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Electromechanical Dynamics

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PROBLEMS

9.1. A long thin steel cable of unstressed length l is hanging from a fixed support, as illustrated in Fig. 9P.1. Assume that the origin of coordinates is at the support and that x measures positive as shown. Assume that the steel cable has the following constants:

Cross-sectional area	$A = 10^{-4} \text{ m}^2$
Young's modulus	$E = 2.0 \times 10^{11} \text{ N/m}^2$
Mass density	$\rho = 7.8 \times 10^3 \text{ kg/m}^3$
Maximum allowable stress	$T_{\text{max}} = 2 \times 10^9 \text{ N/m}^2$

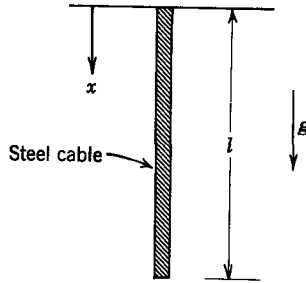


Fig. 9P.1

- (a) Find the length of cable l for which the maximum stress in the cable just equals the maximum allowable stress.
- (b) Find the displacement δ and stress T in the cable as functions of x .
- (c) Find the total elongation of the cable.

9.2. Two thin elastic rods are arranged as shown in Fig. 9P.2. The first rod has modulus of elasticity E_1 , density ρ_1 , and cross-sectional area A_1 . It is attached at one end to a rigid wall and at the other to a very thin rigid plate of mass m and area A_m . On the other side of

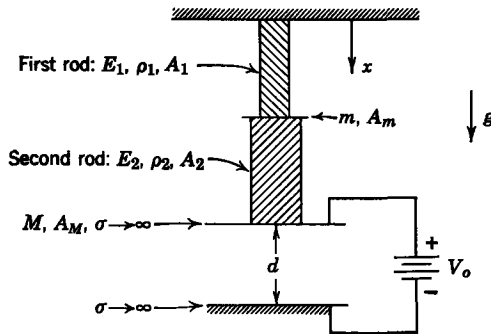


Fig. 9P.2

this plate is attached a second thin elastic rod with elastic modulus E_2 , density ρ_2 , and cross-sectional area A_2 . The other end of the second rod is fixed to a perfectly conducting thin plate with mass M and area A_M . This plate is held at a potential V_0 with respect to a second capacitor plate a distance d away. In the absence of gravity and with $V_0 = 0$, the length of the first rod is L_1 and the length of the second is L_2 . Assuming now that the system is immersed in a gravitational field g and that $V_0 \neq 0$, find the following:

- (a) The stress in the first rod $T^{(1)}(x)$ and the displacement in the first rod $\delta^{(1)}(x)$.
- (b) The stress in the second rod $T^{(2)}(x)$ and the displacement in the second rod $\delta^{(2)}(x)$.

9.3. In Fig. 9P.3 a thin elastic rod of cross-sectional area A , equilibrium length l , elastic modulus E , and mass density ρ is fixed at one end ($x = 0$) and attached to a rigid mass M

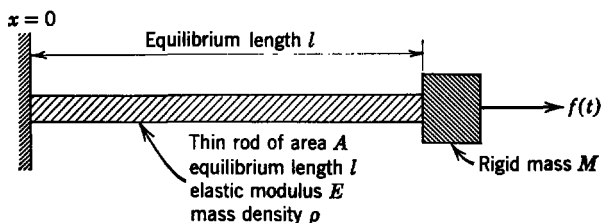


Fig. 9P.3

at the other ($x = l$). The mass is driven by a force source $f(t)$. The system constants are such that the mass M is much greater than the mass of the elastic rod; that is,

$$M \gg \rho Al.$$

The force source is constrained to be

$$f(t) = \text{Re} (f_0 e^{j\omega t}),$$

where f_0 and ω are positive real constants. The system is operating in the sinusoidal steady state. Neglect gravity.

- (a) Find the displacement $\delta(x, t)$ and stress $T(x, t)$ in the elastic rod.
- (b) Show a lumped-parameter mechanical system that represents the behavior of the system in Fig. 9P.3 for low frequencies (from $\omega = 0$ up to and including the lowest resonance frequency). Evaluate the equivalent elements in terms of the given parameters.

9.4. A long thin elastic rod with cross-sectional area A , unstressed length l , modulus of elasticity E , and mass density ρ is constrained at one end by the three ideal, lumped elements

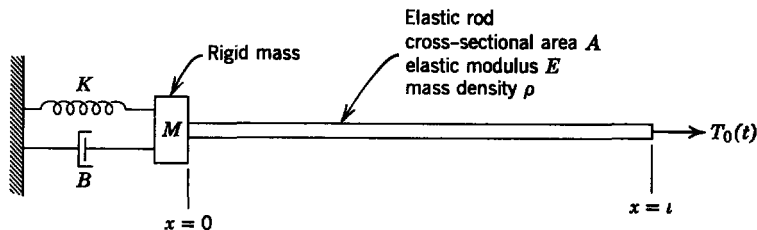


Fig. 9P.4

B , K , and M and at the other by a stress source $T_0(t)$, as shown in Fig. 9P.4. Determine the values of B , K , and M that are necessary for the response of the rod to $T_0(t)$ to be representable purely as a wave traveling in the negative x -direction. (Zero is an acceptable value for an element.)

9.5. A long thin rod of elastic modulus E , mass density ρ , and cross-sectional area A is fixed at one end ($x = 0$) and constrained at the other ($x = l$) by a force source $f(t)$ and a

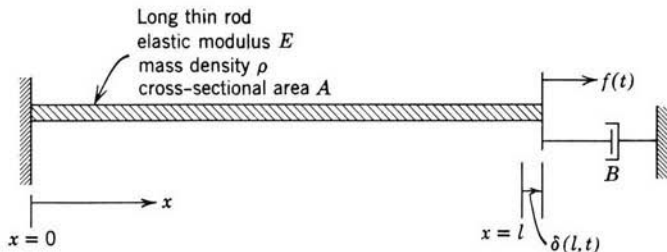


Fig. 9P.5

lumped linear damper of coefficient B , as illustrated in Fig. 9P.5. The applied force is sinusoidal $f(t) = \text{Re}(F_0 e^{j\omega t})$, where F_0 and ω are positive constants. The system is operating in the sinusoidal steady state.

- Write the boundary condition at $x = l$ in terms of the stress $T(l, t)$, the displacement $\delta(l, t)$, and the applied force $f(t)$.
- Assume that the displacement has the form $\delta(x, t) = \text{Re}[\hat{\delta}(x)e^{j\omega t}]$. Find the complex amplitude $\hat{\delta}(x)$ in terms of given data.
- If the damper coefficient B is positive and the frequency ω is real, can the system exhibit a resonance? That is, can the displacement δ be infinite with a zero applied force? Give justification for your answer.

9.6. A thin, circular magnetic rod is fixed at one end and constrained at the other end by a transducer (Fig. 9P.6). In the absence of an excitation, the transducer is simply biased by

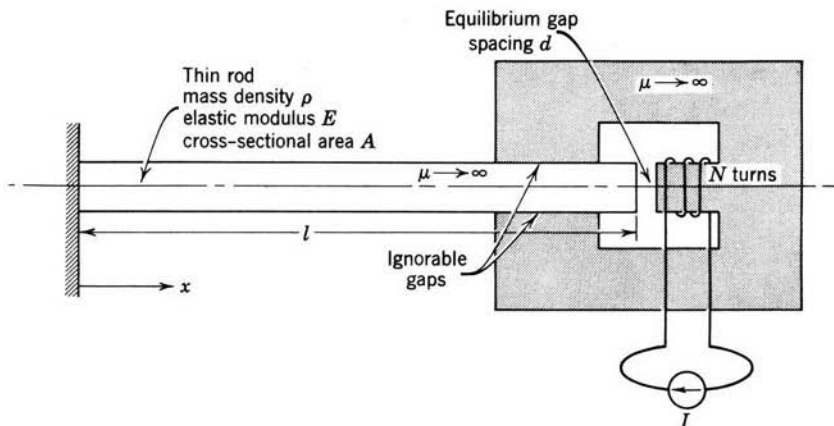


Fig. 9P.6

the constant current source I . When the rod is in static equilibrium, its length is l and the gap spacing is d . Compute the natural frequencies of the system under the assumption that the magnetization force on the rod acts on the end surface. A graphical representation of the eigenfrequencies is an adequate solution.

9.7. In Fig. 9P.7 two identical thin elastic rods are connected by a thin plate of mass M and area A_M . The plate is positioned between four springs, each having constant K . All

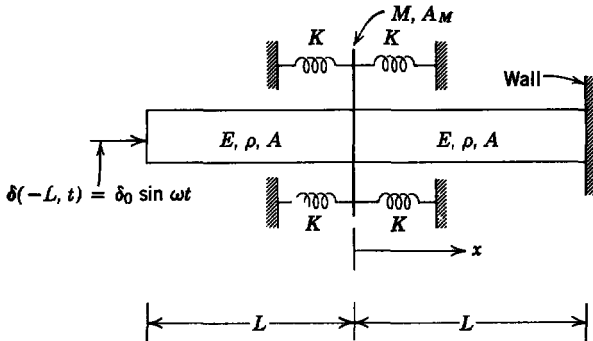


Fig. 9P.7

springs are relaxed when the plate is at $x = 0$. The system is driven on the left with a displacement $\delta(-L, t) = \delta_0 \sin \omega t$. Assume that steady state has been established.

- Write the general solution for the stress $\hat{T}(x)$ and the displacement $\delta(x)$ everywhere in both rods in terms of arbitrary constants.
- What are the boundary conditions that determine the constants in (a)?
- Find the stress $T(x, t)$ everywhere in both rods.

9.8. Example 9.1.5 considers the response of the delay line shown in Fig. 9.1.14 to a transient input signal. In this example design approximations were made concerning the effect of the self-inductance in the output circuit [see approximation following (t)]. You wish to compute the sinusoidal steady-state response of this system without making this approximation. Confine your attention to the sinusoidal steady-state response $v_o = \text{Re}(\hat{v}_o e^{j\omega t})$, to the input $i_i = \text{Re}(\hat{i}_i e^{j\omega t})$, and find the transfer function $H(\omega) = \hat{v}_o / \hat{i}_i$.

9.9. A magnetic transducer is used to excite a thin elastic rod, as shown in Fig. 9P.9. The mass M is attached to the rod, which in turn is fixed to a rigid support at $x = 0$. The driving current $I(t)$ is much smaller than I_0 and is given by $I(t) = \text{Re}[\hat{I} \exp(j\omega t)]$. Under the assumption that the displacements of the mass from its equilibrium position are small, complete the following:

- Find an expression for the position of the mass $y(t)$. You may assume that when there is no applied current the mass is centered between the pole faces and the rod has a length L .
- Under what conditions would you say that it is meaningful to consider the magnetic yoke and plunger as rigid but to recognize that the rod is made of an elastic material?

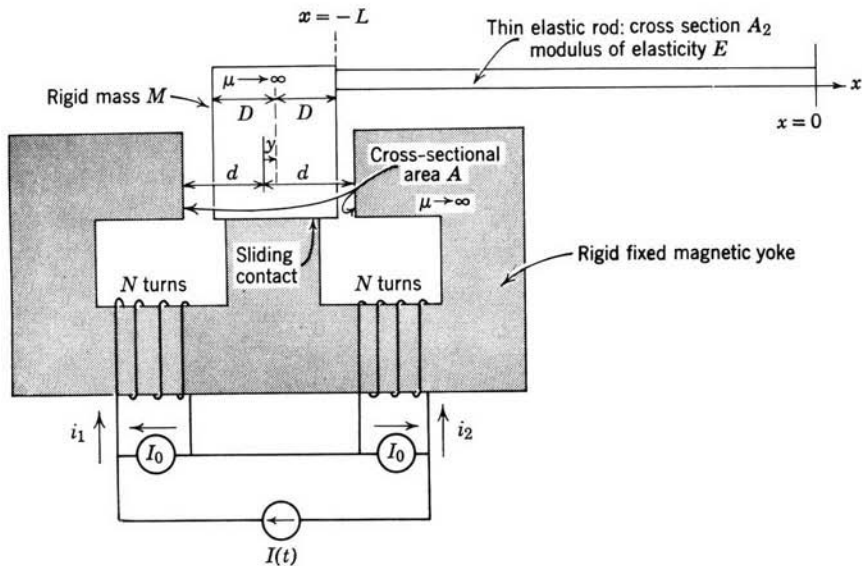


Fig. 9P.9

9.10. A long thin rod is fixed at $x = 0$ and driven at $x = l$, as shown in Fig. 9P.10. The driving transducer consists of a rigid plate with area A attached to the end of the rod, where it undergoes the displacement $\delta(l, t)$ from an equilibrium position exactly between two fixed plates. These fixed plates are biased by potentials V_0 and driven by the voltage $v = \text{Re}(\hat{V}e^{j\omega t})$, as shown. ($|\hat{V}| \ll V_0$)

- Derive a boundary condition relating $\delta(l, t)$, $(\partial\delta/\partial x)(l, t)$ and $v(t)$.
- Compute the driven deflection of the rod $\delta(x, t)$.

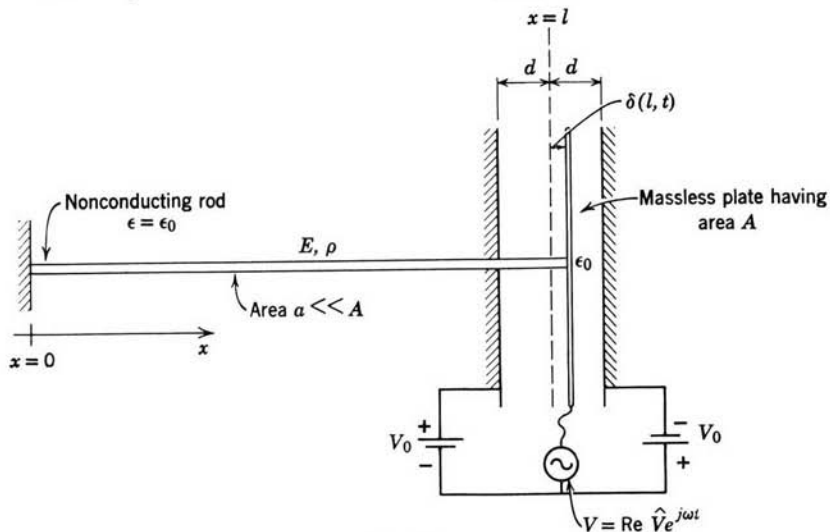


Fig. 9P.10

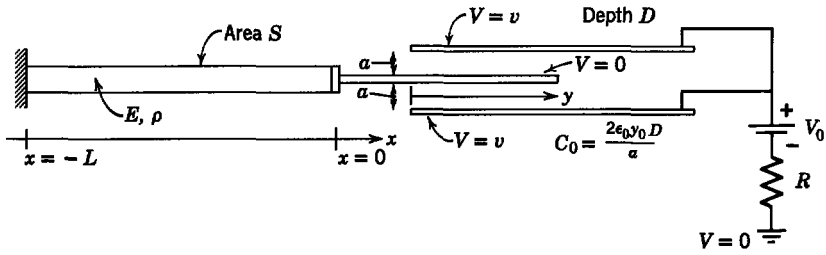


Fig. 9P.11

9.11. A long thin elastic rod (Fig. 9P.11) is fixed at $x = -L$. At $x = 0$ the rod is connected to a movable perfectly conducting plate which has negligible mass. The plate is electrically grounded and constrained to move only in the x -direction. Assume

$$v = V_0 + v_s, \quad v_s \ll V_0,$$

$$y = y_0 + y_s, \quad y_s \ll y_0,$$

where $y_s = \delta(t) =$ small signal displacement. Find the dc voltage (in terms of given parameters) for which there is no elastic wave reflection at the right-hand boundary for wave frequencies such that $\omega \ll 1/RC_0$.

9.12. The system of Fig. 9P.12 consists of two electric field transducers coupled mechanically by a long, thin rod of material with Young's modulus E , mass density ρ , and dielectric permittivity $\epsilon > \epsilon_0$. With the bias voltages V_0 applied and the system at rest, the rod has

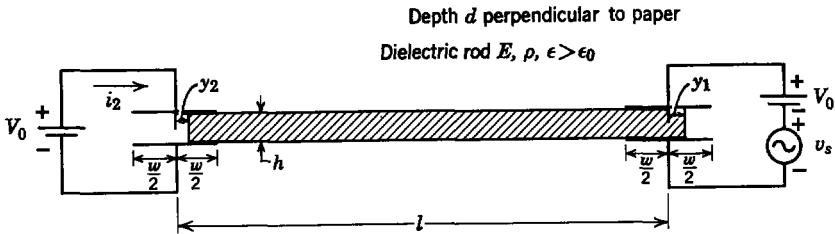


Fig. 9P.12

length l and is in the equilibrium position $y_1 = y_2 = 0$. The dimensions are defined in the figure and you can neglect fringing fields and mechanical friction. For sinusoidal excitation $v_s = \text{Re}(\hat{v}_s e^{j\omega t})$, such that $|\hat{v}_s| \ll V_0$ and steady-state operation about the equilibrium condition, assume that the current i_2 is given by $i_2(t) = \text{Re}(\hat{i}_2 e^{j\omega t})$.

- Find the transfer admittance $Y(j\omega) = \hat{i}_2/\hat{v}_s$.
- Specify the mathematical relation that defines the poles of this admittance. Find the lowest nonzero frequency at which a pole occurs.

9.13. Figure 9P.13 shows an electromechanical filter constructed with two magnetic transducers and a long (length L) thin rod. The transducers, which are alike, are symmetric about the axis of the rod. The ends of the rod form the plungers of magnetic circuits, with axially symmetric air gaps of length g . The left transducer is driven by a constant bias current I_0 and a small current $i = \text{Re}(j e^{j\omega t})$. This signal is transmitted by the rod to the right transducer, which is also biased by a constant current I_0 .

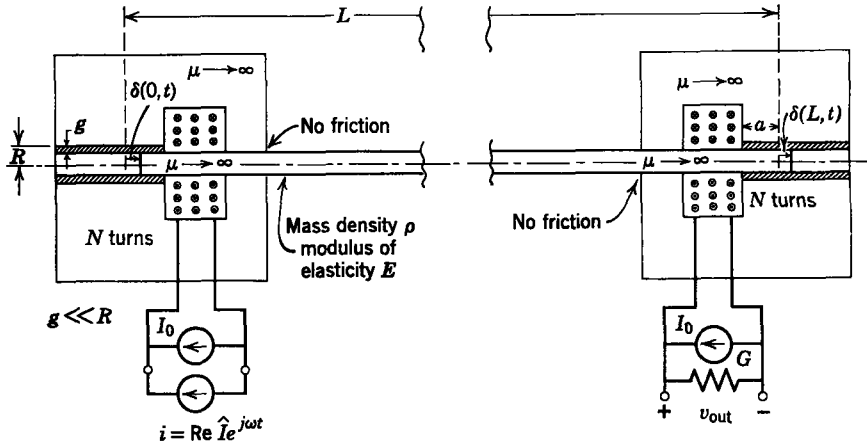


Fig. 9P.13

- (a) Use the energy method to find the magnetic forces acting on the ends of the rod.
- (b) Check the result of part (a) by using the magnetic stress tensor $T_{ij} = \mu H_i H_j - \frac{1}{2} \delta_{ij} \mu H_k H_k$ to find the forces acting on the ends of the rods.
- (c) Assume that the magnetic forces act just on the ends and compute the transfer function $G(\omega) = \hat{v}_{out}/\hat{I}$, where $v_{out} = \text{Re}(\hat{v}_{out} e^{j\omega t})$.
- (d) How would you adjust the system parameters so that $|\hat{v}_{out}|$ is proportional to $|\hat{I}|$, independent of frequency (over some range of frequencies)?

9.14. The electromechanical delay line shown in Fig. 9P.14 consists of a thin elastic rod terminated at both ends by capacitor plates that are massless and very thin. The elastic rod

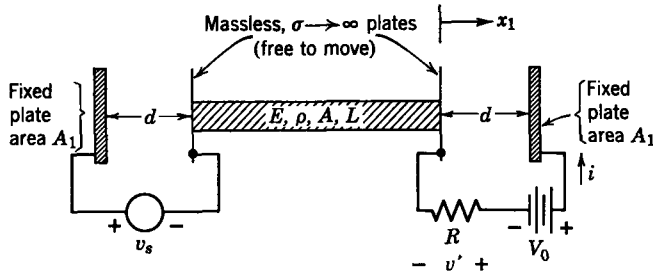


Fig. 9P.14

has length L in the absence of electrical excitations ($v_s = 0$ and $V_0 = 0$). Assume that $\delta(0) \ll d$ and that $\delta(-L) \ll d$.

- (a) If $v_s = V_0$ and the battery at the other capacitor is connected as shown, find the stress everywhere in the rod.
- (b) Suppose that $v_s = V_0 + v_o(t)$, where $v_o(t)$ is a short pulse. How long will it be before the signal is detected as a pulse in the output current i ? Compute a numerical value for the delay, assuming that the rod is steel with length 1 m.

- (c) What must the value of the resistance R be, so that if $v_s = V_0 + v_o(t)$, where v_o is again a short pulse, no pulse will travel back in the $-x_1$ -direction after encountering the capacitor plate at $x_1 = 0$? You may wish to do this by assuming that $v_o(t)$ is a sinusoid and requiring that there be no reflected wave in the rod for all frequencies of excitation. Assume that

$$v' \gg \frac{\epsilon_0 A_1 R}{d} \frac{dv'}{dt} \quad \text{or} \quad \frac{\omega \epsilon_0 A_1 R}{d} \ll 1,$$

where v' is the voltage across R .

9.15. This problem is intended to help you apply the techniques presented in Section 9.1 by considering an analogous situation, namely, the *torsional vibrations* of a thin, cylindrical elastic rod.

- (a) A static experiment is performed on the rod as follows. First, imagine that a line has been painted along one radius at every cross section of the rod when it is unstressed (see Fig. 9P.15). Now suppose that a *constant*, twisting torque τ is applied to the rod. If we single out a small length Δz of the rod (Fig. 9P.15b), we

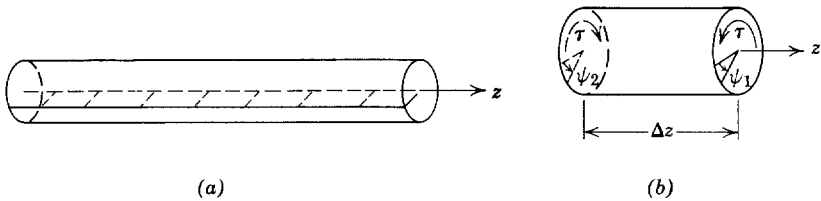


Fig. 9P.15

find that the *difference* between the angular deflections ψ_1 and ψ_2 of the painted line on the two faces is a constant β times the product of the applied torque and the length Δz . Find the relation between the torque τ and the deflection angle ψ in the limit as $\Delta z \rightarrow 0$.

- (b) We wish to find the dynamic equations for the rod. Assume that the twisting torque τ is now a function of z and t , $\tau = \tau(z, t)$. Also, $\psi = \psi(z, t)$. The rod has a moment of inertia J (per unit length) about the z -axis. Write an equation of motion for a length Δz of the rod. Then take the limit of this expression as Δz approaches zero.
- (d) Find a single partial differential equation for $\psi(z, t)$.
- 9.16.** The elastic bar shown in Fig. 9P.16a extends to infinity in the z -direction and has free surfaces at $y = 0$ and $y = a$. Shearing stresses T_{zx} are applied uniformly over the surface at $x = 0$ to set the bar into a mode of vibration, where the material is displaced from equilibrium by the amount $\delta_z(x, t)\mathbf{i}_z$. We wish to find a differential equation of motion for δ_z .
- (a) Write the differential force equation in the z -direction by using the differential slice of material shown in Fig. 9P.16b and the stress function $T_{zx}(x, t)$ N/m².
- (b) Define a shear strain e_{zx} and relate it to the displacement function δ_z . Your strain function should be defined such that we would expect $T_{zx} = 2Ge_{zx}$ (G -N/m²), where G is a constant property of the material. Figure 9P.16c provides a starting point in the derivation.
- (c) Combine the results of (a) and (b) to obtain a single differential equation for δ_z . What is the propagational velocity of disturbances in the x -direction?

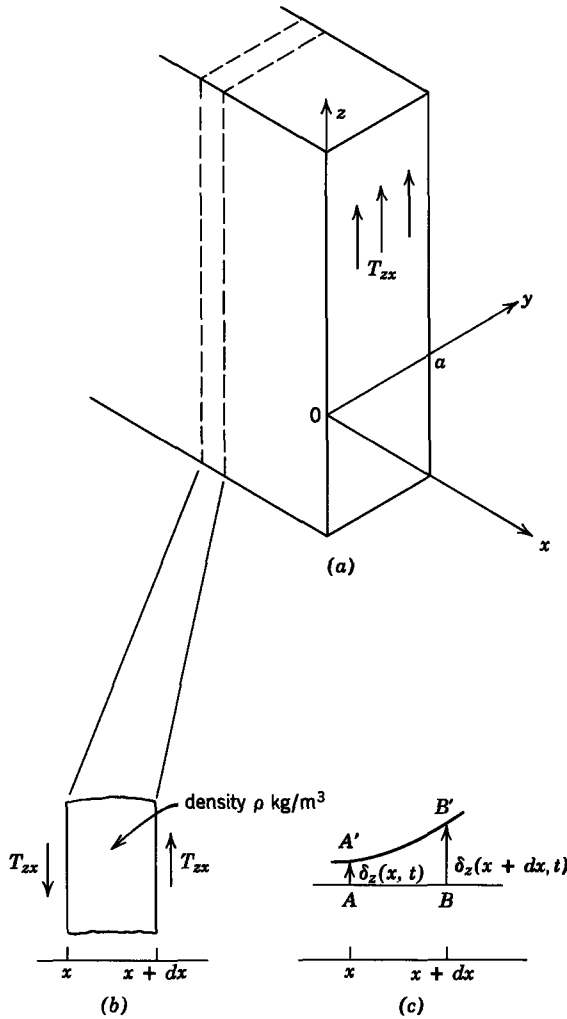


Fig. 9P.16

9.17. In general, an inviscid fluid differs grossly in its dynamics from an elastic solid. If, however, only normal stresses are involved and particle displacements are small, the dynamics of fluids and solids become similar. You are familiar with the derivation of the equation of motion for material in a thin elastic rod. The following derivation is somewhat similar. A tube filled with an initially stationary gas is shown in Fig. 9P.17a. We wish to derive an equation for the particle velocity $v(x, t)$ within the tube. To do this write an equation expressing conservation of mass for a slice of the material, as shown in Fig. 9P.17b. The fluid pressure $p(x, t)$ is a simple form of the stress tensor $T_{ij} = -\delta_{ij}p(x, t)$. We can write a differential equation to express conservation of momentum ($f = ma$) by using the slice of material shown in Fig. 9P.17c.

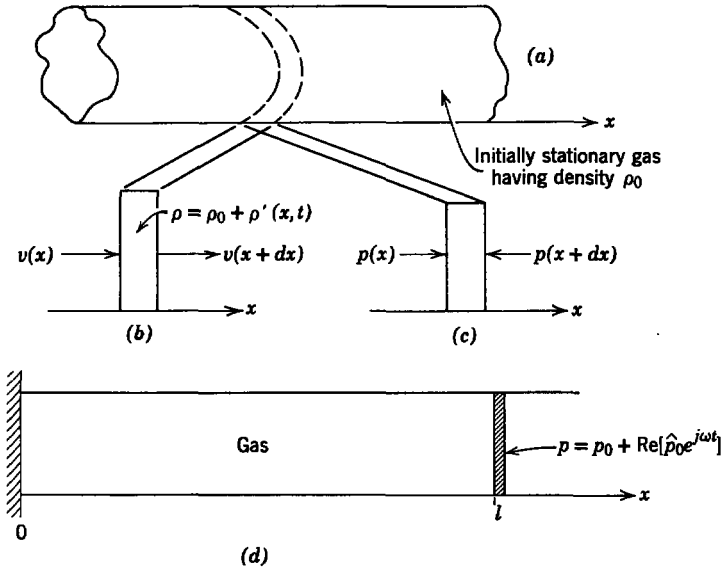


Fig. 9P.17

(a) Show that conservation of mass is expressed by

$$\rho_0 \frac{\partial v}{\partial x} = - \frac{\partial \rho'}{\partial t},$$

where we ignore products of perturbation quantities and $\rho = \rho_0 + \rho'(x, t)$; ρ_0 is the density of the gas when it is stationary.

(b) Show that conservation of momentum is expressed by

$$\rho_0 \frac{\partial v}{\partial t} = - \frac{\partial p'}{\partial x},$$

where $p = p_0 + p'(x, t)$ and p_0 is the pressure when the gas is stationary.

(c) Physical measurements show that the pressure and density are related by a given function $p = p(\rho)$ or

$$p = p_0(\rho_0) + \left(\frac{\partial p}{\partial \rho} \right)_{\rho_0} \rho'.$$

Use the results of (a) and (b) to find one equation for the velocity.

(d) Figure 9P.17d shows the tube driven at $x = l$ by a piston and terminated at $x = 0$ in a rigid wall. What is $v(x, t)$?

9.18. Figure 9P.18 shows the distribution of velocity $\partial \delta / \partial t$ at $t = 0$ in a thin elastic rod of infinite length. Assume that the rod is characterized by the constants E , ρ , and A , where E is Young's modulus, ρ is the density and A is the cross-sectional area of the rod. Given that $T(x, 0) = 0$, make a plot of $T(x, t)$ in the x - t plane. This plot should be similar to that shown in Fig. 9.1.8.

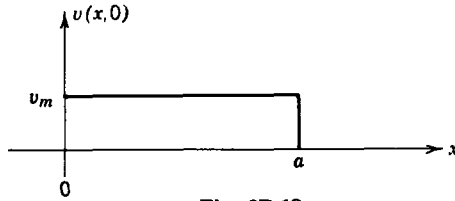


Fig. 9P.18

9.19. Figure 9P.19 shows the distribution of stress $T(x, 0) = E(\partial\delta/\partial x)$ at $t = 0$ in a thin elastic rod of infinite length. Assume that the rod is characterized by the constants E , ρ , and A , where E is Young's modulus, ρ is the density, and A is the cross-sectional area of the rod. Given that $v(x, 0) = 0 = \partial\delta/\partial t$, make a plot of $T(x, t)$ and $v(x, t)$ in the $x-t$ plane. This plot should be similar to that shown in Fig. 9.1.8.

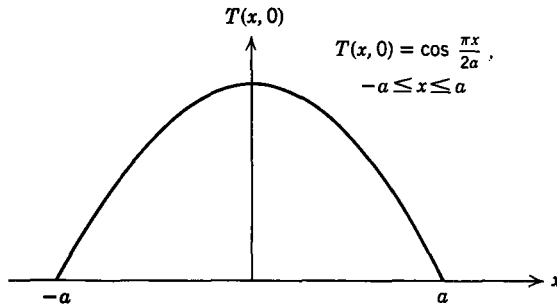


Fig. 9P.19

9.20. A long thin rod supports longitudinal motions $\delta(x, t)$.

- Consider first the case in which the material initially has the velocity distribution $(\partial\delta/\partial t)(x, 0)$ shown in Fig. 9P.20a and initially $(\partial\delta/\partial x)(x, 0) = 0$. The rod has a free end at $x = 0$ but extends to infinity in the positive x -direction. Sketch the resulting velocity $v(x, t)$ in the $x-t$ plane.
- Now, in addition to the initial velocity of Fig. 9P.20a, the end of the rod at $x = 0$ is driven by a force that constrains the stress $T(0, t)$ as shown in Fig. 9P.20b. Sketch the resulting velocity $v(x, t)$ in the $x-t$ plane.

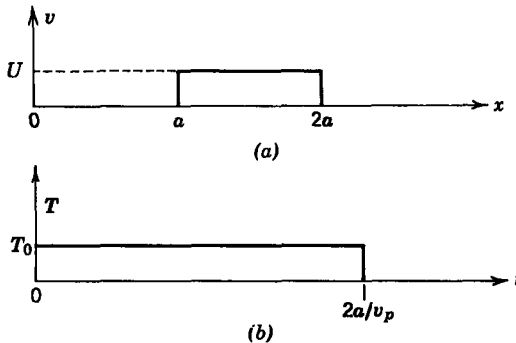


Fig. 9P.20

9.21. An electromechanical device that transduces a pulse of current $I(t)$ into a delayed stress pulse to rupture a diaphragm is shown in Fig. 9P.21. The current $I(t)$ passes through parallel, highly conducting plates shorted by a movable block with conductivity σ . The resulting motion initiates a stress at the left end of the rod, which propagates to the right end to provide the rupturing stress. The current has the form

$$I(t) = \begin{cases} \frac{I_0}{2} \left(1 - \cos \frac{2\pi t}{\tau} \right), & 0 < t < \tau, \\ 0, & t < 0, t > \tau. \end{cases}$$

The time required for the pulse to traverse the length of the rod is long compared with τ .

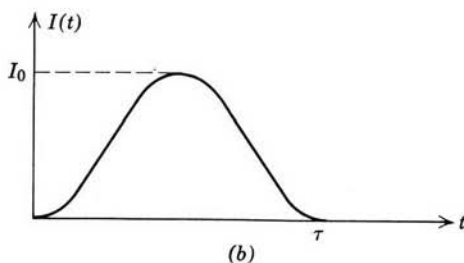
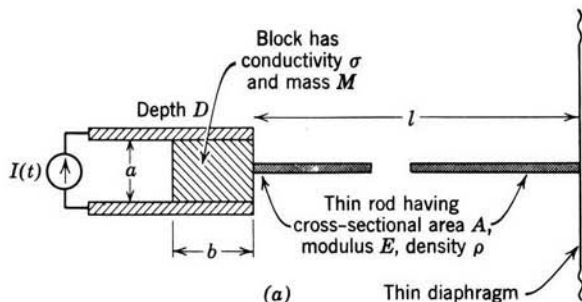


Fig. 9P.21

- Model the electromechanics by assuming that the current is returned on the inside surface of the block and write the boundary condition imposed on the left end of the rod by the magnetic force.
- For what values of τ (in terms of b , σ , etc.) would the surface current model be appropriate?
- Under what conditions can the mass M of the block be ignored in the boundary condition of (a)? Under this condition what is the current I_0 required to produce the peak stress T_r at the right end of the rod? (For this calculation assume that the right end of the rod is fixed in displacement.)

9.22. You are given the following pair of nonlinear differential equations:

$$\frac{\partial W}{\partial t} + W \frac{\partial W}{\partial x} + \frac{\partial U}{\partial x} + \frac{K}{U^3} \frac{\partial U}{\partial x} = 0, \quad (1)$$

$$\frac{\partial U}{\partial t} + \frac{\partial(UW)}{\partial x} = 0, \quad (2)$$

where K is a positive constant. In an equilibrium condition W is zero and U is a positive constant given by $U = C$.

- Linearize (1) and (2), using the given equilibrium conditions.
- By use of the equations of part (a), what is $W(x, t)$ if at $x = 0$, $W = 0$, at $x = -L$, $W = W_0 \cos \omega t$?

9.23. Two identical elastic membranes of mass σ_m per unit area and equilibrium tension S are joined together at $x = 0$ (Fig. 9P.23). Also attached to the membranes at $x = 0$ is a bar of mass M per unit width (kg/m). Both membranes are tied to rigid walls at their other ends ($x = \pm L$).

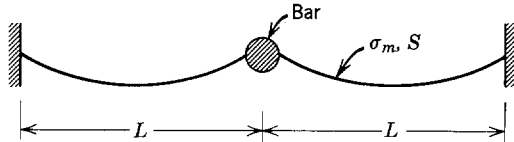


Fig. 9P.23

- Compute the natural frequencies of the system.
 - Explain the effect of the mass/unit length M on the natural frequencies. In particular, give a physical reason for what happens when $M = 0$ and $M \rightarrow \infty$.
- 9.24.** Figure 9P.24 shows an elastic membrane fixed at $x = L$ and $x = -L$ and coupled to a pair of capacitor plates at $x = 0$.

- Find the natural frequencies of the system.
- What is the effect on the natural frequencies of raising the voltage V_0 ?

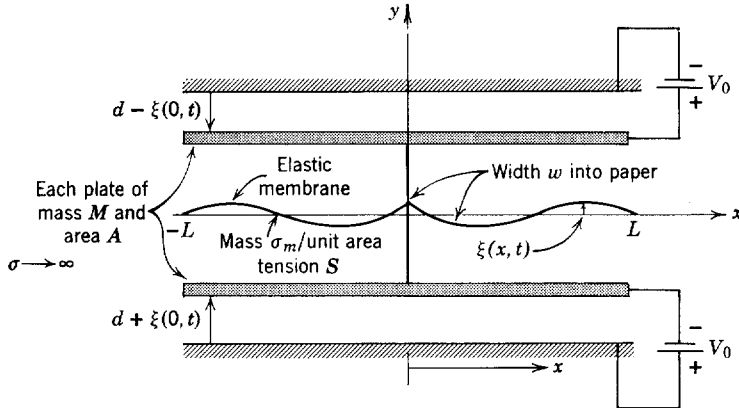


Fig. 9P.24

9.25. A wire is pinned at $x = 0$, where a potentiometer is attached. With the help of an amplifier G , this potentiometer produces a current i_1 proportional to the slope.

$$i_1 = G \frac{\partial \xi}{\partial x}(0, t), \quad G = \text{constant}.$$

The current i_1 is used to drive transducers, which in turn motivate the end of the spring at $x = -l$. The two transducers, which are alike, are connected so that in the absence of i_1

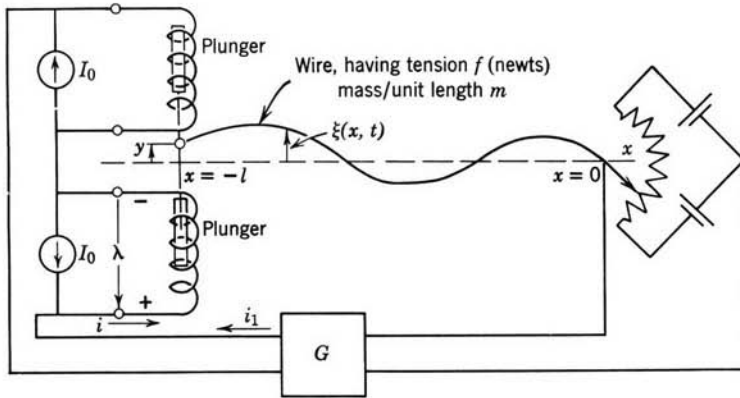


Fig. 9P.25

there is no net force on the wire. The terminal relation for the lower transducer is

$$\lambda = \left(\frac{L_0}{a} y \right) i,$$

where $y = \xi(-l, t)$ as shown in Fig. 9P.25.

- One boundary condition is $\xi(0, t) = 0$. What is the other condition? Ignore the mass of the plungers.
- Find a transcendental expression for the eigenfrequencies (natural frequencies) of the system. Make a plot that shows the graphical solution of this equation.
- What are the natural frequencies when $G = 0$?
- What is the effect of raising the gain G on the first and second nonzero eigenfrequencies?
- Is it possible that this system can be unstable? If so, for what values of G ?

9.26. A magnetic transducer is used to excite a thin elastic membrane of length L , as shown in Fig. 9P.26. The mass M is attached to the membrane, which in turn is fixed to a rigid support at $x = 0$. The driving current $I(t)$ is much smaller than I_0 and is given by

$$I(t) = \text{Re} [\hat{I} \exp(j\omega t)].$$

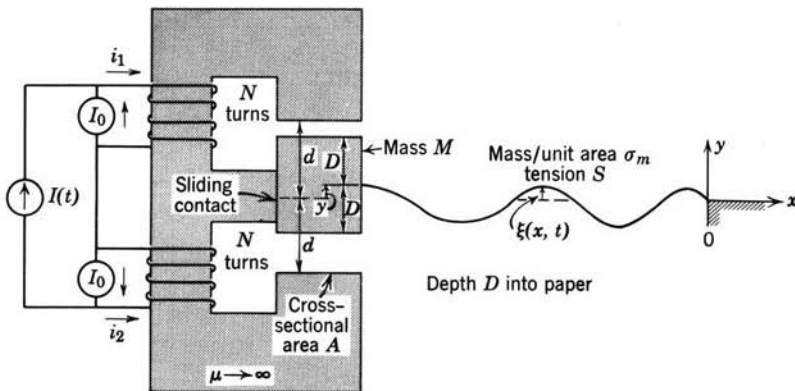


Fig. 9P.26

Work under the assumption that the displacements of the mass from its equilibrium position are small:

- Find an expression for the position of the mass, $y(t)$. You may assume that when there is no applied current the mass is centered between the pole faces.
- Find the resonance frequencies of the system. Your solution may be represented graphically (e.g., see Fig. 9.2.11).

9.27. A string with tension f and a mass per unit length m is fixed at $x = \pm l$, as shown in Fig. 9P.27. At $x = 0$ it is subject to the force $F(t)$ shown in Fig. 9P.27a.

- Write the boundary condition at $x = 0$ which relates $F(t)$ to ξ . For this purpose divide the function $\xi(x, t)$ into two functions valid to the left and to the right of $x = 0$.
- The displacement $\xi(x, t)$ can be divided into an odd and an even function of x . Show that the odd function $\xi(x, t) = -\xi(-x, t)$ is not excited by the driving force.
- We now confine ourselves to displacements that are even functions of x , $\xi(x, t) = \xi(-x, t)$. For $t < 0$ the string assumes a static shape with the force $F(t) = F_0$, where F_0 is a given constant (see Fig. 9P.27b). Use the equation of motion to find $\xi(x)$, $x > 0$, for $t < 0$.
- When $t = 0$, the force $F(t)$ becomes a cosinusoid, as shown in the figure. The string is initially static and has the dependence on x found in part (c). Find the displacement $\xi(x, t)$, $x > 0$, $t > 0$. (Reference. Section 9.2.1.)

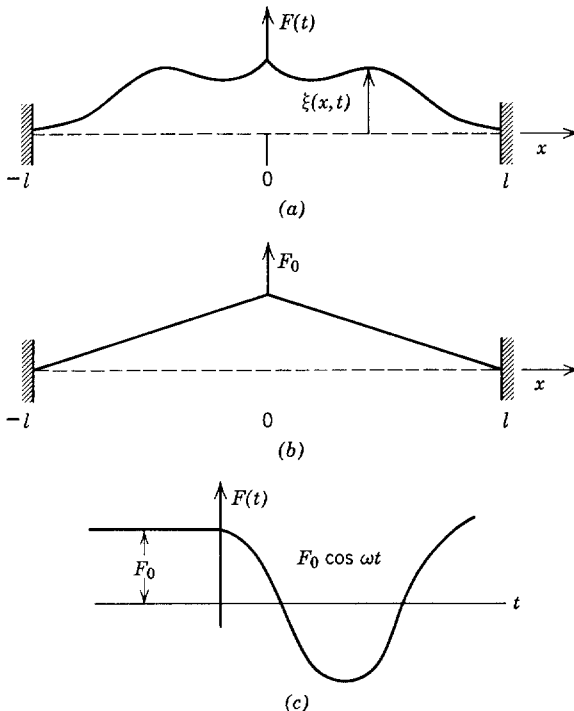


Fig. 9P.27