gives the required equilibrium position X_o

$$\frac{X_o}{a} = [\frac{3}{2a}\frac{L_1I^2}{Mg}]^{1/4} - 1 \tag{f}$$

Part c

For small perturbations from the equilibrium defined by (e),

$$M_o\,\frac{d^2x'}{dt^2} - \frac{6L_1I^2x'}{a^2(1+\frac{X_o}{a})^5} = f(t) \tag{g}$$

where f(t) is an external force acting in the x direction on M.

With the external force an impulse of magnitude I_o and the mass initially at rest, one initial condition is $x(0) = 0$. The second is given by integrating the equation of motion form 0^- to 0^+

$$\int_{0^-}^{0^+} \frac{d}{dt}(M_o\,\frac{dx}{dt})dt - \text{constant}\int_{0^-}^{0^+} x'dt = I_o\int_{0^-}^{0^+} u_{-1}(t)dt \tag{h}$$

The first term is the jump in momentum at t=0, while the second is zero if x is to remain continuous. By definition, the integral on the right is I_o. Hence, from (h) the second initial condition is

PROBLEM 5.8 (Continued)

$$M_o \frac{dx'}{dt}(0) = I_o, \quad x'(o) = 0 \tag{i}$$

In view of these conditions, the response is

$$x'(t) = \frac{I_o}{2\alpha M_o}\left(e^{\alpha t} - e^{-\alpha t}\right) \tag{j}$$

where

$$\alpha = \left[6L_1 I^2 / a^2 M_o \left(1 + \frac{X_o}{a}\right)^5\right]^{1/2}$$

Part d

With proportional feedback through the current i_2, the mutual term in the force equation makes a linear contribution and the force equation becomes

$$M_o \frac{d^2x'}{dt^2} - \left[\frac{6L_1 I^2}{a^2\left(1 + \frac{X_o}{a}\right)^5} - \frac{3MI\alpha}{a}\right]x' = f(t) \tag{k}$$

$$\left(1 + \frac{X_o}{a}\right)^4$$

The effective spring constant is positive if

$$\alpha I > 2L_1 I^2 / a \left(1 + \frac{X_o}{a}\right)^8 M \tag{l}$$

and hence this is the condition for stability. However, once initiated oscillations remain undamped according to this model.

Part e (see insert)

With a damping term introduced by the feedback, the mechanical equation becomes

$$M_o \frac{d^2x'}{dt^2} + \frac{3MI}{a\left(1+\frac{X_o}{a}\right)^4} \beta \frac{dx'}{dt} + K_e x' = f(t) \tag{m}$$

where

$$K_e = \frac{3MI\alpha}{a\left(1+\frac{X_o}{a}\right)^4} - \frac{6L_1 I^2}{a^2\left(1 + \frac{X_o}{a}\right)^5} = \frac{3MI\alpha}{a\left(1+\frac{X_o}{a}\right)^4} - \frac{6L_1 I^2}{a^2\left(1+\frac{X_o}{a}\right)^5}$$

This equation has solutions of the form exp st, where substitution shows that

$$s = -\frac{3MI\beta}{2aM_o\left(1+\frac{X_o}{a}\right)^4} \pm \sqrt{\left(\frac{3MI\beta}{2aM_o}\right)^2 \frac{1}{\left(1+\frac{X_o}{a}\right)^8} - \frac{K_e}{M_o}} \tag{n}$$

For the response to decay, K_e must be positive (the system must be stable without damping) and β must be positive.

$$\frac{3}{2aM_o}\frac{MI\beta}{\left(1+\frac{X_o}{a}\right)^4} > \sqrt{\left(\frac{3MI\beta}{2aM_o\left(1+\frac{X_o}{a}\right)^4}\right)^2 - \frac{K_e}{M_o}}$$

PROBLEM 5.9

Part a

The mechanical equation of motion is

$$M\frac{d^2x}{dt^2} = -K(x-\ell_o)-B\frac{dx}{dt} + f^e \qquad (a)$$

Part b

where the force f^e is found from the coenergy function which is (because the system is electrically linear) $W' = \frac{1}{2}Li^2 = \frac{1}{2}Ax^3i^2$

$$f^e = \frac{\partial W'}{\partial x} = \frac{3}{2}Ax^2i^2 \qquad (b)$$

Part c

We can both find the equilibrium points X_o and determine if they are stable by writing the linearized equation at the outset. Hence, we let $x(t)=X_o+x'(t)$ and (a) and (b) combine to give

$$M\frac{d^2x'}{dt^2} = -K(X_o-\ell_o)-Kx' - B\frac{dx'}{dt} + \frac{3}{2}AI_o^2\,(X_o^2 + 2X_ox') \qquad (c)$$

With the given condition on I_o, the constant (equilibrium) part of this equation is

$$X_o - \ell_o = \frac{3X_o^2}{16\ell_o} \qquad (d)$$

which can be solved for X_o/ℓ_o to obtain

$$\frac{X_o}{\ell_o} = \begin{bmatrix} 12/3 \\ 4/3 \end{bmatrix} \qquad (e)$$

That is, there are two possible equilibrium positions. The perturbation part of (c) tells whether or not these are stable. That equation, upon substitution of X_o and the given value of I_o, becomes

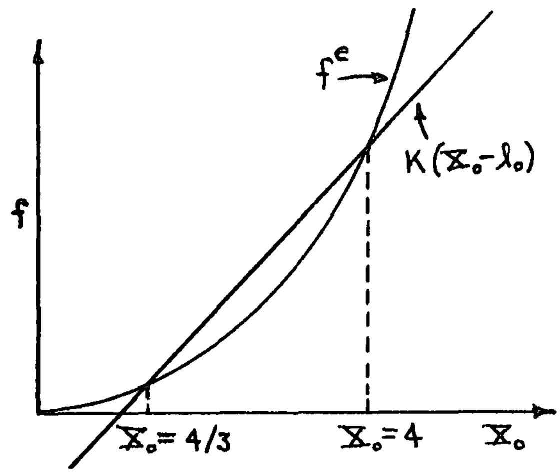
$$M\frac{d^2x'}{dt^2} = -K[1-\binom{3/2}{1/2}]x' - B\frac{dx'}{dt} \qquad (f)$$

where the two possibilities correspond to the two equilibrium points. Hence, we conclude that the effective spring constant is positive (and the system is stable) at $X_o/\ell_o = 4/3$ and the effective spring constant is negative (and hence the equilibrium is unstable) at $X_o/\ell_o = 4$.

Part d

The same conclusions as to the stability of the equilibrium points can be made from the figure.

PROBLEM 5.9 (Continued)



Consider the equilibrium at $X_o = 4$. A small displacement to the right makes the force f^e dominate the spring force, and this tends to carry the mass further in the x direction. Hence, this point is unstable. Similar arguments show that the other point is stable.

PROBLEM 5.10

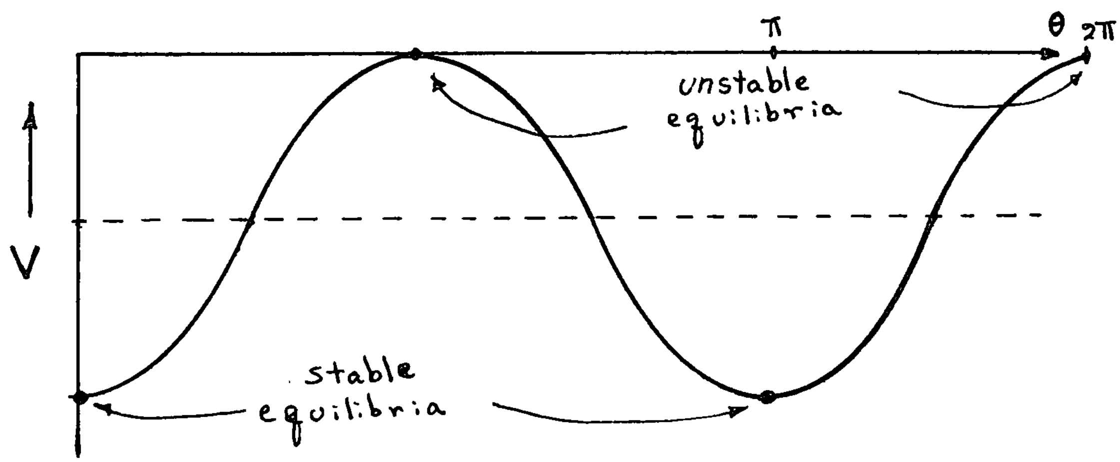
Part a

The terminals are constrained to constant potential, so use coenergy found from terminal equation as

$$W' = \int q dv = \frac{1}{2} C_o(1 + \cos 2\theta)V_o^2 \tag{a}$$

Then, since $T^e = \partial W'/\partial\theta$ and there are no other torques acting on the shaft, the total torque can be found by taking the negative derivative of a potential $V = -W'$, where V is the potential well. A sketch of this well is as shown in the figure.

PROBLEM 5.10 (Continued)



Here it is clear that there are points of zero slope (and hence zero torque and possible static equilibrium) at

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots \qquad \text{(b)}$$

Part b

From the potential well it is clear that the first and third equilibria are stable, while the second and fourth are unstable.

PROBLEM 5.11

Part a

From the terminal pair relation, the coenergy is given by

$$W_m'(i_1,i_2,\theta) = \frac{1}{2}(L_o+M\cos 2\theta)i_1^2 + \frac{1}{2}(L_o-M\cos 2\theta)i_2^2 + M\sin 2\theta\, i_1 i_2 \qquad \text{(a)}$$

so that the torque of electrical origin is

$$T^e = M[\sin 2\theta(i_2^2-i_1^2) + 2\cos 2\theta\, i_1 i_2] \qquad \text{(b)}$$

Part b

For the two phase currents, as given,

$$i_2^2 - i_1^2 = -I^2 \cos 2\omega_s t$$

$$i_1 i_2 = I^2 \frac{1}{2} \sin 2\omega_s t \qquad \text{(c)}$$

so that the torque T^e becomes

PROBLEM 5.11 (Continued)

$$T^e = MI^2[-\sin 2\theta \cos 2\omega_s t + \sin 2\omega_s t \cos 2\theta] \qquad \text{(d)}$$

or

$$T^e = MI^2 \sin(2\omega_s t - 2\theta) \qquad \text{(e)}$$

Substitution of $\theta = \omega_m t + \delta$ obtains

$$T^e = -MI^2 \sin[2(\omega_m - \omega_s)t + 2\delta] \qquad \text{(f)}$$

and for this torque to be constant, we must have the frequency condition

$$\omega_m = \omega_s \qquad \text{(g)}$$

under which condition, the torque can be written as

$$T^e = -MI^2 \sin 2\delta \qquad \text{(h)}$$

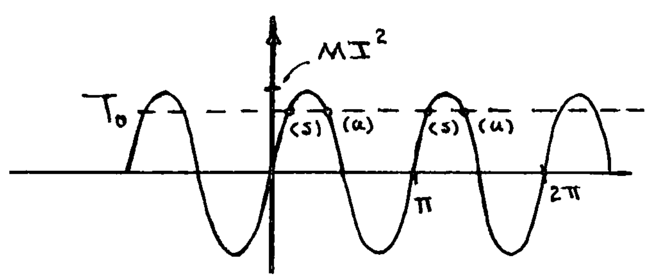
Part c

To determine the possible equilibrium angles δ_o, the perturbations and time derivatives are set to zero in the mechanical equations of motion.

$$T_o = MI^2 \sin 2\delta_o \qquad \text{(i)}$$

Here, we have written the time dependence in a form that is convenient if $\cos 2\delta_o > 0$, as it is at the points marked (s) in the figure. Hence, these points are stable. At the points marked (u), the argument of the sin function and the denominator are imaginary, and the response takes the form of a sinh function. Hence, the equilibrium points indicated by (u) are unstable.

Graphical solutions of this expression are shown in the figure. For there to be equilibrium values of δ the currents must be large enough that the torque can be maintained with the rotor in synchronism with the rotating field. ($MI^2 > T_o$)



PROBLEM 5.11 (Continued)

Returning to the perturbation part of the equation of motion with $\omega_m = \omega_s$,

$$J \frac{d^2}{dt^2}(\omega_m t + \delta_o + \delta') = T_o + T' - MI^2 \sin(2\delta_o + 2\delta') \quad \text{(j)}$$

linearization gives

$$J \frac{d^2\delta'}{dt^2} + (2MI^2 \cos 2\delta_o)\delta' = T' \quad \text{(k)}$$

where the constant terms cancel out by virtue of (i). With $T' = \tau_o u_o(t)$ and initial rest conditions, the initial conditions are

$$\frac{d\delta'}{dt}(0^+) = \frac{\tau_o}{J} \quad \text{(l)}$$

$$\delta'(0^+) = 0 \quad \text{(m)}$$

and hence the solution for $\delta'(t)$ is

$$\delta'(t) = \frac{\tau_o}{J\sqrt{\dfrac{2MI^2\cos 2\delta_o}{J}}} \sin\left[\sqrt{\frac{2MI^2\cos 2\delta_o}{J}}\, t\right] \quad \text{(n)}$$

PROBLEM 5.12

Part a

The magnitude of the field intensity (H) in the gaps is the same. Hence, from Ampere's law,

$$H = Ni/2x \quad \text{(a)}$$

and the flux linked by the terminals is N times that passing across either of the gaps.

$$\lambda = \frac{\mu_o adN^2}{2x} i = L(x)i \quad \text{(b)}$$

Because the system is electrically linear, $W'(i,x) = \frac{1}{2} Li^2$, and we have

$$f^e = \frac{\partial W'}{\partial x} = -\frac{N^2 ad\mu_o}{4x^2} i^2 \quad \text{(c)}$$

as the required force of electrical origin acting in the x direction.

Part b

Taking into account the forces due to the springs, gravity and the magnetic field, the force equation becomes

PROBLEM 5.12 (Continued)

$$M \frac{d^2x}{dt^2} = -2Kx + Mg - \frac{N^2 ad\mu_o}{4x^2} i^2 + f(t) \tag{d}$$

where the last term accounts for the driving force.

The electrical equation requires that the currents sum to zero at the electrical node, where the voltage is $d\lambda/dt$, with λ given by (b).

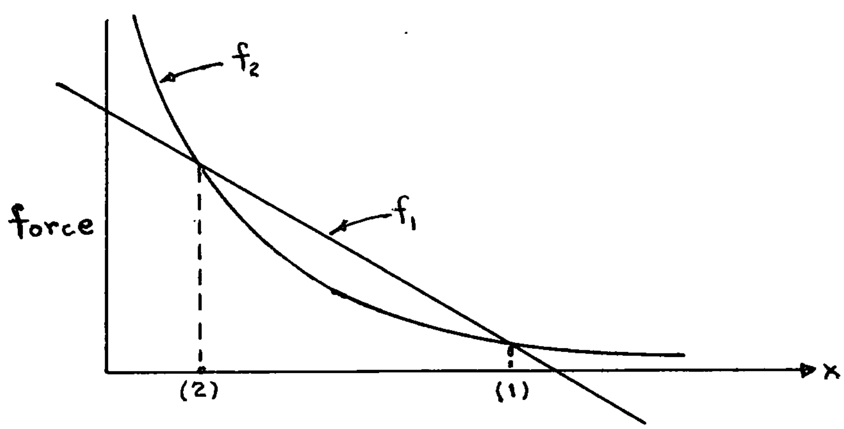
$$I = \frac{1}{R}\frac{d}{dt}\left[\frac{\mu_o adN^2}{2x} i\right] + i \tag{e}$$

Part c

In static equilibrium, the electrical equation reduces to i=I, while the mechanical equation which takes the form $f_1 = f_2$ is satisfied if

$$-2KX + Mg = \frac{N^2 ad\mu_o I^2}{4X^2} \tag{f}$$

Here, f_2 is the negative of the force of electrical origin and therefore (if positive) acts in the $-x$ direction. The respective sides of (f) are shown in the sketch, where the points of possible static equilibrium are indicated. Point (1) is stable, because a small excursion to the right makes f_2 dominate over f_1 and this tends to return the mass in the minus x direction toward the equilibrium point. By contrast, equilibrium point (2) is characterized by having a larger force f_2 and f_1 for small excursions to the left. Hence, the dominate force tends to carry the mass even further from the point of equilibrium and the situation is unstable. In what follows, $x = X$ will be used to indicate the position of stable static equilibrium (1).



PROBLEM 5.12 (Continued)

Part d

If R is very large, then

$$i \approx I$$

even under dynamic conditions. This approximation allows the removal of the characteristic time L/R from the analysis as reflected in the reduction in the order of differential equation required to define the dynamics. The mechanical response is determined by the mechanical equation ($x = X + x'$)

$$M \frac{d^2x'}{dt^2} = -2Kx' + \frac{N^2 ad\mu_o}{2X^3} I^2 x' + f(t) \tag{g}$$

where the constant terms have been balanced out and small perturbations are assumed. In view of the form taken by the excitation, assume $x = \text{Re}\,\hat{x}\, e^{j\omega t}$ and define $K_e \equiv 2K - N^2 ad\mu_o I^2/2X^3$. Then, (g) shows that

$$\hat{x} = \hat{f}/(K_e - \omega^2 M) \tag{h}$$

To compute the output voltage

$$v_o \simeq \frac{d\lambda}{dt}\Bigg|_{i=I} = -\frac{\mu_o adN^2 I}{2X^2}\frac{dx'}{dt} \tag{i}$$

or

$$\hat{v}_o = -j\frac{\omega\mu_o adN^2 I}{2X^2}\hat{x} \tag{j}$$

Then, from (h), the transfer function is

$$\frac{\hat{v}_o}{\hat{f}} = -j\frac{\omega\mu_o adN^2 I}{2X^2(K_e - \omega^2 M)} \tag{k}$$

PROBLEM 5.13

Part a

The system is electrically linear. Hence, the coenergy takes the standard form

$$W' = \frac{1}{2}L_{11}i_1^2 + L_{12}i_1 i_2 + \frac{1}{2}L_{22}i_2^2 \tag{a}$$

and it follows that the force of electrical origin on the plunger is

$$f^e = \frac{\partial W'}{\partial x} = \frac{1}{2}i_1^2\frac{\partial L_{11}}{\partial x} + i_1 i_2\frac{\partial L_{12}}{\partial x} + \frac{1}{2}i_2^2\frac{\partial L_{22}}{\partial x} \tag{b}$$

PROBLEM 5.13 (Continued)

which, for the particular terminal relations of this problem becomes

$$f^e = L_o \{ \frac{-i_1^2}{d} (1+ \frac{x}{d}) - \frac{i_1 i_2}{d} \frac{2x}{d} + \frac{i_2^2}{d} (1- \frac{x}{d}) \} \tag{c}$$

Finally, in terms of this force, the mechanical equation of motion is

$$M \frac{d^2x}{dt^2} = -Kx - B \frac{dx}{dt} + f^e \tag{d}$$

The circuit connections show that the currents i_1 and i_2 are related to the source currents by

$$\begin{aligned} i_1 &= I_o + i \\ i_2 &= I_o - i \end{aligned} \tag{e}$$

Part b

If we use (e) in (b) and linearize, it follows that

$$f^e = -\frac{4L_o I_o}{d} i - \frac{4L_o I_o^2}{d} \frac{x}{d} \tag{f}$$

and the equation of motion is

$$\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + \omega_o^2 x = - Ci \tag{g}$$

where

$$\omega_o = \sqrt{[K + \frac{4L_o I_o^2}{d^2}]/M}$$

$$\alpha = B/M$$

$$C = 4L_o I_o/dM$$

Part c

Both the spring constant and damping in the equation of motion are positive, and hence the system is always stable.

Part d

The homogeneous equation has solutions of the form e^{pt} where

$$p^2 + \alpha p + \omega_o^2 = 0 \tag{h}$$

or, since the system is underdamped

$$p = -\frac{\alpha}{2} \pm j \sqrt{\omega_o^2 - (\frac{\alpha}{2})^2} = -\frac{\alpha}{2} \pm j\omega_p \tag{i}$$

PROBLEM 5.13 (Continued)

The general solution is

$$x(t) = -\frac{CI_o}{\omega_o^2} + e^{-\frac{\alpha}{2}t}[A \sin \omega_p t + D \cos \omega_p t] \tag{j}$$

where the constants are determined by the initial conditions $x(0) = 0$ and $dx/dt(0) = 0$

$$D = \frac{CI_o}{\omega_o^2}; \quad A = \frac{\alpha CI_o}{2\omega_p\omega_o^2} \tag{k}$$

Part e

With a sinusoidal steady state condition, assume $x = \text{Re}\, \hat{x}\, e^{j\omega t}$ and write $i(t) = \text{Re}(-jI_o)e^{j\omega t}$ and (g) becomes

$$\hat{x}(-\omega^2 + j\omega\alpha + \omega_o^2) = CjI_o \tag{l}$$

Thus, the required solution is

$$x(t) = \frac{\text{Re}\, jCI_o e^{j\omega t}}{(\omega_o^2-\omega^2)+j\omega\alpha} \tag{m}$$

PROBLEM 5.14

Part a

From the terminal equations, the current i_1 is determined by Kirchhoff's current law

$$G\, L_1 \frac{di_1}{dt} + i_1 = I + GMI_2\Omega \sin \Omega t \tag{a}$$

The first term in this expression is the current which flows through G because of the voltage developed across the self inductance of the coil, while the last is a current through G induced by the rotational motion. The terms on the right are known functions of time, and constitute a driving function for the linear equation.

Part b

We can divide the solution into particular solutions due to the two driving terms and a homogeneous solution. From the constant drive I we have the solution

$$i_1 = I \tag{b}$$

Because $\sin \Omega t = \text{Re}(-je^{j\Omega t})$, if we assume a particular solution for the sinusoidal drive of the form $i_1 = \text{Re}(\hat{I}_1 e^{j\Omega t})$, we have

PROBLEM 5.14 (Continued)

$$\hat{I}_1\ (j\Omega GL_1 + 1) = -\ j\Omega GMI_2 \tag{c}$$

or, rearranging

$$\hat{I}_1 = \frac{-\Omega GMI_2(\Omega GL_1 + j)}{1 + (\Omega GL_1)^2} \tag{d}$$

We now multiply this complex amplitude by $e^{j\Omega t}$ and take the real part to obtain the particular solution due to the sinusoidal drive

$$i_1 = \frac{-GMI_2\Omega}{1+(\Omega GL_1)^2}\ (\Omega GL_1\ \cos\ \Omega t\ -\ \sin\ \Omega t) \tag{e}$$

The homogeneous solution is

$$i_1 = A\ e^{-t/GL_1} \tag{f}$$

and the total solution is the sum of (b), (e) and (f) with the constant A determined by the initial conditions.

In view of the initial conditions, the complete solution for i_1, normalized to the value necessary to produce a flux equal to the maximum mutual flux, is then

$$\frac{L_1 i_1}{MI_2} = \left[\frac{(\Omega GL_1)^2}{1+(\Omega GL_1)^2} - \frac{L_1 I}{MI_2}\right] e^{-\frac{\Omega t}{\Omega GL_1}}$$

$$+ \frac{GL_1\Omega}{1+(\Omega GL_1)^2}\ (\sin\ \Omega t\ -\ \Omega GL_1\ \cos\ \Omega t) + \frac{L_1 I}{MI_2} \tag{g}$$

Part c

The terminal relation is used to find the flux linking coil 1

$$\frac{\lambda_1}{MI_2} = \left[\frac{(\Omega GL_1)^2}{1+(\Omega GL_1)^2} - \frac{L_1 I}{MI_2}\right] e^{-\frac{\Omega t}{\Omega GL_1}}$$

$$+ \frac{GL_1\Omega}{1+(\Omega GL_1)^2}\ \sin\ \Omega t + \frac{\cos\ \Omega t}{1+(\Omega GL_1)^2} + \frac{L_1 I}{MI_2} \tag{h}$$

The flux has been normalized with respect to the maximum mutual flux (MI_2).

PROBLEM 5.14 (Continued)

Part d

In order to identify the limiting cases and the appropriate approximations it is useful to plot (g) and (h) as functions of time. These equations contain two constants, ΩGL_1 and L_1I/MI_2. The time required for one rotation is $2\pi/\Omega$ and GL_1 is the time constant of the inductance L_1 and conductance G in series. Thus, ΩGL_1 is essentially the ratio of an electrical time constant to the time required for the coil to traverse the applied field one time. The quantity MI_2 is the maximum flux of the externally applied field that can link the rotatable coil and L_1I is the self flux of the coil due to current I acting alone. Thus, L_1I/MI_2 is the ratio of self excitation to mutual excitation.

To first consider the limiting case that can be approximated by a current source we require that

$$\Omega GL_1 \ll 1 \text{ and } \Omega GL_1 \ll \frac{L_1I}{MI_2} \tag{i}$$

To demonstrate this set

$$\Omega GL_1 = 0.1 \text{ and } \frac{L_1I}{MI_2} = 1 \tag{j}$$

and plot current and flux as shown in Fig. (a). We note first that the transient dies out very quickly compared to the time of one rotation. Furthermore, the flux varies appreciably while the current varies very little compared to its average value. In the ideal limit ($G\to 0$) the transient would die out instantaneously and the current would be constant. Thus the approximation of the situation by an ideal current-source excitation would involve a small error; however, the saving in analytical time is often well worth the decrease in accuracy resulting from the approximation.

Part e

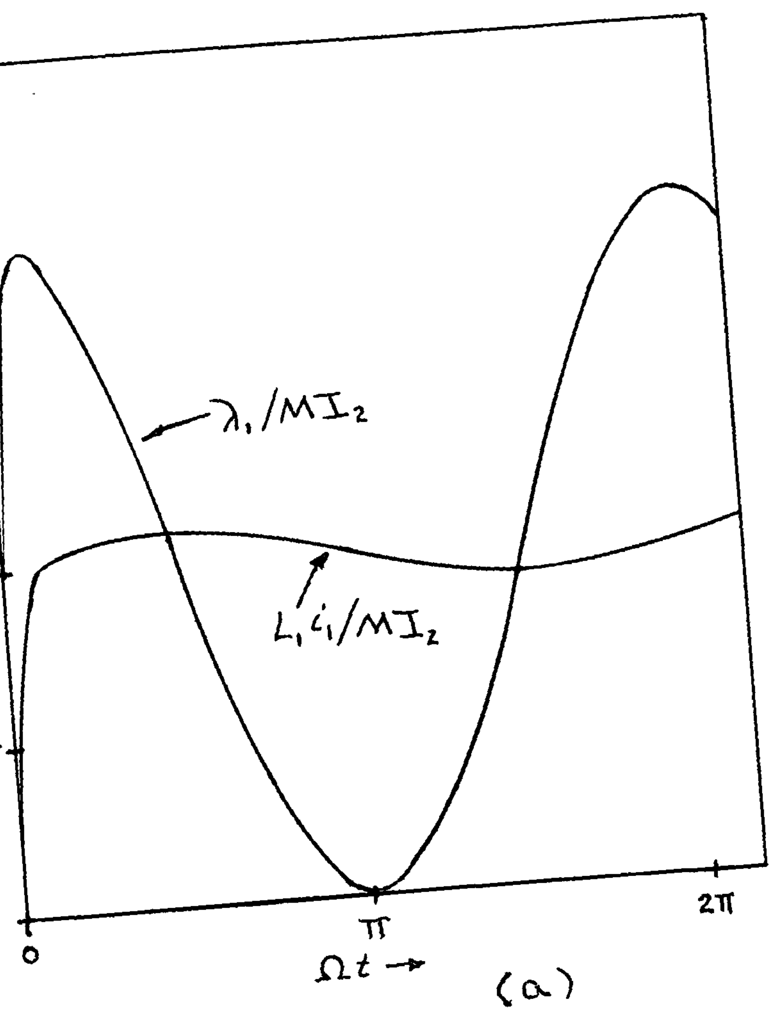
We next consider the limiting case that can be approximated by a constant-flux constraint. This requires that

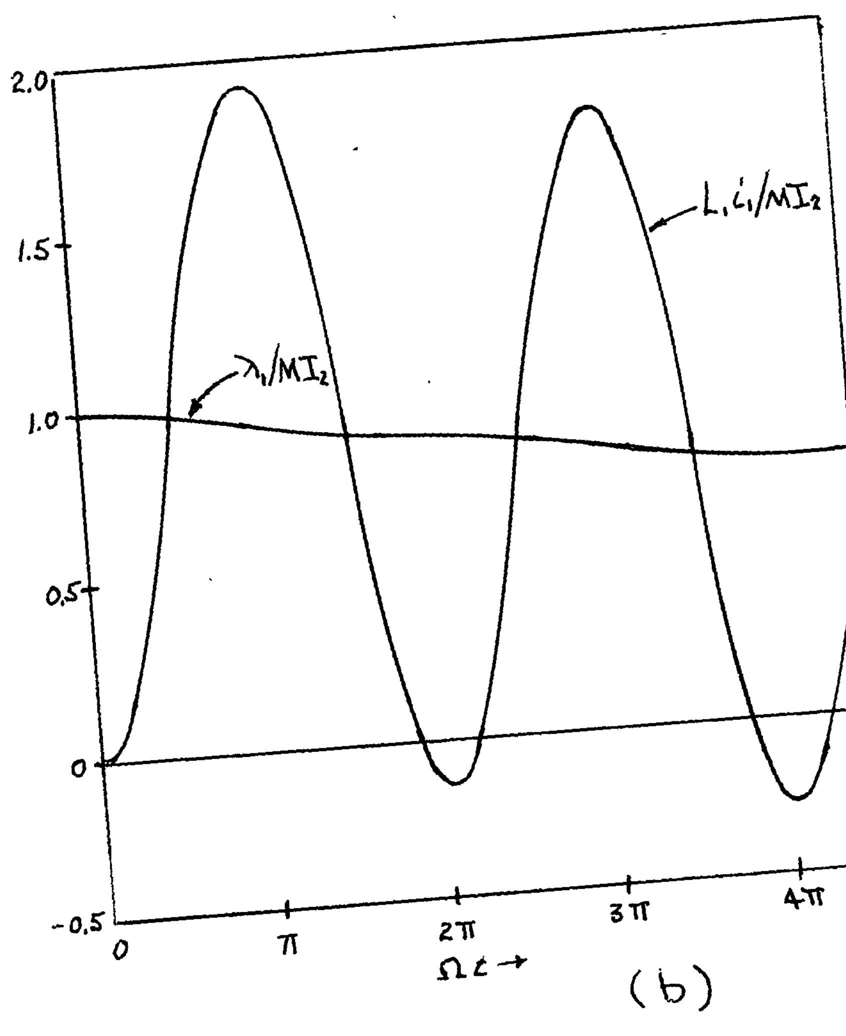
$$\Omega GL_1 \gg 1 \tag{k}$$

To study this case, set

$$\Omega GL_1 = 50 \text{ and } I = 0 \tag{l}$$

The resulting curves of flux and current are shown plotted in Fig. (b). Note that with this constraint the current varies drastically but the flux pulsates only slightly about a value that decays slowly compared to a rotational period. Thus, when considering events that occur in a time interval comparable


λ_1/MI_2
$L_1 i_1/MI_2$
0
π
2π
Ωt →
(a)


2.0
1.5
1.0
0.5
0
-0.5
$L_1 i_1/MI_2$
λ_1/MI_2
0
π
2π
3π
4π
Ωt →
(b)

PROBLEM 5.14 (Continued)

with the rotational period, we can approximate this system with a constant-flux constraint. In the ideal, limiting case, which can be approached with superconductors, $G \to \infty$ and λ_1 stays constant at its initial value. This initial value is the flux that links the coil at the instant the switch S is closed.

In the limiting cases of constant-current and constant flux constraints the losses in the electrical circuit go to zero. This fact allows us to take advantage of the conservative character of lossless systems, as discussed in Sec. 5.2.1.

Part f

Between the two limiting cases of constant-current and constant flux constraints the conductance G is finite and provides electrical damping on the mechanical system. We can show this by demonstrating that mechanical power supplied by the speed source is dissipated in the conductance G. For this purpose we need to evaluate the torque supplied by the speed source. Because the rotational velocity is constant, we have

$$T^m = - T^e \tag{m}$$

The torque of electrical origin T^e is in turn

$$T^e = \frac{\partial W'(i_1, i_2, \theta)}{\partial \theta} \tag{n}$$

Because the system is electrically linear, the coenergy W' is

$$W' = \frac{1}{2} L_1 i_1^2 + M\, i_1 i_2 \cos\theta + \frac{1}{2} L_2\, i_2^2 \tag{o}$$

and therefore,

$$T^e = - M\, i_1\, I_2 \sin\theta \tag{p}$$

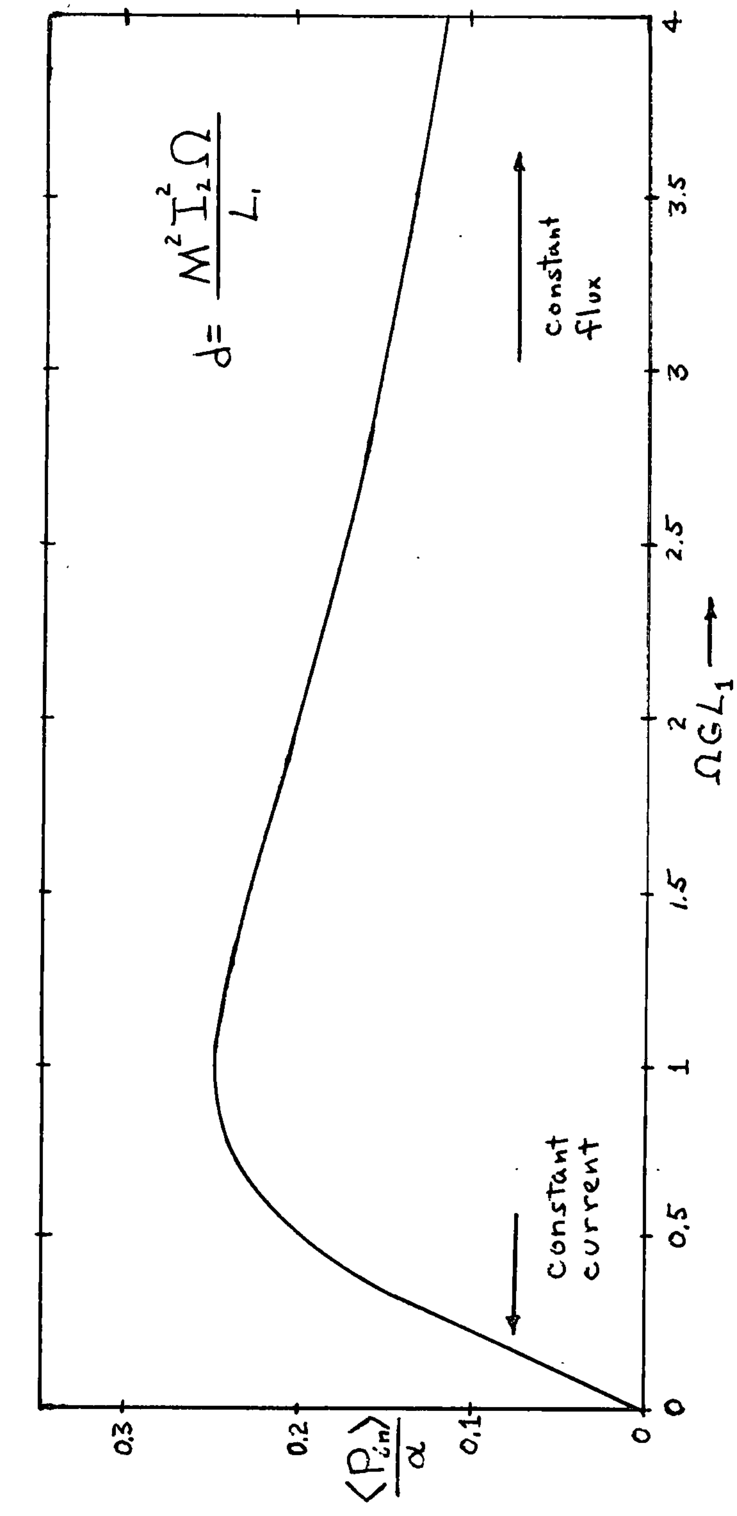
The power supplied by the torque T^m to rotate the coil is

$$P_{in} = - T^e \frac{d\theta}{dt} = M\Omega i_1 I_2 \sin \Omega t \tag{q}$$

Part g

Hence, from (p) and (q), it follows that in the sinusoidal steady state the average power $\langle P_{in} \rangle$ supplied by the external torque is

$$\langle P_{in} \rangle = \frac{1}{2} \left[\frac{G\, M^2 I_2^2\, \Omega^2}{1+(\Omega G L_1)^2} \right] \tag{r}$$



$\alpha = \frac{M^2 I_2^2 \Omega}{L_1}$
constant flux
constant current
$\frac{\langle P_{in} \rangle}{\alpha}$
$\Omega G L_1$
0
0.1
0.2
0.3
0.5
1
1.5
2
2.5
3
3.5
4

PROBLEM 5.14 (Continued)

This power, which is dissipated in the conductance G, is plotted as a function of ΩGL_1 in Fig. (c). Note that because Ω and L_1 are used as normalizing constants, ΩGL_1 can only be varied by varying G. Note that for both large and small values of ΩGL_1 the average mechanical power dissipated in G becomes small. The maximum in $<P_{in}>$ occurs at $\Omega GL_1 = 1$.

PROBLEM 5.15

Part a

The coenergy of the capacitor is

$$W_e' = \frac{1}{2} C(x)\, V^2 = \frac{1}{2}\left(\varepsilon_o \frac{A}{x}\right)V^2$$

The electric force in the x direction is

$$f_e = \frac{\partial W_e'}{\partial x} = -\frac{1}{2}\frac{\varepsilon_o A}{x^2} V^2$$

If this force is linearized around $x = x_o$, $V = V_o$

$$f_e(x) = -\frac{1}{2}\frac{\varepsilon_o AV_o^2}{x_o^2} - \frac{\varepsilon_o AV_o v}{x_o^2} + \frac{\varepsilon_o AV_o^2 x}{x_o^3}$$

The linearized equation of motion is then

$$B\frac{dx}{dt} + \left(K - \frac{\varepsilon_o AV_o^2}{x_o^3}\right)x = -\frac{\varepsilon_o A}{x_o^2} V_o v + f(t)$$

The equation for the electric circuit is

$$V + R\frac{d}{dt}(C(x)V) = V_o$$

Part b

We can keep the voltage constant if

$$R \longrightarrow 0$$

In this case

$$B\frac{dx}{dt} + K'x = f(t) = F\,u_{-1}(t);\quad K' = K - \frac{\varepsilon_o AV_o^2}{x_o^3}$$

The particular solution is

$$x(t) = F/K'$$

PROBLEM 5.15 (Continued)

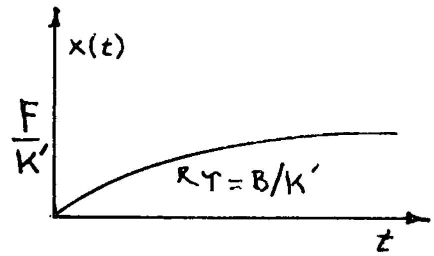
The natural frequency S is the solution to

$$sB \; + K'x = 0 \qquad\qquad s = -K'/B$$

Notice that since

$$K'/B = (K - \frac{\varepsilon_o A V_o^2}{x_o^3})/B$$

there is voltage V_o above which the system is unstable. Assuming V_o is less than this voltage



$$x(t) = F/K' \; (1 - e^{-(K'/B)t})$$

Now we can be more specific about the size of R. We want the time constant of the RC circuit to be small compared to the "action time" of the mechanical system

$$RC(x_o) << B/K'$$

$$R << \frac{B}{K'C(x_o)}$$

Part c

From part a we suspect that

$$RC(x_o) >> \tau_{mech}$$

where τ_{mech} can be found by letting $R \to \infty$. Since the charge will be constant

$$\frac{dq}{dt} = 0 \qquad q = C(x_o)V_o = C(x_o + x)(V_o + v)$$

$$\simeq C(x_o)V_o + C(x_o)v + V_o \frac{dC}{dx}(x_o)x$$

$$v = -\frac{V_o}{C(x_o)} \frac{dC}{dx}(x_o)x = + \frac{V_o x}{\varepsilon_o A} \frac{\varepsilon_o A}{x_o^2} x = V_o \frac{x}{x_o}$$

Using this expression for induced v, the linearized equation of motion becomes

$$B \frac{dx}{dt} + (K - \frac{\varepsilon_o A V_o^2}{x_o^3})x = -\frac{\varepsilon_o A}{x_o^3} V_o^2 \, x + f(t)$$

$$B \frac{dx}{dt} + Kx = f(t)$$

PROBLEM 5.15 (Continued)

The electric effect disappears because the force of a capacitor with constant charge is independent of the plate separation. The solutions are the same as part (a) except that $K' = K$. The constraint on the resistor is then

$$R >> \frac{1}{C(x_o)} \quad B/K$$

PROBLEM 5.16

We wish to write the sum of the forces in the form

$$f = f_1 + f_2 = -\frac{\partial V}{\partial x} \tag{a}$$

For $x > 0$, this is done by making

$$V = -\frac{1}{2} Kx^2 + F_o x \tag{b}$$

as shown in the figure. The potential is symmetric about the origin. The largest value of v_o that can be contained by the potential well is determined by the peak value of potential which, from (b), comes at

$$x = F_o/K \tag{c}$$

where the potential is

$$V = \frac{1}{2} F_o^2/K \tag{d}$$

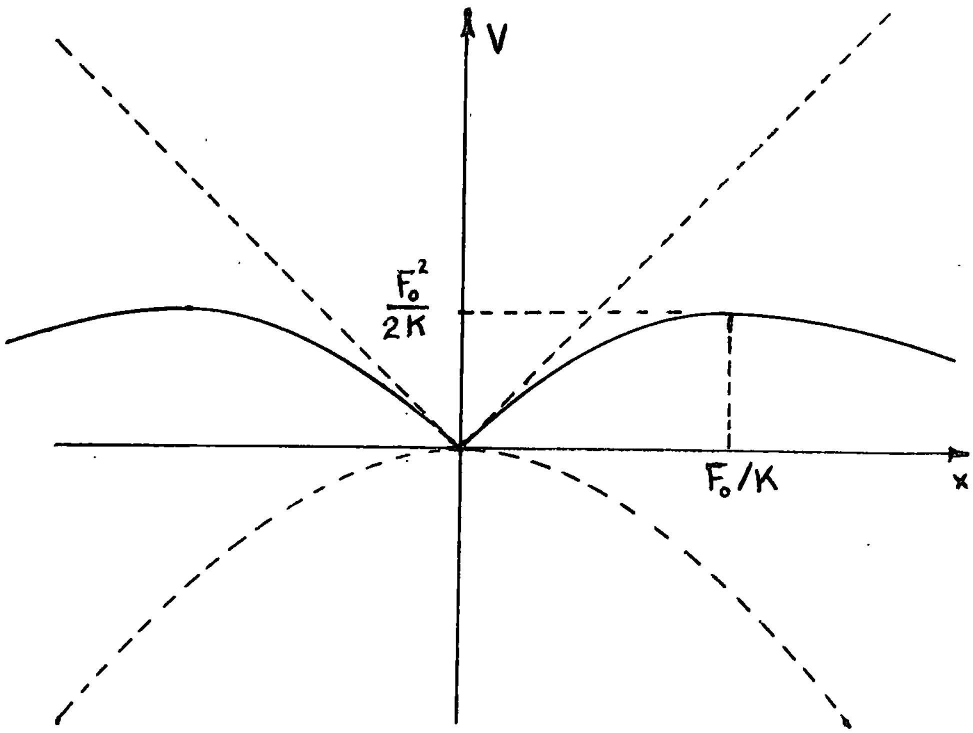
Because the minimum value of the potential is zero, this means that the kinetic energy must exceed this peak value to surmount the barrier. Hence,

$$\frac{1}{2} Mv_o^2 = \frac{1}{2} F_o^2/K \tag{e}$$

or

$$v_o = \sqrt{\frac{F_o^2}{KM}} \tag{f}$$

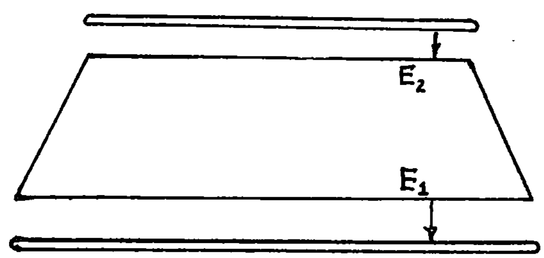
PROBLEM 5.16 (Continued)



PROBLEM 5.17

Part a

The electric field intensities defined in the figure are



$$E_2 = (v_2-v_1)/(d-x) \tag{a}$$

$$E_1 = v_1/(d+x) \tag{b}$$

Hence, the total charge on the respective electrodes is

$$q_1 = v_1\left[\frac{A_2\varepsilon_o}{d+x} + \frac{A_1\varepsilon_o}{d-x}\right] - \frac{v_2A_1\varepsilon_o}{d-x} \tag{c}$$

$$q_2 = \frac{A_1\varepsilon_o(v_2-v_1)}{d-x} \tag{d}$$

Part b

Conservation of energy requires

$$v_1dq_1 + v_2dq_2 = dW + f^e dx \tag{e}$$

and since the charge q_1 and voltage v_2 are constrained, we make the transformation $v_2dq_2 = d(v_2q_2)-q_2dv_2$ to obtain

$$v_1dq_1-q_2dv_2 = dW'' + f^e dx \tag{f}$$

It follows from this form of the conservation of energy equation that $f^e = -\frac{\partial W''}{\partial x}$ and hence $W'' \equiv U$. To find the desired function we integrate (f) using the terminal relations.

$$U = W'' = \int v_1dq_1 - q_2dv_2 \tag{g}$$

The integration on q_1 makes no contribution since q_1 is constrained to be zero. We require $v_2(q_1=0,v_2)$ to evaluate the remaining integral

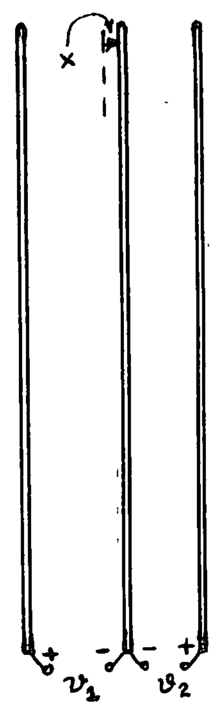
$$q_2(q_1=0,v_2) = \frac{v_2A_1\varepsilon_o}{d-x}\left[1 - \frac{1}{[\frac{A_2(d-x)}{A_1(d+x)} + 1]}\right] \tag{h}$$

Then, from (g),

$$U = -\frac{1}{2}\frac{V_o^2A_1\varepsilon_o}{d-x}\left[1 - \frac{1}{[\frac{A_2(d-x)}{A_1(d+x)} + 1]}\right] \tag{i}$$

PROBLEM 5.18

Part a



Because the two outer plates are constrained differently once the switch is opened, it is convenient to work in terms of two electrical terminal pairs, defined as shown in the figure. The plane parallel geometry makes it straightforward to compute the terminal relations as being those for simple parallel plate capacitors, with no mutual capacitance.

$$q_1 = v_1\varepsilon_oA/a + x \tag{a}$$

$$q_2 = v_2\varepsilon_oA/a-x \tag{b}$$

PROBLEM 5.18 (Continued)

Conservation of energy for the electromechanical coupling requires

$$v_1 dq_1 + v_2 dq_2 = dW + f^e dx \tag{c}$$

This is written in a form where q_1 and v_2 are the independent variables by using the transformation $v_2 dq_2 = d(v_2 q_2) - q_2 dv_2$ and defining $W'' = W - v_2 q_2$

$$v_1 dq_1 - q_2 dv_2 = dW'' + f^e dx \tag{d}$$

This is done because after the switch is opened it is these variables that are conserved. In fact, for $t > 0$,

$$v_2 = V_o \text{ and (from (a))} q_1 = V_o \varepsilon_o A/a \tag{e}$$

The energy function W" follows from (d) and the terminal conditions, as

$$W'' = \int v_1 dq_1 - \int q_2 dv_2 \tag{f}$$

or

$$\frac{1}{2}\frac{(a+x)}{\varepsilon_o A} q_1^2 - \frac{1}{2}\frac{\varepsilon_o A v_2^2}{a-x} \tag{g}$$

Hence, for $t > 0$, we have (from (e))

$$W'' = \frac{1}{2}\frac{(a+x)}{a^2}\varepsilon_o A V_o^2 - \frac{1}{2}\frac{\varepsilon_o A V_o^2}{a-x} \tag{h}$$

Part b

The electrical force on the plate is $f^e = -\frac{\partial W''}{\partial x}$. Hence, the force equation is (assuming a mass M for the plate)

$$M\frac{d^2x}{dt^2} = -Kx - \frac{1}{2}\frac{\varepsilon_o A V_o^2}{a^2} + \frac{1}{2}\frac{\varepsilon_o A V_o^2}{(a-x)^2} \tag{i}$$

For small excursions about the origin, this can be written as

$$M\frac{d^2x}{dt^2} = -Kx - \frac{1}{2}\frac{\varepsilon_o A V_o^2}{a^2} + \frac{1}{2}\frac{\varepsilon_o A V_o^2}{a^2} + \frac{\varepsilon_o A V_o^2}{a^3}x \tag{j}$$

The constant terms balance, showing that a static equilibrium at the origin is possible. Then, the system is stable if the effective spring constant is positive.

$$K > \varepsilon_o A V_o^2/a^3 \tag{k}$$

Part c

The total potential V(x) for the system is the sum of W" and the potential energy stored in the springs. That is,

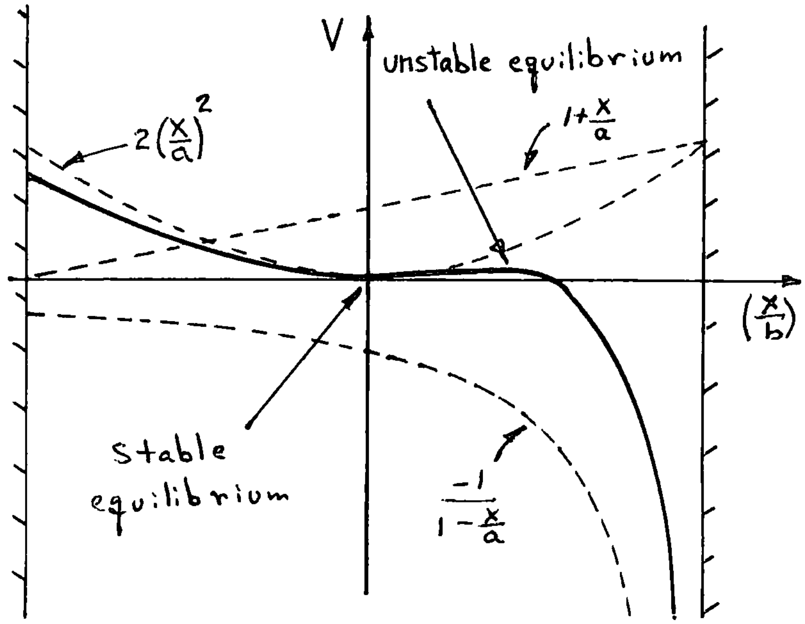
PROBLEM 5.18 (Continued)

$$V = \frac{1}{2} Kx^2 + \frac{1}{2} \frac{(a+x)}{a^2} \varepsilon_o AV_o^2 - \frac{1}{2} \frac{\varepsilon_o AV_o^2}{a-x} \tag{l}$$

or

$$\frac{a^2K}{2} \left(\frac{x}{a}\right)^2 + \frac{1}{2} \frac{\varepsilon_o AV_o^2}{a} \left[\left(1 + \frac{x}{a}\right) - \frac{1}{\left(1-\frac{x}{a}\right)}\right] \tag{m}$$

This is sketched in the figure for $a^2K/2 = 2$ and $1/2\ \varepsilon_o AV_o^2/a = 1$. In addition to the point of stable equilibrium at the origin, there is also an unstable equilibrium point just to the right of the origin.



PROBLEM 5.19

Part a

The coenergy is

$$W' = \frac{1}{2} Li^2 = \frac{1}{2} L_o i^2/[1 - \frac{x}{a}]^4 \tag{a}$$

and hence the force of electrical origin is

$$f^e = \frac{\partial W'}{\partial x} = 2L_o i^2/a[1 - \frac{x}{a}]^5 \tag{b}$$

Hence, the mechanical equation of motion, written as a function of (i,x) is

$$M \frac{d^2x}{dt^2} = - Mg + \frac{2L_o i^2}{a[1- \frac{x}{a}]^5} \tag{c}$$

PROBLEM 5.19 (Continued)

while the electrical loop equation, written in terms of these same variables (using the terminal relation for λ) is

$$V_o + v = Ri + \frac{d}{dt}\left[\frac{L_o i}{(1-\frac{x}{a})^4}\right] \qquad \text{(d)}$$

These last two expressions are the equations of motion for the mass.

Part b

In static equilibrium, the above equations are satisfied by (x,v,i) having the respective values (X_o,V_o,I_o). Hence, we assume that

$$x = X_o + x'(t): \quad v = V_o + v(t): \quad i = I_o + i'(t) \qquad \text{(e)}$$

The equilibrium part of (c) is then

$$0 = -Mg + \frac{2L_oI_o^2}{a}/(1 - \frac{X_o}{a})^5 \qquad \text{(f)}$$

while the perturbations from this equilibrium are governed by

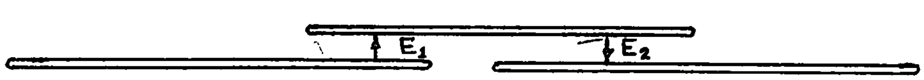
$$M\frac{d^2x'}{dt^2} = + \frac{10\,L_oI_o^2\,x'}{a^2(1-\frac{X_o}{a})^6} + \frac{4\,L_oI_o\,i'}{a(1-\frac{X_o}{a})^5} \qquad \text{(g)}$$

The equilibrium part of (d) is simply $V_o = I_oR$, and the perturbation part is

$$v = Ri' + \frac{L_o}{[1-\frac{X_o}{a}]^4}\frac{di'}{dt} + \frac{4\,L_oI_o}{a[1-\frac{X_o}{a}]^5}\frac{dx'}{dt} \qquad \text{(h)}$$

Equations (g) and (h) are the linearized equations of motion for the system which can be solved given the driving function $v(t)$ and (if the transient is of interest) the initial conditions.

PROBLEM 5.20



Part a

The electric field intensities, defined as shown, are

$$E_1 = (v_1-v_2)/s; \quad E_2 = v_2/s \qquad \text{(a)}$$

PROBLEM 5.20 (Continued)

In terms of these quantities, the charges are

$$q_1 = \varepsilon_o(\frac{a}{2} - x)dE_1;\; q_2 = -\varepsilon_o(\frac{a}{2} - x)dE_1 + \varepsilon_o(\frac{a}{2} + x)dE_2 \qquad \text{(b)}$$

Combining (a) and (b), we have the required terminal relations

$$\begin{aligned} q_1 &= v_1C_{11} - v_2C_{12} \\ q_2 &= -v_1C_{12} + v_2C_{22} \end{aligned} \qquad \text{(c)}$$

where

$$C_{11} = \frac{\varepsilon_o d}{s}(\frac{a}{2} - x);\quad C_{22} = \frac{\varepsilon_o ad}{s}$$

$$C_{12} = \frac{\varepsilon_o d}{s}(\frac{a}{2} - x)$$

For the next part it is convenient to write these as $q_1(v_1,q_2)$ and $v_2(v_1,q_2)$.

$$\begin{aligned} q_1 &= v_1\left[C_{11} - \frac{C_{12}^2}{C_{22}}\right] - q_2\frac{C_{12}}{C_{22}} \\ v_2 &= \frac{q_2}{C_{22}} + v_1\frac{C_{12}}{C_{22}} \end{aligned} \qquad \text{(d)}$$

Part b

Conservation of energy for the coupling requires

$$v_1dq_1 + v_2dq_2 = dW + f^e dx \qquad \text{(e)}$$

To treat v_1 and q_2 as independent variables (since they are constrained to be constant) we let $v_1dq_1 = d(v_1q_1) - q_1dv_1$, and write (e) as

$$-q_1\,dv_1 + v_2dq_2 = -dW'' + f^e\,dx \qquad \text{(f)}$$

From this expression it is clear that $f^e = \partial W''/\partial x$ as required. In particular, the function W" is found by integrating (f)

$$W'' = \int_o^{V_o} q_1(v_1',0)dv_1' - \int_o^{Q} v_2(V_o,q_2')dq_2' \qquad \text{(g)}$$

to obtain

$$W'' = \frac{1}{2}V_o^2\left[C_{11} - \frac{C_{12}^2}{C_{22}}\right] - \frac{Q^2}{2C_{22}} - \frac{V_oQC_{12}}{C_{22}} \qquad \text{(h)}$$

Of course, C_{11}, C_{22} and C_{12} are functions of x as defined in (c).

PROBLEM 5.21

Part a

The equation of motion as developed in Prob. 3.8 but with $I(t)=I_o$=constant, is

$$J\frac{d^2\theta}{dt^2} = -\frac{I\,L_m^2\,I_o}{L_2}(1-\cos\theta)\sin\theta \tag{a}$$

This has the required form if we define

$$V = -\frac{I\,L_m^2}{L_2}I_o(\cos\theta + \frac{1}{2}\sin^2\theta) \tag{b}$$

as can be seen by differentiating (b) and recovering the equation of motion. This potential function could also have been obtained by starting directly with the thermodynamic energy equation and finding a hybred energy function (one having i_1, λ_2,θ as independent variables). See Example 5.2.2 for this more fundamental approach.

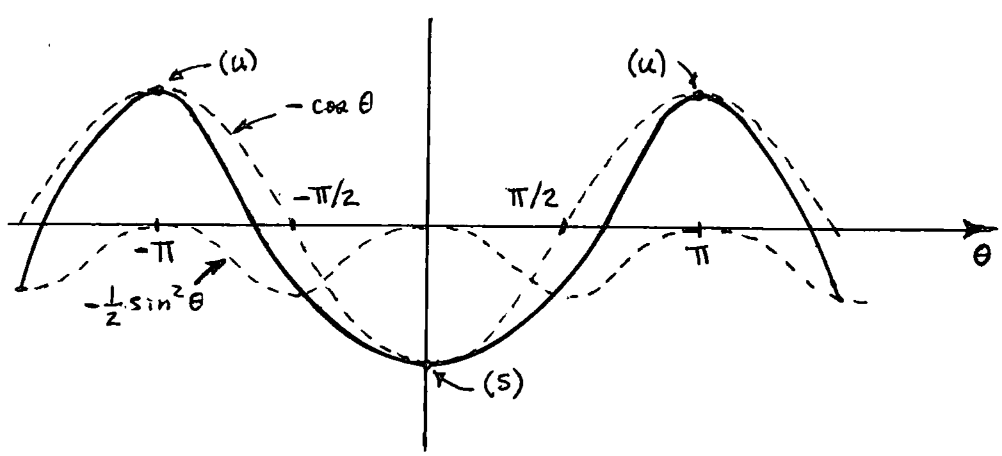
Part b

A sketch of the potential well is as shown below. The rotor can be in stable static equilibrium at $\theta = 0$ (s) and unstable static equilibrium at $\theta = \pi$(u).

Part c

For the rotor to execute continuous rotory motion from an initial rest position at $\theta = 0$, it must have sufficient kinetic energy to surmount the peak in potential at $\theta = \pi$. To do this,

$$\frac{1}{2}J(\frac{d\theta}{dt})^2 \geq \frac{2IL_m^2I_o}{L_2} \tag{c}$$



PROBLEM 5.22

Part a

The coenergy stored in the magnetic coupling is simply

$$W' = \frac{1}{2}L_o(1 + 0.2\cos\theta + 0.05\cos 2\theta)i^2 \qquad \text{(a)}$$

Since the gravitational field exerts a torque on the pendulum given by

$$T_p = \frac{\partial}{\partial\theta}(-Mg\,\ell\cos\theta) \qquad \text{(b)}$$

and the torque of electrical origin is $T^e = \partial W'/\partial\theta$, the mechanical equation of motion is

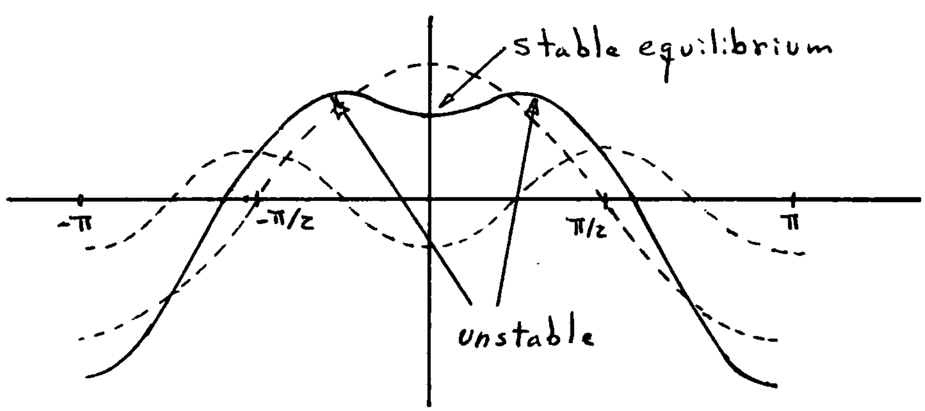
$$\frac{d}{dt}\left[\frac{1}{2}M\ell^2\left(\frac{d\theta}{dt}\right)^2 + V\right] = 0 \qquad \text{(c)}$$

where (because $I^2L_o = 6Mg\ell$)

$$V = Mg\ell[0.4\cos\theta - 0.15\cos 2\theta - 3]$$

Part b

The potential distribution V is plotted in the figure, where it is evident that there is a point of stable static equilibrium at $\theta = 0$ (the pendulum straight up) and two points of unstable static equilibrium to either side of center. The constant contribution has been ignored in the plot because it is arbitrary.



PROBLEM 5.23

Part a

The magnetic field intensity is uniform over the cross section and equal to the surface current flowing around the circuit. Define H as into the paper and $H = i/D$. Then λ is H multiplied by μ_o and the area xd.

$$\lambda = \frac{\mu_o xd}{D} i \tag{a}$$

The system is electrically linear and so the energy is $W = \frac{1}{2} \lambda^2/L$. Then, since $f^e = - \partial W/\partial x$, the equation of motion is

$$M \frac{d^2x}{dt^2} = f = - Kx + \frac{1}{2} \frac{\Lambda^2 D}{\mu_o x^2 d} \tag{b}$$

Part b

Let $x = X_o + x'$ where x' is small and (b) becomes approximately

$$M \frac{d^2x'}{dt^2} = -KX_o - Kx' + \frac{1}{2} \frac{\Lambda^2 D}{\mu_o X_o^2 d} - \frac{\Lambda^2 Dx'}{\mu_o X_o^3 d} \tag{c}$$

The constant terms define the static equilibrium

$$X_o = [\frac{1}{2} \frac{\Lambda^2 D}{\mu_o dK}]^{1/3} \tag{d}$$

and if we use this expression for X_o, the perturbation equation becomes,

$$M \frac{d^2x'}{dt^2} = -Kx' - 2Kx' \tag{e}$$

Hence, the point of equilibrium at X_o as given by (d) is stable, and the magnetic field is equivalent to the spring constant 2K.

Part c

The total force is the negative derivative with respect to x of V where

$$V = \frac{1}{2} Kx^2 + \frac{1}{2} \frac{\Lambda^2 D}{\mu_o xd} \tag{f}$$

This makes it possible to integrate the equation of motion (b) once to obtain

$$\frac{dx}{dt} = \pm \sqrt{\frac{2}{M} (E-V)} \tag{g}$$

The potential well is as shown in figure (a). Here again it is apparent that the equilibrium point is one where the mass can be static and stable. The constant of integration E is established physically by releasing the mass from static

PROBLEM 5.23 (Continued)

positions such as (1) or (2) shown in Fig. (a). Then the bounded excursions of the mass can be pictured as having the level E shown in the diagram. The motions are periodic in nature regardless of the initial position or velocity.

Part d

The constant flux dynamics can be contrasted with those occurring at constant current simply by replacing the energy function with the coenergy function. That is, with the constant current constraint, it is appropriate to find the electrical force from $W' = \frac{1}{2} Li^2$, where $f^e = \partial W'/\partial x$. Hence, in this case

$$V = \frac{1}{2} Kx^2 - \frac{1}{2} \frac{\mu_o xd}{D} I^2 \tag{h}$$

A plot of this potential well is shown in Fig. (b). Once again there is a point X_o of stable static equilibrium given by

$$X_o = \frac{1}{2} \frac{\mu_o dI^2}{DK} \tag{i}$$

However, note that if oscillations of sufficiently large amplitude are initiated that it is now possible for the plate to hit the bottom of the parallel plate system at x = 0.

PROBLEM 5.25

Part a

Force on the capacitor plate is simply

$$f^e = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x} [\frac{1}{2} \frac{\pi a^2 \varepsilon_o v^2}{x}] \tag{a}$$

due to the electric field and a force f due to the attached string.

Part b

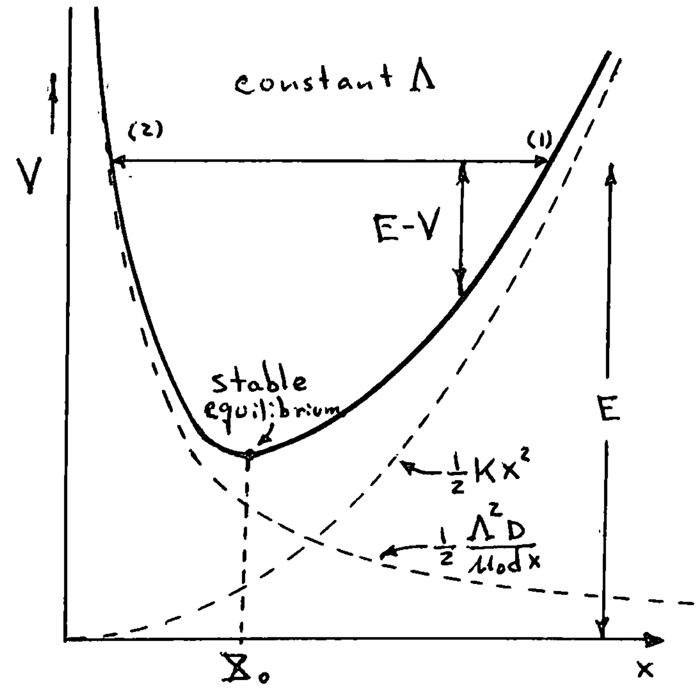
With the mass M_1 rotating at a constant angular velocity, the force f^e must balance the centrifugal force $\omega_m^2 rM_1$ transmitted to the capacitor plate by the string.

$$\frac{1}{2} \frac{\pi a^2 \varepsilon_o V_o^2}{\ell^2} = \omega_m^2 \ell M_1 \tag{b}$$

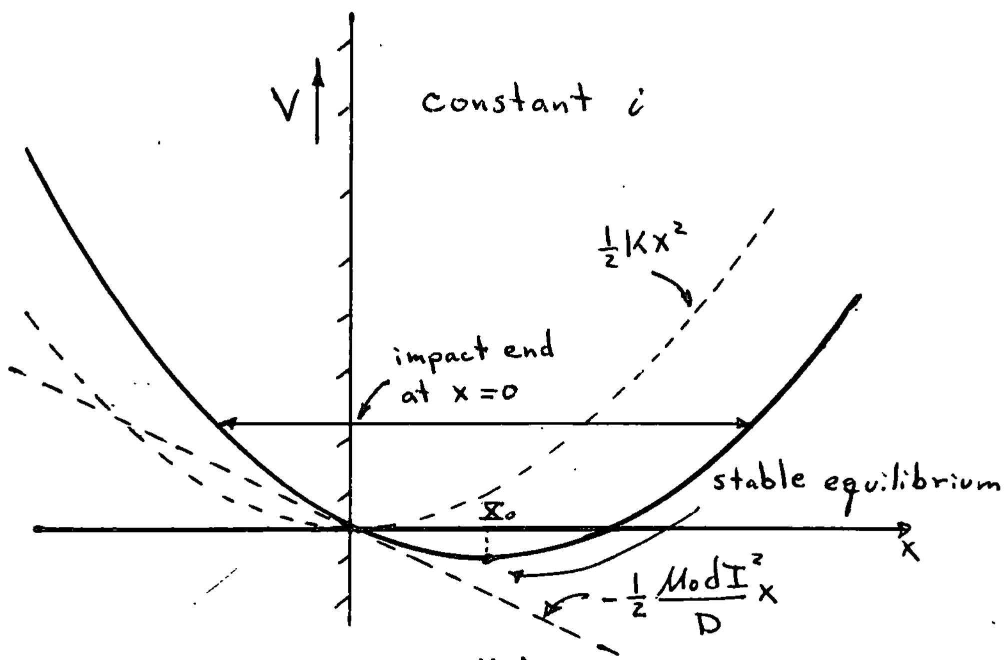
or

$$\omega_m = \sqrt{\frac{\pi a^2 \varepsilon_o V_o^2}{2 \ell^3 M_1}} \tag{c}$$

where ℓ is both the equilibrium spacing of the plates and the equilibrium radius of the trajectory for M_1.


constant Λ
(2)
(1)
V
E-V
E
stable equilibrium
$\frac{1}{2}Kx^2$
$\frac{1}{2}\frac{\Lambda^2 D}{\mu_0 d x}$
X_0
x

(a)


V
constant i
$\frac{1}{2}Kx^2$
impact end at x = 0
stable equilibrium
X_0
x
$-\frac{1}{2}\frac{\mu_0 d I^2}{D}x$

(b)

PROBLEM 5.25 (Continued)

Part c

The θ directed force equation is (see Prob. 2.8) for the accleration on a particle in circular coordinates)

$$M_1[r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}]=0 \tag{d}$$

which can be written as

$$\frac{d}{dt}[M_1r^2\frac{d\theta}{dt}]=0 \tag{e}$$

This shows that the angular momentum is constant even as the mass M_1 moves in and out

$$M_1r^2\frac{d\theta}{dt}=M_1\ell^2\omega_m = \text{constant of the motion} \tag{f}$$

This result simply shows that if the radius increases, the angular velocity must decrease accordingly

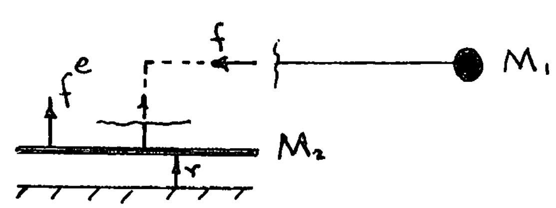
$$\frac{d\theta}{dt}=\frac{\ell^2}{r^2}\omega_m \tag{g}$$

Part d

The radial component of the force equation for M_1 is

$$M_1[\frac{d^2r}{dt^2}-r(\frac{d\theta}{dt})^2]=-f \tag{h}$$

where f is the force transmitted by the string, as shown in the figure.



The force equation for the capacitor plate is

$$M_2\frac{d^2r}{dt^2}=f^e+f \tag{i}$$

where f^e is supplied by (a) with $v = V_o$ = constant. Hence, these last two expressions can be added to eliminate f and obtain

PROBLEM 5.25 (Continued)

$$(M_1 + M_2)\frac{d^2r}{dt^2} - M_1 r\left(\frac{d\theta}{dt}\right)^2 = \frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\pi a^2\varepsilon_o V_o^2}{r}\right) \qquad (j)$$

If we further use (g) to eliminate $d\theta/dt$, we obtain an expression for $r(t)$ that can be written in the standard form

$$(M_1 + M_2)\frac{d^2r}{dt^2} + \frac{\partial}{\partial r} V = 0 \qquad (k)$$

where

$$V = \frac{M_1 \ell^4 \omega_m^2}{2r^2} - \frac{1}{2}\frac{\pi a^2\varepsilon_o V_o^2}{r} \qquad (l)$$

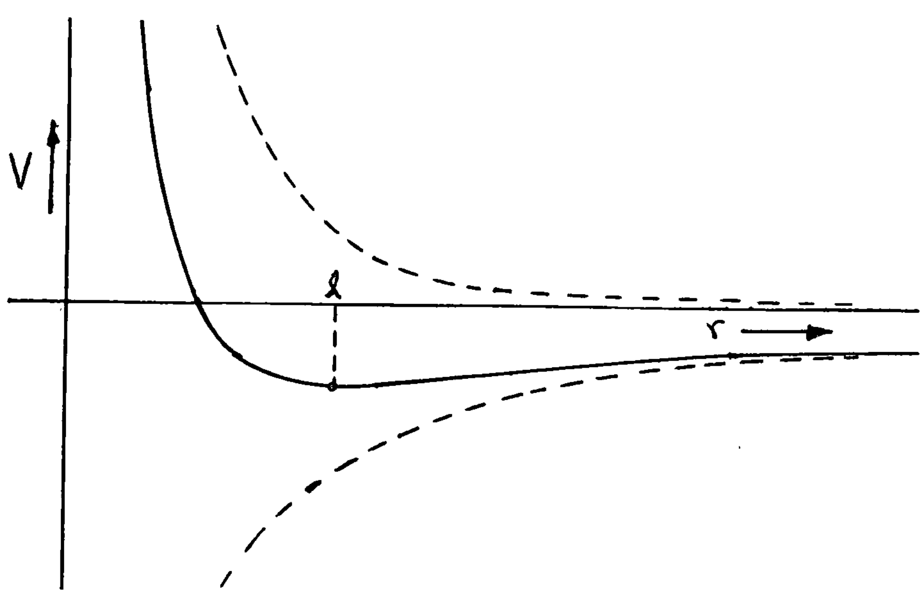
Of course, (k) can be multiplied by dr/dt and written in the form

$$\frac{d}{dt}\left[\frac{1}{2}(M_1 + M_2)\left(\frac{dr}{dt}\right)^2 + V\right] = 0 \qquad (m)$$

to show that V is a potential well for the combined mass of the rotating particle and the plate.

Part e

The potential well of (l) has the shape shown in the figure. The minimum represents the equilibrium position found in (c), as can be seen by differentiating (l) with respect to r, equating the expression to zero and solving for ω_m assuming that $r = \ell$. In this example, the potential well is the result of a combination of the negative coenergy for the electromechanical system, constrained to constant potential, and the dynamic system with angular momentum conserved.



PROBLEM 5.26

Part a

To begin the analysis we first write the Kirchhoff voltage equations for the two electric circuits with switch S closed

$$V = i_1R_1 + \frac{d\lambda_1}{dt} \tag{a}$$

$$0 = i_2R_2 + \frac{d\lambda_2}{dt} \tag{b}$$

To obtain the electrical terminal relations for the system we neglect fringing fields and assume infinite permeability for the magnetic material to obtain*

$$\lambda_1 = N_1\phi \ , \quad \lambda_2 = N_2\phi \tag{c}$$

where the flux ϕ through the coils is given by

$$\phi = \frac{2\mu_o \, wd \, (N_1i_1 + N_2i_2)}{g(1 + \frac{x}{g})} \tag{d}$$

We can also use (c) and (d) to calculate the stored magnetic energy as**

$$W_m = \frac{g(1 + \frac{x}{g})\phi^2}{4\mu_o \, wd} \tag{e}$$

We now multiply (a) by N_1/R_1 and (b) by N_2/R_2, add the results and use (c) and (d) to obtain

$$\frac{N_1V}{R_1} = \frac{g(1 + \frac{x}{g})}{2\mu_o \, wd}\phi + \left(\frac{N_1^2}{R_1} + \frac{N_2^2}{R_2}\right)\frac{d\phi}{dt} \tag{f}$$

Note that we have only one electrical unknown, the flux ϕ, and if the plunger is at rest (x = constant) this equation has constant coefficients.

*The neglect of fringing fields makes the two windings unity coupled. In practice there will be small fringing fields that cause leakage inductances. However, these leakage inductances affect only the initial part of the transient and neglecting them causes negligible error when calculating the closing time of the relay.

**Here we have used the equation (System is electrically linear)

$$W_m = \frac{1}{2} L_1 \, i_1^2 + L_{12} \, i_1 \, i_2 + \frac{1}{2} L_2 \, i_2^2$$

PROBLEM 5.26 (Continued)

Part b

Use the given definitions to write (f) in the form

$$\phi_o = (1 + \frac{x}{g})\phi + \tau_o \frac{d\phi}{dt} \tag{g}$$

Part c

During interval 1 the flux is determined by (g) with $x = x_o$ and the initial condition is $\phi = 0$. Thus the flux undergoes the transient

$$\phi = \frac{\phi_o}{1 + \frac{x_o}{g}}\left[1 - e^{-(1 + \frac{x_o}{g}) \frac{t}{\tau_o}}\right] \tag{h}$$

To determine the time at which interval 1 ends and to describe the dynamics of interval 2 we must write the equation of motion for the mechanical node. Neglecting inertia and damping forces this equation is

$$K(x - \ell) = f^e \tag{i}$$

In view of (c) (λ_1 and λ_2 are the independent variables implicit in ϕ) we can use (e) to evaluate the force f^e as

$$f^e = -\frac{\partial W_m(\lambda_1, \lambda_2, x)}{\partial x} = -\frac{\phi^2}{4\mu_o wd} \tag{j}$$

Thus, the mechanical equation of motion becomes

$$K(x - \ell) = -\frac{\phi^2}{4\mu_o wd} \tag{k}$$

The flux level ϕ_1 at which interval 1 ends is given by

$$K(x_o - \ell) = -\frac{\phi_1^2}{4\mu_o wd} \tag{l}$$

Part d

During interval 2, flux and displacement are related by (k), thus we eliminate x between (k) and (g) and obtain

$$\phi_o = \left[(1 + \frac{\ell}{g}) - (\frac{\ell - x_o}{g}) \frac{\phi^2}{\phi_1^2}\right] \phi + \tau_o \frac{d\phi}{dt} \tag{m}$$

were we have used (ℓ) to write the equation in terms of ϕ_1. This is the nonlinear differential equation that must be solved to find the dynamical behavior during interval 2.

PROBLEM 5.26 (Continued)

To illustrate the solution of (m) it is convenient to normalize the equation as follows

$$\frac{d(\frac{\phi}{\phi_o})}{d(\frac{t}{\tau_o})} = (\frac{\ell-x_o}{g})(\frac{\phi_o}{\phi_1})^2 (\frac{\phi}{\phi_o})^3 - (1+\frac{\ell}{g})\frac{\phi}{\phi_o} + 1 \tag{n}$$

We can now write the necessary integral formally as

$$\int_{\frac{\phi_1}{\phi_o}}^{\frac{\phi}{\phi_o}} \frac{d(\frac{\phi'}{\phi_o})}{\left[(\frac{\ell-x_o}{g})(\frac{\phi_o}{\phi_1})^2 (\frac{\phi'}{\phi_o})^3 - (1+\frac{\ell}{g})\frac{\phi'}{\phi_o} + 1\right]} = \int_o^{\frac{t}{\tau_o}} d(\frac{t}{\tau_o}) \tag{o}$$

where we are measuring time t from the start of interval 2.

Using the given parameter values,

$$\int_{0.1}^{\frac{\phi}{\phi_o}} \frac{d(\frac{\phi'}{\phi_o})}{\left[400\,(\frac{\phi'}{\phi_o})^3 - 9\,\frac{\phi'}{\phi_o} + 1\right]} = \frac{t}{\tau_o} \tag{p}$$

We factor the cubic in the denominator into a first order and a quadratic factor and do a partial-fraction expansion* to obtain

$$\int_{0.1}^{\frac{\phi}{\phi_o}} \left[\frac{0.156}{5.29\,\frac{\phi'}{\phi_o} + 1} + \frac{(-2.23\,\frac{\phi'}{\phi_o} + 0.844)}{75.7\,(\frac{\phi'}{\phi_o})^2 - 14.3\,\frac{\phi'}{\phi_o} + 1}\right] d(\frac{\phi'}{\phi_o}) = \frac{t}{\tau_o} \tag{q}$$

Integration of this expression yields

*Phillips, H.B., Analytic Geometry and Calculus, second edition, John Wiley and Sons, New York, 1946, pp. 250-253.

PROBLEM 5.26 (Continued)

$$\frac{t}{\tau_o} = 0.0295 \ln\left[3.46\left(\frac{\phi}{\phi_o}\right) + 0.654\right] - 0.0147 \ln\left[231\left(\frac{\phi}{\phi_o}\right)^2 - 43.5\left(\frac{\phi}{\phi_o}\right) + 3.05\right]$$

$$+ 0.127 \tan^{-1}\left[15.1\left(\frac{\phi}{\phi_o}\right) - 1.43\right] - 0.0108 \qquad \text{(r)}$$

Part e

During interval 3, the differential equation is (g) with $x = 0$, for which the solution is

$$\phi = \phi_2 + (\phi_o - \phi_2)(1 - e^{-\frac{t}{\tau_o}}) \qquad \text{(s)}$$

where t is measured from the start of interval 3 and where ϕ_2 is the value of flux at the start of interval 3 and is given by (k) with $x = 0$

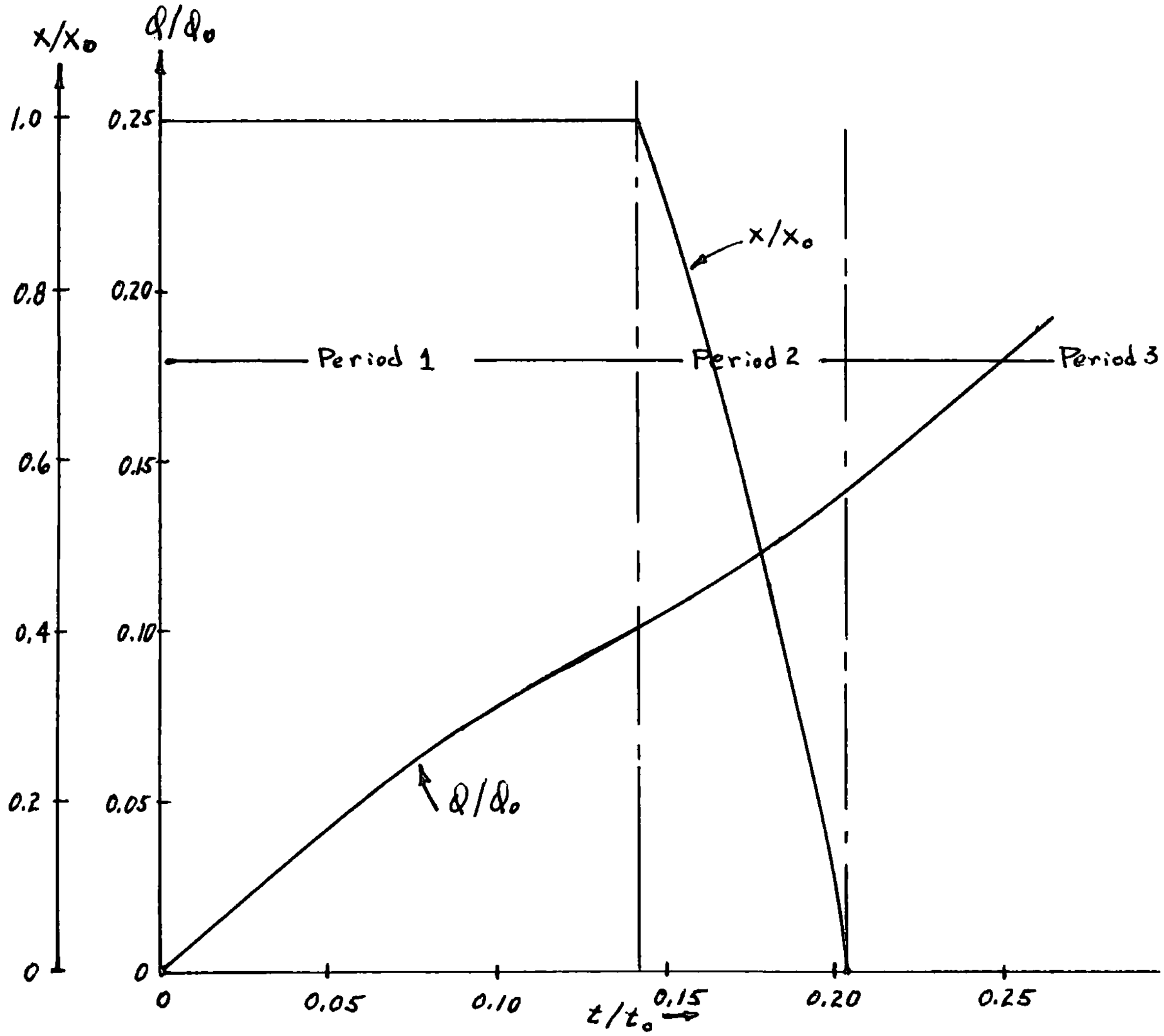
$$K\ell = \frac{\phi_2^2}{4\mu_o wd} \qquad \text{(t)}$$

Part f

For the assumed constants in this problem

$$\frac{\phi_2}{\phi_1} = \sqrt{2} \qquad \text{(u)}$$

The transients in flux and position are plotted in Fig. (a) as functions of time. Note that the mechanical transient occupies only a fraction of the time interval of the electrical transient. Thus, this example represents a case in which the electrical time constant is purposely made longer than the mechanical transient time.



x/x₀
Q/Q₀
1.0
0.8
0.6
0.4
0.2
0
0.25
0.20
0.15
0.10
0.05
0
x/x₀
Period 1
Period 2
Period 3
Q/Q₀
0
0.05
0.10
0.15
0.20
0.25
t/t₀

PROBLEM 6.1

Part a

From Fig. 6P.1 we see the geometric relations

$$r' = r, \ \theta' = \theta - \Omega t, \quad z' = z, \ t' = t \tag{a}$$

There is also a set of back transformations

$$r = r', \quad \theta = \theta' + \Omega t', \quad z = z', \quad t = t' \tag{b}$$

Part b

Using the chain rule for partial derivatives

$$\frac{\partial\psi}{\partial t'} = (\frac{\partial\psi}{\partial r})(\frac{\partial r}{\partial t'}) + (\frac{\partial\psi}{\partial\theta})(\frac{\partial\theta}{\partial t'}) + (\frac{\partial\psi}{\partial z})(\frac{\partial z}{\partial t'}) + (\frac{\partial\psi}{\partial t})(\frac{\partial t}{\partial t'}) \tag{c}$$

From (b) we learn that

$$\frac{\partial r}{\partial t'} = 0, \ \frac{\partial\theta}{\partial t'} = \Omega, \ \frac{\partial z}{\partial t'} = 0, \ \frac{\partial t}{\partial t'} = 1 \tag{d}$$

Hence,

$$\frac{\partial\psi}{\partial t'} = \frac{\partial\psi}{\partial t} + \Omega\frac{\partial\psi}{\partial\theta} \tag{e}$$

We note that the remaining partial derivatives of ψ are

$$\frac{\partial\psi}{\partial r'} = \frac{\partial\psi}{\partial r}, \quad \frac{\partial\psi}{\partial\theta'} = \frac{\partial\psi}{\partial\theta}, \quad \frac{\partial\psi}{\partial z'} = \frac{\partial\psi}{\partial z} \tag{f}$$

PROBLEM 6.2

Part a

The geometric transformation laws between the two inertial systems are

$$x_1' = x_1 - Vt, \ x_2' = x_2, \ x_3' = x_3, \ t' = t \tag{a}$$

The inverse transformation laws are

$$x_1 = x_1' + Vt', \ x_2 = x_2', \ x_3 = x_3', \ t = t' \tag{b}$$

The transformation of the magnetic field when there is no electric field present in the laboratory faame is

$$\bar{B}' = \bar{B} \tag{c}$$

Hence the time rate of change of the magnetic field seen by the moving observer is

$$\frac{\partial B'}{\partial t'} = \frac{\partial B}{\partial t'} = (\frac{\partial B}{\partial x_1})(\frac{\partial x_1}{\partial t'}) + (\frac{\partial B}{\partial x_2})(\frac{\partial x_2}{\partial t'}) + (\frac{\partial B}{\partial x_3})(\frac{\partial x_3}{\partial t'}) + (\frac{\partial B}{\partial t})(\frac{\partial t}{\partial t'}) \tag{d}$$

PROBLEM 6.2 (Continued)

From (b) we learn that

$$\frac{\partial x_1}{\partial t'} = V, \; \frac{\partial x_2}{\partial t'} = 0, \; \frac{\partial x_3}{\partial t'} = 0, \; \frac{\partial t}{\partial t'} = 1 \qquad \text{(e)}$$

While from the given field we learn that

$$\frac{\partial B}{\partial x_1} = kB_o \cos kx_1, \; \frac{\partial B}{\partial x_2} = \frac{\partial B}{\partial x_3} = \frac{\partial B}{\partial t} = 0 \qquad \text{(f)}$$

Combining these results

$$\frac{\partial B'}{\partial t'} = \frac{\partial B}{\partial t'} = V\frac{\partial B}{\partial x_1} = VkB_o \cos kx_1 \qquad \text{(g)}$$

which is just the convective derivative of B.

Part b

Now (b) becomes

$$x_1 = x_1', \; x_2 = x_2' + Vt, \; x_3 = x_3', \; t = t' \qquad \text{(h)}$$

When these equations are used with (d) we learn that

$$\frac{\partial B'}{\partial t'} = \frac{\partial B}{\partial t'} = V\frac{\partial B}{\partial x_2} + \frac{\partial B}{\partial t} = 0 \qquad \text{(i)}$$

because both $\frac{\partial B}{\partial x_2}$ and $\frac{\partial B}{\partial t}$ are naught. The convective derivative is zero.

PROBLEM 6.3

Part a

The quasistatic magnetic field transformation is

$$\bar{B}' = \bar{B} \qquad \text{(a)}$$

The geometric transformation laws are

$$x = x' + Vt', \; y = y', \; z = z', \; t = t' \qquad \text{(b)}$$

This means that

$$\bar{B}' = \bar{B}(t,x) = \bar{B}(t', x' + Vt') = \bar{i}_y B_o \cos(\omega t' - k(x' + Vt'))$$

$$= \bar{i}_y B_o \cos[(\omega - kV)t' - kx'] \qquad \text{(c)}$$

From (c) it is possible to conclude that

$$\omega' = \omega - kV \qquad \text{(d)}$$

Part b

If $\omega' = 0$ the wave will appear stationary in time, although it will still have a spacial distribution; it will not appear to move.

PROBLEM 6.3 (Continued)

$$\omega' = 0 = \omega - kV; \; V = \omega/k = v_p \tag{e}$$

The observer must move at the phase velocity v_p to make the wave appear stationary.

PROBLEM 6.4

These three laws were determined in an inertial frame of reference, and since there is no a priori reason to prefer one inertial frame more than another, they should have the same form in the primed inertial frame.

We start with the geometric laws which relate the coordinates of the two frames

$$\bar{r}' = \bar{r} - \bar{v}_r t, \; t = t', \; \bar{r} = \bar{r}' + \bar{v}_r t' \tag{a}$$

We recall from Chapter 6 that as a consequence of (a) and the definitions of the operators

$$\nabla = \nabla', \; \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \bar{v}_r \cdot \nabla', \; \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \bar{v}_r \cdot \nabla \tag{b}$$

In an inertial frame of reference moving with the velocity $\bar{v}_r$ we expect the equation to take the same form as in the fixed frame. Thus,

$$\rho' \frac{\partial \bar{v}'}{\partial t'} + \rho'(\bar{v}'\cdot\nabla')\bar{v}' + \nabla' p' = 0 \tag{c}$$

$$\frac{\partial \rho'}{\partial t'} + \nabla'\cdot\rho'\bar{v}' = 0 \tag{d}$$

$$p' = p'(\rho') \tag{e}$$

However, from (b) these become

$$\rho' \frac{\partial \bar{v}'}{\partial t} + \rho'(\bar{v}'+\bar{v}_r)\cdot\nabla(\bar{v}'+\bar{v}_r) + \nabla p' = 0 \tag{f}$$

$$\frac{\partial \rho'}{\partial t} + \nabla\cdot\rho'(\bar{v}'+\bar{v}_r) = 0 \tag{g}$$

$$p' = p'(\rho') \tag{h}$$

where we have used the fact that $\bar{v}_r\cdot\nabla\rho' = \nabla\cdot(\bar{v}_r\rho')$. Comparison of (1)-(3) with (f)-(h) shows that a self consistent transformation that leaves the equations invariant in form is

$$\rho' = \rho; \; p' = p; \; \bar{v}' = \bar{v} - \bar{v}_r$$

PROBLEM 6.5

Part a

$$\rho'(\bar{r}',t') = \rho(\bar{r},t) = \rho_o(1-\frac{r}{a}) = \rho_o(1-\frac{r'}{a}) \tag{a}$$

$$\bar{J}' = \rho'\bar{v}' = 0 \tag{b}$$

Where we have chosen $\bar{v}_r = v_o\bar{i}_z$ so that

$$\bar{v}' = \bar{v} - \bar{v}_r = 0 \tag{c}$$

Since there are no currents, there is only an electric field in the primed frame

$$\bar{E}' = (\rho_o/\varepsilon_o)(\frac{r'}{2} - \frac{r'^2}{3a})\bar{i}_r \tag{d}$$

$$\bar{H}' = 0,\ \bar{B}' = \mu_o\bar{H}' = 0 \tag{e}$$

Part b

$$\rho(r,t) = \rho_o(1-\frac{r}{a}) \tag{f}$$

This charge distribution generates an electric field

$$\bar{E} = (\rho_o/\varepsilon_o)(\frac{r}{2} - \frac{r^2}{3a})\bar{i}_r \tag{g}$$

In the stationary frame there is an electric current

$$\bar{J} = \rho\bar{v} = \rho_o(1-\frac{r}{a})v_o\bar{i}_z \tag{h}$$

This current generates a magnetic field

$$\bar{H} = \rho_o v_o(\frac{r}{2} - \frac{r^2}{3a})\bar{i}_\theta \tag{i}$$

Part c

$$\bar{J} = \bar{J}' - \rho'\bar{v}_r = \rho_o(1-\frac{r'}{a})v_o\bar{i}_z \tag{j}$$

$$\bar{E} = \bar{E}' - \bar{v}_r\times\bar{B}' = \bar{E}' = (\rho_o/\varepsilon_o)(\frac{r'}{2} - \frac{r'^2}{3a})\bar{i}_r \tag{k}$$

$$\bar{H} = \bar{H}' + \bar{v}_r\times\bar{D}' = v_o\rho_o(\frac{r'}{2} - \frac{r'^2}{3a})\bar{i}_\theta \tag{l}$$

If we include the geometric transformation $r' = r$, (j), (k), and (l) become (h), (g), and (i) of part (b) which we derived without using transformation laws. The above equations apply for $r<a$. Similar reasoning gives the fields in each frame for $r>a$.

PROBLEM 6.6

Part a

In the frame rotating with the cylinder

$$\bar{E}'(r') = \frac{K}{r'}\,\bar{i}_r \qquad (a)$$

$$\bar{H}' = 0,\ \bar{B}' = \mu_o\bar{H}' = 0 \qquad (b)$$

But then since $r' = r$, $\bar{v}_r(r) = r\omega\bar{i}_\theta$

$$\bar{E} = \bar{E}' - \bar{v}_r \times \bar{B}' = \bar{E}' = \frac{K}{r}\,\bar{i}_r \qquad (c)$$

$$V = \int_a^b \bar{E}\cdot d\bar{\ell} = \int_a^b \frac{K}{r}\,dr = K\ \ln(b/a) \qquad (d)$$

$$\bar{E} = \frac{V}{\ln(b/a)}\,\frac{1}{r}\,\vec{i}_r = \bar{E}' = \frac{V}{\ln(b/a)}\,\frac{1}{r'}\,\vec{i}_r \qquad (e)$$

The surface charge density is then

$$\sigma_a' = \vec{i}_r\cdot\varepsilon_o\bar{E}' = \frac{\varepsilon_o V}{\ln(b/a)}\,\frac{1}{a} = \sigma_a \qquad (f)$$

$$\sigma_b' = -\vec{i}_r\cdot\varepsilon_o\bar{E}' = -\frac{\varepsilon_o V}{\ln(b/a)}\,\frac{1}{b} = \sigma_b \qquad (g)$$

Part b

$$\bar{J} = \bar{J}' + \bar{v}_r\,\rho' \qquad (h)$$

But in this problem we have only surface currents and charges

$$\bar{K} = \bar{K}' + \bar{v}_r\,\sigma' = \bar{v}_r\,\sigma' \qquad (i)$$

$$\bar{K}(a) = \frac{a\omega\varepsilon_o V}{a\ \ln(b/a)}\,\vec{i}_\theta = \frac{\omega\varepsilon_o V}{\ln(b/a)}\,\vec{i}_\theta \qquad (j)$$

$$\bar{K}(b) = -\frac{b\omega\varepsilon_o V}{b\ \ln(b/a)}\,\vec{i}_\theta = -\frac{\omega\varepsilon_o V}{\ln(b/a)}\,\vec{i}_\theta \qquad (k)$$

Part c

$$\bar{H} = -\frac{\omega\varepsilon_o V}{\ln(b/a)}\,\vec{i}_z \qquad (l)$$

Part d

$$\bar{H} = \bar{H}' + \bar{v}_r \times \bar{D}' = \bar{v}_r \times \bar{D}' \qquad (m)$$

$$\bar{H} = r'\omega\Big(\frac{\varepsilon_o V}{\ln(b/a)}\,\frac{1}{r'}\Big)(\vec{i}_\theta \times \vec{i}_r) \qquad (n)$$

PROBLEM 6.6 (Continued)

$$\bar{H} = -\frac{\omega\varepsilon_o V}{\ln(b/a)}\vec{i}_z \tag{o}$$

This result checks with the calculation of part (c).

PROBLEM 6.7

Part a

The equation of the top surface is

$$f(x,y,t) = y - a\sin(\omega t)\cos(kx) + d = 0 \tag{a}$$

The normal to this surface is then

$$\bar{n} = \frac{\nabla f}{|\nabla f|} \approx ak\sin(\omega t)\sin(kx)\bar{i}_x + \bar{i}_y \tag{b}$$

Applying the boundary condition $\bar{n}\cdot\bar{B} = 0$ at each surface and keeping only linear terms, we learn that

$$h_y(x,d,t) = -ak\sin(\omega t)\sin(kx)\frac{\Lambda}{\mu_o d} \tag{c}$$

$$h_y(x,0,t) = 0 \tag{d}$$

We look for a solution for $\bar{h}$ that satisfies

$$\nabla \times \bar{h} = 0, \quad \nabla\cdot\bar{h} = 0 \tag{e}$$

Let $\bar{h} = \nabla\psi$, $\nabla^2\psi = 0$ (f)

Now we must make an intelligent guess for a Laplacian ψ using the periodicity of the problem and the boundary condition $h_y = \partial\psi/\partial y = 0$ at $y = 0$. Try

$$\psi = \frac{A}{k}\cosh(ky)\sin(kx)\sin(\omega t) \tag{g}$$

$$\vec{h} = A\sin(\omega t)[\cos(kx)\cosh(ky)\bar{i}_x + \sin(kx)\sinh(ky)\bar{i}_y] \tag{h}$$

Equation (c) then requires the constant A to be

$$A = \frac{-ak\quad\Lambda}{\sinh(kd)\mu_o d} \tag{i}$$

Part b

$$\nabla \times \bar{E} = \bar{i}_x(\frac{\partial E}{\partial y}z) - \bar{i}_y(\frac{\partial E}{\partial y}z) = -\frac{\partial\bar{B}}{\partial t} \tag{j}$$

$$\frac{\partial\bar{B}}{\partial t} = \omega\mu_o A\cos(\omega t)[\cos(kx)\cosh(ky)\bar{i}_x + \sin(kx)\sinh(ky)\bar{i}_y] \tag{k}$$

PROBLEM 6.7 (Continued)

$$\bar{E} = -\omega\mu_o \frac{A}{k}\cos(\omega t)[\cos(kx)\sinh(ky)]\bar{i}_z \qquad (l)$$

Now we check the boundary conditions. Because $\bar{v}(y=0) = 0$

$$\bar{n} \times \bar{E} = (\bar{n}\cdot\bar{v})\bar{B} = 0 \quad (y=0) \qquad (m)$$

But $\bar{E}(y=0) = 0$, so (m) is satisfied.

If a particle is on the top surface, its coordinates x,y,t must satisfy (a). It follows that

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \bar{v}\cdot\nabla f = 0 \qquad (n)$$

Since $\bar{n} = \frac{\nabla f}{|\nabla f|}$ we have that

$$(\bar{n}\cdot\bar{v}) = \frac{-1}{|\nabla f|}\frac{\partial f}{\partial t} \sim a\omega\cos(\omega t)\cos(kx) \qquad (o)$$

Now we can check the boundary condition at the top surface

$$\bar{n}\times\bar{E} = -\omega\mu_o\frac{A}{k}\cos(\omega t)\cos(kx)\sinh(kd)[\bar{i}_x - ak\sin(\omega t)\sin(kx)\bar{i}_y] \qquad (p)$$

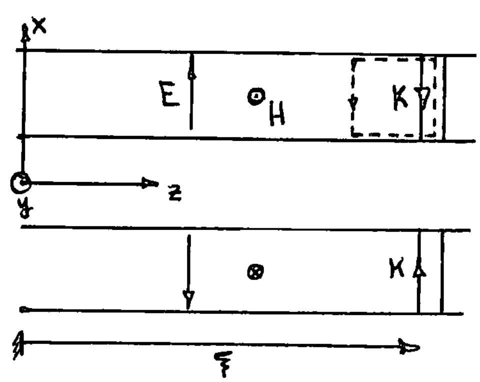
$$(\bar{n}\cdot\bar{v})\bar{B} = a\omega\cos(\omega t)\cos(kx)[\frac{-\sinh(kd)}{ak}\mu_o A\bar{i}_x + \qquad (q)$$

$$\mu_o A\sin(\omega t)\sin(kx)\sinh(kd)\bar{i}_y]$$

Comparing (p) and (q) we see that the boundary condition is satisfied at the top surface.

PROBLEM 6.8

Part a



Since the plug is perfectly conducting we expect that the current I will return as a surface current on the left side of the plug. Also E', H' will be zero in the plug and the transformation laws imply that E,H will then also be zero.

Using ampere's law

$$\bar{H} = \begin{cases} \frac{-I}{2\pi r}\bar{i}_\theta & 0 < z < \xi \\ 0 & \xi < z \end{cases} \qquad (a)$$

PROBLEM 6.8 (Continued)

Also we know that

$$\nabla\cdot\bar{E} = 0,\ \nabla \times \bar{E} = -\frac{\partial\bar{B}}{\partial t} = 0 \qquad 0 < z < \xi \tag{b}$$

We choose a simple Laplacian $\bar{E}$ field consistent with the perfectly conducting boundary conditions

$$\bar{E} = \frac{K}{r}\vec{i}_r \tag{c}$$

K can be evaluated from

$$\oint_C \bar{E}''\cdot d\bar{\ell} = -\frac{d}{dt}\int_S \bar{B}\cdot da \tag{d}$$

If we use the deforming contour shown above which has a fixed left leg at z = z and a moving right leg in the conductor. The notation $\bar{E}''$ means the electric field measured in a frame of reference which is stationary with respect to the local element of the deforming contour. Here

$$\bar{E}''(z) = \bar{E}(z),\quad \bar{E}''(\xi+\Delta) = \bar{E}'(\xi+\Delta) = 0 \tag{e}$$

$$\oint E''\cdot d\bar{\ell} = -\int_a^b E(z,r)dr = -K\ln(b/a) \tag{f}$$

The contour contains a flux

$$\int_S \bar{B}\cdot d\bar{a} = (\xi - z)\int_a^b \mu_o H_\theta dr = -\mu_o\frac{I}{2\pi}\ln(b/a)(\xi - z) \tag{g}$$

So that

$$-K\ln(b/a) = -\frac{d}{dt}\int_S \bar{B}\cdot\bar{da} = +\mu_o\frac{I}{2\pi}\ln(b/a)\frac{d\xi}{dt} \tag{h}$$

Since $v = \frac{d\xi}{dt}$,

$$\bar{E} = \begin{cases} -\dfrac{v\mu_o I}{2\pi}\dfrac{1}{r}\vec{i}_r & 0 < z < \xi \\ 0 & \xi < z \end{cases} \tag{j}$$

Part b

The voltage across the line at z = 0 is

$$V = -\int_a^b E_r dr = \frac{v\mu_o I}{2\pi}\ln(b/a) \tag{k}$$

PROBLEM 6.8 (Continued)

$$I\left(R + \frac{v\mu_o}{2\pi} \ln(b/a)\right) = V_o \tag{l}$$

$$I = \frac{V_o}{R + \frac{v\mu_o}{2\pi} \ln(b/a)} \tag{m}$$

$$V = \left[\frac{1}{\frac{2\pi R}{v\mu_o \ln(b/a)} + 1}\right] V_o \tag{n}$$

$$\bar{H} = \begin{cases} \dfrac{-V_o}{R + \frac{v\mu_o}{2\pi}\ln(b/a)} \dfrac{1}{2\pi r} \vec{i}_\theta & 0 < z < \xi \\ 0 & \xi < z \end{cases} \tag{o}$$

$$\bar{E} = \begin{cases} -\left[\dfrac{1}{\frac{2\pi R}{v\mu_o} + (\ln b/a)}\right] \dfrac{V_o}{r} \vec{i}_r & 0 < z < \xi \\ 0 & \xi < z \end{cases} \tag{p}$$

Part c

Since $\bar{E} = 0$ to the right of the plug the voltmeter reads zero. The terminal voltage V is not zero because of the net change of magnetic flux in the loop connecting these two voltage points.

Part d

Using the results of part (b)

$$P_{in} = VI = \frac{v\mu_o \ln(b/a)}{2\pi} \left[\frac{1}{R + \frac{v\mu_o}{2\pi} \ln(b/a)}\right]^2 V_o^2$$

$$\frac{dW_m}{dt} = v \int_a^b \frac{\mu_o}{2} H^2(r) 2\pi r \, dr$$

$$= \frac{1}{2} \left[\frac{v\mu_o \ln(b/a)}{2\pi} \left(\frac{1}{R + \frac{v\mu_o}{2\pi} \ln(b/a)}\right)^2 V_o^2\right]$$

PROBLEM 6.8 (Continued)

There is a net electrical force on the block, the mechanical system that keeps the block traveling at constant velocity receives power at the rate

$$\frac{1}{2}\frac{v\mu_o \ln(b/a)}{2\pi}\left[\frac{1}{R+\frac{v\mu_o \ln(b/a)}{2\pi}}\right]^2 V_o^2$$

from the electrical system.

Part e

$$L(x) = \int \frac{\mu_o H(r,I)x\, dr}{I} = \frac{\mu_o}{2\pi} \ln\ (b/a)x$$

$$f^e = \frac{\partial W_m'}{\partial x}\ ;\ W_m' = \frac{1}{2} L(x)i^2$$

$$f^e = \frac{1}{2}\frac{\partial L}{\partial x} i^2 = \frac{1}{2}\frac{\mu_o}{2\pi} \ln(b/a)i^2$$

The power converted from electrical to mechanical is then

$$\bar{f}_e' \frac{\bar{d}x}{dt} = f_e\ v = \frac{1}{2}\frac{\mu_o v}{2\pi} \ln(b/a)\ \left[\frac{V_o}{R+\frac{v\mu_o}{2\pi}\ln(b/a)}\right]^2$$

as predicted in Part (d).

PROBLEM 6.9

The surface current circulating in the system must remain

$$K = \frac{B_o}{\mu_o} \tag{a}$$

Hence the electric field in the finitely conducting plate is

$$E' = \frac{B_o}{\mu_o\sigma_s} \tag{b}$$

But then

$$E = E' - \bar{V} \times \bar{B} \tag{c}$$

$$= B_o\left(\frac{1}{\mu_o\sigma_s} - v\right)$$

v must be chosen so that E = 0 to comply with the shorted end, hence

$$v = \frac{1}{\mu_o\sigma_s} \tag{d}$$

PROBLEM 6.10

Part a

Ignoring the effect of the induced field we must conclude that

$$\bar{E} = 0 \tag{a}$$

everywhere in the stationary frame. But then

$$E' = \bar{E} + \bar{V} \times \bar{B} = \bar{V} \times \bar{B} \tag{b}$$

Since the plate is conducting

$$\bar{J}' = \bar{J} = \sigma\bar{V} \times \bar{B} \tag{c}$$

The force on the plate is then

$$F = \int \bar{J} \times B \, dv = DWd(\sigma\bar{V} \times \bar{B}) \times \bar{B} \tag{d}$$

$$F_x = - DWd \, \sigma v \, B_o^2 \tag{e}$$

Part b

$$M \frac{dv}{dt} + (DWd\sigma B_o^2)v = 0 \tag{f}$$

$$v = v_o \, e^{-\frac{DWd\sigma B_o^2 \, t}{M}} \tag{g}$$

Part c

The additional induced field must be small. From (e)

$$J' \simeq \sigma B_o v_o \tag{h}$$

Hence $K' \simeq \sigma B_o d v_o$ (i)

The induced field then has a magnitude

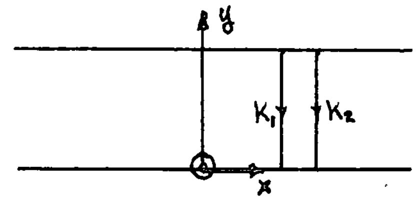
$$\frac{B'}{B_o} \simeq \frac{\mu_o K'}{B_o} = \mu_o \sigma d v_o << 1 \tag{j}$$

$$\sigma d << \frac{1}{\mu_o v_o} \tag{k}$$

It must be a very thin plate or a poorly conducting one.

PROBLEM 6.11

Part a



The condition $\frac{i}{W} << H_o$ means that the field induced by the current can be ignored. Then the magnetic field in the stationary frame is

$$H \simeq -H_o\vec{i}_z \quad \text{everywhere outside the perfect conductors} \tag{a}$$

The surface currents on the sliding conductor are such that

$$K_1 + K_2 = i/W \tag{b}$$

The force on the conductor is then

$$F = \int \bar{J} \times \bar{B}\, dv = [(K_1 + K_2)\vec{i}_y \times B_o\vec{i}_z]WD$$

$$= \mu_o H_o di\, \vec{i}_x \tag{c}$$

Part b

The circuit equation is

$$Ri + \frac{d\lambda}{dt} = V_o \tag{d}$$

$$\frac{d\lambda}{dt} \simeq \mu_o H_o dv \tag{e}$$

Since $F = M\frac{dv}{dt}$ (f)

$$(\frac{MR}{\mu_o H_o d})\frac{dv}{dt} + (\mu_o H_o d)v = V_o \tag{g}$$

$$v = \frac{V_o}{\mu_o H_o d}(1 - e^{-\frac{(\mu_o H_o d)^2}{MR}t})u_{-1}(t) \tag{h}$$

PROBLEM 6.12

Part a

We assume the simple magnetic field

$$\bar{H} = \begin{cases} -\frac{i}{D}\vec{i}_3 & 0 < x_1 < x \\ 0 & x < x_1 \end{cases} \tag{a}$$

$$\lambda(x) = \int \bar{B}\cdot\overline{da} = \frac{\mu_o Wx}{D} i \tag{b}$$

Part b

$$L(x) = \frac{\lambda(x,i)}{i} = \frac{\mu_o Wx}{D} \tag{c}$$

PROBLEM 6.12 (Continued)

Since the system is linear

$$W_m'(i,x) = \frac{1}{2} L(x)i^2 = \frac{1}{2} \frac{\mu_o Wx}{D} i^2 \tag{d}$$

Part c

$$f^e = \frac{\partial W_m'}{\partial x} = \frac{1}{2} \frac{\mu_o W}{D} i^2 \tag{e}$$

Part d

The mechanical equation is

$$M \frac{dx^2}{dt^2} + B \frac{dx}{dt} = \frac{1}{2} \frac{\mu_o W}{D} i^2 \tag{f}$$

The electrical circuit, equation is

$$\frac{d\lambda}{dt} = \frac{d}{dt} \left(\frac{\mu_o Wx}{D} i\right) = V_o \tag{g}$$

Part e

From (f) we learn that

$$\frac{dx}{dt} = \frac{\mu_o W}{2BD} i^2 = \text{const} \tag{h}$$

while from (g) we learn that

$$\frac{\mu_o Wi}{D} \frac{dx}{dt} = V_o \tag{i}$$

Solving these two simultaneously

$$\frac{dx}{dt} = \left[\frac{DV_o^2}{2\mu_o WB}\right]^{1/3} \tag{j}$$

Part f

From (e)

$$i = \sqrt{\frac{2BD}{\mu_o W} \frac{dx}{dt}} = \left(\frac{D}{\mu_o W}\right)^{2/3} (2B)^{1/3} V_o^{1/3} \tag{k}$$

Part g

As in part (a)

$$\bar{H} = \begin{cases} -\dfrac{i(t)\vec{i}_3}{D} & 0 < x_1 < x \\ 0 & x < x_1 \end{cases} \tag{l}$$

Part h

The surface current $\bar{K}$ is

PROBLEM 6.12 (Continued)

$$\bar{K} = -\frac{i(t)}{D}\vec{i}_2 \tag{m}$$

The force on the short is

$$\bar{F} = \int \bar{J} \times \bar{B}\, dv = DW\,\bar{K} \times \left(\frac{\mu_o \bar{H}_1 + \mu_o \bar{H}_2}{2}\right) \tag{n}$$

$$= \frac{\mu_o W}{2D} i^2(t)\vec{i}_1$$

Part i

$$\nabla \times \bar{E} = \frac{\partial E_2}{\partial x_1}\vec{i}_3 = -\frac{\partial \bar{B}}{\partial t} = \frac{\mu_o}{D}\frac{di}{dt}\vec{i}_3 \tag{o}$$

$$\bar{E}_2 = [\frac{\mu_o}{D} x \frac{di}{dt} + C]\vec{i}_3 \tag{p}$$

$$= [\frac{\mu_o}{D} x \frac{di}{dt} - \frac{V(t)}{W}] i_3$$

Part j

Choosing a contour with the right leg in the moving short, the left leg fixed at $x_1 = 0$

$$\oint_C \vec{E}' \cdot d\vec{\ell} = -\frac{d}{dt}\int_S \bar{B} \cdot d\bar{a} \tag{q}$$

Since $E' = 0$ in the short and we are only considering quasistatic fields

$$\oint \vec{E}' \cdot d\vec{\ell} = V(t) = W x \mu_o \frac{\partial H_o}{\partial t} + W \frac{dx}{dt}\mu_o H_o \tag{r}$$

$$= \frac{d}{dt}\left(\frac{\mu_o W x}{D} i(t)\right) \tag{s}$$

Part k

$$\bar{n} \times (\bar{E}^b) = v_n \bar{B}^b \tag{t}$$

Here

$$\bar{n} = i_1,\ v_n = \frac{dx}{dt},\ \bar{B}^b = -\frac{\mu_o i}{D}\vec{i}_3 \tag{u}$$

$$\bar{E}_b = \left(\frac{\mu_o x}{D}\frac{di}{dt} - \frac{V(t)}{W}\right)\vec{i}_2 = \left(-\frac{dx}{dt}\frac{\mu_o W}{D} i\right)\vec{i}_2 \tag{v}$$

$$-\frac{dx}{dt}\frac{\mu_o W}{D} i = \left(\frac{dx}{dt}\right)\left(-\frac{\mu_o i}{D}\right) \tag{w}$$

Part l

Equations (n) and (e) are identical. Equations (s) and (g) are identical if $V(t) = V_o$. Since we used (e) and (g) to solve the first part we would get the same answer using (n) and (s) in the second part.

PROBLEM 6.12 (Continued)

Part m

Since $\frac{di}{dt} = 0$,

$$\bar{E}_2(x) = -\frac{V(t)}{w}\,\vec{i}_y = -\frac{V_o}{W}\,\vec{i}_y \qquad (x)$$

PROBLEM 6.13

Part a

$$K\,\frac{d^2\psi}{dt^2} = T_1^e(\psi) + T_2(\psi) \qquad (a)$$

Part b

$$J_1 = \frac{i_1 \vec{i}_r}{D2\alpha r}\ ;\ \bar{F}_1 = \bar{J}_1 \times \bar{B} = -\frac{\mu_o H_o i_1}{D2\alpha R}\,\vec{i}_\theta \qquad (b)$$

Similarly

$$\bar{F}_2 = -\frac{\mu_o H_o i_2}{D2\alpha R}\,\vec{i}_\theta \qquad (c)$$

Part c

$$T_1^e = [\int (\bar{r} \times \bar{f})dv]_z = -\mu_o H_o (R_2 - R_1) i_1 \qquad (d)$$

$$T_2^e = -\mu_o H_o (R_2 - R_1) i_2 \qquad (e)$$

Part d

$$v_1 = E_1(R_2 - R_1);\ v_2 = E_2(R_2 - R_1) \qquad (f)$$

Part e

$$J_1 = J_1' = \sigma E_1' = \sigma(\bar{E}_1 + \bar{V}\times\bar{B}) = \sigma(E_1 + R\mu_o H_o \frac{d\psi}{dt}) \qquad (g)$$

$$E_1 = \frac{1}{\sigma}\frac{i_1}{2\alpha DR} - R\mu_o H_o \frac{d\psi}{dt} \qquad (h)$$

$$v_1 = \frac{1}{\sigma}\frac{(R_2 - R_1)}{2\alpha RD}\,i_1 - \mu_o H_o R(R_2 - R_1)\,\frac{d\psi}{dt} \qquad (i)$$

$$v_2 = \frac{1}{\sigma}\frac{R_2 - R_1}{2\alpha RD}\,i_2 - \mu_o H_o R(R_2 - R_1)\,\frac{d\psi}{dt} \qquad (j)$$

Part f

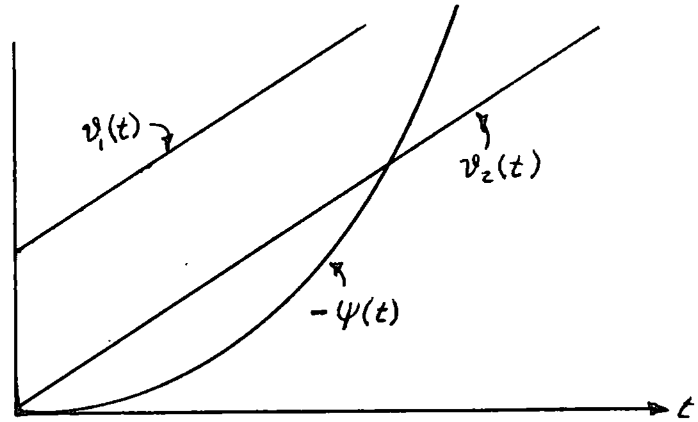
$$K\,\frac{d^2\psi}{dt^2} = -\mu_o H_o (R_2 - R_1) i_0\ u_{-1}(t) \qquad (k)$$

$$\psi(t) = -\frac{\mu_o H_o}{2K}(R_2 - R_1) i_0\ t^2\ u_{-1}(t) \qquad (l)$$

$$v_2(t) = (\mu_o H_o (R_2 - R_1))^2\,\frac{R}{K}\,i_0 t\ u_{-1}(t) \qquad (m)$$

PROBLEM 6.13 (Continued)

$$v_1(t) = [\frac{1}{\sigma}\frac{(R_2-R_1)}{2\alpha RD} + (\mu_o H_o(R_2-R_1))^2 \frac{R}{K} t] i_o u_{-1}(t) \qquad (n)$$



Part g

$$K \frac{d^2\psi}{dt^2} = -\mu_o H_o(R_2-R_1)i_1 \qquad (o)$$

$$= -\frac{\mu_o H_o(R_2-R_1)\sigma 2\alpha RD}{(R_2-R_1)} [v_1 + \mu_o H_o R(R_2-R_1)\frac{d\psi}{dt}]$$

$$\frac{d^2\psi}{dt^2} + K_1 \frac{d\psi}{dt} = -K_2 v_1(t) \qquad (p)$$

$$K_1 = [(\mu_o H_o R)^2 \, 2\alpha D(R_2-R_1)\sigma]/K \qquad (q)$$

$$K_2 = \frac{\mu_o H_o 2\alpha DR\sigma}{K}$$

Find the particular solution

$$\psi_p(\omega,t) = R_e[\frac{-jK_2 v_o}{\omega^2 - K_1 j\omega} e^{j\omega t}] \qquad (r)$$

$$= \frac{K_2 v_o}{\omega\sqrt{K_1^2+\omega^2}} \sin(\omega t + \tan^{-1}\frac{K_1}{\omega})u_{-1}(t) \qquad (s)$$

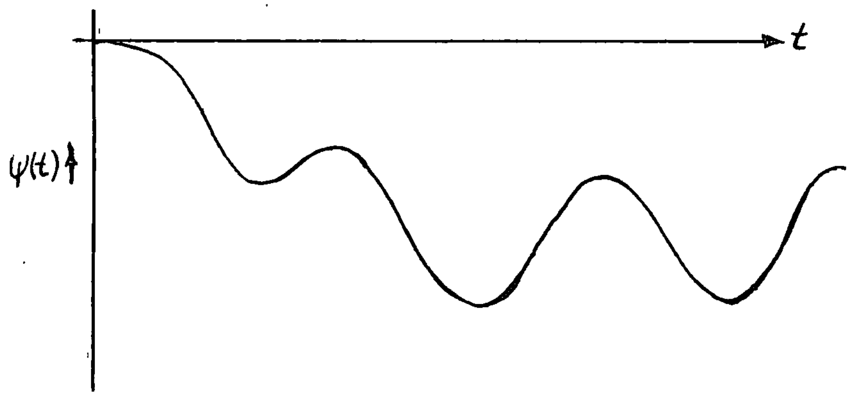
$$\psi(t) = A + \frac{B}{K_1} e^{-K_1 t} + \psi_p(\omega,t) \qquad (t)$$

We must choose A and B so that

$$\psi(0) = 0; \frac{d\psi}{dt}(0) = 0 \qquad (u)$$

$$A = -\frac{K_2}{K_1\omega} v_o \qquad B = +\frac{K_2\omega}{(K_1^2+\omega^2)} v_o \qquad (v)$$

PROBLEM 6.13(Continued)



Part h

The secondary terminals are constrained so that $v_2=-i_2R_2$. Thus, (j) becomes

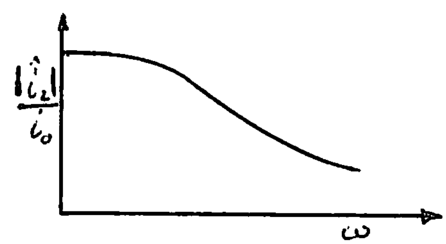
$$\frac{d\psi}{dt}=\frac{R_3}{RK_4}i_2;\ R_3=R_o+\frac{1}{\sigma}\frac{(R_2-R_1)}{2\ RD\alpha};\ K_4=\mu_oH_o(R_2-R_1) \qquad (w)$$

Then, it follows from (a), (d) and (e) that

$$\frac{di_2}{dt}+\frac{RK_4^2}{KR_3}i_2=-\frac{K_4^2Ri_o}{KR_3}\cos\omega t$$

from which it follows that

$$\frac{|\hat{i}_2|}{i_o}=\frac{K_4^2R}{KR_3\sqrt{\omega^2+(\frac{RK_4^2}{KR_3})^2}}$$



PROBLEM 6.14

Part a

The electric field in the moving laminations is

$$E'=\frac{J'}{\sigma}=\frac{J}{\sigma}=\frac{i}{\sigma A}\vec{i}_z \qquad (a)$$

The electric field in the stationary frame is

$$\vec{E}=\vec{E}'-\bar{V}\times\bar{B}=(\frac{i}{\sigma A}+r\omega B_y)\vec{i}_z \qquad (b)$$

$$B_y=-\frac{\mu_oNi}{S} \qquad (c)$$

$$V=(\frac{2D}{\sigma A}-\frac{\mu_o2Dr\omega N}{S})i \qquad (d)$$

PROBLEM 6.14 (Continued)

Now we have the V-i characteristic of the device. The device is in series with an inductance and a load resistor $R_t=R_L+R_{int}$.

$$[R_t + \frac{2D}{\sigma A} - \frac{\mu_o 2DrN}{S}\omega]i + \frac{\mu_o N^2 aD}{S}\frac{di}{dt} = 0 \qquad (e)$$

Part b

Let

$$R_1 = R_t + \frac{2D}{\sigma A} - \frac{2D\mu_o rN\omega}{S}, \quad L = \frac{\mu_o N^2 aD}{S} \qquad (f)$$

$$i = I_o e^{-(R_1/L)t} \qquad (g)$$

$$P_d = i^2/R_L = \frac{I_o^2}{R_L}\left[e^{-(R_1/L)t}\right]^2$$

If

$$R_1 = R_t + \frac{2D}{\sigma A} - \frac{2D\mu_o rN\omega}{S} < 0 \qquad (h)$$

the power delivered is unbounded as $t \to \infty$.

Part c

As the current becomes large, the electrical nonlinearity of the magnetic circuit will limit the exponential growth and determine a level of stable steady state operation (see Fig. 6.4.12).

PROBLEM 6.15

After the switch is closed, the armature circuit equation is*

$$(R_L + R_a)i_L + L_a\frac{di_L}{dt} = G\dot{\theta}i_f \qquad (a)$$

Since $G\dot{\theta}i_f$ is a constant and $i_L(0) = 0$ we can solve for the load current and shaft torque

$$i_L(t) = \frac{G\dot{\theta}i_f}{(R_L+R_a)}\left(1-e^{-\frac{(R_L+R_a)}{L_a}t}\right)u_{-1}(t) \qquad (b)$$

$$T^e(t) = i_L(t)\,Gi_f$$

$$= \frac{(Gi_f)^2\,\dot{\theta}}{(R_L+R_a)}\left(1-e^{-\frac{(R_L+R_a)}{L_a}t}\right)u_{-1}(t) \qquad (c)$$

* Note: $\dot{i}_a = -i_L$

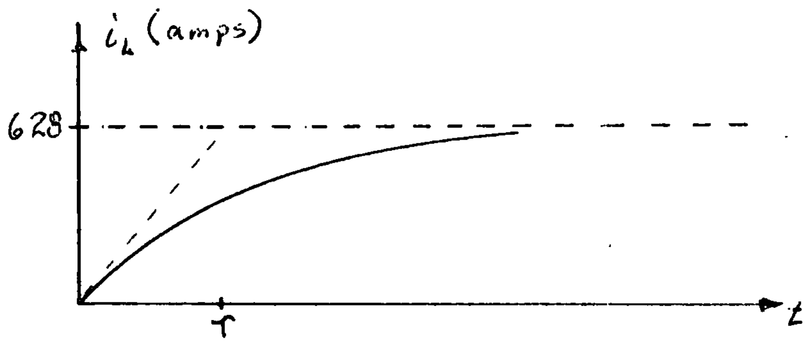
PROBLEM 6.15 (Continued)

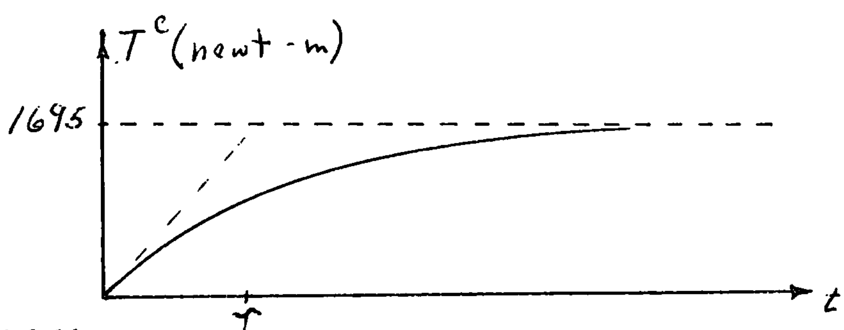
From the data given

$$\tau = L_a/R_L+R_a \approx 2.5 \times 10^{-3} \text{ sec} \tag{d}$$

$$i_{L_{max}} = \frac{G\dot{\theta}i_f}{R_L+R_a} \quad 628 \text{ amps} \tag{e}$$

$$T_{max} = \frac{(Gi_f)^2\dot{\theta}}{R_L+R_a} \approx 1695 \text{ newton-meters} \tag{f}$$





PROBLEM 6.16

Part a

With S_1 closed the equation of the field circuit is

$$R_f i_f + L_f \frac{di_f}{dt} = V_f \tag{a}$$

Since $i_f(0) = 0$

$$i_f(t) = \frac{V_f}{R_f}\left(1-e^{-\frac{R_f}{L_f}t}\right)u_{-1}(t) \tag{b}$$

Since the armature circuit is open

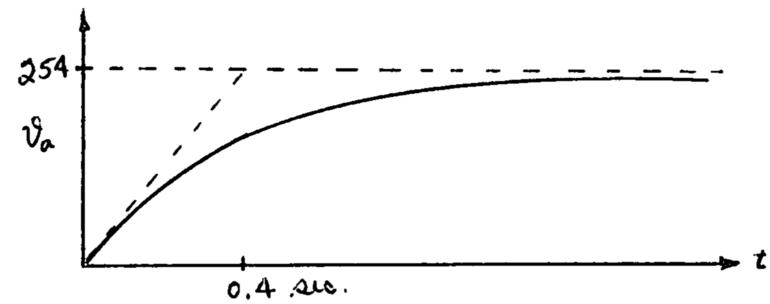
$$V_a = G\dot{\theta}i_f = \frac{V_f\, G\dot{\theta}}{R_f}\left(1-e^{-\frac{R_f}{L_f}t}\right)u_{-1}(t) \tag{c}$$

PROBLEM 6.16 (Continued)

From the given data

$$\tau = L_f/R_f = 0.4 \text{ sec}$$

$$V_{a_{max}} = \frac{V_f\, G\dot{\theta}}{R_f} = 254 \text{ volts}$$



Part b

Since there is no coupling of the armature circuit to the field circuit i_f is still given by (b).

Because S_2 is closed, the armature circuit equation is

$$(R_L+R_a)V_L + L_a \frac{dV_L}{dt} = R_L G\dot{\theta} i_f \qquad \text{(d)}$$

Since the field current rises with a time constant

$$\tau = 0.4 \text{ sec} \qquad \text{(e)}$$

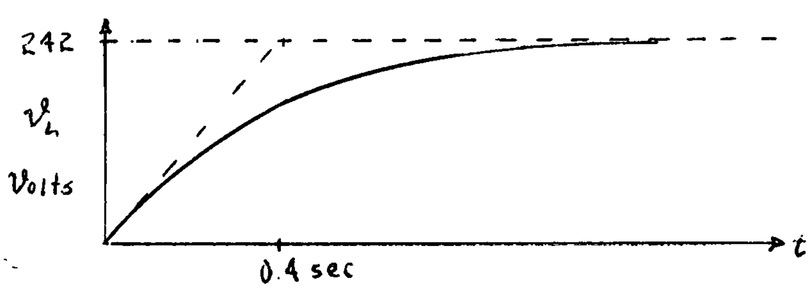
while the time constant of the armature circuit is

$$\tau = L_a/R_L+R_a = 0.0025 \text{ sec} \qquad \text{(f)}$$

we will only need the particular solution for $V_L(t)$

$$V_L(t) \approx \frac{R_L\, G\dot{\theta}}{R_L+R_a} i_f = \left(\frac{R_L}{R_L+R_a}\right)G\dot{\theta}\,\frac{V_f}{R_f}\left(1-e^{-\frac{R_f}{L_f}t}\right)u_{-1}(t) \qquad \text{(g)}$$

$$V_{L_{max}} = \left(\frac{R_L}{R_L+R_a}\right)\left(\frac{G\dot{\theta}}{R_f}\right)V_f = 242 \text{ volts} \qquad \text{(h)}$$



PROBLEM 6.17

The equation of motion of the shaft is

$$J_r \frac{d\omega}{dt} + \frac{T_o}{\omega_o} \omega = T_o + T_e(t) \qquad (a)$$

If $T_e(t)$ is thought of as a driving term, the response time of the mechanical circuit is

$$\tau = \frac{J_r \omega_o}{T_o} = 0.0785 \text{ sec} \qquad (b)$$

In Probs. 6.15 to 6.16 we have already calculated the armature circuit time constant to be

$$\tau = \frac{L_a}{R_a + R_L} \simeq 2.5 \times 10^{-3} \text{ sec} \qquad (c)$$

We conclude that the rise time of the armature circuit may be neglected, this is equivalent to ignoring the armature inductance. The circuit equation for the armature is then

$$(R_a + R_L) i_L = G\omega i_f \qquad (d)$$

Then

$$T_e = -G i_f i_L = \frac{-(G i_f)^2 \omega}{R_a + R_L} \qquad (e)$$

Plugging into (a)

$$J_r \frac{d\omega}{dt} + K\omega = T_o \qquad (f)$$

Here

$$K = \left(\frac{T_o}{\omega_o} + \frac{(G i_f)^2}{R_a + R_L}\right); \; i_f = \frac{V_f}{R_f} \qquad (g)$$

Using the initial condition that $\omega(0) = \omega_o$

$$\omega(t) = \frac{T_o}{K} + \left(\omega_o - \frac{T_o}{K}\right) e^{-(K/J_r)t} \qquad t \geq 0 \qquad (h)$$

From which we can calculate the net torque on the shaft as

$$T = J_r \frac{d\omega}{dt} = (T_o - K\omega_o) e^{-(K/J_r)t} \, u_{-1}(t) \qquad (i)$$

and the armature current $i_L(t)$

$$i_L(t) = \left(\frac{G i_f}{R_a + R_L}\right) \omega(t) \qquad t \geq 0 \qquad (j)$$

PROBLEM 6.17 (Continued)

From the given data

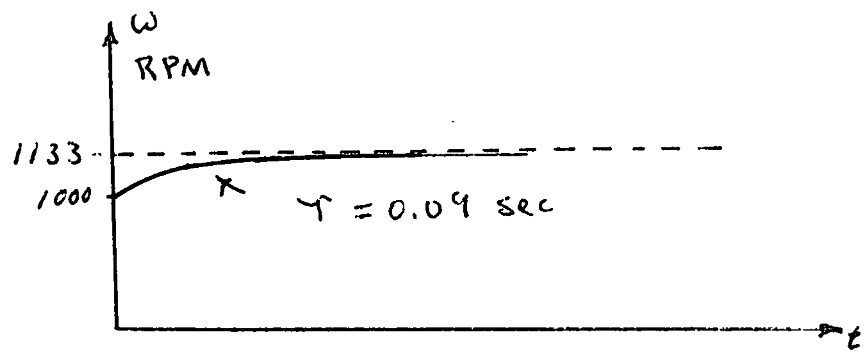
$$\omega_{final} = \frac{T_o}{K} = 119.0 \text{ rad/sec} = 1133 \text{ RPM} \tag{k}$$

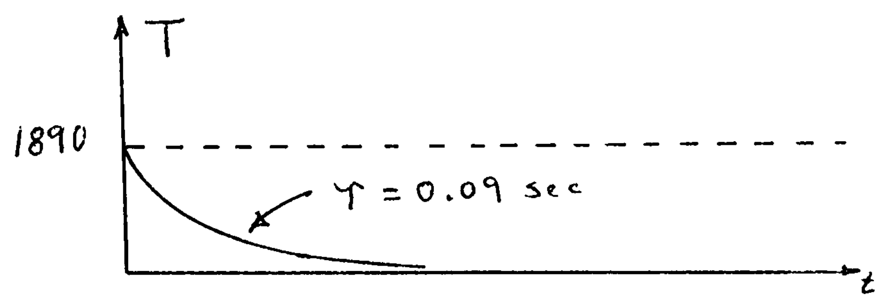
$$T_{max} = (T_o - K\omega_o) \simeq 1890 \text{ newton-m} \tag{l}$$

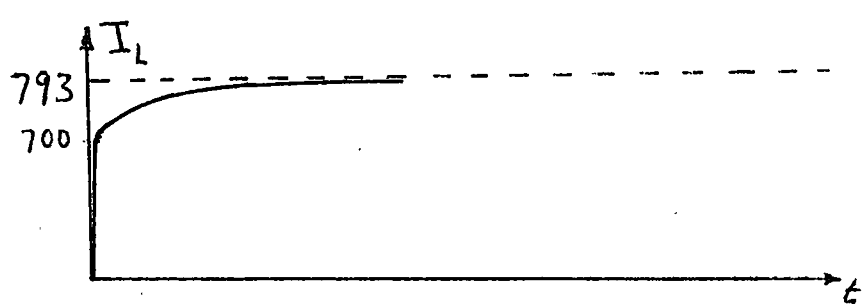
$$i_{L_{min}} = \frac{Gi_f}{R_a+R_L}\,\omega_o \simeq 700 \text{ amps} \tag{m}$$

$$i_{L_{max}} = \left(\frac{Gi_f}{R_a+R_L}\right)\omega_{final} \simeq 793 \text{ amps} \tag{n}$$

$$K = 134.5 \text{ newton-meters}, \ \tau = J_r/K \simeq 0.09 \text{ sec} \tag{o}$$







PROBLEM 6.18

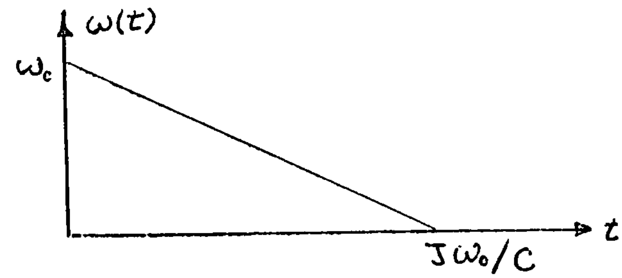
Part a

Let the coulomb torque be C, then the equation of motion is

$$J \frac{d\omega}{dt} + C = 0 \tag{a}$$

Since $\omega(0) = \omega_o$

$$\omega(t) = \omega_o(1 - \frac{C}{J\omega_o} t) \qquad 0 \leq t \leq (J/C)\,\omega_o \tag{b}$$

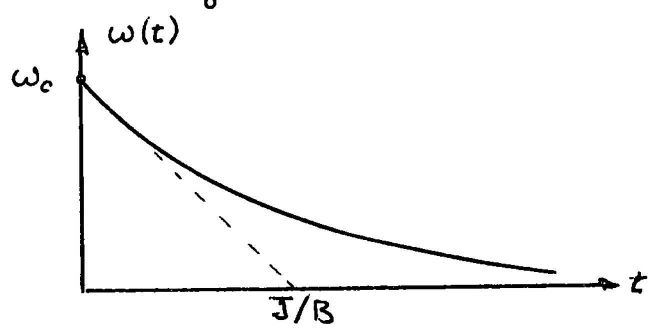


Part b

Now the equation of motion is

$$J \frac{d\omega}{dt} + B\omega = 0 \tag{c}$$

$$\omega(t) = \omega_o\, e^{-(B/J)t} \tag{d}$$



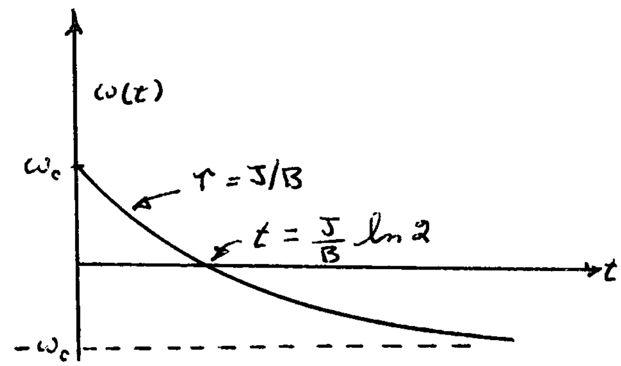
Part c

Let $C = B\omega_o$, the equation of motion is now

$$J \frac{d\omega}{dt} + B\omega = -B\omega_o \tag{e}$$

$$\left\{\omega(t) = -\omega_o + 2\omega_o e^{-\frac{B}{J}t} \qquad 0 < t \leq \frac{J}{B} \ln 2\right\} \tag{f}$$

PROBLEM 6.18 (Continued)



PROBLEM 6.19

Part a

The armature circuit equation is

$$R_a i_L + L_a \frac{di_L}{dt} = G\omega i_f - V_a u_{-1}(t) \tag{a}$$

Differentiating

$$L_a \frac{d^2 i_L}{dt^2} + R_a \frac{di_L}{dt} = G i_f \frac{d\omega}{dt} - V_a u_o(t) \tag{b}$$

The mechanical equation of motion is

$$J_r \frac{d\omega}{dt} = - G i_L i_f \tag{c}$$

Thus, (b) becomes

$$L_a \frac{d^2 i_L}{dt^2} + R_a \frac{di_L}{dt} + \frac{(G i_f)^2}{J_r} i_L = - V_a u_o(t) \tag{d}$$

Initial conditions are

$$i_L(0^+) = 0, \quad \frac{di_L}{dt}(0^+) = - \frac{V_a}{L_a} \tag{e}$$

and it follows from (d) that

$$i_L(t) = (- \frac{V_a}{L_a \beta} e^{-\alpha t} \sin \beta t) u_{-1}(t) \tag{f}$$

where

$$\alpha = \frac{R_a}{2L_a} = 7.5/\text{sec} \tag{g}$$

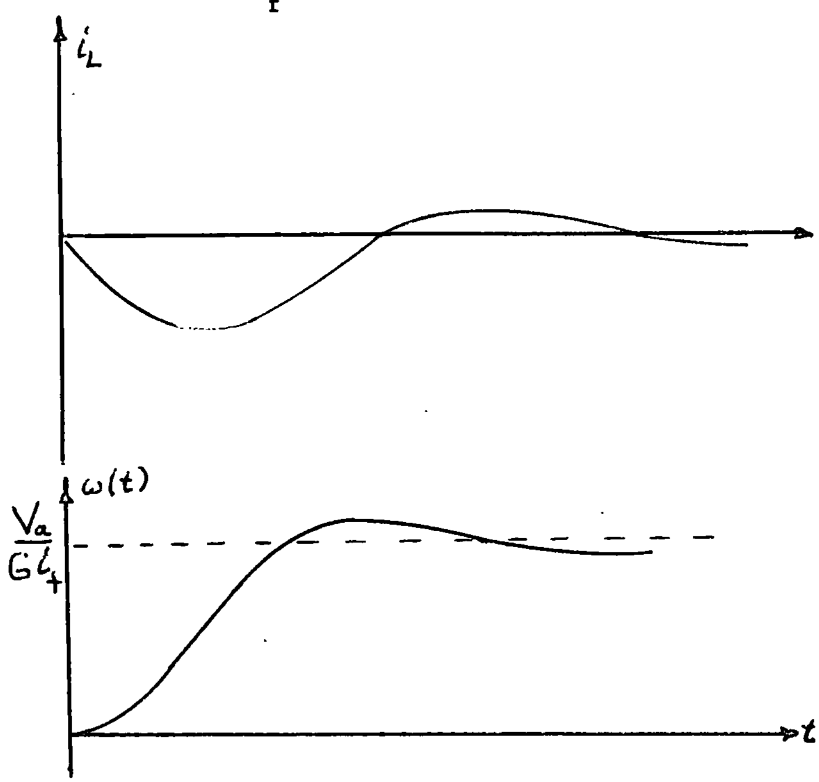
$$\beta = \sqrt{\frac{(G i_f)^2}{J_r L_a} - (\frac{R_a}{2L_a})^2} \simeq 19.9 \text{ rad/sec} \tag{h}$$

PROBLEM 6.19 (Continued)

$$\frac{V_a}{L_a\beta} \simeq 1160 \text{ amps} \tag{i}$$

$$\omega(t) = -\frac{V_a}{Gi_f}\left[\frac{\alpha}{\beta}\, e^{-\alpha t}\sin\beta t + (e^{-\alpha t}\cos\beta t - 1)\right]$$

$$\frac{V_a}{Gi_f} = 153.3 \text{ rad/sec} \tag{k}$$



Part b

Now we replace R_a by R_a+R_L in part (a). Because of the additional damping

$$i_L(t) = -\frac{V_a}{2L_a\gamma}\left(e^{-(\alpha-\gamma)t} - e^{-(\alpha+\gamma)t}\right)u_{-1}(t) \tag{l}$$

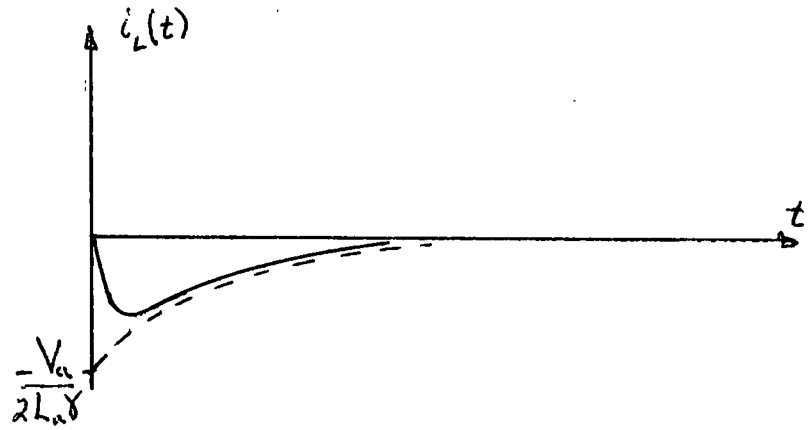
where

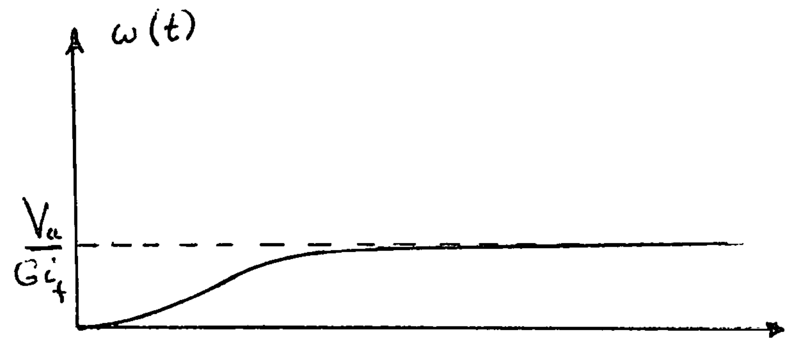
$$\alpha = \frac{R_a + R_L}{2L_a} = 75/\text{sec} \tag{m}$$

$$\gamma = \sqrt{\left(\frac{R_a + R_L}{2L_a}\right)^2 - \frac{(Gi_f)^2}{J_r L_a}} = 10.6/\text{sec.} \tag{n}$$

PROBLEM 6.19 (Continued)

$$\omega(t) = \frac{V_a}{2L_a\gamma}\frac{GI_f}{J_r}\left[\frac{1}{(\gamma-\alpha)}e^{-(\alpha-\gamma)t} + \frac{1}{\gamma+\alpha}e^{-(\alpha+\gamma)t} + \frac{2\gamma}{\alpha^2-\gamma^2}\right] \qquad (o)$$





PROBLEM 6.20

Part a

The armature circuit equation is

$$v_a = R_a i_a + GI_f\omega \qquad (a)$$

The equation of motion is

$$J\frac{d\omega}{dt} = GI_f i_a \qquad (b)$$

Which may be integrated to yield

$$\omega(t) = \frac{G}{J}\int_{-\infty}^{t} i_a(t) \qquad (c)$$

PROBLEM 6.20 (Continued)

Combining (c) with (a)

$$v_a = R_a i_a + \frac{(GI_f)^2}{J_r} \int_{-\infty}^{t} i_a(t) \tag{d}$$

We recognize that

$$C = \frac{J_r}{(GI_f)^2} \tag{e}$$

Part b

$$C = \frac{J_r}{(GI_f)^2} = \frac{(0.5)}{(1.5)^2(1)} = 0.22 \text{ farads}$$

PROBLEM 6.21

According to (6.4.30) the torque of electromagnetic origin is

$$T^e = Gi_f i_a$$

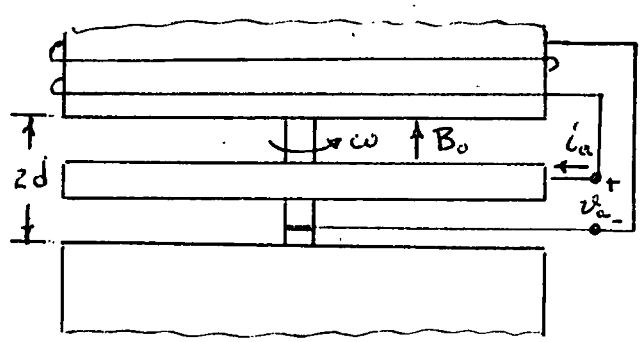
For operation on a-c, maximum torque is produced when i_f and i_a are in phase, a situation assured for all loading conditions by a series connection of field and armature. Parallel operation, on the other hand, will yield a phase relation between i_f and i_a that varies with loading. This gives reduced performance unless phase connecting means are employed. This is so troublesome and expensive that the series connection is used almost exclusively.

PROBLEM 6.22

From (6.4.50) et. seq. the homopolar machine, viewed from the disk terminals in the steady state, has the volt ampere relation

$$v_a = R_a i_a + G\omega i_f$$

$$R_a = \frac{\ln(b/a)}{2\pi\sigma d}$$



For definition of v_a and i_a shown to the right and with the interconnection with the coil shown in Fig. 6P.22

$$B_o = \frac{\mu_o N i_a}{2d}$$

Then from (6.4.52)

$$G\omega i_f = \frac{\omega B_o}{2}(b^2-a^2) = \frac{\omega\mu_o N i_a}{4d}(b^2-a^2)$$

PROBLEM 6.22 (Continued)

Substitution of this into the voltage equation yields for steady state (because the coil resistance is zero).

$$0 = R_a i_a + \frac{\omega\mu_o N i_a}{4d}(b^2-a^2)$$

for self-excitation with $i_a \neq 0$

$$\frac{\omega\mu_o N}{4d}(b^2-a^2) = -R_a$$

Because all terms on the left are positive except for ω, we specify $\omega < 0$ (it rotates in the direction opposite to that shown). With this provision the number of turns must be

$$N = \frac{4dR_a}{|\omega|\mu_o(b^2-a^2)} = \frac{4d\ln(b/a)}{2\pi\sigma d|\omega|\mu_o(b^2-a^2)}$$

$$N = \frac{2\ln(b/a)}{\pi\sigma\mu_o|\omega|(b^2-a^2)}$$

PROBLEM 6.23

Part a

Denoting the left disk and magnet as 1 and the right one as 2, the flux densities defined as positive upward are

$$B_1 = -\frac{\mu_o N}{\ell}(i_1-i_2)$$

$$B_2 = -\frac{\mu_o N}{\ell}(i_1+i_2)$$

Adding up voltage drops around the loop carrying current i_1 we have:

$$-N\pi a^2\frac{dB_2}{dt} - N\pi a^2\frac{dB_1}{dt} + i_1R_L + i_1R_a \underset{-}{+} \frac{\Omega B_1}{2}(b^2-a^2) = 0$$

where $R_a = \dfrac{\ln(\overset{a/b}{\cancel{b/a}})}{2\pi\sigma h}$

Part b

Substitution of the expression for B_1 and B_2 into this voltage expression and simplification yield

$$L\frac{di_1}{dt} + i_1(R_L+R_a) - G\Omega i_1 + G\Omega i_2 = 0$$

PROBLEM 6.23 (Continued)

where

$$L = \frac{2\mu_o N^2 \pi a^2}{\ell}$$

$$G = \frac{-\mu_o N(b^2-a^2)}{2\ell}$$

The equation for the circuit carrying current i_2 can be written similarly as

$$L\frac{di_2}{dt} + i_2(R_L+R_a) - G\Omega i_2 - G\Omega i_1 = 0$$

These are linear differential equations with constant coefficients, hence, assume

$$i_1 = I_1 e^{st}; \quad i_2 = I_2 e^{st}$$

Then

$$[Ls + R_L+R_a-G\Omega]I_1 + G\Omega I_2 = 0$$

$$[Ls + R_L+R_a-G\Omega]I_2 - G\Omega I_1 = 0$$

Elimination of I_1 yields

$$\left[\frac{[Ls + R_L+R_a-G\Omega]^2}{G\Omega} + G\Omega\right] I_2 = 0$$

If $I_2 \neq 0$ as it must be if we are to supply current to the load resistances, then

$$[Ls + R_L+R_a-G\Omega]^2 + (G\Omega)^2 = 0$$

For steady-state sinusoidal operation s must be purely imaginary. This requires

$$R_L + R_a - G\Omega = 0$$

or

$$G = \frac{-\mu_o N(b^2-a^2)}{2\ell} = \frac{R_L + \dfrac{\ln \overset{a\|b}{(\cancel{b/a})}}{2\pi\sigma h}}{\Omega}$$

This is the condition required.

Part c

When the condition of (b) is satisfied

$$s = \pm j\omega = \pm j\,\frac{G\Omega}{L}$$

$$\omega = \frac{-\mu_o N(b^2-a^2)\ell\Omega}{2\ell\mu_o N^2\pi a^2} = \frac{-(\frac{b^2}{a^2}-1)\Omega}{2\cdot 2\pi N}$$

PROBLEM 6.23 (Continued)

Thus the system will operate in the sinusoidal steady-state with amplitudes determined by initial conditions. With the condition of part (b) satisfied the voltage equations show that

$$I_1 = jI_2$$

and the currents form a balanced two-phase set.