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Electromechanical Dynamics

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Chapter 2

LUMPED ELECTROMECHANICAL ELEMENTS

2.0 INTRODUCTION

The purpose of this chapter is to present the techniques of making mathematical models (writing differential equations) for lumped-parameter electromechanical systems. In the context used here lumped-parameter systems are defined as follows: the electromagnetic fields are quasi-static and electrical terminal properties can be described as functions of a finite number of electrical variables. Also, the mechanical effects can be described by a finite number of mechanical variables. Thus the general feature of lumped-parameter electromechanical systems is that field equations can be integrated throughout space to obtain ordinary differential equations.

Electrical parts of the systems are treated by circuit theory generalized to include the effects of electromechanical coupling; the mechanical parts of the systems are treated by the techniques of rigid body mechanics with electromechanical forces included.

The approach followed here is best illustrated by considering the block diagram in Fig. 2.0.1 in which an electromechanical system is separated for analytical purposes into a purely electrical part, a purely mechanical part, and a coupling part. The equations that describe the electrical part of the system are based on Kirchhoff's laws; the equations for the mechanical part of the system are obtained from Newton's laws and the continuity of space. Both sets of equations contain electromechanical coupling terms that arise from the interconnection of the coupling system.

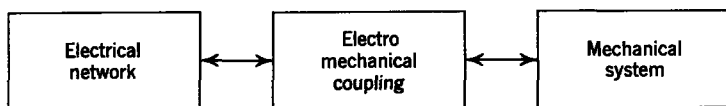


Fig. 2.0.1 An electromechanical system.

In what follows we review concepts of circuit theory and derive lumped parameters in a general way to include electromechanical coupling terms. We then review the concepts of rigid-body mechanics. Electromechanical coupling is discussed in Chapter 3.

2.1 CIRCUIT THEORY

The mathematical description of a circuit essentially involves two steps. First, we must be able to describe mathematically the physical properties of each element in the circuit in order to produce expressions for the terminal properties of the elements. Second, we must combine the equations for the elements in a manner prescribed by the interconnections of the elements. This step is performed by using Kirchhoff's laws and the topology of the circuit.* Thus we need only to generalize the description of circuit elements to include the effects of electromechanical interactions.

Conventional circuit theory is the special case of stationary systems in which quasi-static electromagnetic field theory applies. All the concepts of circuit theory can be derived from field theory†; for example, Kirchhoff's current law is derived from the conservation of charge. When we postulate a node that is an interconnection of wires at which no charge can accumulate, the conservation of charge [see (1.1.22) or (1.1.26) with $\rho_f = 0$] becomes

$$\oint_S \mathbf{J}_f \cdot \mathbf{n} \, da = 0, \quad (2.1.1)$$

where the surface S encloses the node. Because current is restricted to the wires, (2.1.1) yields Kirchhoff's current law

$$\sum_k i_k = 0, \quad (2.1.2)$$

where i_k is the current flowing away from the node on the k th wire.

Kirchhoff's voltage law is obtained by recognizing that a voltage is uniquely defined only in a region in which the time rate of change of magnetic flux density is negligible. Thus either (1.1.23) or (1.1.24) becomes

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0. \quad (2.1.3)$$

This leads to the Kirchhoff voltage equation which requires that the sum of the voltage drops around a closed loop (contour C) be zero,

$$\sum_k v_k = 0, \quad (2.1.4)$$

* E. A. Guillemin, *Introductory Circuit Theory*, Wiley, New York, 1953, Chapters 2 and 4.

† Electrical Engineering Staff, M.I.T., *Electric Circuits*, Technology Press and Wiley, New York, 1943, Chapters 1 and 2.

where v_k is the voltage drop across the k th element in the loop taken in the direction of summation.*

In conventional circuit theory there are three basic types of passive elements: (a) resistances that dissipate electric energy as heat; (b) inductances that store magnetic energy; and (c) capacitances that store electric energy. It is a fact of life that electromechanical coupling of practical significance occurs in elements with appreciable electric or magnetic energy storage. Consequently, we shall consider electromechanical effects in circuit elements that are generalizations of the inductances and capacitances of circuit theory. To be sure, our systems have resistances, but they are treated as purely electrical circuit elements and considered as external to the coupling network.

We proceed now to generalize the concepts of inductance and capacitance to include electromechanical effects. As stated before, we wish to obtain terminal equations suitable for inclusion in a Kirchhoff-law description of a circuit.

2.1.1 Generalized Inductance

From a field point of view an inductor is a quasi-static magnetic field system, as defined in Section 1.1.1*a*. Thus we start with the field description of a quasi-static magnetic field system and derive the terminal characteristics when parts of the system are in motion.

First it is essential to recognize that in an ideal, lossless magnetic field system there is a perfectly conducting path between the two terminals of each terminal pair, as illustrated schematically in Fig. 2.1.1. We assume that the

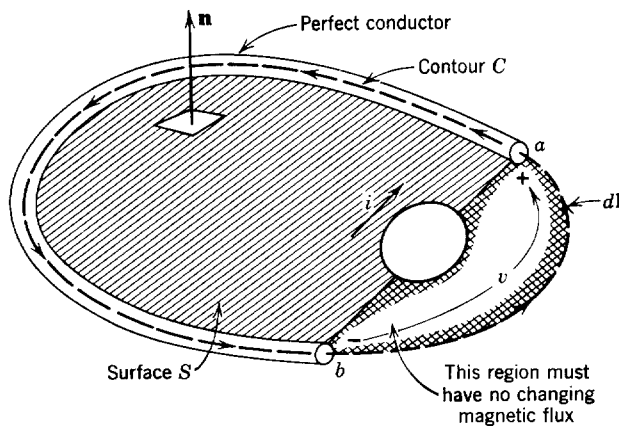


Fig. 2.1.1 Configuration for defining terminal voltage.

* For a discussion of the definition and use of the concept of voltage see Section B.1.4.

terminal pair is excited by the current source i and that the terminal pair is in a region of space in which the time rate of change of magnetic flux density is negligible. This restriction is necessary if we are to be able to describe a terminal voltage unambiguously. The perfect conductor that connects the two terminals is often wound into a coil and the coil may encircle an iron core. The drawing in Fig. 2.1.1 is simplified to illustrate the principles involved.

We must include the possibility that the perfect conductor in Fig. 2.1.1 is moving. We define a contour C that passes through and is fixed to the perfect conductor. That portion of the contour which goes from b to a outside the perfect conductor is fixed and in a region of negligible magnetic flux density. The *terminal voltage* v (see Fig. 2.1.1) is defined in the usual way* as

$$v = - \int_b^a \mathbf{E} \cdot d\mathbf{l}; \quad (2.1.5)$$

it is understood that this line integral is evaluated along the path from b to a that is external to the perfect conductor. We now consider the line integral

$$\oint_C \mathbf{E}' \cdot d\mathbf{l}$$

around the contour of Fig. 2.1.1. The electric field intensity \mathbf{E}' is the field that an observer will measure when he is fixed with respect to the contour. The contour is fixed to the perfect conductor, and, by definition, a perfect conductor can support no electric field.† Consequently, we reach the conclusion that

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = \int_b^a \mathbf{E} \cdot d\mathbf{l} = -v. \quad (2.1.6)$$

Thus Faraday's law, (1.1.23) of Table 1.2, yields the terminal voltage

$$v = \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da, \quad (2.1.7)$$

where the surface S is enclosed by the contour C in Fig. 2.1.1 and the positive direction of the normal vector \mathbf{n} is defined by the usual right-hand rule, as shown.

Equation 2.1.7 indicates why the external path from b to a in Fig. 2.1.1 must be in a region of negligible time rate of change of magnetic flux density. If it is not, the terminal voltage will depend on the location of the external

* See Section B.1.4.

† A more complete discussion of conductors in motion is given in Chapter 6.

contour and will not be defined unambiguously. For convenience we define the *flux linkage* λ of the circuit as

$$\lambda = \int_S \mathbf{B} \cdot \mathbf{n} \, da \quad (2.1.8)$$

and rewrite (2.1.7) as

$$v = \frac{d\lambda}{dt}. \quad (2.1.9)$$

In a quasi-static magnetic-field system the magnetic flux density is determined by (1.1.20) to (1.1.22) of Table 1.2 and a constitutive law,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n} \, da, \quad (1.1.20)$$

$$\oint_S \mathbf{B} \cdot \mathbf{n} \, da = 0, \quad (1.1.21)$$

$$\oint_S \mathbf{J}_f \cdot \mathbf{n} \, da = 0, \quad (1.1.22)$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \quad (1.1.4)$$

(The differential forms of these equations can also be employed.) In the solution of any problem the usual procedure is to use (1.1.22) first to relate the terminal current to current density in the system and then (1.1.20), (1.1.21), and (1.1.4) to solve for the flux density \mathbf{B} . The resulting flux density is a function of terminal current, material properties (1.1.4), and system geometry. The use of this result in (2.1.8) shows that the flux linkage λ is also a function only of terminal current, material properties, and system geometry.

We are interested in evaluating terminal voltage by using (2.1.9); thus we are interested in time variations of flux linkage λ . If we assume that the system geometry is fixed, except for one movable part whose position can be described instantaneously by a displacement x with respect to a fixed reference, and we further assume that \mathbf{M} is a function of field quantities alone (and therefore a function of current), we can write

$$\lambda = \lambda(i, x). \quad (2.1.10)$$

In this expression we have indicated explicit functional dependence only on those variables (i and x) that may be functions of time.

We can now use (2.1.10) in (2.1.9) and expand the time derivative to obtain

$$v = \frac{d\lambda}{dt} = \frac{\partial \lambda}{\partial i} \frac{di}{dt} + \frac{\partial \lambda}{\partial x} \frac{dx}{dt}. \quad (2.1.11)$$

This expression illustrates some general terminal properties of magnetic

field systems. We note that the first term on the right of (2.1.11) is proportional to di/dt and is the result of changing current. This term can exist when the system is mechanically stationary and is often referred to as a *transformer voltage*. The second term on the right of (2.1.11) is proportional to dx/dt , which is a mechanical speed. This term exists only when there is relative motion in the system and is conventionally referred to as a *speed voltage*. No matter how many terminal pairs or mechanical displacements a system may have, the voltage at each terminal pair will have terms of the two types contained in (2.1.11).

If we now restrict our system (with one electrical terminal pair and one mechanical displacement) to materials whose magnetization densities are linear with field quantities, we have an *electrically linear* system whose flux linkage can be expressed in terms of an inductance L as

$$\lambda = L(x)i. \quad (2.1.12)$$

This system is electrically linear because the flux linkage is a linear function of current. The variation of flux linkage with geometry, as indicated in general in (2.1.10), is included in (2.1.12) in the function $L(x)$. When the flux linkage is written in the form of (2.1.12), the terminal voltage becomes

$$v = L(x) \frac{di}{dt} + i \frac{dL}{dx} \frac{dx}{dt}. \quad (2.1.13)$$

Once again the first term on the right is the transformer voltage and the second term is the speed voltage.

In the special case of fixed geometry (x constant) the second term on the right of (2.1.13) goes to zero and we obtain

$$v = L \frac{di}{dt}, \quad (2.1.14)$$

which is the terminal relation of an inductance that is conventional in linear circuit theory.

Electromechanical systems often have more than one electrical terminal pair and more than one mechanical displacement. For such a situation the process described is still valid. To illustrate this generalization assume a quasi-static magnetic field system with N electrical terminal pairs and M mechanical variables that are functions of time. There are N electrical currents,

$$i_1, i_2, \dots, i_N,$$

and M mechanical displacements,

$$x_1, x_2, \dots, x_M.$$

Because this is a quasi-static magnetic field system, there is a perfectly conducting path between the two terminals of each terminal pair, as illustrated in Fig. 2.1.1. Thus the voltage for any terminal pair is determined by using the contour for that terminal pair with (2.1.6). Then the flux linkage for any terminal pair (say the k th) is given by (2.1.8):

$$\lambda_k = \int_{S_k} \mathbf{B} \cdot \mathbf{n} \, da, \quad (2.1.15)$$

where S_k is the surface enclosed by the contour used with (2.1.6) to evaluate voltage v_k at the k th terminal pair. The voltage v_k is then given by (2.1.7) as

$$v_k = \frac{d\lambda_k}{dt}. \quad (2.1.16)$$

The fields in this more general situation are again described by (1.1.20) to (1.1.22) and (1.1.4). Consequently, the generalization of (2.1.10) is

$$\begin{aligned} \lambda_k &= \lambda_k(i_1, i_2, \dots, i_N; x_1, x_2, \dots, x_M), \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.17)$$

We can now write the generalization of (2.1.11) by using (2.1.17) in (2.1.16) to obtain

$$\begin{aligned} v_k &= \sum_{j=1}^N \frac{\partial \lambda_k}{\partial i_j} \frac{di_j}{dt} + \sum_{j=1}^M \frac{\partial \lambda_k}{\partial x_j} \frac{dx_j}{dt}, \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.18)$$

Once again the terms in the first summation are referred to as transformer voltages and the terms in the second summation are referred to as speed voltages.

The preceding development has indicated the formalism by which we obtain lumped-parameter descriptions of quasi-static magnetic field systems. We have treated ideal lossless systems. In real systems losses are primarily resistive losses in wires and losses in magnetic materials.* Even though they may be quite important in system design and operation (efficiency, thermal limitations, etc.), they usually have little effect on the electromechanical interactions. Consequently, the effects of losses are accounted for by electrical resistances external to the lossless electromechanical coupling system.

* Losses in magnetic materials result from hysteresis and eddy currents. For a discussion of these effects and their mathematical models see Electrical Engineering Staff, M.I.T., *Magnetic Circuits and Transformers*, Technology Press and Wiley, New York, 1943, Chapters 5, 6, and 13. Eddy currents are discussed in Chapter 7 of this book.

Example 2.1.1. As an example of the calculation of lumped parameters, consider the magnetic field system of Fig. 2.1.2. It consists of a fixed structure made of highly permeable magnetic material with an excitation winding of N turns. A movable plunger, also made of highly permeable magnetic material, is constrained by a nonmagnetic sleeve to move in the x -direction. This is the basic configuration used for tripping circuit breakers, operating valves, and other applications in which a relatively large force is applied to a member that moves a relatively small distance.*

We wish to calculate the flux linkage λ at the electrical terminal pair (as a function of current i and displacement x) and the terminal voltage v for specified time variation of i and x .

To make the analysis of the system of Fig. 2.1.2 more tractable but still quite accurate it is conventional to make the following assumptions:

1. The permeability of the magnetic material is high enough to be assumed infinite.
2. The air-gap lengths g and x are assumed small compared with transverse dimensions $g \ll w$, $x \ll 2w$, so that fringing at the gap edges can be ignored.
3. Leakage flux is assumed negligible; that is, the only appreciable flux passes through the magnetic material except for gaps g and x .

Needed to solve this problem are the quasi-static magnetic field equations (1.1.20) through (1.1.22) and (1.1.4).

We first assume that the terminal current is i . Then by using (1.1.22) we establish that the current at each point along the winding is i . Next, we recognize that the specification of infinitely permeable magnetic material implies that we can write (1.1.4) as

$$\mathbf{B} = \mu \mathbf{H}$$

with $\mu \rightarrow \infty$. Thus with finite flux density \mathbf{B} the field intensity \mathbf{H} is zero inside the magnetic

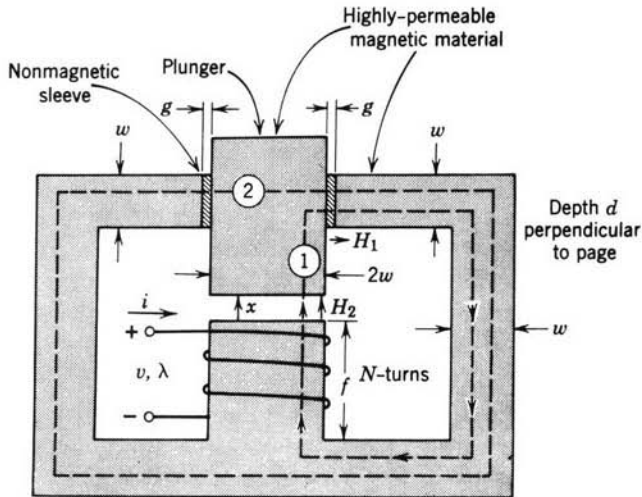


Fig. 2.1.2 A magnetic field system.

* A. E. Knowlton, ed., *Standard Handbook for Electrical Engineers*, 9th ed. McGraw-Hill, New York, 1957, Section 5-39 through 5-52.

material. Thus the only nonzero \mathbf{H} occurs in the air gaps g and x , where $\mathbf{M} = 0$, and (1.1.4) becomes

$$\mathbf{B} = \mu_0 \mathbf{H}.$$

The use of (1.1.20) with contour (2) in Fig. 2.1.2 shows that the field intensities in the two gaps g are equal in magnitude and opposite in direction. This is expected from the symmetry of the system. Denoting the magnitude of the field intensity in the gaps g as H_1 and the field intensity in gap x by H_2 , we can integrate (1.1.20) around contour (1) in Fig. 2.1.2 to obtain

$$H_1 g + H_2 x = Ni, \quad (a)$$

where H_2 is taken positive upward and H_1 is taken positive to the right. We now use (1.1.21) with a surface that encloses the plunger and passes through the gaps to obtain

$$\mu_0 H_1 (2wd) - \mu_0 H_2 (2wd) = 0. \quad (b)$$

We combine (a) and (b) to obtain

$$H_1 = H_2 = \frac{Ni}{g + x}.$$

The flux through the center leg of the core is simply the flux crossing the air gap x and is

$$\phi = \mu_0 H_2 (2wd) = \frac{2wd\mu_0 Ni}{g + x}.$$

In the absence of leakage flux this same flux links the N -turn winding N times; that is, when we evaluate

$$\int_S \mathbf{B} \cdot \mathbf{n} \, da$$

over a surface enclosed by the wire of the N -turn winding, we obtain the flux linkage λ as

$$\lambda = N\phi = \frac{2wd\mu_0 N^2 i}{g + x}. \quad (c)$$

Note that because λ is a linear function of i the system is electrically linear and we can write (c) as

$$\lambda = L(x)i, \quad (d)$$

where

$$L(x) = \frac{2wd\mu_0 N^2}{g + x}. \quad (e)$$

When we assume that the current i and displacement x are specified functions of time, we can use (d) with (2.1.13) to evaluate the terminal voltage as

$$v = \frac{2wd\mu_0 N^2}{g + x} \frac{di}{dt} - \frac{2wd\mu_0 N^2 i}{(g + x)^2} \frac{dx}{dt}.$$

The first term is the transformer voltage that will exist if x is fixed and i is varying. The second term is the speed voltage that will exist if i is constant and x is varying.

Example 2.1.2. As a second example, consider the system in Fig. 2.1.3 which has two electrical terminal pairs and the mechanical displacement is rotational. This system consists

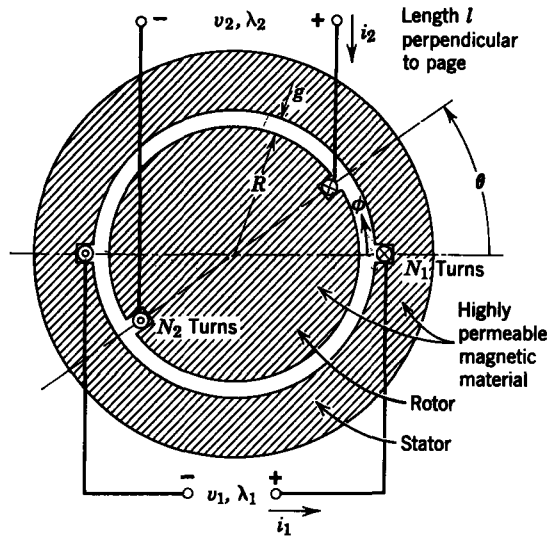


Fig. 2.1.3 Doubly excited magnetic field system.

of a fixed annular section of highly permeable magnetic material that is concentric with a cylindrical piece of the same material of the same axial length. Mounted in axial slots in the material are coils labeled with the numbers of turns and current directions. The angular position of the inner structure (rotor) relative to the outer structure (stator) is indicated by an angle θ which can vary with time. Current is fed to the coil on the rotor through sliding contacts (brushes that make contact with slip rings).

The system in Fig. 2.1.3 represents the basic method of construction of many rotating machines. In our solution we discuss how this configuration is used with some variations to achieve the lumped parameters desired for rotating-machine operation.

We wish to calculate the two flux linkages λ_1 and λ_2 as functions of the currents i_1 and i_2 and the angular displacement θ . The voltages at the two terminal pairs are also to be found, assuming that i_1 , i_2 , and θ are specified functions of time.

Electromechanical systems of the type illustrated in Fig. 2.1.3 are normally constructed with relative dimensions and materials that allow reasonably accurate calculation of lumped parameters when the following assumptions are made:

1. The permeability of the magnetic material is high enough to be assumed infinite.
2. The radial air-gap length g is small enough compared with the radius R and axial length l to allow the neglect of fringing fields at the ends and of radial variation of magnetic field intensity in the air gap.
3. The slots containing the windings are small enough both radially and circumferentially to perturb the fields a negligible amount; that is, the coils are considered to be infinitely thin.

Equations 1.1.20 and 1.1.21 are used to write the radial fields in the air gap in terms of the angular variable ϕ defined in Fig. 2.1.3. If we define the magnetic field as positive when

directed radially outward and consider $0 < \theta < \pi$,

$$H_r = \frac{N_1 i_1 - N_2 i_2}{2g}, \quad \text{for } 0 < \phi < \theta,$$

$$H_r = \frac{N_1 i_1 + N_2 i_2}{2g}, \quad \text{for } \theta < \phi < \pi,$$

$$H_r = -\frac{N_1 i_1 - N_2 i_2}{2g}, \quad \text{for } \pi < \phi < \pi + \theta,$$

$$H_r = -\frac{N_1 i_1 + N_2 i_2}{2g}, \quad \text{for } \pi + \theta < \phi < 2\pi.$$

The flux linkages with the two windings can be found from the integrals

$$\lambda_1 = \int_0^\pi N_1 \mu_0 H_r IR d\phi,$$

$$\lambda_2 = \int_\theta^{\pi+\theta} N_2 \mu_0 H_r IR d\phi.$$

Evaluation of these integrals yields

$$\lambda_1 = L_1 i_1 + L_m i_2,$$

$$\lambda_2 = L_m i_1 + L_2 i_2,$$

where

$$L_1 = N_1^2 L_0, \quad L_2 = N_2^2 L_0,$$

$$L_m = L_0 N_1 N_2 \left(1 - \frac{2\theta}{\pi}\right), \quad \text{for } 0 < \theta < \pi,$$

$$L_0 = \frac{\mu_0 IR \pi}{2g}.$$

Similar arguments show that for $-\pi < \theta < 0$ the terminal relations have the same form except that

$$L_m = L_0 N_1 N_2 \left(1 + \frac{2\theta}{\pi}\right)$$

Note that only the mutual inductance L_m is a function of angular displacement θ because the geometry seen by each coil individually does not change with θ ; thus the self-inductances are constants. The mutual inductance is sketched as a function of θ in Fig. 2.1.4.

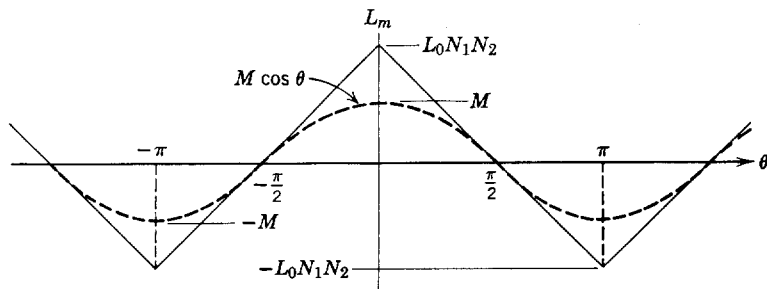


Fig. 2.1.4 Mutual inductance L_m as a function of θ .

In the design of rotating machines, especially for operation on alternating currents, it is desirable to have a system similar to that in Fig. 2.1.3 but to modify it in such a way that the mutual inductance varies cosinusoidally with θ ($L_m = M \cos \theta$). This is accomplished by putting additional slots and windings at different positions around the periphery of both members. By using a proper distribution of slots and numbers of turns the dependence of L_m can be made the cosinusoidal function shown by the dashed curve in Fig. 2.1.4. In many later examples we assume that this design process has been followed.

When the two currents i_1 and i_2 and the angular position θ are functions of time and the mutual inductance is expressed as $L_m = M \cos \theta$, we can write the terminal voltages as

$$v_1 = \frac{d\lambda_1}{dt} = L_1 \frac{di_1}{dt} + M \cos \theta \frac{di_2}{dt} - i_2 M \sin \theta \frac{d\theta}{dt},$$

$$v_2 = \frac{d\lambda_2}{dt} = L_2 \frac{di_2}{dt} + M \cos \theta \frac{di_1}{dt} - i_1 M \sin \theta \frac{d\theta}{dt}.$$

Note that the first term in each expression is a derivative with a constant coefficient, whereas the last two terms are derivatives with time-varying coefficients.

So far we have described lumped-parameter magnetic field systems by expressing the flux linkages as functions of the currents and displacements. Although this is a natural form for deriving lumped parameters, we shall find it convenient later to express lumped parameters in different forms. For example, we often use the lumped parameters in a form that expresses the current as a function of flux linkages and displacements. Assuming that the flux linkages are known as functions of currents and displacements, we are merely required to solve a set of simultaneous algebraic equations.

Consider the situation of an electrically linear system with one electrical terminal pair and one displacement. The flux linkage for this system has been described in (2.1.12) and it is a simple matter to solve this expression for i to obtain

$$i = \frac{\lambda}{L(x)}. \quad (2.1.19)$$

For an electrically linear system with two electrical terminal pairs we can write the expressions for the two flux linkages as

$$\lambda_1 = L_1 i_1 + L_m i_2, \quad (2.1.20)$$

$$\lambda_2 = L_m i_1 + L_2 i_2. \quad (2.1.21)$$

It is assumed that the three inductances are functions of the mechanical displacements. Solutions of these equations for i_1 and i_2 are

$$i_1 = \frac{L_2}{L_1 L_2 - L_m^2} \lambda_1 - \frac{L_m}{L_1 L_2 - L_m^2} \lambda_2, \quad (2.1.22)$$

$$i_2 = -\frac{L_m}{L_1 L_2 - L_m^2} \lambda_1 + \frac{L_1}{L_1 L_2 - L_m^2} \lambda_2. \quad (2.1.23)$$

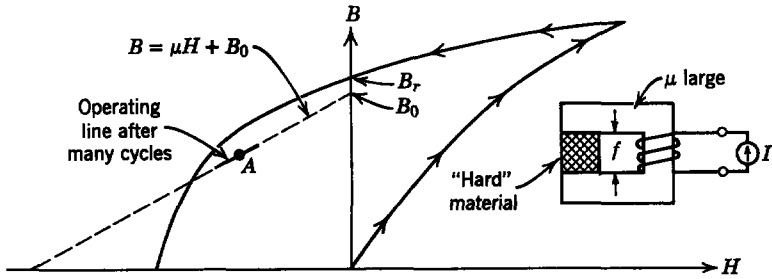


Fig. 2.1.5 Hard magnetic material magnetized and then demagnetized by the current I .

For the more general (and possibly not electrically linear) case for which flux linkages are expressed by (2.1.17) we can, at least in theory, solve these N simultaneous equations to obtain the general functional form

$$\begin{aligned} i_k &= i_k(\lambda_1, \lambda_2, \dots, \lambda_N; x_1, x_2, \dots, x_M), \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.24)$$

Although we have generalized the concept of inductance to the point at which we can describe an electrically nonlinear system, the full, nonlinear expressions are rarely used in the analysis of electromechanical devices. This is so because devices that involve mechanical motion normally have air gaps (where the material is electrically linear) and the fields in the air gaps predominate (see the last two examples). It is worthwhile, however, to understand the origins of lumped parameters and how to describe nonlinear systems because there are some devices in which nonlinearities predominate.

In our examples of magnetic field systems we have considered only a "soft" magnetic material with a flux density (or more accurately a magnetization density \mathbf{M}) that is ideally a linear function of a magnetic field intensity. A different type of magnetic material often used in electromechanical systems is the "hard" or permanent magnet material. In these materials the flux density is, in general, not given by $\mathbf{B} = \mu\mathbf{H}$. Figure 2.1.5 shows the type of curve that would result if a hard magnetic material were subjected to a field intensity H by means of an extremely large current applied as shown.* When the current (i.e., H) is removed, there is a residual flux density B_r . Then, if the sample is removed from the magnetic circuit and subjected to other magnetic fields (of limited strength), the B - H curve settles down to operate about some point such as A in Fig. 2.1.5. To a good approximation we can often model this relation by the straight line shown in Fig. 2.1.5.

$$B = \mu H + B_0. \quad (2.1.25)$$

* See, for example, Electrical Engineering Staff, M.I.T., *op. cit.*, Chapter 4.

The calculation of terminal relations is now the same as described, except that in the permanent magnet (2.1.25) is used rather than (1.1.8).

Example 2.1.3. The system shown in Fig. 2.1.2 is excited electrically by removing the N turns and placing a permanent magnet of length f in the magnetic circuit. Then the analysis of Example 2.1.1 is altered by the integration of (1.1.20) around the contour 1. If, in the section of length f , the material is characterized by (2.1.25), (a) of Example 2.1.1 is replaced by

$$H_1 g + H_2 x + \frac{fB}{\mu} = \frac{fB_0}{\mu}, \quad (\text{a})$$

where B is the flux density in the magnet. Equation 1.1.21, however, again shows that B is the same in the magnet as it is in each of the air gaps. Hence

$$B = \mu_0 H_1 = \mu_0 H_2 \quad (\text{b})$$

and (a) shows that

$$B = \frac{fB_0}{(\mu/\mu_0)(g+x) + f}. \quad (\text{c})$$

Note from (a) that we can replace the permanent magnet with an equivalent current source I driving N turns, as shown in Fig. 2.1.2, but in which

$$i = I = \frac{fB_0}{N\mu},$$

and in the magnetic circuit there is a magnetic material of permeability μ and length f . This model allows us to compute forces of electrical origin for systems involving permanent magnets on the same basis as those excited by currents through electrical terminal pairs.

2.1.2 Generalized Capacitance

To derive the terminal characteristics of lumped-parameter electric field systems we start with the quasi-static equations given in Table 1.2. The equations we need are (1.1.24) to (1.1.26) and (1.1.13):

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0, \quad (1.1.24)$$

$$\oint_S \mathbf{D} \cdot \mathbf{n} \, da = \int_V \rho_f \, dV, \quad (1.1.25)$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad (1.1.13)$$

$$\oint_S \mathbf{J}'_f \cdot \mathbf{n} \, da = - \frac{d}{dt} \int_V \rho_f \, dV. \quad (1.1.26)$$

We can use the differential forms of these equations alternatively, although the integral forms are more appropriate for the formalism of this section.

It is essential to recognize that an ideal, lossless electric field system consists of a set of equipotential bodies with no conducting paths between them.

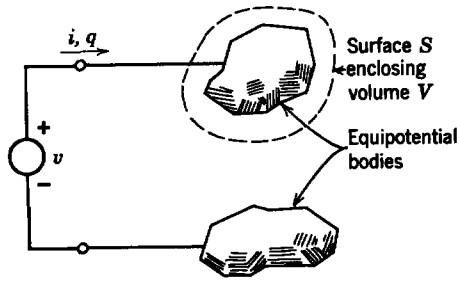


Fig. 2.1.6 A simple electric field system.

Terminals are brought out so that excitation may be applied to the equipotential bodies. It is conventional to select one equipotential body as a reference and designate its voltage as zero. The potentials of the other bodies are then specified with respect to the reference.

As a simple example of finding the terminal relations for an electric field system, consider the two equipotential bodies in Fig. 2.1.6. We assume that the voltage v is impressed between the two equipotential bodies and wish to find the current i . We choose a surface S (see Fig. 2.1.6) which encloses only the upper equipotential body and apply the conservation of charge (1.1.26). The only current density on the surface S occurs where the wire cuts through it. At this surface there is no free charge density ρ_f , hence $J_f = J'_f$, so that

$$\oint_S \mathbf{J}'_f \cdot \mathbf{n} \, da = -i. \quad (2.1.26)$$

The minus sign results because the normal vector \mathbf{n} is directed outward from the surface.

The total charge q on the upper equipotential body in Fig. 2.1.6 is

$$q = \int_V \rho_f \, dV, \quad (2.1.27)$$

where V is a volume that includes the body and is enclosed by the surface S . Use of the conservation of charge (1.1.26) with (2.1.26) and (2.1.27) yields the terminal current

$$i = \frac{dq}{dt}. \quad (2.1.28)$$

Equation 2.1.28 simply expresses the fact that a current i leads to an accumulation of charge on the body. For a quasi-static system in which we impose voltage constraints the field quantities and the charge density ρ_f are determined by (1.1.24), (1.1.25) and (1.1.13), and all are functions of the

applied voltages, the material properties (polarization), and the geometry. Thus, because (2.1.27) is an integral over space, the charge q is a function of the applied voltages, material properties, and geometry.

If we again consider the system in Fig. 2.1.6 and specify that the time variation of the geometry is uniquely specified by a mechanical displacement x with respect to a fixed reference, we can write the charge in the general functional form

$$q = q(v, x). \quad (2.1.29)$$

In writing the charge in this way we have indicated explicit functional dependence on only those variables (v and x) that may be functions of time.

We can now use (2.1.29) in (2.1.28) to obtain the terminal current as

$$i = \frac{\partial q}{\partial v} \frac{dv}{dt} + \frac{\partial q}{\partial x} \frac{dx}{dt}. \quad (2.1.30)$$

The first term exists only when the voltage is changing with time and the second term exists only when there is relative mechanical motion.

If we consider a system whose polarization density \mathbf{P} is a linear function of field quantities, the system is *electrically linear* and the functional dependence of (2.1.29) can be written in the form

$$q = C(x)v. \quad (2.1.31)$$

The capacitance C contains the dependence on geometry. For a system whose charge is expressible by (2.1.31) the terminal current is

$$i = C(x) \frac{dv}{dt} + v \frac{dC}{dx} \frac{dx}{dt}. \quad (2.1.32)$$

In the special case in which the geometry does not vary with time (x is constant), (2.1.32) reduces to

$$i = C \frac{dv}{dt}, \quad (2.1.33)$$

which is the terminal equation used for a capacitance in linear circuit theory.

Now that we have established the formalism for calculating the terminal properties of lumped-parameter electric field systems by treating the simplest case, we can, as we did for the magnetic field system in the preceding section, generalize the equations to describe systems with any number of electrical terminal pairs and mechanical displacements. We assume a system with $N + 1$ equipotential bodies. We select one of them as a reference (zero) potential and apply voltages to the other N terminals. There are then N terminal voltages,

$$v_1, v_2, \dots, v_N.$$

We assume that there are M mechanical displacements that uniquely specify the time variation of the geometry

$$x_1, x_2, \dots, x_M.$$

Although we write these displacements as if they were translational, they can equally well be rotational (angular).

For the integration of (2.1.26) we select a surface S_k that encloses the k th equipotential body. Then (2.1.27) is integrated over the enclosed volume V_k and the conservation of charge expression (1.1.26) is used to express the current into the terminal connected to the k th equipotential as

$$i_k = \frac{dq_k}{dt}, \quad (2.1.34)$$

where

$$q_k = \int_{V_k} \rho_f dV. \quad (2.1.35)$$

Because the system is quasi-static, the fields are functions only of the applied voltages, the material properties, and the displacements. Thus we can generalize the functional form of (2.1.29) for our multivariable problem to

$$\begin{aligned} q_k &= q_k(v_1, v_2, \dots, v_N; x_1, x_2, \dots, x_M), \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.36)$$

From (2.1.36) and (2.1.34), the k 'th terminal current follows as

$$\begin{aligned} i_k &= \sum_{j=1}^N \frac{\partial q_k}{\partial v_j} \frac{dv_j}{dt} + \sum_{j=1}^M \frac{\partial q_k}{\partial x_j} \frac{dx_j}{dt}, \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.37)$$

If we specify that our multivariable system is electrically linear (a situation that occurs when polarization \mathbf{P} is a linear function of electric field intensity) we can write the function of (2.1.36) in the form

$$\begin{aligned} q_k &= \sum_{j=1}^N C_{kj}(x_1, x_2, \dots, x_M)v_j, \\ k &= 1, 2, \dots, N. \end{aligned} \quad (2.1.38)$$

Equations 2.1.36 and 2.1.38 can be inverted to express the voltages as functions of the charges and displacements. This process was illustrated for magnetic field systems by (2.1.19) through (2.1.24).

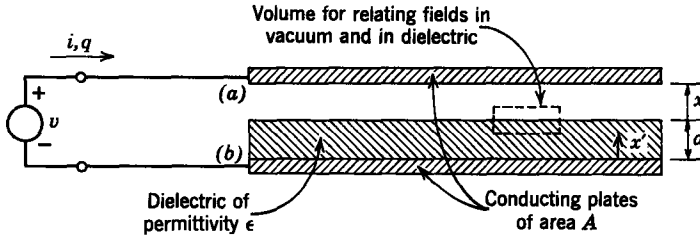


Fig. 2.1.7 A parallel-plate capacitor.

Example 2.1.4. Consider the simple parallel-plate capacitor of Fig. 2.1.7. It consists of two rectangular, parallel highly conducting plates of area A . Between the plates is a rectangular slab of dielectric material with constant permittivity ϵ ,

$$\mathbf{D} = \epsilon \mathbf{E}.$$

The lower plate and the dielectric are fixed and the upper plate can move and has the instantaneous position x with respect to the top of the dielectric. The transverse dimensions are large compared with the plate separation. Thus fringing fields can be neglected. The terminal voltage is constrained by the source v which is specified as a function of time.

We wish to calculate the instantaneous charge on the upper plate and the current to the upper plate.

To solve this problem we need the given relation between \mathbf{D} and \mathbf{E} , (1.1.24) and (1.1.25), and the definition of the voltage of point a with respect to point b

$$v = - \int_b^a \mathbf{E} \cdot d\mathbf{l}$$

With the neglect of fringing fields, the field quantities \mathbf{D} and \mathbf{E} will have only vertical components. We take them both as being positive upward. In the vacuum space

$$D_v = \epsilon_0 E_v$$

and in the dielectric

$$D_d = \epsilon E_d.$$

We assume that the dielectric has no free charge; consequently, we use (1.1.25) with a rectangular box enclosing the dielectric-vacuum interface as illustrated in Fig. 2.1.7 to obtain

$$\epsilon_0 E_v = \epsilon E_d.$$

We now use the expression for the voltage to write

$$v = - \int_0^d \frac{\epsilon_0}{\epsilon} E_v dx' - \int_d^{x+d} E_v dx'.$$

Integration of these expressions yields the vacuum electric field intensity

$$E_v = - \frac{v}{x + (\epsilon_0/\epsilon)d}.$$

We now use (1.1.25) with a rectangular surface enclosing the upper plate to obtain

$$q = A\epsilon_0 E_v = \frac{\epsilon_0 A v}{x + (\epsilon_0/\epsilon)d}$$

As would be expected from the linear constitutive law used in the derivation, the system is electrically linear. The charge can be expressed as

$$q = C(x)v,$$

where

$$C(x) = \frac{\epsilon_0 A}{x + (\epsilon_0/\epsilon)d}.$$

When voltage v and displacement x are specified functions of time, we can write the terminal current as

$$i = \frac{dq}{dt} = \frac{\epsilon_0 A}{x + (\epsilon_0/\epsilon)d} \frac{dv}{dt} - \frac{\epsilon_0 A v}{[x + (\epsilon_0/\epsilon)d]^2} \frac{dx}{dt}.$$

Note that the first term will exist when the geometry (x) is fixed and the voltage is varying and that the second term will exist when the voltage is constant and the geometry is varying. This illustrates once again how mechanical motion can generate a time-varying current.

Example 2.1.5. As an example of a multiply excited electric field system, consider the system in Fig. 2.1.8 which is essentially a set of three parallel-plate capacitors immersed in vacuum. Of the three plates, one is fixed and two are movable, as indicated. The dimensions and variables are defined in Fig. 2.1.8.

We wish to calculate the charges q_1 and q_2 on the two movable plates and the currents i_1 and i_2 to those plates. We neglect fringing at the edges of the plates and assume that the voltages v_1 and v_2 are specified.

We designate the electric field intensities in the three regions as E_1 , E_2 , and E_m , with the positive directions as indicated in Fig. 2.1.8. The electric field intensities have constant

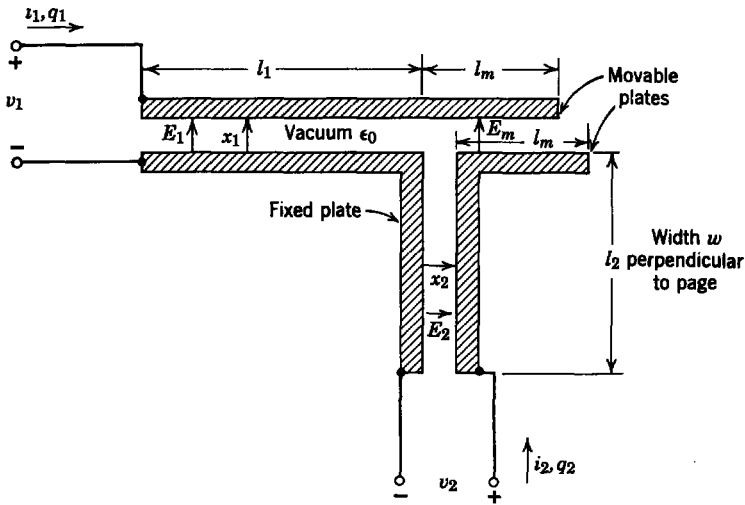


Fig. 2.1.8 A multiply excited electric field system.

magnitudes in the three regions; consequently, our definition of voltage applied in the three regions yields

$$E_1 = -\frac{v_1}{x_1},$$

$$E_2 = -\frac{v_2}{x_2},$$

$$E_m = \frac{v_2 - v_1}{x_1}.$$

We now use (1.1.25) with a rectangular surface enclosing the top plate to obtain

$$q_1 = -l_1 w \epsilon_0 E_1 - (l_m - x_2) w \epsilon_0 E_m$$

and with a rectangular box enclosing the right-hand movable plate to obtain

$$q_2 = -l_2 w \epsilon_0 E_2 + (l_m - x_2) w \epsilon_0 E_m.$$

These two expressions are electrically linear and can be written in the forms

$$q_1 = C_1 v_1 - C_m v_2, \quad (\text{a})$$

$$q_2 = -C_m v_1 + C_2 v_2, \quad (\text{b})$$

where

$$C_1 = \frac{\epsilon_0 w [l_1 + (l_m - x_2)]}{x_1}, \quad (\text{c})$$

$$C_2 = \epsilon_0 w \left(\frac{l_2}{x_2} + \frac{l_m - x_2}{x_1} \right), \quad (\text{d})$$

$$C_m = \frac{\epsilon_0 w (l_m - x_2)}{x_1}. \quad (\text{e})$$

With v_1 , v_2 , x_1 , and x_2 given as specified functions of time, we can write the terminal currents as

$$i_1 = C_1 \frac{dv_1}{dt} - C_m \frac{dv_2}{dt} + v_1 \left(\frac{\partial C_1}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial C_1}{\partial x_2} \frac{dx_2}{dt} \right) - v_2 \left(\frac{\partial C_m}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial C_m}{\partial x_2} \frac{dx_2}{dt} \right),$$

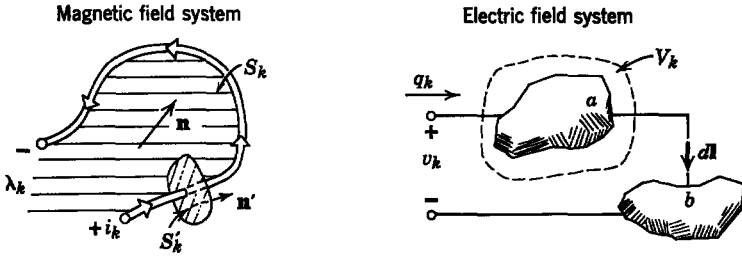
$$i_2 = -C_m \frac{dv_1}{dt} + C_2 \frac{dv_2}{dt} - v_1 \left(\frac{\partial C_m}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial C_m}{\partial x_2} \frac{dx_2}{dt} \right) + v_2 \left(\frac{\partial C_2}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial C_2}{\partial x_2} \frac{dx_2}{dt} \right).$$

A comparison of these results with those of the preceding example illustrates how quickly the expressions become longer and more complex as the numbers of electrical terminal pairs and mechanical displacements are increased.

2.1.3 Discussion

In the last two sections we specified the process by which we can obtain the electrical terminal properties of lumped-parameter, magnetic field and electric field systems. The general forms of the principal equations are summarized in Table 2.1. The primary purpose of obtaining terminal relations is to be able to include electromechanical coupling terms when writing circuit

Table 2.1 Summary of Terminal Variables and Terminal Relations



Definition of Terminal Variables

Flux

$$\lambda_k = \int_{S_k} \mathbf{B} \cdot \mathbf{n} \, da$$

Current

$$i_k = \int_{S_k'} \mathbf{J}_f \cdot \mathbf{n}' \, da$$

Charge

$$q_k = \int_{V_k} \rho_f \, dV$$

Voltage

$$v_k = \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

Terminal Conditions

$$v_k = \frac{d\lambda_k}{dt}$$

$$\lambda_k = \lambda_k(i_1 \cdots i_N; \text{geometry})$$

$$i_k = i_k(\lambda_1 \cdots \lambda_N; \text{geometry})$$

$$i_k = \frac{dq_k}{dt}$$

$$q_k = q_k(v_1 \cdots v_N; \text{geometry})$$

$$v_k = v_k(q_1 \cdots q_N; \text{geometry})$$

equations. After a review of rigid-body mechanics and a look at some energy considerations, we shall address ourselves to the problem of writing coupled equations of motion for electromechanical systems.

2.2 MECHANICS

We now discuss lumped-parameter modeling of the mechanical parts of systems. In essence, we shall consider the basic notions of rigid-body mechanics, including the forces of electric origin.

Just as in circuit theory, there are two steps in the formulation of equations of motion for rigid-body mechanical systems. First, we must specify the kinds of elements and their mathematical descriptions. This is analogous to defining terminal relations for circuit elements in circuit theory. Next, we must specify the laws that are used for combining the mathematical descriptions of elements into equations of motion. In mechanics these are Newton's second law and the continuity of space (often called geometrical compatibility) and they are analogous to Kirchhoff's laws in circuit theory.

In the two sections to follow we first define mechanical elements and then specify the laws and illustrate the formulation of equations of motion.

2.2.1 Mechanical Elements

In general, a mechanical system, which, for the moment, we define as an interconnected system of ponderable bodies in relative motion, will exhibit kinetic energy storage in moving masses, potential energy storage due to gravitational fields or elastic deformations, and mechanical losses due to friction of various types. Any component of a system will exhibit all three effects to varying degrees; as a practical matter of analysis and design, however, we can often represent mechanical systems as interconnections of ideal elements, each of which exhibits only one of these effects. This process is analogous to that of defining ideal circuit elements in circuit theory and has its justification in derivations from continuum mechanics. These derivations are not made in a formal way; however, in our treatment of continuum mechanics in later chapters the connections between the continuum theory and ideal, lumped-parameter, mechanical elements will become clear.

We consider three types of ideal mechanical elements: (a) elements that store kinetic energy—for translational systems these are masses and for rotational systems they are moments of inertia; (b) elements that store potential energy—for both translational and rotational systems these are springs; (c) elements that dissipate mechanical energy as heat—for both translational and rotational systems these are mechanical dampers. We define the nomenclature by which we describe each element before we combine them into systems. In the process we introduce the concept of mechanical circuits and mechanical circuit elements, which are simply pictorial representations of mathematical relations somewhat analogous to the circuits of electrical systems.* Mechanical circuits are of value for two reasons: they provide a formalism for writing the equations of motion and they emphasize the concept of mechanical terminal pairs.

The mechanical variables used in describing mechanical elements and systems are force and displacement. The displacement is always measured with respect to a reference position. In a way analogous to that used in circuit theory, we define ideal mechanical sources.

First, we define a position source as in Fig. 2.2.1. Here we represent a simple physical situation in which the motion of the two objects (Fig. 2.2.1a) is restricted to the vertical. Thus there are two *mechanical nodes* whose positions x_1 and x_2 are measured with respect to a fixed reference. The

* M. F. Gardner, and J. L. Barnes, *Transients in Linear Systems*, Vol. 1, Wiley, New York, 1942, Chapter 2. (Various conventions are used for drawing mechanical circuits. One alternative is given in the above.)

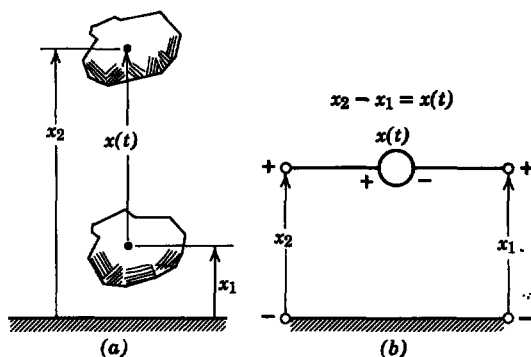


Fig. 2.2.1 A position source: (a) physical; (b) schematic.

position source $x(t)$ constrains the relative positions of the two nodes to

$$x_2 - x_1 = x(t); \quad (2.2.1)$$

the sign is determined by the + and - signs associated with the source. In Fig. 2.2.1b we give the circuit representation of the position source. The circuit is a pictorial representation of the scalar equation (2.2.1) and as such is completely analogous to the representations in circuit theory. Because (2.2.1) is valid regardless of other mechanical elements attached to the nodes, the ideal position source can supply an arbitrary amount of force.

In a similar way we can define a velocity source $v(t)$ that constrains the relative velocity of the two nodes to

$$\frac{dx_2}{dt} - \frac{dx_1}{dt} = v(t). \quad (2.2.2)$$

The circuit is as shown in Fig. 2.2.1b with x replaced by v .

A different kind of ideal mechanical source is a force source for which nomenclature is given in Fig. 2.2.2. In the physical representation of Fig. 2.2.2a motion is constrained to the vertical and the force $f(t)$ is vertical. The position of the arrow indicates the direction of positive force, and the convention we use here is that, with the arrow as shown and a positive $f(t)$, the force tends to push the two mechanical nodes apart exactly as if one were standing on node x_1 and pushing upward on node x_2 with the hands. The circuit representation of this force source is given in Fig. 2.2.2b. In the circuit our convention is that with the arrow as shown [$f(t)$ positive] the force tends to increase x_2 and decrease x_1 .

The sources of Figs. 2.2.1 and 2.2.2 have been specified for translational systems. We can also specify analogous sources and circuits for rotational

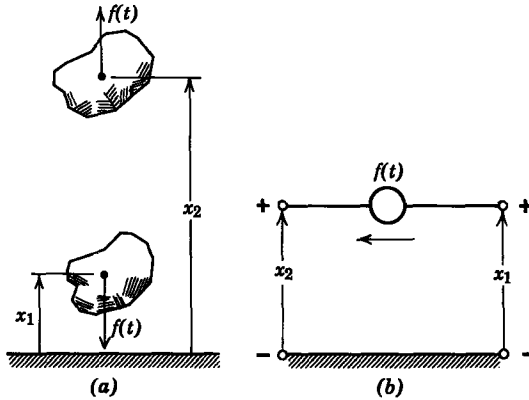


Fig. 2.2.2 A force source: (a) physical; (b) schematic.

systems. These extensions of our definitions should be clear from Figs. 2.2.3 and 2.2.4, in which θ_1 and θ_2 measure the angular displacements of the two disks, with respect to fixed references, and the torque T is applied between them.

In the examples we shall consider we shall encounter either pure translation or rotation about a fixed axis. Consequently, the geometry of motion as described so far is adequate for our purposes. We now describe ideal, passive, lumped-parameter, mechanical elements.

2.2.1a The Spring*

An *ideal* spring is a device with negligible mass and mechanical losses whose deformation is a single-valued function of the applied force. A *linear*

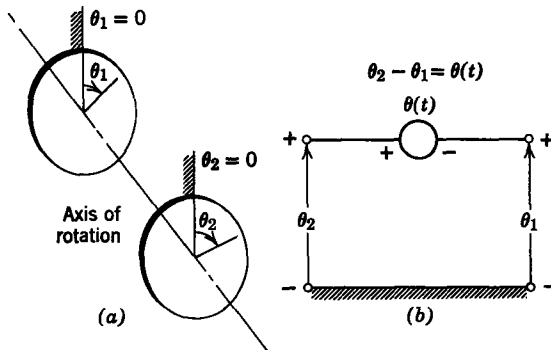


Fig. 2.2.3 An angular position source: (a) physical; (b) schematic.

* For a comprehensive treatise on the subject see A. M. Wahl, *Mechanical Springs*, 2nd ed., McGraw-Hill, New York, 1963.

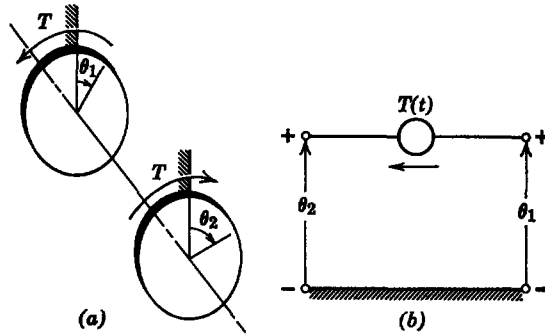


Fig. 2.2.4 A torque source: (a) physical; (b) schematic.

ideal spring has deformation *proportional* to force. In our treatment we are concerned almost exclusively with linear springs. We represent a spring physically as in Fig. 2.2.5a and in mechanical circuits as the symbol of Fig. 2.2.5b. The force f at one end of the spring must always be balanced by an equal and opposite force f at the other end. The force is thus transmitted through the spring much as current is transmitted through an inductance. In the circuit of Fig. 2.2.5b the applied force f is represented as a force source.

The spring of Fig. 2.2.5 has a spring constant K and the force is a linear function of the *relative* displacement of the two ends of the spring. Thus

$$f = K(x_2 - x_1 - l), \quad (2.2.3)$$

where l is the value of the relative displacement for which the force is zero.

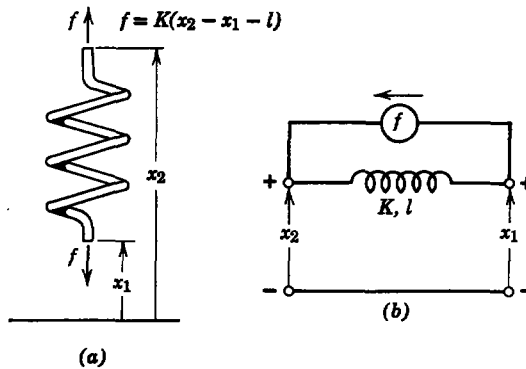


Fig. 2.2.5 A linear ideal spring for translational motion held in equilibrium by a force f : (a) physical system; (b) circuit.

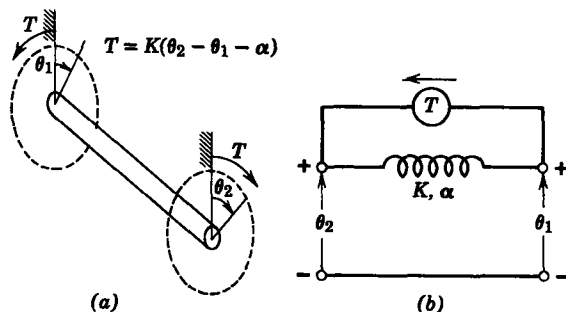


Fig. 2.2.6 A linear ideal torsional spring: (a) physical system; (b) circuit.

It is always possible, although in many cases not convenient, to define reference positions for measuring x_1 and x_2 such that $l = 0$.

We can also have linear ideal torsional springs in rotational systems. The mathematical and circuit representations are analogous to those of a translational spring and are evident in Fig. 2.2.6. The torque is a linear function of the relative angular displacement of the two ends

$$T = K(\theta_2 - \theta_1 - \alpha). \quad (2.2.4)$$

Note that the K in (2.2.3) has different dimensions than the K in (2.2.4).

2.2.1b The Mechanical Damper

The mechanical damper is analogous to electrical resistance in that it dissipates energy as heat. An *ideal* damper is a device that exhibits no mass or spring effect and exerts a force that is a function of the relative velocity between its two nodes. A *linear* ideal damper has a force proportional to the relative velocity of the two nodes. In all cases a damper produces a force that opposes the relative motion of the two nodes.

A linear damper (often called a viscous damper) is usually constructed in such a way that friction forces result from the viscous drag of a fluid under laminar flow conditions.* Two examples of viscous dampers, one for linear and one for rotary motion, are shown in Fig. 2.2.7 along with the mechanical circuits. Note that the force f (or torque T) is the force (or torque) that must be applied by an external agent to produce a positive relative velocity of the two nodes. For the linear-motion damper the terminal relation is

$$f = B \frac{d}{dt} (x_2 - x_1) \quad (2.2.5)$$

* For more detail on viscous laminar flow see Chapter 14.

and for the rotary-motion damper it is analogous (Fig. 2.2.7):

$$T = B \frac{d}{dt} (\theta_2 - \theta_1). \quad (2.2.6)$$

Note that the damping constant B has different dimensions for the two systems. In each case both displacements are measured with respect to references that are fixed.

Mechanical friction occurs in a variety of situations under many different physical conditions. Sometimes friction is unwanted but must be tolerated and accounted for analytically, as, for example, in bearings, sliding electrical contacts, and the aerodynamic drag on a moving body. In other cases friction is desired and is designed into equipment. Examples are vibration dampers and shock absorbers. Although in some cases a linear model is a useful approximation, in many others it is inadequate. The subject of friction

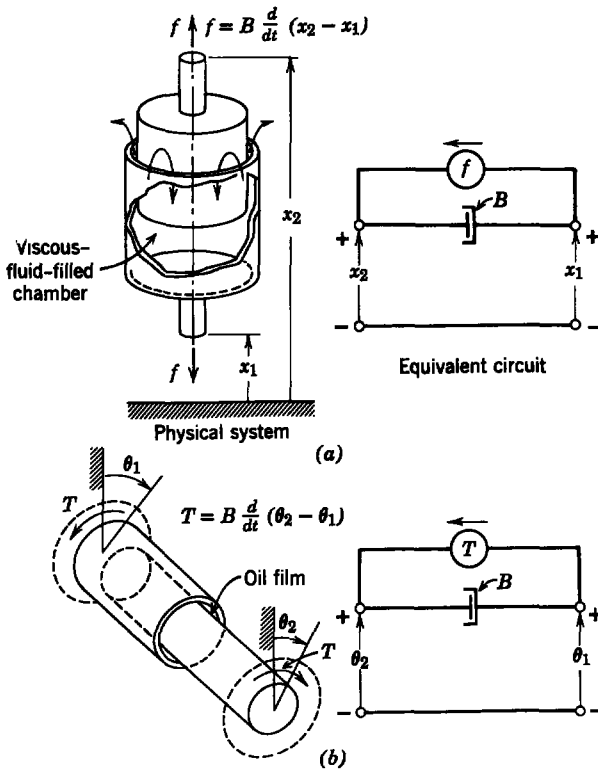


Fig. 2.2.7 Mechanical dampers: (a) translational system; (b) rotational system.

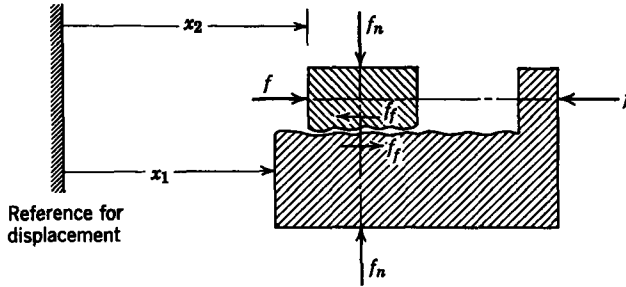


Fig. 2.2.8 Coulomb friction between members in contact.

is lengthy and complex*; most practical devices, however, can be modeled as described or by one of two nonlinear models that we now discuss.

The first of these additional models is coulomb friction which is characteristic of sliding contacts between dry materials. See Fig. 2.2.8 in which the blocks are assumed to have negligible mass. If we apply constant, equal and opposite, normal forces f_n , as shown, and then apply equal and opposite forces f , as shown, the blocks may or may not move relatively, depending on the friction coefficient of the surface. If we vary the force f , which must be balanced by the friction force f_f for steady motion, and measure the resultant steady relative velocity, we can plot the friction coefficient (f_f/f_n) as a function of relative velocity (see Fig. 2.2.9). The quantity μ_s is the coefficient of static friction and μ_d is the coefficient of sliding friction. When we

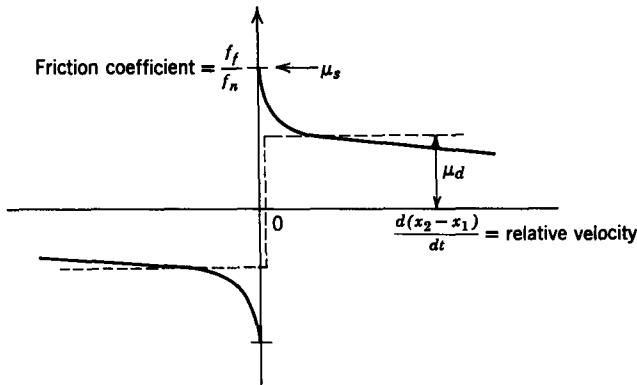


Fig. 2.2.9 Typical coulomb friction characteristic.

* See, for example, G. W. Van Stanten, *Introduction to a Study of Mechanical Vibration*, 3rd ed., Macmillan, New York, 1961, Chapter 14.

approximate this curve by the piecewise linear dashed line of Fig. 2.2.9, we can represent coulomb friction mathematically by the relation

$$f = \frac{f_n \mu_a (d/dt)(x_2 - x_1)}{|(d/dt)(x_2 - x_1)|}. \quad (2.2.7)$$

It is important to remember that coulomb friction, like all other forms of friction, produces a force that tends to oppose the relative motion of the nodes in the system.

Coulomb friction can also occur in rotational systems, in which case an expression analogous to (2.2.7) can be used for a mathematical description.

The final model of friction that we shall consider is that resulting primarily from the drag of a viscous fluid in turbulent flow.* This type of friction can be represented with fair accuracy by a model that makes the force (or torque) proportional to the square of relative velocity (or relative angular velocity). Such an expression is

$$f = \pm B_s \left[\frac{d}{dt} (x_2 - x_1) \right]^2. \quad (2.2.8)$$

Once again the force produced by the friction opposes the relative mechanical motion. Two examples in which square-law damping occurs are given in Fig. 2.2.10. The linear motion damper of Fig. 2.2.10a has the configuration characteristic of dashpots for making time-delay relays and for automobile shock absorbers. The square-law rotational damping of Fig. 2.2.10b occurs frequently in high-speed rotating machines in which the fluid is air or some other gaseous coolant.

2.2.1c The Mass

The final ideal mechanical element we need to consider is the element that stores kinetic energy but has no spring or damping effects. For translational

* See, for example, H. Schlichting, *Boundary Layer Theory*, 4th ed., McGraw-Hill, New York, 1960, Chapter 21.

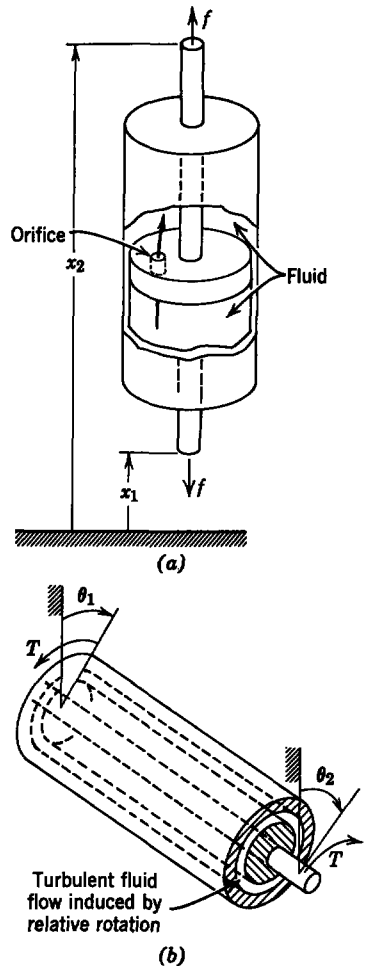


Fig. 2.2.10 Typical square-law dampers: (a) orifice and piston damper; (b) damping due to rotation.

systems this is a mass and for rotational systems a moment of inertia. After we have reviewed the plane motion of a point mass, we shall generalize to translation of a rigid body of finite size and to rotation of a rigid body about a fixed axis.

The motion of a point that has associated with it a constant amount of mass M is described by Newton's second law*:

$$\mathbf{f} = M \frac{d\mathbf{v}}{dt}, \quad (2.2.9)$$

where \mathbf{f} is the force vector acting on the mass point and \mathbf{v} is the absolute vector velocity of the point. It should be clearly recognized that \mathbf{v} must be measured in relation to a fixed or nonaccelerating point or frame of reference. Such a reference system is called an *inertial reference*.

In motion occurring on the surface of the earth the earth is often considered to be approximately an inertial reference and \mathbf{v} is measured in relation to the earth. When dealing with the motion of long-range missiles, orbital vehicles, and spacecraft, the earth cannot be considered to be a nonaccelerating reference and velocities are then measured with respect to the fixed stars.

When the velocity of a mass begins to approach the velocity of light, the Newtonian equation of motion in the form of (2.2.9) becomes invalid because the mass of a given particle of matter increases with velocity, according to the theory of relativity.† The present considerations are limited to velocity levels small compared with the velocity of light, so that Newton's law (2.2.9) will apply. This is consistent with the approximations we have made in defining the quasi-static electromagnetic field equations.

If we now consider the two-dimensional motion of a point mass M , we can define an orthogonal inertial reference system as in Fig. 2.2.11a and write Newton's second law in component form as

$$f_x = M \frac{d^2x}{dt^2}, \quad (2.2.10)$$

$$f_y = M \frac{d^2y}{dt^2}. \quad (2.2.11)$$

We can also represent these two equations by the mechanical circuit of Fig. 2.2.11b. Each degree of freedom of a point mass M is represented by a node with mass M . A force that tends to increase the displacement of a

* For a review of the fundamental concepts involved in Newton's laws, see for example R. R. Long, *Engineering Science Mechanics*, Prentice-Hall, Englewood Cliffs, N.J., 1963, Chapter 2.

† Long, *ibid.*, Chapter 10.

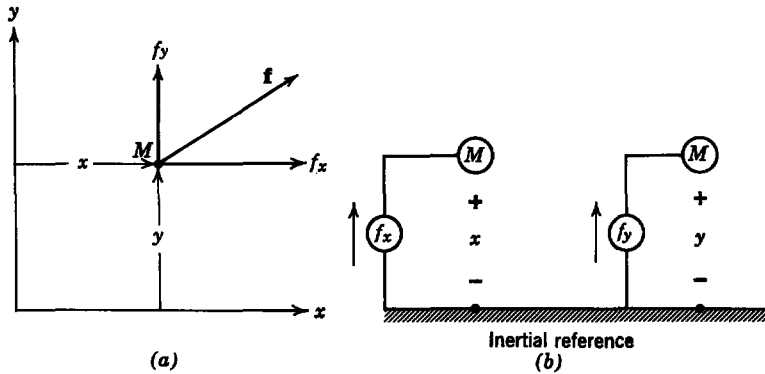


Fig. 2.2.11 Plane motion of a point mass: (a) physical; (b) circuit representation.

node is represented by an arrow pointing toward the node. The extension of these representations to three-dimensional motion is straightforward.

In order to obtain a representation for a mass element more general than a point mass, we need to review briefly the dynamics of rigid bodies. We consider first the translational motion of a rigid body.

A rigid body, by definition, is one in which any line drawn in or on the body remains constant in length and all angles drawn in or on the body remain constant. In Fig. 2.2.12 we represent a rigid body with a mass density ρ (kilograms per cubic meter) that may vary from point to point in the body but remains constant in time at any point in the body. We define an inertial coordinate system (x, y, z) and specify the position vector \mathbf{r} of an arbitrary point p that is fixed in the body.

The instantaneous acceleration of the point p is

$$\mathbf{a}_p = \frac{d^2\mathbf{r}}{dt^2}. \quad (2.2.12)$$

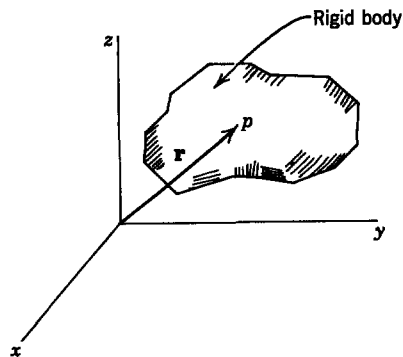


Fig. 2.2.12 Geometry for analyzing the translation of a rigid body.

Thus the infinitesimal element of mass ρdV at point p will have an acceleration force

$$d\mathbf{f}_n = \rho dV \frac{d^2\mathbf{r}}{dt^2}. \quad (2.2.13)$$

The force $d\mathbf{f}_n$ consists of two components: $d\mathbf{f}$ from sources external to the body and $d\mathbf{f}_i$ from sources within the body. Thus (2.2.13) can be written as

$$d\mathbf{f} + d\mathbf{f}_i = \rho dV \frac{d^2\mathbf{r}}{dt^2}. \quad (2.2.14)$$

The mass density associated with each point p in the rigid body is constant. Hence, when we integrate this expression throughout the volume of the body and recognize that the internal forces integrate to zero,* we obtain the result

$$\mathbf{f} = M \frac{d^2\mathbf{r}_m}{dt^2}, \quad (2.2.15)$$

where \mathbf{f} is the total external force applied to the body,

$$M = \int_V \rho dV \text{ is a constant and is the total mass of the body,}$$

$$\mathbf{r}_m = \frac{\int_V \rho \mathbf{r} dV}{M} \text{ is the position vector of the center of mass of the body.}$$

From the result of (2.2.15) it is clear that the translational motion of a rigid body can be described completely by treating the body as if all the mass were concentrated at the center of mass. Consequently, (2.2.10) and (2.2.11)

* To illustrate that the internal forces integrate to zero consider an ensemble of N interacting particles. An internal force is applied between two particles: $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$, where \mathbf{f}_{ij} is the force on the i th particle due to the source connected between the i th and j th particles. The total internal force on the i th particle is

$$\mathbf{f}_i = \sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{f}_{ij}.$$

The total internal force on the ensemble is

$$\mathbf{f} = \sum_{i=1}^N \mathbf{f}_i = \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{f}_{ij}.$$

Because $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$, we conclude that each term in this summation is canceled by an equal and opposite term and the net internal force is zero. If we let $N \rightarrow \infty$, the nature of the result is unchanged; thus we conclude that the integral of internal forces over the volume of a rigid body must be zero.

and Fig. 2.2.11, which were defined for a point mass, hold equally well for describing the motion of the center of mass of a rigid body.

Our rotational examples involve rotation about a fixed axis only. Thus we treat only the mechanics of rigid bodies rotating about fixed axes.* For this purpose we consider the system of Fig. 2.2.13. The body has mass density ρ that may vary with space in the body but at a point p fixed in the material is constant. We select a rectangular coordinate system whose z -axis coincides with the axis of rotation. The instantaneous angular velocity of the body is

$$\boldsymbol{\omega} = \mathbf{i}_z \frac{d\theta}{dt}. \quad (2.2.16)$$

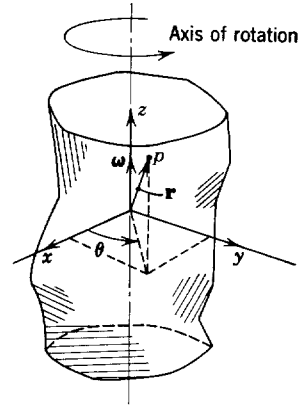


Fig. 2.2.13 Rigid-body rotation.

At the point p with coordinates (x, y, z) the element of mass ρdV will have the instantaneous velocity

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \quad (2.2.17)$$

where $\mathbf{r} = \mathbf{i}_x x + \mathbf{i}_y y + \mathbf{i}_z z$ is the radius vector from the origin to the mass element.

The acceleration force on this mass element is

$$d\mathbf{f}_n = \rho dV \frac{d\mathbf{v}}{dt} = \rho dV \left[\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \right], \quad (2.2.18)$$

where $d\mathbf{f}_n$ contains both internal and external forces [see (2.2.14)] and the last term has been written by using (2.2.17). We use the identity for the triple vector product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \quad (2.2.19)$$

to write (2.2.18) in the form

$$d\mathbf{f}_n = \rho dV \left[\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \mathbf{r}) - \mathbf{r}(\boldsymbol{\omega} \cdot \boldsymbol{\omega}) \right]. \quad (2.2.20)$$

To find the acceleration torque on this mass element we write

$$d\mathbf{T}_n = \mathbf{r} \times d\mathbf{f}_n. \quad (2.2.21)$$

* For a treatment of the general case of simultaneous translation and rotation in three dimensions, see, for example, Long, *op. cit.*, Chapter 6.

Using (2.2.20) in (2.2.21) and simplifying, we find that

$$d\mathbf{T} + d\mathbf{T}_i = \rho dV \left\{ \mathbf{i}_x(x^2 + y^2) \frac{d^2\theta}{dt^2} - \mathbf{i}_x \left[xz \frac{d^2\theta}{dt^2} - yz \left(\frac{d\theta}{dt} \right)^2 \right] - \mathbf{i}_y \left[yz \frac{d^2\theta}{dt^2} + xz \left(\frac{d\theta}{dt} \right)^2 \right] \right\}, \quad (2.2.22)$$

where $d\mathbf{T}$ = the torque from external sources,
 $d\mathbf{T}_i$ = the torque from internal sources.

To find the total acceleration torque on the body we must integrate (2.2.22) throughout the volume V of the body. For this purpose we find it convenient to define the moment of inertia about the z -axis as

$$J_z = \int_V (x^2 + y^2) \rho dV \quad (2.2.23)$$

and the products of inertia*

$$J_{xz} = \int_V xz \rho dV, \quad (2.2.24)$$

$$J_{yz} = \int_V yz \rho dV. \quad (2.2.25)$$

When we use these parameters and integrate (2.2.22) throughout the volume V of the body, the internal torque integrates to zero [see (2.2.14) and (2.2.15) and the associated footnote for arguments similar to those required here], and we obtain for the total acceleration torque \mathbf{T} applied by external sources

$$\mathbf{T} = \mathbf{i}_z J_z \frac{d^2\theta}{dt^2} - \mathbf{i}_x \left[J_{xz} \frac{d^2\theta}{dt^2} - J_{yz} \left(\frac{d\theta}{dt} \right)^2 \right] - \mathbf{i}_y \left[J_{yz} \frac{d^2\theta}{dt^2} + J_{xz} \left(\frac{d\theta}{dt} \right)^2 \right]. \quad (2.2.26)$$

With the restriction to rotation about a fixed axis, only the first term in this expression affects the dynamics of the body. Thus we write the z component of (2.2.26) as

$$T_z = J_z \frac{d^2\theta}{dt^2} \quad (2.2.27)$$

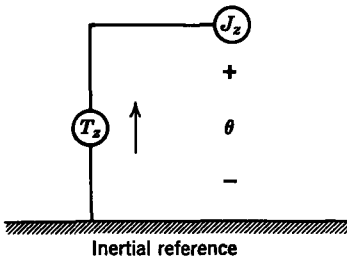


Fig. 2.2.14 Circuit representation of a moment of inertia.

and represent this element in a mechanical circuit as in Fig. 2.2.14. Note that this circuit has exactly the same form as that adopted earlier for representing mass in a translational system (see Fig. 2.2.11).

* See, for example, I. H. Shames, *Engineering Mechanics*, Prentice-Hall, Englewood Cliffs, N.J., 1960, pp. 187-188.

The last two terms on the right of (2.2.26) (the x - and y -components) represent a torque that must be applied by bearings and support structure to maintain the axis of rotation fixed. It should be clear from the definitions of the products of inertia in (2.2.24) and (2.2.25) that certain axes of symmetry make these products of inertia zero. Such axes are called principal axes.* When rotation occurs about a principal axis (the body is dynamically balanced), no bearing torque is necessary to maintain the axis of rotation fixed.

Now that we have completed the definitions of the elements that will make up the mechanical parts of our electromechanical systems, we have only to describe how we combine elements to obtain complete equations of motion.

2.2.2 Mechanical Equations of Motion

When electrical circuit elements are interconnected, Kirchhoff's loop and node relations must be satisfied. The sum of the voltage differences around any loop must be zero and the sum of all currents into any node must be zero.

Similar relations must hold in networks composed of interconnected ideal mechanical elements. Consider first a *mechanical node*. A mechanical node is a location in a mechanical system which has a certain position relative to a reference position and a particular mass associated with it. A node appears in the mechanical circuit as a circle in which two or more terminals of ideal elements are connected. Figure 2.2.15 shows a simple mechanical system with one node p having mass M connected to two springs, a damper, and a force source. The motion of this system is completely specified by one coordinate x which is the instantaneous position of node p . The system is said to have one *degree of freedom*. In general, the number of degrees of freedom in a mechanical system equals the number of nodes it has.

The forces acting in the passive elements of Fig. 2.2.15a are all vertical and may be described by the scalar quantities f_1 , f_2 , and f_3 and shown also in the free-body diagram of the mass shown in Fig. 2.2.15b and in the circuit of Fig. 2.2.15c. The convention used here is that forces are drawn as *applied to the node*. The arrow directions on f_1 , f_2 , and f_3 are such that when they are positive they tend to *decrease* the displacement of the node. Hence they act out of the node in the circuit. This convention is adopted because we can express f_1 and f_2 [see (2.2.3)] as

$$f_1 = K_1(x - l_1) \quad (2.2.28)$$

$$f_2 = K_2(x - l_2) \quad (2.2.29)$$

and can express f_3 [see (2.2.5)] as

$$f_3 = B \frac{dx}{dt}. \quad (2.2.30)$$

* *Ibid.*, pp. 195–197.

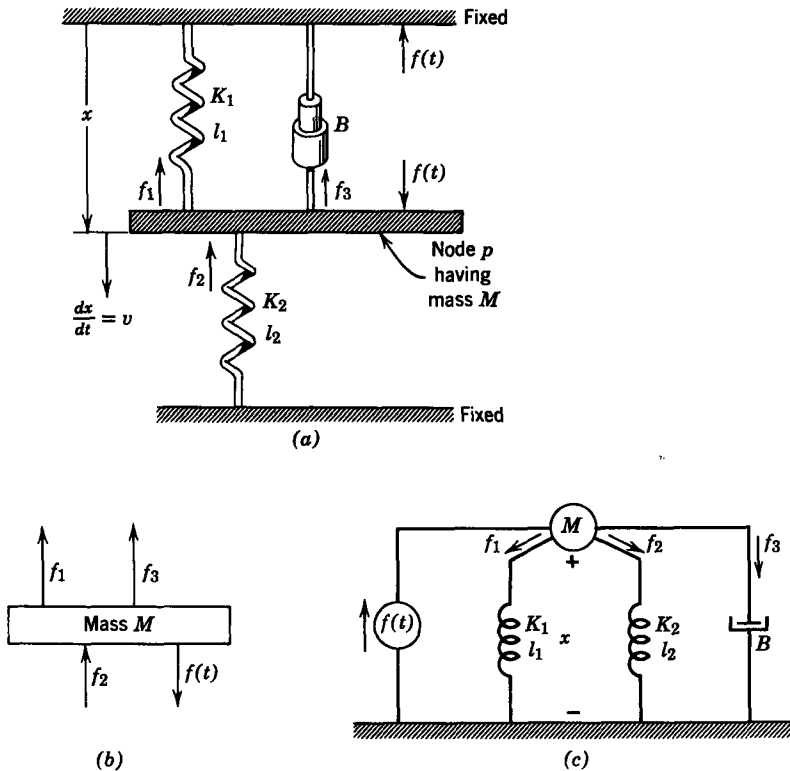


Fig. 2.2.15 System showing forces at a node: (a) system; (b) free body diagram of mass M ; (c) circuit.

Newton's second law (2.2.10) requires that the *algebraic sum of all forces applied to a node in the positive x -direction must equal the acceleration force for the mass of the node*. For the example in Fig. 2.2.15 this requires that

$$f(t) - f_1 - f_2 - f_3 = M \frac{d^2x}{dt^2}. \quad (2.2.31)$$

Note that forces acting in the $+x$ -direction "flow" into the node in Fig. 2.2.15c. Substitution from (2.2.28) through (2.2.30) into (2.2.31) yields the differential equation

$$f(t) - K_1(x - l_1) - K_2(x - l_2) - B \frac{dx}{dt} = M \frac{d^2x}{dt^2}. \quad (2.2.32)$$

Thus, if the system constants are known and $f(t)$ is specified, this differential equation can be solved to find $x(t)$; (2.2.32) is the equation of motion for the system in Fig. 2.2.15.

We can generalize (2.2.31) to describe any mechanical node with mass M and displacement x and with n forces applied by sources and passive elements.

$$\sum_{i=1}^n f_i = M \frac{d^2x}{dt^2}. \tag{2.2.33}$$

We must exercise caution to include the correct sign on each force in the summation.

In a rotational system the nodes have moments of inertia; thus the summation of torques applied to a node must equal the acceleration torque of the moment of inertia associated with the node. For a node with n torques applied the expression is

$$\sum_{i=1}^n T_i = J \frac{d^2\theta}{dt^2}. \tag{2.2.34}$$

Once again care must be exercised in attaching the correct sign to each torque in the summation.

As an example of the application of (2.2.34), consider the rotational system in Fig. 2.2.16a for which the mechanical circuit is shown in Fig. 2.2.16b.

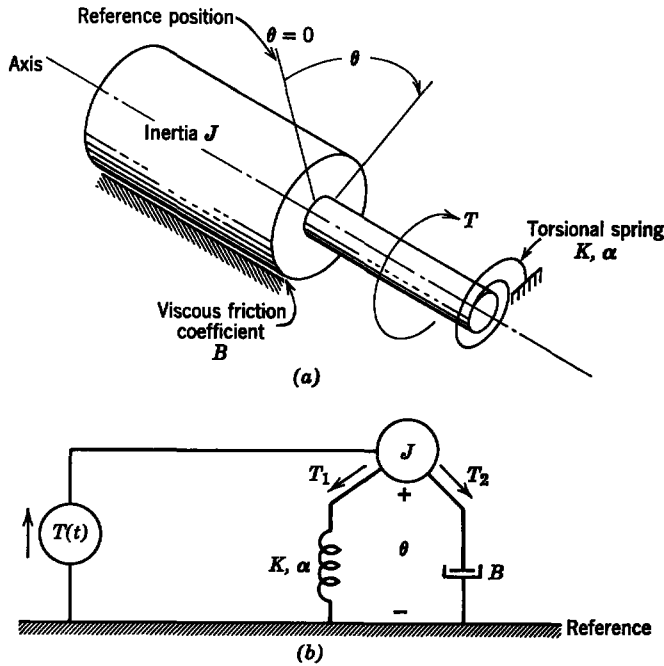


Fig. 2.2.16 Rotational mechanical system: (a) system; (b) mechanical circuit.

We represent the torques *applied to the node in the $-\theta$ -direction* by the passive elements as T_1 and T_2 . Reference to (2.2.4) and (2.2.6) shows that these torques can be expressed as

$$T_1 = K(\theta - \alpha), \quad (2.2.35)$$

$$T_2 = B \frac{d\theta}{dt}. \quad (2.2.36)$$

By use of these expressions with the source and arrow directions in Fig. 2.2.16*b* (2.2.34) yields the equation of motion

$$T(t) - K(\theta - \alpha) - B \frac{d\theta}{dt} = J \frac{d^2\theta}{dt^2}. \quad (2.2.37)$$

Once again, torques acting in the $+\theta$ -direction “flow” into the node. When the constants are known and $T(t)$ is specified, this differential equation can be used to find the response $\theta(t)$.

An important point should be made here. In a mechanical system the reference or “ground” is usually considered fixed; this is necessary if any masses are involved. If the reference is fixed and elements are attached to the reference, it implies that forces not usually considered are available to prevent the ground point from moving. The surface of the earth, for example, is often taken as a reference, and although the earth will move a certain amount when a net force is exerted on it this movement is extremely small and can be neglected.

Equations 2.2.33 and 2.2.34, which are analogous for our forms of mechanical equivalent circuits to Kirchhoff’s current law, are used almost exclusively for formulating equations of motion for mechanical systems. A second relation for mechanical systems analogous to Kirchhoff’s voltage law is seldom used in a formal way but must at all times be satisfied. This second relation is called geometrical compatibility or continuity of space. To illustrate this concept with an example we use the system in Fig. 2.2.17. This system has three mechanical nodes (p , q , and r) whose velocities (v_1 , v_2 , and v_3) are measured in relation to the fixed point g .

The mechanical circuit of Fig. 2.2.17*b* has two independent mechanical loops. From examination of Fig. 2.2.17*a* it is evident that the velocity of p must be equal to the velocity of q , plus the velocity of p relative to q .

$$v_1 + (v_2 - v_1) - v_2 = 0. \quad (2.2.38)$$

This condition, which is identically satisfied, states that in the left-hand loop of Fig. 2.2.17*b* the sum of the velocity differences around the loop must be zero. Similarly, for the right-hand loop of Fig. 2.2.17*b*

$$v_2 + (v_3 - v_2) - v_3 = 0. \quad (2.2.39)$$

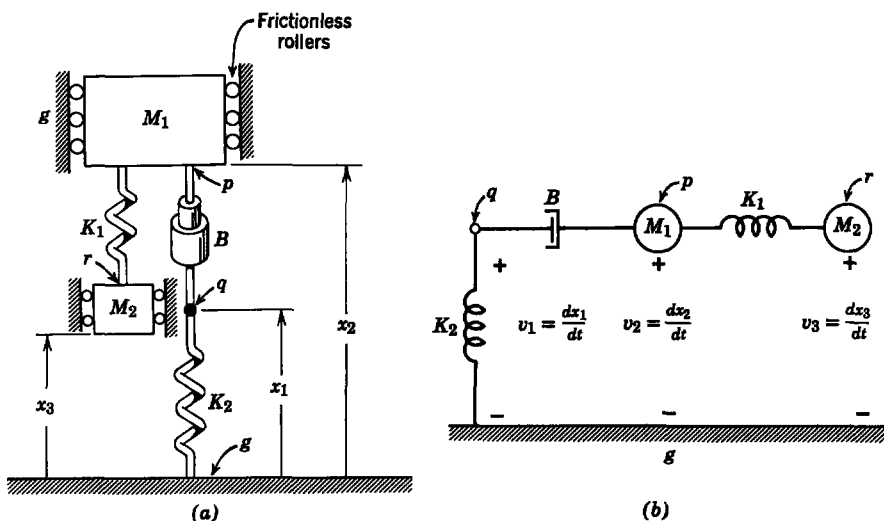


Fig. 2.2.17 Mechanical system and its network diagram: (a) system; (b) mechanical equivalent circuit.

We can generalize from this example to state that for a mechanical loop with n elements geometrical compatibility requires that

$$\sum_{i=1}^n (v_{i+1} - v_i) = 0, \quad (2.2.40)$$

where $(v_{i+1} - v_i)$ is the velocity difference across the i th element taken positive in the direction of summation. An equivalent relation for rotational systems can be obtained by summing angular velocity differences around a loop.

In establishing the loop equations it is preferable to work with *velocity differences* as above. If displacements are used, the geometric compatibility equations will contain constant terms, such as unstretched lengths of springs, and will be more complicated than (2.2.40).

As stated earlier, the geometric compatibility relation is not often explicitly used in formulating equations of motion. Nonetheless, it must be satisfied.

Example 2.2.1. Consider the simplified model of an automobile suspension system shown in Fig. 2.2.18a. The tire surface is excited by variations in the road surface. We wish to formulate equations suitable for determining the motion of the auto mass M_1 and the force on the tires.

The mechanical circuit is shown in Fig. 2.2.18b. The road acts as a position source applied to node 1, as shown. We assume that the references for measuring displacements of the two springs are chosen so that the equilibrium lengths are zero. In the circuit of

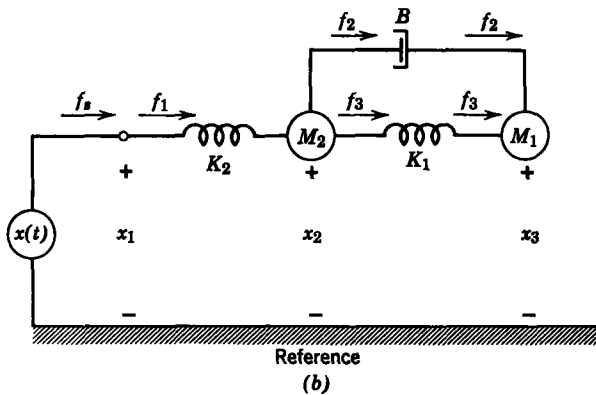
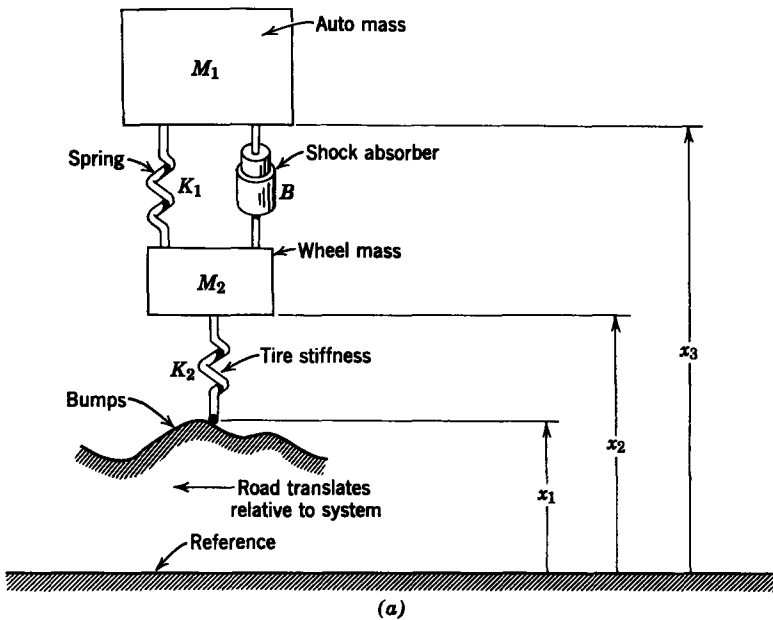


Fig. 2.2.18 One-dimensional model of auto suspension: (a) system; (b) equivalent circuit.

Fig. 2.2.18b three passive ideal mechanical elements (excluding masses) appear. The equations for these ideal elements are

$$f_1 = K_2(x_1 - x_2), \quad (a)$$

$$f_2 = B \left(\frac{dx_2}{dt} - \frac{dx_3}{dt} \right), \quad (b)$$

$$f_3 = K_1(x_2 - x_3). \quad (c)$$

These equations may be combined at nodes 2 and 3 according to (2.2.33) to eliminate the forces at those nodes (which are not of interest).

$$K_2(x_1 - x_2) = B\left(\frac{dx_2}{dt} - \frac{dx_3}{dt}\right) + K_1(x_2 - x_3) + M_2 \frac{d^2x_2}{dt^2}, \quad (d)$$

$$B\left(\frac{dx_2}{dt} - \frac{dx_3}{dt}\right) + K_1(x_2 - x_3) = M_1 \frac{d^2x_3}{dt^2}. \quad (e)$$

With the specified position source

$$x_1 = x(t), \quad (f)$$

(d) and (e) can be solved for x_2 and x_3 . Then the force f_s applied to the tires by the road can be found from (a) as

$$f_s = f_1 = K_2(x_1 - x_2). \quad (g)$$

Note that the forces acting on the reference in the network diagram *do not balance* but equal f_s . It is presumed that the force transmitted to the earth by the automobile tires will not move the earth.

2.3 DISCUSSION

In this chapter we have laid the foundation for studying lumped-parameter electromechanics by reviewing the derivations of lumped electric circuit elements, by generalizing the derivations to include the effects of mechanical motion, and by reviewing the basic definitions and techniques of rigid-body mechanics. The stage is now set to include the electromechanical coupling network of Fig. 2.0.1 and to study some general properties of electromechanical systems, including the techniques for obtaining complete equations of motion.