chapter 6

electromagnetic induction

In our development thus far, we have found the electric and magnetic fields to be uncoupled. A net charge generates an electric field while a current is the source of a magnetic field. In 1831 Michael Faraday experimentally discovered that a time varying magnetic flux through a conducting loop also generated a voltage and thus an electric field, proving that electric and magnetic fields are coupled.

6-1 FARADAY'S LAW OF INDUCTION

6-1-1 The Electromotive Force (EMF)

Faraday's original experiments consisted of a conducting loop through which he could impose a dc current via a switch. Another short circuited loop with no source attached was nearby, as shown in Figure 6-1. When a dc current flowed in loop 1, no current flowed in loop 2. However, when the voltage was first applied to loop 1 by closing the switch, a transient current flowed in the opposite direction in loop 2.

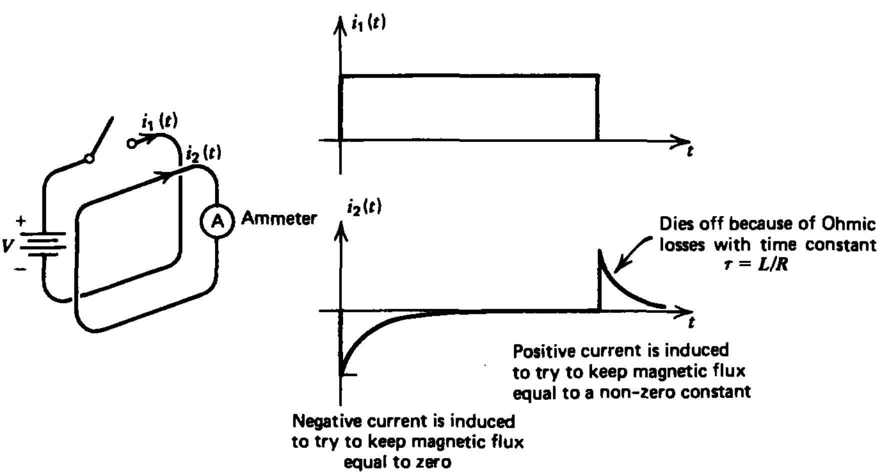


Figure 6-1 Faraday's experiments showed that a time varying magnetic flux through a closed conducting loop induced a current in the direction so as to keep the flux through the loop constant.

When the switch was later opened, another transient current flowed in loop 2, this time in the same direction as the original current in loop 1. Currents are induced in loop 2 whenever a time varying magnetic flux due to loop 1 passes through it.

In general, a time varying magnetic flux can pass through a circuit due to its own or nearby time varying current or by the motion of the circuit through a magnetic field. For any loop, as in Figure 6-2, Faraday's law is

$$\text{EMF} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{1}$$

where EMF is the electromotive force defined as the line integral of the electric field. The minus sign is introduced on the right-hand side of (1) as we take the convention that positive flux flows in the direction perpendicular to the direction of the contour by the right-hand rule.

6-1-2 Lenz's Law

The direction of induced currents is always such as to oppose any changes in the magnetic flux already present. Thus in Faraday's experiment, illustrated in Figure 6-1, when the switch in loop 1 is first closed there is no magnetic flux in loop 2 so that the induced current flows in the opposite direction with its self-magnetic field opposite to the imposed field. The induced current tries to keep a zero flux through

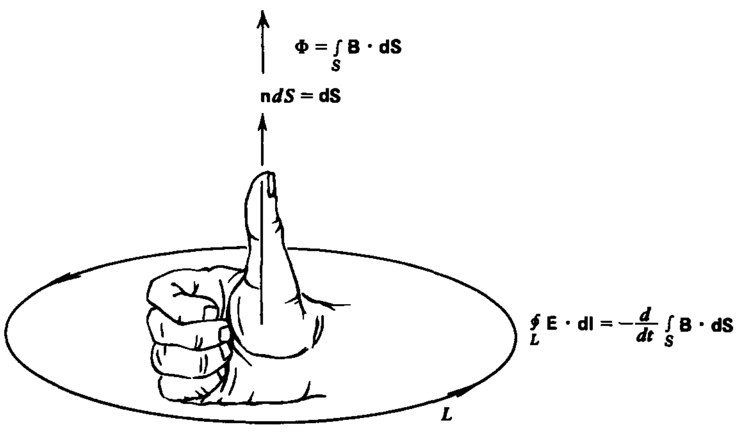


Figure 6-2 Faraday's law states that the line integral of the electric field around a closed loop equals the time rate of change of magnetic flux through the loop. The positive convention for flux is determined by the right-hand rule of curling the fingers on the right hand in the direction of traversal around the loop. The thumb then points in the direction of positive magnetic flux.

loop 2. If the loop is perfectly conducting, the induced current flows as long as current flows in loop 1, with zero net flux through the loop. However, in a real loop, resistive losses cause the current to exponentially decay with an L/R time constant, where L is the self-inductance of the loop and R is its resistance. Thus, in the dc steady state the induced current has decayed to zero so that a constant magnetic flux passes through loop 2 due to the current in loop 1.

When the switch is later opened so that the current in loop 1 goes to zero, the second loop tries to maintain the constant flux already present by inducing a current flow in the same direction as the original current in loop 1. Ohmic losses again make this induced current die off with time.

If a circuit or any part of a circuit is made to move through a magnetic field, currents will be induced in the direction such as to try to keep the magnetic flux through the loop constant. The force on the moving current will always be opposite to the direction of motion.

Lenz's law is clearly demonstrated by the experiments shown in Figure 6-3. When a conducting ax is moved into a magnetic field, eddy currents are induced in the direction where their self-flux is opposite to the applied magnetic field. The Lorentz force is then in the direction opposite to the motion of the ax. This force decreases with time as the currents decay with time due to Ohmic dissipation. If the ax was slotted, effectively creating a very high resistance to the eddy currents, the reaction force becomes very small as the induced current is small.

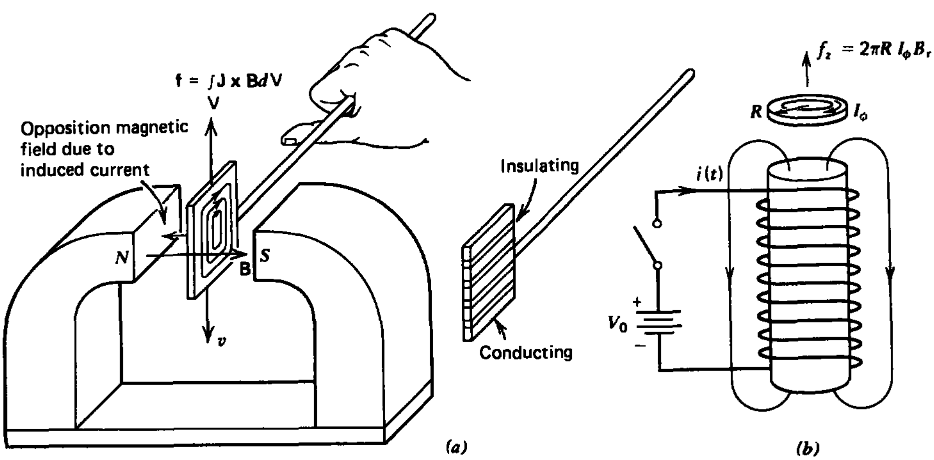


Figure 6-3 Lenz's law. (*a*) Currents induced in a conductor moving into a magnetic field exert a force opposite to the motion. The induced currents can be made small by slotting the ax. (*b*) A conducting ring on top of a coil is flipped off when a current is suddenly applied, as the induced currents try to keep a zero flux through the ring.

When the current is first turned on in the coil in Figure 6-3*b*, the conducting ring that sits on top has zero flux through it. Lenz's law requires that a current be induced opposite to that in the coil. Instantaneously there is no z component of magnetic field through the ring so the flux must return radially. This creates an upwards force:

$$\mathbf{f} = 2\pi R\mathbf{I}\times\mathbf{B} = 2\pi RI_\phi B_r \mathbf{i}_z \tag{2}$$

which flips the ring off the coil. If the ring is cut radially so that no circulating current can flow, the force is zero and the ring does not move.

(a) Short Circuited Loop

To be quantitative, consider the infinitely long time varying line current $I(t)$ in Figure 6-4, a distance r from a rectangular loop of wire with Ohmic conductivity σ, cross-sectional area A, and total length $l = 2(D+d)$. The magnetic flux through the loop due to $I(t)$ is

$$\begin{aligned}\Phi_m &= \int_{z=-D/2}^{D/2}\int_{r}^{r+d} \mu_0 H_\phi(r')\,dr'\,dz\\ &= \frac{\mu_0 ID}{2\pi}\int_r^{r+d}\frac{dr'}{r'} = \frac{\mu_0 ID}{2\pi}\ln\frac{r+d}{r}\end{aligned} \tag{3}$$

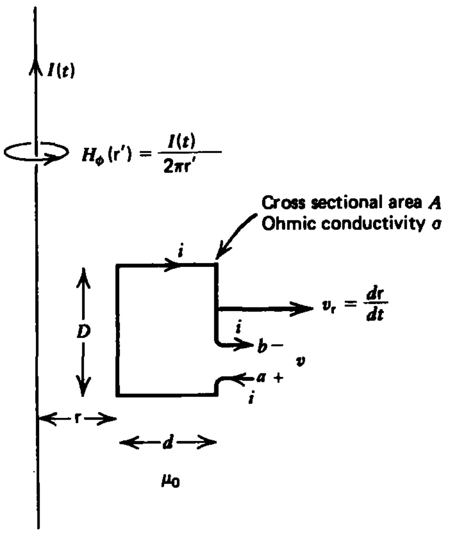


Figure 6-4 A rectangular loop near a time varying line current. When the terminals are short circuited the electromotive force induces a current due to the time varying mutual flux and/or because of the motion of the circuit through the imposed nonuniform magnetic field of the line current. If the loop terminals are open circuited there is no induced current but a voltage develops.

The mutual inductance M is defined as the flux to current ratio where the flux through the loop is due to an external current. Then (3) becomes

$$\Phi_m = M(\mathrm{r})I, \qquad M(\mathrm{r}) = \frac{\mu_0 D}{2\pi}\ln\frac{\mathrm{r}+d}{\mathrm{r}} \tag{4}$$

When the loop is short circuited ($v = 0$), the induced Ohmic current i gives rise to an electric field $[E = J/\sigma = i/(A\sigma)]$ so that Faraday's law applied to a contour within the wire yields an electromotive force just equal to the Ohmic voltage drop:

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \frac{il}{\sigma A} = iR = -\frac{d\Phi}{dt} \tag{5}$$

where $R = l/(\sigma A)$ is the resistance of the loop. By convention, the current is taken as positive in the direction of the line integral.

The flux in (5) has contributions both from the imposed current as given in (3) and from the induced current proportional to the loop's self-inductance L, which for example is given in Section 5-4-3c for a square loop ($D = d$):

$$\Phi = M(\mathrm{r})I + Li \tag{6}$$

If the loop is also moving radially outward with velocity $v_r = d\mathrm{r}/dt$, the electromotively induced Ohmic voltage is

$$\begin{aligned} -iR &= \frac{d\Phi}{dt} \\ &= M(\mathrm{r})\frac{dI}{dt} + I\frac{dM(\mathrm{r})}{dt} + L\frac{di}{dt} \\ &= M(\mathrm{r})\frac{dI}{dt} + I\frac{dM}{d\mathrm{r}}\frac{d\mathrm{r}}{dt} + L\frac{di}{dt} \end{aligned} \tag{7}$$

where L is not a function of the loop's radial position.

If the loop is stationary, only the first and third terms on the right-hand side contribute. They are nonzero only if the currents change with time. The second term is due to the motion and it has a contribution even for dc currents.

Turn-on Transient. If the loop is stationary ($d\mathrm{r}/dt = 0$) at $\mathrm{r} = \mathrm{r}_0$, (7) reduces to

$$L\frac{di}{dt} + iR = -M(\mathrm{r}_0)\frac{dI}{dt} \tag{8}$$

If the applied current I is a dc step turned on at $t = 0$, the solution to (8) is

$$i(t) = -\frac{M(\mathrm{r}_0)I}{L}e^{-(R/L)t}, \quad t > 0 \tag{9}$$

where the impulse term on the right-hand side of (8) imposes the initial condition $i(t=0)=-M(\mathrm{r}_0)I/L$. The current is negative, as Lenz's law requires the self-flux to oppose the applied flux.

Turn-off Transient. If after a long time T the current I is instantaneously turned off, the solution is

$$i(t)=\frac{M(\mathrm{r}_0)I}{L}e^{-(R/L)(t-T)},\quad t>T \tag{10}$$

where now the step decrease in current I at $t=T$ reverses the direction of the initial current.

Motion with a dc Current. With a dc current, the first term on the right-hand side in (7) is zero yielding

$$L\frac{di}{dt}+iR=\frac{\mu_0 IDd}{2\pi\mathrm{r}(\mathrm{r}+d)}\frac{d\mathrm{r}}{dt} \tag{11}$$

To continue, we must specify the motion so that we know how r changes with time. Let's consider the simplest case when the loop has no resistance ($R=0$). Then (11) can be directly integrated as

$$Li=-\frac{\mu_0 ID}{2\pi}\ln\frac{1+d/\mathrm{r}}{1+d/\mathrm{r}_0} \tag{12}$$

where we specify that the current is zero when $\mathrm{r}=\mathrm{r}_0$. This solution for a lossless loop only requires that the total flux of (6) remain constant. The current is positive when $\mathrm{r}>\mathrm{r}_0$ as the self-flux must aid the decreasing imposed flux. The current is similarly negative when $\mathrm{r}<\mathrm{r}_0$ as the self-flux must cancel the increasing imposed flux.

The force on the loop for all these cases is only due to the force on the z-directed current legs at r and $\mathrm{r}+d$:

$$\begin{aligned} f_\mathrm{r}&=\frac{\mu_0 DiI}{2\pi}\left(\frac{1}{\mathrm{r}+d}-\frac{1}{\mathrm{r}}\right)\\ &=-\frac{\mu_0 DiId}{2\pi\mathrm{r}(\mathrm{r}+d)}\end{aligned} \tag{13}$$

being attractive if $iI>0$ and repulsive if $iI<0$.

(b) Open Circuited Loop

If the loop is open circuited, no induced current can flow and thus the electric field within the wire is zero ($\mathbf{J}=\sigma\mathbf{E}=0$). The electromotive force then only has a contribution from the gap between terminals equal to the negative of the voltage:

$$\oint_L \mathbf{E}\cdot d\mathbf{l}=\int_b^a \mathbf{E}\cdot d\mathbf{l}=-v=-\frac{d\Phi}{dt}\Rightarrow v=\frac{d\Phi}{dt} \tag{14}$$

Note in Figure 6-4 that our convention is such that the current i is always defined positive flowing out of the positive voltage terminal into the loop. The flux Φ in (14) is now only due to the mutual flux given by (3), as with $i=0$ there is no self-flux. The voltage on the moving open circuited loop is then

$$v=M(\mathrm{r})\frac{dI}{dt}+I\frac{dM}{d\mathrm{r}}\frac{d\mathrm{r}}{dt} \tag{15}$$

(c) Reaction Force

The magnetic force on a short circuited moving loop is always in the direction opposite to its motion. Consider the short circuited loop in Figure 6-5, where one side of the loop moves with velocity v_x. With a uniform magnetic field applied normal to the loop pointing out of the page, an expansion of the loop tends to link more magnetic flux requiring the induced current to flow clockwise so that its self-flux is in the direction given by the right-hand rule, opposite to the applied field. From (1) we have

$$\oint_L \mathbf{E}\cdot d\mathbf{l}=\frac{il}{\sigma A}=iR=-\frac{d\Phi}{dt}=B_0D\frac{dx}{dt}=B_0Dv_x \tag{16}$$

where $l=2(D+x)$ also changes with time. The current is then

$$i=\frac{B_0Dv_x}{R} \tag{17}$$

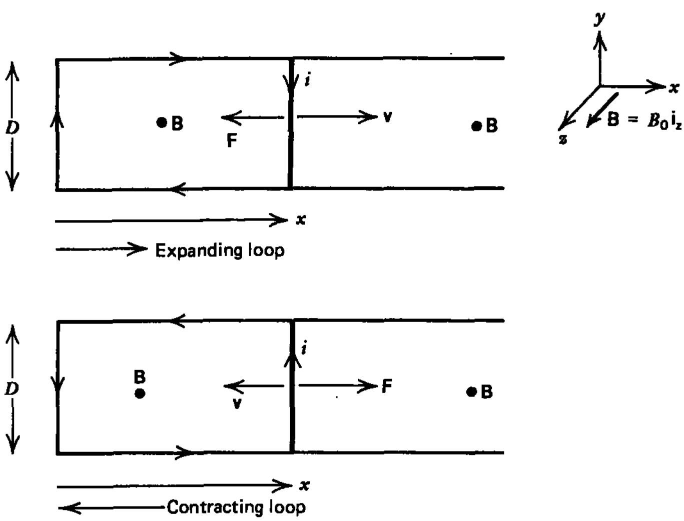


Figure 6-5 If a conductor moves perpendicular to a magnetic field a current is induced in the direction to cause the Lorentz force to be opposite to the motion. The total flux through the closed loop, due to both the imposed field and the self-field generated by the induced current, tries to remain constant.

where we neglected the self-flux generated by i, assuming it to be much smaller than the applied flux due to B_0. Note also that the applied flux is negative, as the right-hand rule applied to the direction of the current defines positive flux into the page, while the applied flux points outwards.

The force on the moving side is then to the left,

$$\mathbf{f} = -iD\mathbf{i}_y \times B_0\mathbf{i}_z = -iDB_0\mathbf{i}_x = -\frac{B_0^2D^2v_x}{R}\mathbf{i}_x \tag{18}$$

opposite to the velocity.

However if the side moves to the left ($v_x < 0$), decreasing the loop's area thereby linking less flux, the current reverses direction as does the force.

6-1-3 Laminations

The induced eddy currents in Ohmic conductors results in Ohmic heating. This is useful in induction furnaces that melt metals, but is undesired in many iron core devices. To reduce this power loss, the cores are often sliced into many thin sheets electrically insulated from each other by thin oxide coatings. The current flow is then confined to lie within a thin sheet and cannot cross over between sheets. The insulating laminations serve the same purpose as the cuts in the slotted ax in Figure 6-3*a*.

The rectangular conductor in Figure 6-6*a* has a time varying magnetic field $B(t)$ passing through it. We approximate the current path as following the rectangular shape so that

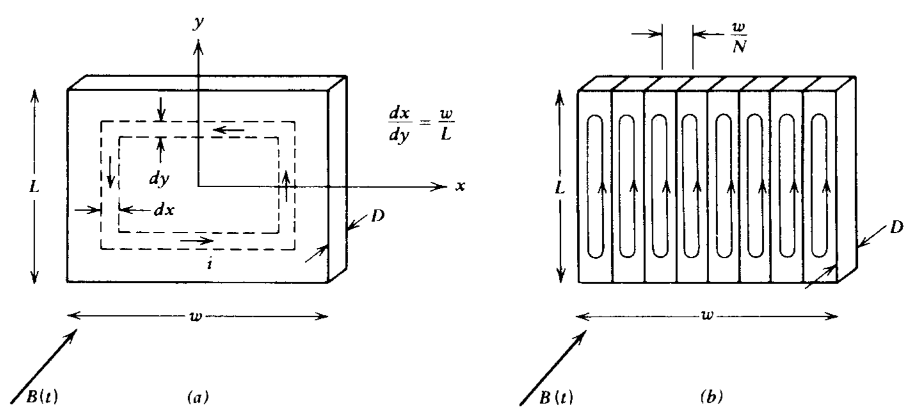


Figure 6-6 (*a*) A time varying magnetic field through a conductor induces eddy currents that cause Ohmic heating. (*b*) If the conductor is laminated so that the induced currents are confined to thin strips, the dissipated power decreases.

the flux through the loop of incremental width dx and dy of area $4xy$ is

$$\Phi = -4xyB(t) \tag{19}$$

where we neglect the reaction field of the induced current assuming it to be much smaller than the imposed field. The minus sign arises because, by the right-hand rule illustrated in Figure 6-2, positive flux flows in the direction opposite to $B(t)$. The resistance of the loop is

$$R_x = \frac{4}{\sigma D}\left(\frac{y}{dx}+\frac{x}{dy}\right) = \frac{4}{\sigma D}\frac{L}{w}\frac{x}{dx}\left[1+\left(\frac{w}{L}\right)^2\right] \tag{20}$$

The electromotive force around the loop then just results in an Ohmic current:

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = iR_x = \frac{-d\Phi}{dt} = 4xy\frac{dB}{dt} = \frac{4L}{w}x^2\frac{dB}{dt} \tag{21}$$

with dissipated power

$$dp = i^2R_x = \frac{4Dx^3\sigma L(dB/dt)^2\,dx}{w[1+(w/L)^2]} \tag{22}$$

The total power dissipated over the whole sheet is then found by adding the powers dissipated in each incremental loop:

$$\begin{aligned} P &= \int_0^{w/2} dp \\ &= \frac{4D(dB/dt)^2\sigma L}{w[1+(w/L)^2]}\int_0^{w/2} x^3\,dx \\ &= \frac{LDw^3\sigma(dB/dt)^2}{16[1+(w/L)^2]} \end{aligned} \tag{23}$$

If the sheet is laminated into N smaller ones, as in Figure 6-6b, each section has the same solution as (23) if we replace w by w/N. The total power dissipated is then N times the power dissipated in a single section:

$$P = \frac{LD(w/N)^3\sigma(dB/dt)^2N}{16[1+(w/NL)^2]} = \frac{\sigma LDw^3(dB/dt)^2}{16N^2[1+(w/NL)^2]} \tag{24}$$

As N becomes large so that $w/NL \ll 1$, the dissipated power decreases as $1/N^2$.

6-1-4 Betatron

The cyclotron, discussed in Section 5-1-4, is not used to accelerate electrons because their small mass allows them to reach relativistic speeds, thereby increasing their mass and decreasing their angular speed. This puts them out of phase with the applied voltage. The betatron in Figure 6-7 uses the transformer principle where the electrons circulating about the evacuated toroid act like a secondary winding. The imposed time varying magnetic flux generates an electric field that accelerates the electrons.

Faraday's law applied to a contour following the charge's trajectory at radius R yields

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = E_\phi 2\pi R = -\frac{d\Phi}{dt} \tag{25}$$

which accelerates the electrons as

$$m\frac{dv_\phi}{dt} = -eE_\phi = \frac{e}{2\pi R}\frac{d\Phi}{dt} \Rightarrow v_\phi = \frac{e}{2\pi m R}\Phi \tag{26}$$

The electrons move in the direction so that their self-magnetic flux is opposite to the applied flux. The resulting Lorentz force is radially inward. A stable orbit of constant radius R is achieved if this force balances the centrifugal force:

$$m\frac{dv_r}{dt} = \frac{mv_\phi^2}{R} - ev_\phi B_z(R) = 0 \tag{27}$$

which from (26) requires the flux and magnetic field to be related as

$$\Phi = 2\pi R^2 B_z(R) \tag{28}$$

This condition cannot be met by a uniform field (as then $\Phi = \pi R^2 B_z$) so in practice the imposed field is made to approximately vary with radial position as

$$B_z(r) = B_0\left(\frac{R}{r}\right) \Rightarrow \Phi = 2\pi \int_{r=0}^{R} B_z(r) r\,dr = 2\pi R^2 B_0 \tag{29}$$

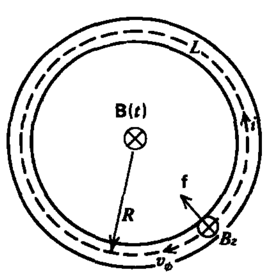


Figure 6-7 The betatron accelerates electrons to high speeds using the electric field generated by a time varying magnetic field.

where R is the equilibrium orbit radius, so that (28) is satisfied.

The magnetic field must remain curl free where there is no current so that the spatial variation in (29) requires a radial magnetic field component:

$$\nabla\times\mathbf{B}=\left(\frac{\partial B_r}{\partial z}-\frac{\partial B_z}{\partial r}\right)\mathbf{i}_\phi=0\Rightarrow B_r=-\frac{B_0R}{r^2}z \tag{30}$$

Then any z-directed perturbation displacements

$$\frac{d^2z}{dt^2}=\frac{ev_\phi}{m}B_r(R)=-\left(\frac{eB_0}{m}\right)^2 z$$

$$\Rightarrow z=A_1\sin\omega_0 t+A_2\cos\omega_0 t,\qquad \omega_0=\frac{eB_0}{m} \tag{31}$$

have sinusoidal solutions at the cyclotron frequency $\omega_0=eB_0/m$, known as betatron oscillations.

6-1-5 Faraday's Law and Stokes' Theorem

The integral form of Faraday's law in (1) shows that with magnetic induction the electric field is no longer conservative as its line integral around a closed path is non-zero. We may convert (1) to its equivalent differential form by considering a stationary contour whose shape does not vary with time. Because the area for the surface integral does not change with time, the time derivative on the right-hand side in (1) may be brought inside the integral but becomes a partial derivative because $\mathbf{B}$ is also a function of position:

$$\oint_L \mathbf{E}\cdot d\mathbf{l}=-\int_S\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S} \tag{32}$$

Using Stokes' theorem, the left-hand side of (32) can be converted to a surface integral,

$$\oint_L \mathbf{E}\cdot d\mathbf{l}=\int_S\nabla\times\mathbf{E}\cdot d\mathbf{S}=-\int_S\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S} \tag{33}$$

which is equivalent to

$$\int_S\left(\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}\right)\cdot d\mathbf{S}=0 \tag{34}$$

Since this last relation is true for any surface, the integrand itself must be zero, which yields Faraday's law of induction in differential form as

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \tag{35}$$

6-2 MAGNETIC CIRCUITS

Various alloys of iron having very high values of relative permeability are typically used in relays and machines to constrain the magnetic flux to mostly lie within the permeable material.

6-2-1 Self-Inductance

The simple magnetic circuit in Figure 6-8 has an N turn coil wrapped around a core with very high relative permeability idealized to be infinite. There is a small air gap of length s in the core. In the core, the magnetic flux density $\mathbf{B}$ is proportional to the magnetic field intensity $\mathbf{H}$ by an infinite permeability μ. The $\mathbf{B}$ field must remain finite to keep the flux and coil voltage finite so that the $\mathbf{H}$ field in the core must be zero:

$$\lim_{\mu\to\infty} \mathbf{B}=\mu\mathbf{H} \Rightarrow \begin{cases} \mathbf{H}=0 \\ \mathbf{B} \text{ finite} \end{cases} \tag{1}$$

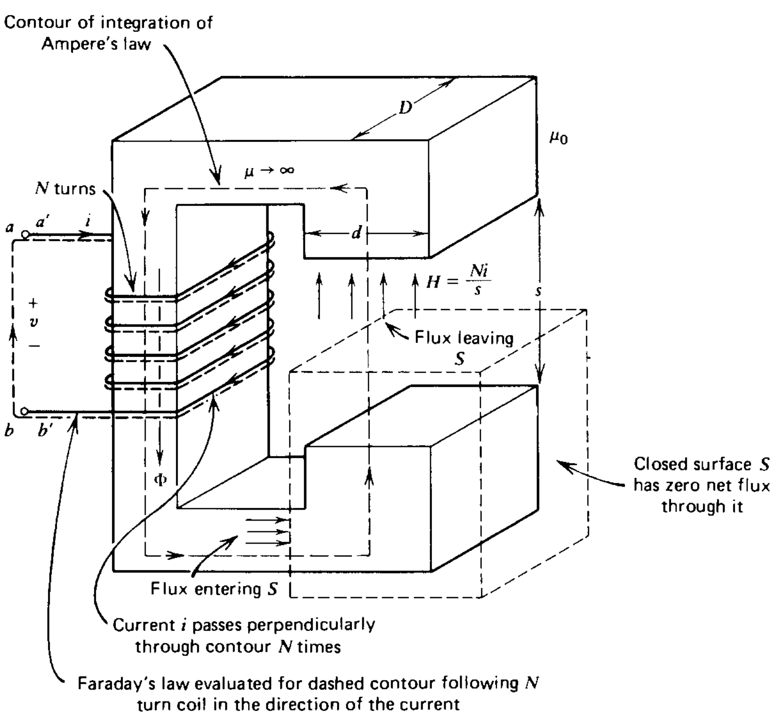


Figure 6-8 The magnetic field is zero within an infinitely permeable magnetic core and is constant in the air gap if we neglect fringing. The flux through the air gap is constant at every cross section of the magnetic circuit and links the N turn coil N times.

The **H** field can then only be nonzero in the air gap. This field emanates perpendicularly from the pole faces as no surface currents are present so that the tangential component of **H** is continuous and thus zero. If we neglect fringing field effects, assuming the gap s to be much smaller than the width d or depth D, the **H** field is uniform throughout the gap. Using Ampere's circuital law with the contour shown, the only nonzero contribution is in the air gap,

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = Hs = I_{\text{total enclosed}} = Ni \tag{2}$$

where we realize that the coil current crosses perpendicularly through our contour N times. The total flux in the air gap is then

$$\Phi = \mu_0 HDd = \frac{\mu_0 NDd}{s} i \tag{3}$$

Because the total flux through any closed surface is zero,

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0 \tag{4}$$

all the flux leaving S in Figure 6-8 on the air gap side enters the surface through the iron core, as we neglect leakage flux in the fringing field. The flux at any cross section in the iron core is thus constant, given by (3).

If the coil current i varies with time, the flux in (3) also varies with time so that a voltage is induced across the coil. We use the integral form of Faraday's law for a contour that lies within the winding with Ohmic conductivity σ, cross sectional area A, and total length l. Then the current density and electric field within the wire is

$$J = \frac{i}{A}, \qquad E = \frac{J}{\sigma} = \frac{i}{\sigma A} \tag{5}$$

so that the electromotive force has an Ohmic part as well as a contribution due to the voltage across the terminals:

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \underbrace{\int_{a'}^{b'} \frac{i}{\sigma A}\, dl}_{\substack{iR \\ \text{in wire}}} + \underbrace{\int_b^a \mathbf{E}\cdot d\mathbf{l}}_{\substack{-v \\ \text{across} \\ \text{terminals}}} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{6}$$

The surface S on the right-hand side is quite complicated because of the spiral nature of the contour. If the coil only had one turn, the right-hand side of (6) would just be the time derivative of the flux of (3). For two turns, as in Figure 6-9, the flux links the coil twice, while for N turns the total flux

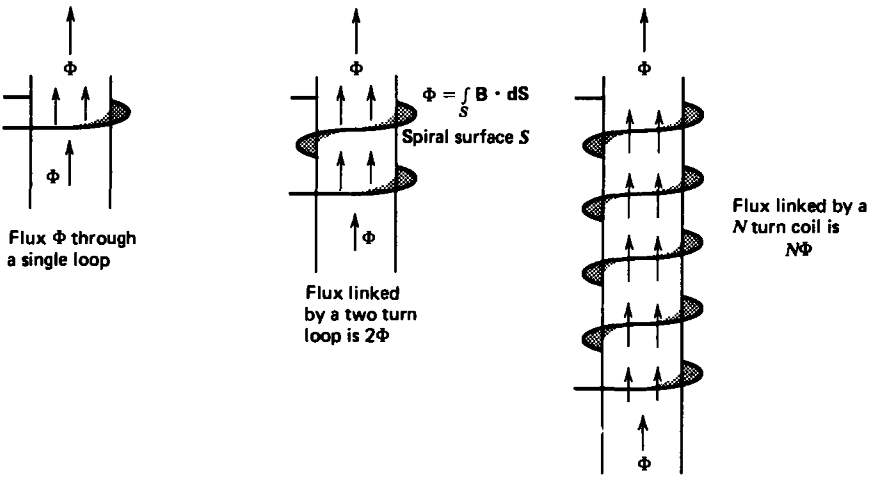


Figure 6-9 The complicated spiral surface for computation of the linked flux by an N turn coil can be considered as N single loops each linking the same flux Φ.

linked by the coil is $N\Phi$. Then (6) reduces to

$$v = iR + L\frac{di}{dt} \tag{7}$$

where the self-inductance is defined as

$$L = \frac{N\Phi}{i} = \frac{N\int_S \mathbf{B}\cdot d\mathbf{S}}{\oint_L \mathbf{H}\cdot d\mathbf{l}} = \frac{\mu_0 N^2 Dd}{s}\ \text{henry}\ [\text{kg-m}^2\text{-A}^{-2}\text{-s}^{-2}] \tag{8}$$

For linearly permeable materials, the inductance is always independent of the excitations and only depends on the geometry. Because of the fixed geometry, the inductance is a constant and thus was taken outside the time derivative in (7). In geometries that change with time, the inductance will also be a function of time and must remain under the derivative. The inductance is always proportional to the square of the number of coil turns. This is because the flux Φ in the air gap is itself proportional to N and it links the coil N times.

EXAMPLE 6-1 SELF-INDUCTANCES

Find the self-inductances for the coils shown in Figure 6-10.

(a) Solenoid

An N turn coil is tightly wound upon a cylindrical core of radius a, length l, and permeability μ.

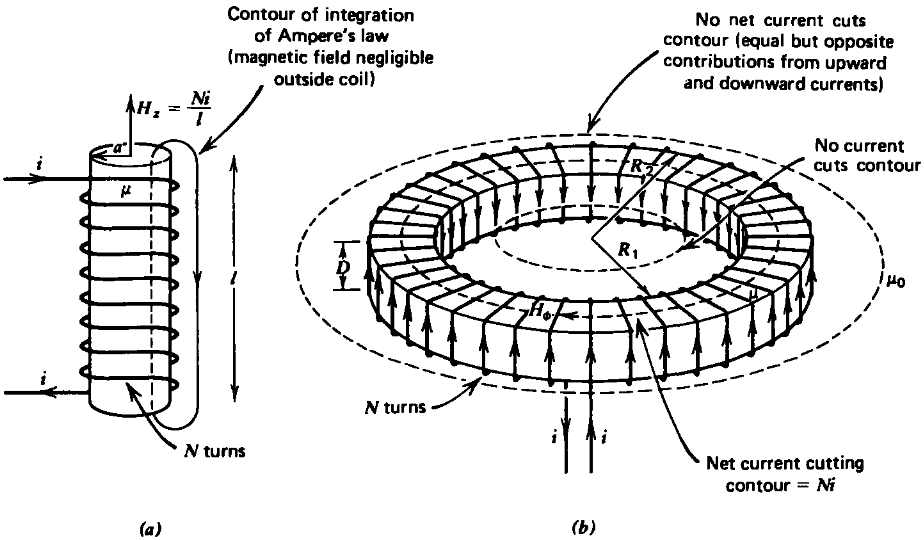


Figure 6-10 Inductances. (*a*) Solenoidal coil; (*b*) toroidal coil.

SOLUTION

A current i flowing in the wire approximates a surface current

$$K_\phi = Ni/l$$

If the length l is much larger than the radius a, we can neglect fringing field effects at the ends and the internal magnetic field is approximately uniform and equal to the surface current,

$$H_z = K_\phi = \frac{Ni}{l}$$

as we assume the exterior magnetic field is negligible. The same result is obtained using Ampere's circuital law for the contour shown in Figure 6-10*a*. The flux links the coil N times:

$$L = \frac{N\Phi}{i} = \frac{N\mu H_z \pi a^2}{i} = \frac{N^2 \mu \pi a^2}{l}$$

(b) Toroid

An N turn coil is tightly wound around a donut-shaped core of permeability μ with a rectangular cross section and inner and outer radii R_1 and R_2.

SOLUTION

Applying Ampere's circuital law to the three contours shown in Figure 6-10*b*, only the contour within the core has a net current passing through it:

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = H_\phi 2\pi r = \begin{cases} 0, & r < R_1 \\ Ni, & R_1 < r < R_2 \\ 0, & r > R_2 \end{cases}$$

The inner contour has no current through it while the outer contour enclosing the whole toroid has equal but opposite contributions from upward and downward currents.

The flux through any single loop is

$$\begin{aligned} \Phi &= \mu D \int_{R_1}^{R_2} H_\phi \, dr \\ &= \frac{\mu D N i}{2\pi} \int_{R_1}^{R_2} \frac{dr}{r} \\ &= \frac{\mu D N i}{2\pi} \ln \frac{R_2}{R_1} \end{aligned}$$

so that the self-inductance is

$$L = \frac{N\Phi}{i} = \frac{\mu D N^2}{2\pi} \ln \frac{R_2}{R_1}$$

6-2-2 Reluctance

Magnetic circuits are analogous to resistive electronic circuits if we define the magnetomotive force (MMF) $\mathcal{F}$ analogous to the voltage (EMF) as

$$\mathcal{F} = Ni \tag{9}$$

The flux then plays the same role as the current in electronic circuits so that we define the magnetic analog to resistance as the reluctance:

$$\mathcal{R} = \frac{\mathcal{F}}{\Phi} = \frac{N^2}{L} = \frac{\text{(length)}}{\text{(permeability)(cross-sectional area)}} \tag{10}$$

which is proportional to the reciprocal of the inductance.

The advantage to this analogy is that the rules of adding reluctances in series and parallel obey the same rules as resistances.

(a) Reluctances in Series

For the iron core of infinite permeability in Figure 6-11*a*, with two finitely permeable gaps the reluctance of each gap is found from (8) and (10) as

$$\mathcal{R}_1 = \frac{s_1}{\mu_1 a_1 D}, \qquad \mathcal{R}_2 = \frac{s_2}{\mu_2 a_2 D} \tag{11}$$

so that the flux is

$$\Phi = \frac{\mathcal{F}}{\mathcal{R}_1 + \mathcal{R}_2} = \frac{Ni}{\mathcal{R}_1 + \mathcal{R}_2} \Rightarrow L = \frac{N\Phi}{i} = \frac{N^2}{\mathcal{R}_1 + \mathcal{R}_2} \tag{12}$$

The iron core with infinite permeability has zero reluctance. If the permeable gaps were also iron with infinite permeability, the reluctances of (11) would also be zero so that the flux

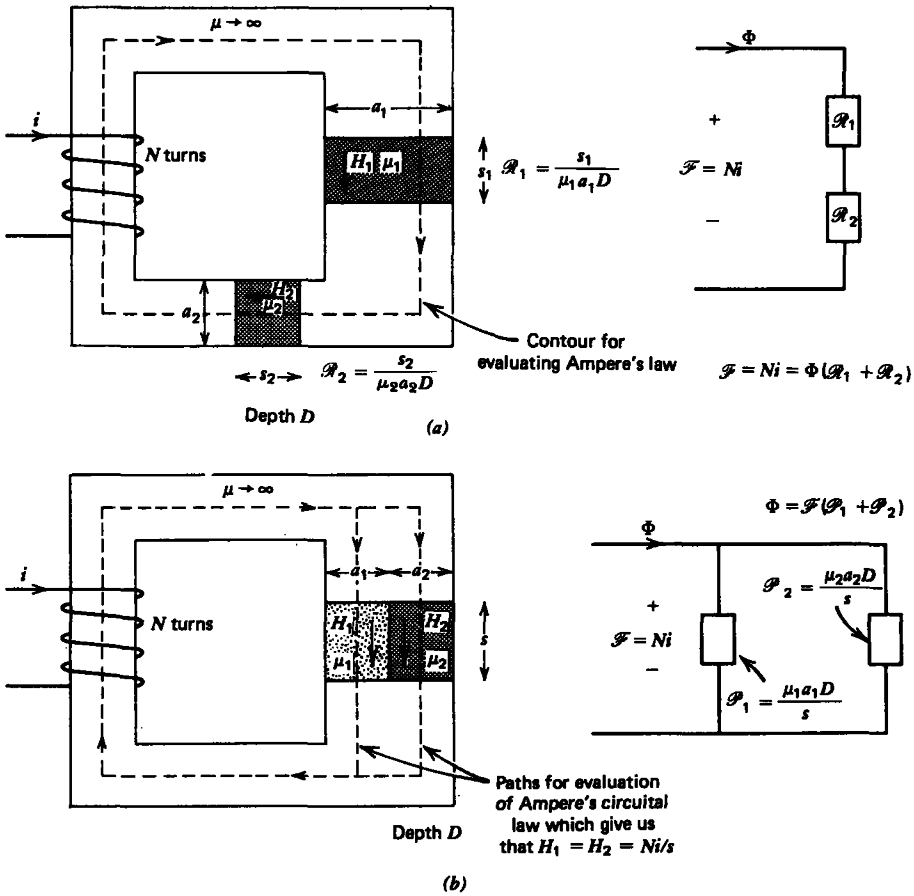


Figure 6-11 Magnetic circuits are most easily analyzed from a circuit approach where (*a*) reluctances in series add and (*b*) permeances in parallel add.

in (12) becomes infinite. This is analogous to applying a voltage across a short circuit resulting in an infinite current. Then the small resistance in the wires determines the large but finite current. Similarly, in magnetic circuits the small reluctance of a closed iron core of high permeability with no gaps limits the large but finite flux determined by the saturation value of magnetization.

The **H** field is nonzero only in the permeable gaps so that Ampere's law yields

$$H_1 s_1 + H_2 s_2 = Ni \tag{13}$$

Since the flux must be continuous at every cross section,

$$\Phi = \mu_1 H_1 a_1 D = \mu_2 H_2 a_2 D \tag{14}$$

we solve for the **H** fields as

$$H_1 = \frac{\mu_2 a_2 Ni}{\mu_1 a_1 s_2 + \mu_2 a_2 s_1}, \qquad H_2 = \frac{\mu_1 a_1 Ni}{\mu_1 a_1 s_2 + \mu_2 a_2 s_1} \tag{15}$$

(b) Reluctances in Parallel

If a gap in the iron core is filled with two permeable materials, as in Figure 6-11*b*, the reluctance of each material is still given by (11). Since each material sees the same magnetomotive force, as shown by applying Ampere's circuital law to contours passing through each material,

$$H_1 s = H_2 s = Ni \Rightarrow H_1 = H_2 = \frac{Ni}{s} \tag{16}$$

the H fields in each material are equal. The flux is then

$$\Phi = (\mu_1 H_1 a_1 + \mu_2 H_2 a_2) D = \frac{Ni(\mathcal{R}_1 + \mathcal{R}_2)}{\mathcal{R}_1 \mathcal{R}_2} = Ni(\mathcal{P}_1 + \mathcal{P}_2) \tag{17}$$

where the permeances $\mathcal{P}_1$ and $\mathcal{P}_2$ are just the reciprocal reluctances analogous to conductance.

6-2-3 Transformer Action

(a) Voltages are not Unique

Consider two small resistors R_1 and R_2 forming a loop enclosing one leg of a closed magnetic circuit with permeability μ, as in Figure 6-12. An N turn coil excited on one leg with a time varying current generates a time varying flux that is approximately

$$\Phi(t) = \mu N A i_1 / l \tag{18}$$

where l is the average length around the core.

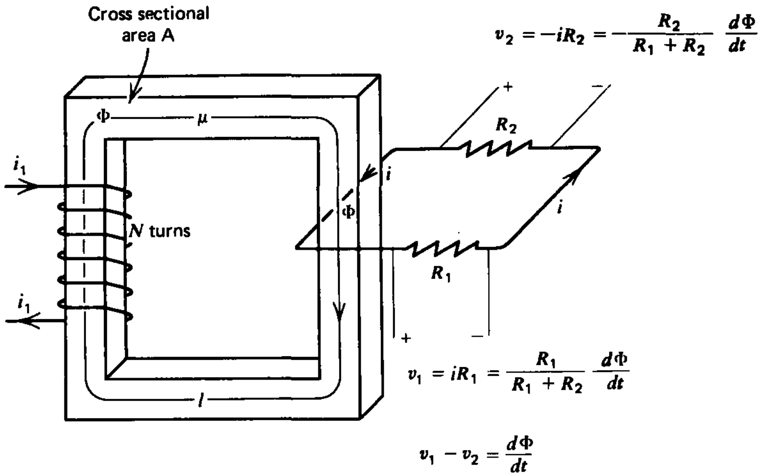


Figure 6-12 Voltages are not unique in the presence of a time varying magnetic field. A resistive loop encircling a magnetic circuit has different measured voltages across the same node pair. The voltage difference is equal to the time rate of magnetic flux through the loop.

Applying Faraday's law to the resistive loop we have

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = i(R_1+R_2) = +\frac{d\Phi(t)}{dt} \Rightarrow i = \frac{1}{R_1+R_2}\frac{d\Phi}{dt} \tag{19}$$

where we neglect the self-flux produced by the induced current i and reverse the sign on the magnetic flux term because Φ penetrates the loop in Figure 6-12 in the direction opposite to the positive convention given by the right-hand rule illustrated in Figure 6-2.

If we now measured the voltage across each resistor, we would find different values and opposite polarities even though our voltmeter was connected to the same nodes:

$$\begin{aligned} v_1 &= iR_1 = +\frac{R_1}{R_1+R_2}\frac{d\Phi}{dt} \\ v_2 &= -iR_2 = \frac{-R_2}{R_1+R_2}\frac{d\Phi}{dt} \end{aligned} \tag{20}$$

This nonuniqueness of the voltage arises because the electric field is no longer curl free. The voltage difference between two points depends on the path of the connecting wires. If any time varying magnetic flux passes through the contour defined by the measurement, an additional contribution results.

(b) Ideal Transformers

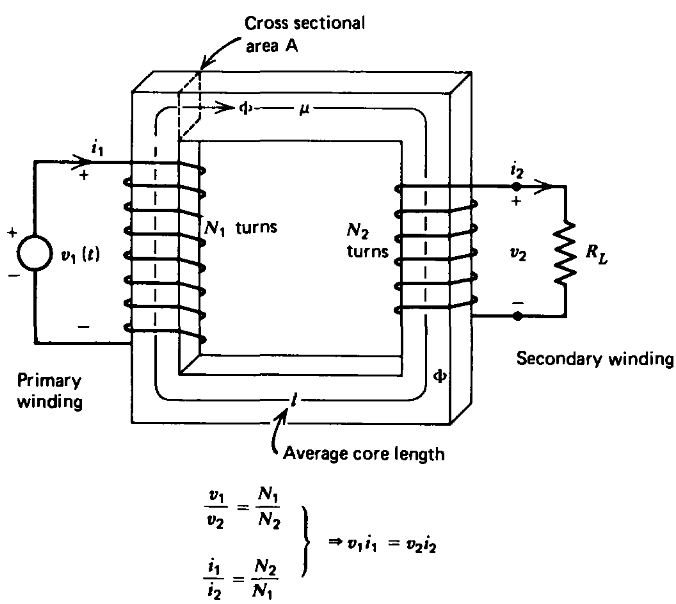
Two coils tightly wound on a highly permeable core, so that all the flux of one coil links the other, forms an ideal transformer, as in Figure 6-13. Because the iron core has an infinite permeability, all the flux is confined within the core. The currents flowing in each coil, i_1 and i_2, are defined so that when they are positive the fluxes generated by each coil are in the opposite direction. The total flux in the core is then

$$\Phi = \frac{N_1 i_1 - N_2 i_2}{\mathcal{R}}, \qquad \mathcal{R} = \frac{l}{\mu A} \tag{21}$$

where $\mathcal{R}$ is the reluctance of the core and l is the average length of the core.

The flux linked by each coil is then

$$\begin{aligned} \lambda_1 &= N_1 \Phi = \frac{\mu A}{l}(N_1^2 i_1 - N_1 N_2 i_2) \\ \lambda_2 &= N_2 \Phi = \frac{\mu A}{l}(N_1 N_2 i_1 - N_2^2 i_2) \end{aligned} \tag{22}$$



(a)

Figure 6-13 (a) An ideal transformer relates primary and secondary voltages by the ratio of turns while the currents are in the inverse ratio so that the input power equals the output power. The **H** field is zero within the infinitely permeable core. (b) In a real transformer the nonlinear B-H hysteresis loop causes a nonlinear primary current i_1 with an open circuited secondary ($i_2 = 0$) even though the imposed sinusoidal voltage $v_1 = V_0 \cos \omega t$ fixes the flux to be sinusoidal. (c) A more complete transformer equivalent circuit.

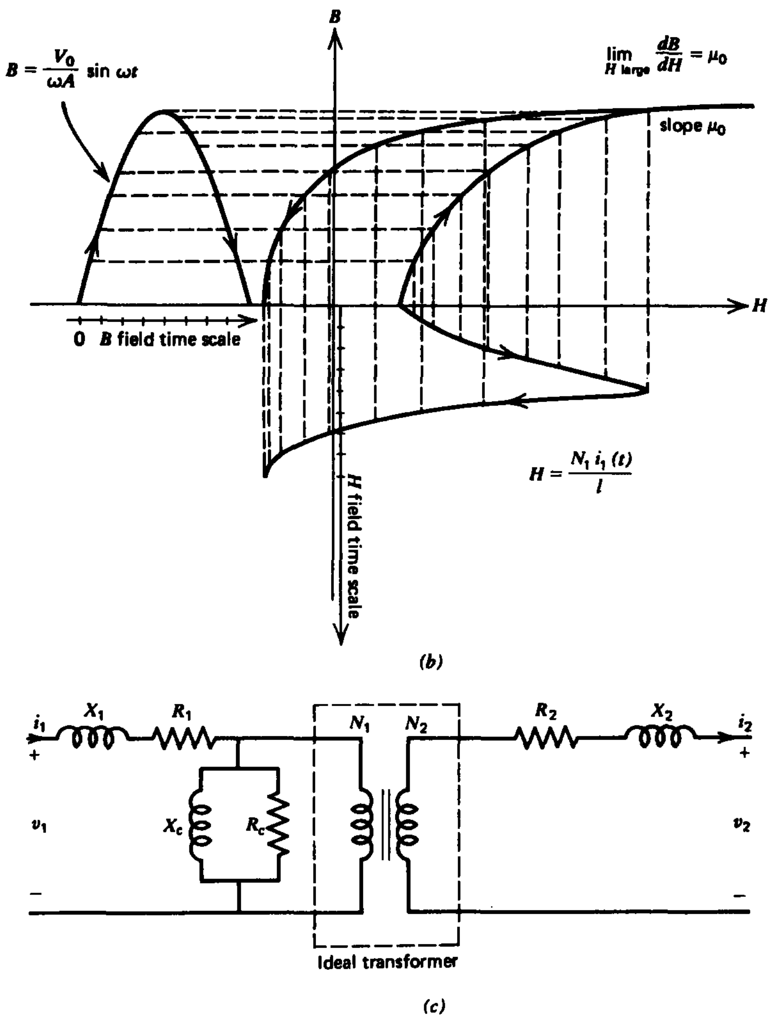


Figure 6.13.

which can be written as

$$\begin{aligned} \lambda_1 &= L_1 i_1 - M i_2 \\ \lambda_2 &= M i_1 - L_2 i_2 \end{aligned} \tag{23}$$

where L_1 and L_2 are the self-inductances of each coil alone and M is the mutual inductance between coils:

$$L_1 = N_1^2 L_0, \qquad L_2 = N_2^2 L_0, \qquad M = N_1 N_2 L_0, \qquad L_0 = \mu A/l \tag{24}$$

In general, the mutual inductance obeys the equality:

$$M = k(L_1 L_2)^{1/2}, \qquad 0 \le k \le 1 \tag{25}$$

where k is called the coefficient of coupling. For a noninfinite core permeability, k is less than unity because some of the flux of each coil goes into the free space region and does not link the other coil. In an ideal transformer, where the permeability is infinite, there is no leakage flux so that $k=1$.

From (23), the voltage across each coil is

$$\begin{aligned} v_1 &= \frac{d\lambda_1}{dt} = L_1\frac{di_1}{dt} - M\frac{di_2}{dt} \\ v_2 &= \frac{d\lambda_2}{dt} = M\frac{di_1}{dt} - L_2\frac{di_2}{dt} \end{aligned} \tag{26}$$

Because with no leakage, the mutual inductance is related to the self-inductances as

$$M = \frac{N_2}{N_1}L_1 = \frac{N_1}{N_2}L_2 \tag{27}$$

the ratio of coil voltages is the same as the turns ratio:

$$\frac{v_1}{v_2} = \frac{d\lambda_1/dt}{d\lambda_2/dt} = \frac{N_1}{N_2} \tag{28}$$

In the ideal transformer of infinite core permeability, the inductances of (24) are also infinite. To keep the voltages and fluxes in (26) finite, the currents must be in the inverse turns ratio

$$\frac{i_1}{i_2} = \frac{N_2}{N_1} \tag{29}$$

The electrical power delivered by the source to coil 1, called the primary winding, just equals the power delivered to the load across coil 2, called the secondary winding:

$$v_1 i_1 = v_2 i_2 \tag{30}$$

If $N_2 > N_1$, the voltage on winding 2 is greater than the voltage on winding 1 but current i_2 is less than i_1 keeping the powers equal.

If primary winding 1 is excited by a time varying voltage $v_1(t)$ with secondary winding 2 loaded by a resistor R_L so that

$$v_2 = i_2 R_L \tag{31}$$

the effective resistance seen by the primary winding is

$$R_{\text{eff}} = \frac{v_1}{i_1} = \frac{N_1}{N_2}\frac{v_2}{(N_2/N_1)i_2} = \left(\frac{N_1}{N_2}\right)^2 R_L \tag{32}$$

A transformer is used in this way as an impedance transformer where the effective resistance seen at the primary winding is increased by the square of the turns ratio.

(c) Real Transformers

When the secondary is open circuited ($i_2=0$), (29) shows that the primary current of an ideal transformer is also zero. In practice, applying a primary sinusoidal voltage $V_0 \cos \omega t$ will result in a small current due to the finite self-inductance of the primary coil. Even though this self-inductance is large if the core permeability μ is large, we must consider its effect because there is no opposing flux as a result of the open circuited secondary coil. Furthermore, the nonlinear hysteresis curve of the iron as discussed in Section 5-5-3*c* will result in a nonsinusoidal current even though the voltage is sinusoidal. In the magnetic circuit of Figure 6.13*a* with $i_2=0$, the magnetic field is

$$\mathbf{H}=\frac{N_1 i_1}{l} \tag{33}$$

while the imposed sinusoidal voltage also fixes the magnetic flux to be sinusoidal

$$v_1=\frac{d\Phi}{dt}=V_0 \cos \omega t \Rightarrow \Phi=BA=\frac{V_0}{\omega}\sin \omega t \tag{34}$$

Thus the upper half of the nonlinear B-H magnetization characteristic in Figure 6-13*b* has the same shape as the flux-current characteristic with proportionality factors related to the geometry. Note that in saturation the B-H curve approaches a straight line with slope μ_0. For a half-cycle of flux given by (34), the nonlinear open circuit magnetizing current is found graphically as a function of time in Figure 6-13*b*. The current is symmetric over the negative half of the flux cycle. Fourier analysis shows that this nonlinear current is composed of all the odd harmonics of the driving frequency dominated by the third and fifth harmonics. This causes problems in power systems and requires extra transformer windings to trap the higher harmonic currents, thus preventing their transmission.

A more realistic transformer equivalent circuit is shown in Figure 6-13*c* where the leakage reactances X_1 and X_2 represent the fact that all the flux produced by one coil does not link the other. Some small amount of flux is in the free space region surrounding the windings. The nonlinear inductive reactance X_c represents the nonlinear magnetization characteristic illustrated in Figure 6-13*b*, while R_c represents the power dissipated in traversing the hysteresis loop over a cycle. This dissipated power per cycle equals the area enclosed by the hysteresis loop. The winding resistances are R_1 and R_2.

6-3 FARADAY'S LAW FOR MOVING MEDIA

6-3-1 The Electric Field Transformation

If a point charge q travels with a velocity $\mathbf{v}$ through a region with electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, it experiences the combined Coulomb–Lorentz force

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{1}$$

Now consider another observer who is travelling at the same velocity $\mathbf{v}$ as the charge carrier so that their relative velocity is zero. This moving observer will then say that there is no Lorentz force, only a Coulombic force

$$\mathbf{F}'=q\mathbf{E}' \tag{2}$$

where we indicate quantities measured by the moving observer with a prime. A fundamental postulate of mechanics is that all physical laws are the same in every inertial coordinate system (systems that travel at constant relative velocity). This requires that the force measured by two inertial observers be the same so that $\mathbf{F}'=\mathbf{F}$:

$$\mathbf{E}'=\mathbf{E}+\mathbf{v}\times\mathbf{B} \tag{3}$$

The electric field measured by the two observers in relative motion will be different. This result is correct for material velocities much less than the speed of light and is called a Galilean field transformation. The complete relativistically correct transformation slightly modifies (3) and is called a Lorentzian transformation but will not be considered here.

In using Faraday's law of Section 6-1-1, the question remains as to which electric field should be used if the contour L and surface S are moving. One uses the electric field that is measured by an observer moving at the same velocity as the convecting contour. The time derivative of the flux term cannot be brought inside the integral if the surface S is itself a function of time.

6-3-2 Ohm's Law for Moving Conductors

The electric field transformation of (3) is especially important in modifying Ohm's law for moving conductors. For nonrelativistic velocities, an observer moving along at the same velocity as an Ohmic conductor measures the usual Ohm's law in his reference frame,

$$\mathbf{J}_f' = \sigma \mathbf{E}' \tag{4}$$

where we assume the conduction process is unaffected by the motion. Then in Galilean relativity for systems with no free charge, the current density in all inertial frames is the same so that (3) in (4) gives us the generalized Ohm's law as

$$\mathbf{J}_f' = \mathbf{J}_f = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{5}$$

where $\mathbf{v}$ is the velocity of the conductor.

The effects of material motion are illustrated by the parallel plate geometry shown in Figure 6-14. A current source is applied at the left-hand side that distributes itself uniformly as a surface current $K_x = \pm I/D$ on the planes. The electrodes are connected by a conducting slab that moves to the right with constant velocity U. The voltage across the current source can be computed using Faraday's law with the contour shown. Let us have the contour continually expanding with the 2–3 leg moving with the conductor. Applying Faraday's law we have

$$\oint_L \mathbf{E}' \cdot d\mathbf{l} = \int_1^2 \cancelto{0}{\mathbf{E}'} \cdot d\mathbf{l} + \underbrace{\int_2^3 \mathbf{E}' \cdot d\mathbf{l}}_{iR} + \int_3^4 \cancelto{0}{\mathbf{E}'} \cdot d\mathbf{l} + \underbrace{\int_4^1 \mathbf{E} \cdot d\mathbf{l}}_{-v}$$

$$= -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{6}$$

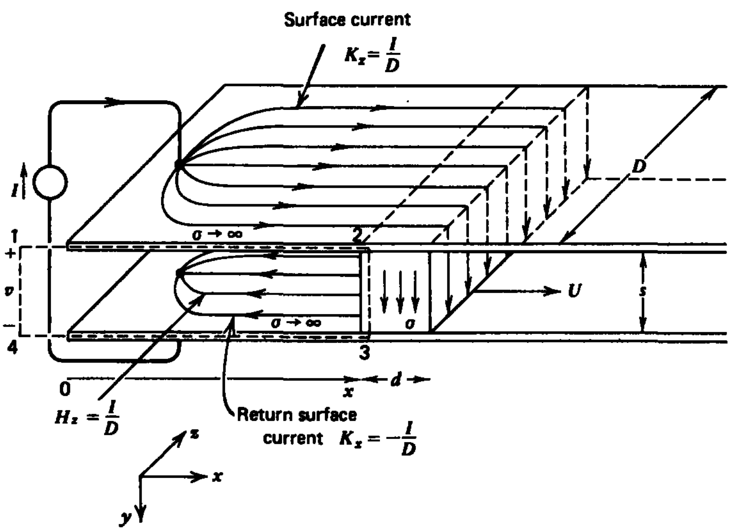


Figure 6-14 A moving, current-carrying Ohmic conductor generates a speed voltage as well as the usual resistive voltage drop.

where the electric field used along each leg is that measured by an observer in the frame of reference of the contour. Along the 1–2 and 3–4 legs, the electric field is zero within the stationary perfect conductors. The second integral within the moving Ohmic conductor uses the electric field **E**′, as measured by a moving observer because the contour is also expanding at the same velocity, and from (4) and (5) is related to the terminal current as

$$\mathbf{E}' = \frac{\mathbf{J}'}{\sigma} = \frac{I}{\sigma Dd}\mathbf{i}_y \tag{7}$$

In (6), the last line integral across the terminals defines the voltage.

$$\frac{Is}{\sigma Dd} - v = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = -\frac{d}{dt}(\mu_0 H_z x s) \tag{8}$$

The first term is just the resistive voltage drop across the conductor, present even if there is no motion. The term on the right-hand side in (8) only has a contribution due to the linearly increasing area ($dx/dt = U$) in the free space region with constant magnetic field,

$$H_z = I/D \tag{9}$$

The terminal voltage is then

$$v = I\left(R + \frac{\mu_0 U s}{D}\right), \qquad R = \frac{s}{\sigma Dd} \tag{10}$$

We see that the speed voltage contribution arose from the flux term in Faraday's law. We can obtain the same solution using a contour that is stationary and does not expand with the conductor. We pick the contour to just lie within the conductor at the time of interest. Because the contour does not expand with time so that both the magnetic field and the contour area does not change with time, the right-hand side of (6) is zero. The only difference now is that along the 2–3 leg we use the electric field as measured by a stationary observer,

$$\mathbf{E} = \mathbf{E}' - \mathbf{v}\times\mathbf{B} \tag{11}$$

so that (6) becomes

$$IR + \frac{\mu_0 U I s}{D} - v = 0 \tag{12}$$

which agrees with (10) but with the speed voltage term now arising from the electric field side of Faraday's law.

This speed voltage contribution is the principle of electric generators converting mechanical work to electric power when moving a current-carrying conductor through a magnetic field. The resistance term accounts for the electric power dissipated. Note in (10) that the speed voltage contribution just adds with the conductor's resistance so that the effective terminal resistance is $v/I = R + (\mu_o Us/D)$. If the slab moves in the opposite direction such that U is negative, the terminal resistance can also become negative for sufficiently large U $(U < -RD/\mu_0 s)$. Such systems are unstable where the natural modes grow rather than decay with time with any small perturbation, as illustrated in Section 6-3-3*b*.

6-3-3 Faraday's Disk (Homopolar Generator)*

(a) Imposed Magnetic Field

A disk of conductivity σ rotating at angular velocity ω transverse to a uniform magnetic field $B_0\mathbf{i}_z$, illustrates the basic principles of electromechanical energy conversion. In Figure 6-15*a* we assume that the magnetic field is generated by an N turn coil wound on the surrounding magnetic circuit,

$$B_0 = \frac{\mu_0 N i_f}{s} \tag{13}$$

The disk and shaft have a permeability of free space μ_0, so that the applied field is not disturbed by the assembly. The shaft and outside surface at $\mathrm{r} = R_0$ are highly conducting and make electrical connection to the terminals via sliding contacts.

We evaluate Faraday's law using the contour shown in Figure 6-15*a* where the 1–2 leg within the disk is stationary so the appropriate electric field to be used is given by (11):

$$E_{\mathrm{r}} = \frac{J_{\mathrm{r}}}{\sigma} - \omega \mathrm{r} B_0 = \frac{i_r}{2\pi\sigma d\mathrm{r}} - \omega \mathrm{r} B_0 \tag{14}$$

where the electric field and current density are radial and i_r is the total rotor terminal current. For the stationary contour with a constant magnetic field, there is no time varying flux through the contour:

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \int_1^2 E_{\mathrm{r}}\, d\mathrm{r} + \underbrace{\int_3^4 \mathbf{E}\cdot d\mathbf{l}}_{-v_r} = 0 \tag{15}$$

* *Some of the treatment in this section is similar to that developed in: H. H. Woodson and J. R. Melcher,* Electromechanical Dynamics, *Part I, Wiley, N.Y., 1968, Ch. 6.*

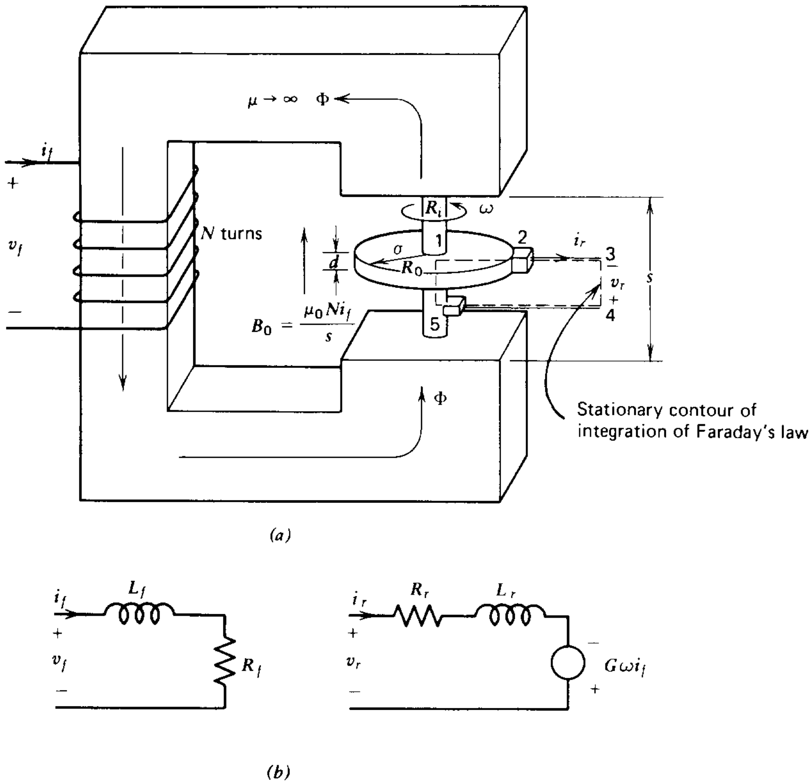


Figure 6-15 (*a*) A conducting disk rotating in an axial magnetic field is called a homopolar generator. (*b*) In addition to Ohmic and inductive voltages there is a speed voltage contribution proportional to the speed of the disk and the magnetic field.

Using (14) in (15) yields the terminal voltage as

$$\begin{aligned} v_r &= \int_{R_i}^{R_0} \left(\frac{i_r}{2\pi r\sigma d} - \omega r B_0\right) dr \\ &= \frac{i_r}{2\pi\sigma d}\ln\frac{R_0}{R_i} - \frac{\omega B_0}{2}(R_0^2 - R_i^2) \\ &= i_r R_r - G\omega i_f \end{aligned} \tag{16}$$

where R_r is the internal rotor resistance of the disk and G is called the speed coefficient:

$$R_r = \frac{\ln(R_0/R_i)}{2\pi\sigma d}, \qquad G = \frac{\mu_0 N}{2s}(R_0^2 - R_i^2) \tag{17}$$

We neglected the self-magnetic field due to the rotor current, assuming it to be much smaller than the applied field B_0, but it is represented in the equivalent rotor circuit in Figure 6-15*b* as the self-inductance L_r in series with a resistor and a speed voltage source linearly dependent on the field current. The stationary field coil is represented by its self-inductance and resistance.

For a copper disk ($\sigma \approx 6\times 10^7$ siemen/m) of thickness 1 mm rotating at 3600 rpm ($\omega = 120\pi$ radian/sec) with outer and inner radii $R_0 = 10$ cm and $R_i = 1$ cm in a magnetic field of $B_0 = 1$ tesla, the open circuit voltage is

$$v_{oc} = -\frac{\omega B_0}{2}(R_0^2 - R_i^2) \approx -1.9\text{ V} \tag{18}$$

while the short circuit current is

$$i_{sc} = \frac{v_{oc}}{\ln(R_0/R_i)} 2\pi\sigma d \approx 3\times 10^5 \text{ amp} \tag{19}$$

Homopolar generators are typically high current, low voltage devices. The electromagnetic torque on the disk due to the Lorentz force is

$$\begin{aligned} \mathbf{T} &= \int_{\phi=0}^{2\pi}\int_{z=0}^{d}\int_{r=R_i}^{R_0} r\mathbf{i}_r \times (\mathbf{J}\times\mathbf{B}) r\, dr\, d\phi\, dz \\ &= -i_r B_0 \mathbf{i}_z \int_{R_i}^{R_0} r\, dr \\ &= -\frac{i_r B_0}{2}(R_0^2 - R_i^2)\mathbf{i}_z \\ &= -G i_f i_r \mathbf{i}_z \end{aligned} \tag{20}$$

The negative sign indicates that the Lorentz force acts on the disk in the direction opposite to the motion. An external torque equal in magnitude but opposite in direction to (20) is necessary to turn the shaft.

This device can also be operated as a motor if a rotor current into the disk ($i_r < 0$) is imposed. Then the electrical torque causes the disk to turn.

(b) Self-Excited Generator

For generator operation it is necessary to turn the shaft and supply a field current to generate the magnetic field. However, if the field coil is connected to the rotor terminals, as in Figure 6-16*a*, the generator can supply its own field current. The equivalent circuit for self-excited operation is shown in Figure 6-16*b* where the series connection has $i_r = i_f$.

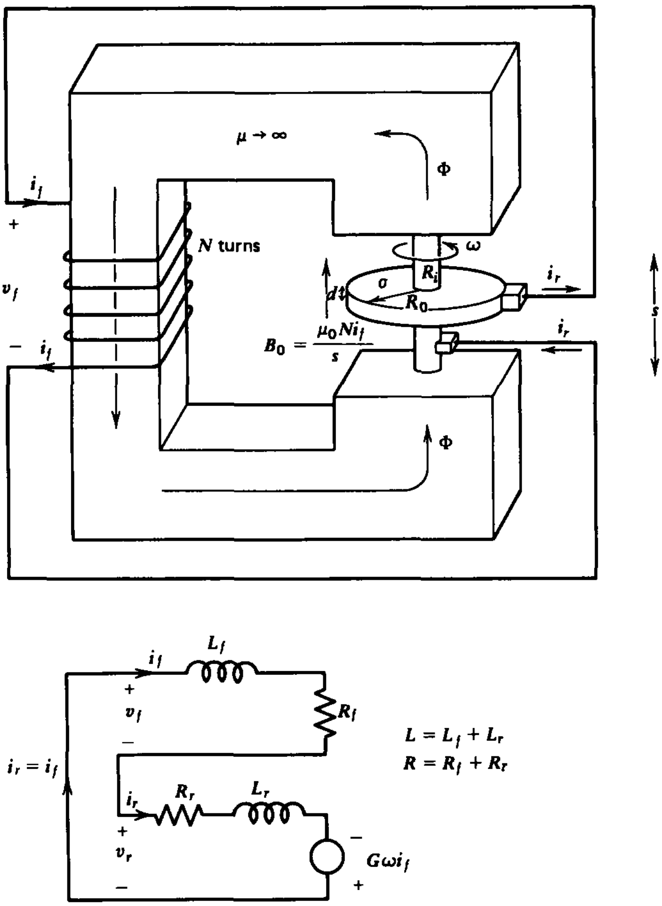


Figure 6-16 A homopolar generator can be self-excited where the generated rotor current is fed back to the field winding to generate its own magnetic field.

Kirchoff's voltage law around the loop is

$$L\frac{di}{dt} + i(R - G\omega) = 0, \qquad R = R_r + R_f, \qquad L = L_r + L_f \tag{21}$$

where R and L are the series resistance and inductance of the coil and disk. The solution to (21) is

$$i = I_0 e^{-[(R-G\omega)/L]t} \tag{22}$$

where I_0 is the initial current at $t = 0$. If the exponential factor is positive

$$G\omega > R \tag{23}$$

the current grows with time no matter how small I_0 is. In practice, I_0 is generated by random fluctuations (noise) due to residual magnetism in the iron core. The exponential growth is limited by magnetic core saturation so that the current reaches a steady-state value. If the disk is rotating in the opposite direction ($\omega<0$), the condition of (23) cannot be satisfied. It is then necessary for the field coil connection to be reversed so that $i_r=-i_f$. Such a dynamo model has been used as a model of the origin of the earth's magnetic field.

(c) Self-Excited ac Operation

Two such coupled generators can spontaneously generate two phase ac power if two independent field windings are connected, as in Figure 6-17. The field windings are connected so that if the flux through the two windings on one machine add, they subtract on the other machine. This accounts for the sign difference in the speed voltages in the equivalent circuits,

$$\begin{aligned} L\frac{di_1}{dt}+(R-G\omega)i_1+G\omega i_2&=0\\ L\frac{di_2}{dt}+(R-G\omega)i_2-G\omega i_1&=0 \end{aligned} \tag{24}$$

where L and R are the total series inductance and resistance. The disks are each turned at the same angular speed ω.

Since (24) are linear with constant coefficients, solutions are of the form

$$i_1=I_1e^{st},\qquad i_2=I_2e^{st} \tag{25}$$

which when substituted back into (24) yields

$$\begin{aligned} (Ls+R-G\omega)I_1+G\omega I_2&=0\\ -G\omega I_1+(Ls+R-G\omega)I_2&=0 \end{aligned} \tag{26}$$

For nontrivial solutions, the determinant of the coefficients of I_1 and I_2 must be zero,

$$(Ls+R-G\omega)^2=-(G\omega)^2 \tag{27}$$

which when solved for s yields the complex conjugate natural frequencies,

$$\begin{aligned} s&=-\frac{(R-G\omega)}{L}\pm j\frac{G\omega}{L}\\ I_1/I_2&=\pm j \end{aligned} \tag{28}$$

where the currents are 90° out of phase. If the real part of s is positive, the system is self-excited so that any perturbation

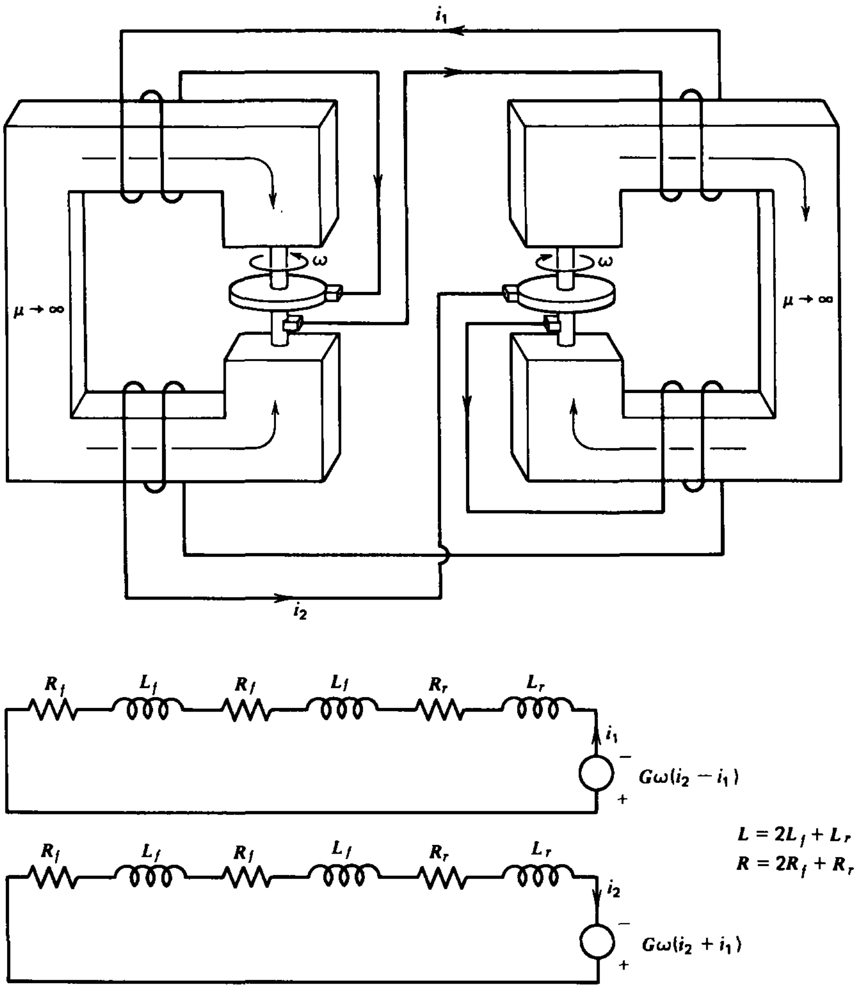


Figure 6-17 Cross-connecting two homopolar generators can result in self-excited two-phase alternating currents. Two independent field windings are required where on one machine the fluxes add while on the other they subtract.

grows at an exponential rate:

$$G\omega > R \tag{29}$$

The imaginary part of s yields the oscillation frequency

$$\omega_0 = \text{Im}(s) = G\omega/L \tag{30}$$

Again, core saturation limits the exponential growth so that two-phase power results. Such a model may help explain the periodic reversals in the earth's magnetic field every few hundred thousand years.

(d) Periodic Motor Speed Reversals

If the field winding of a motor is excited by a dc current, as in Figure 6-18, with the rotor terminals connected to a generator whose field and rotor terminals are in series, the circuit equation is

$$\frac{di}{dt}+\frac{(R-G_g\omega_g)}{L}i=\frac{G_m\omega_m}{L}I_f \tag{31}$$

where L and R are the total series inductances and resistances. The angular speed of the generator ω_g is externally

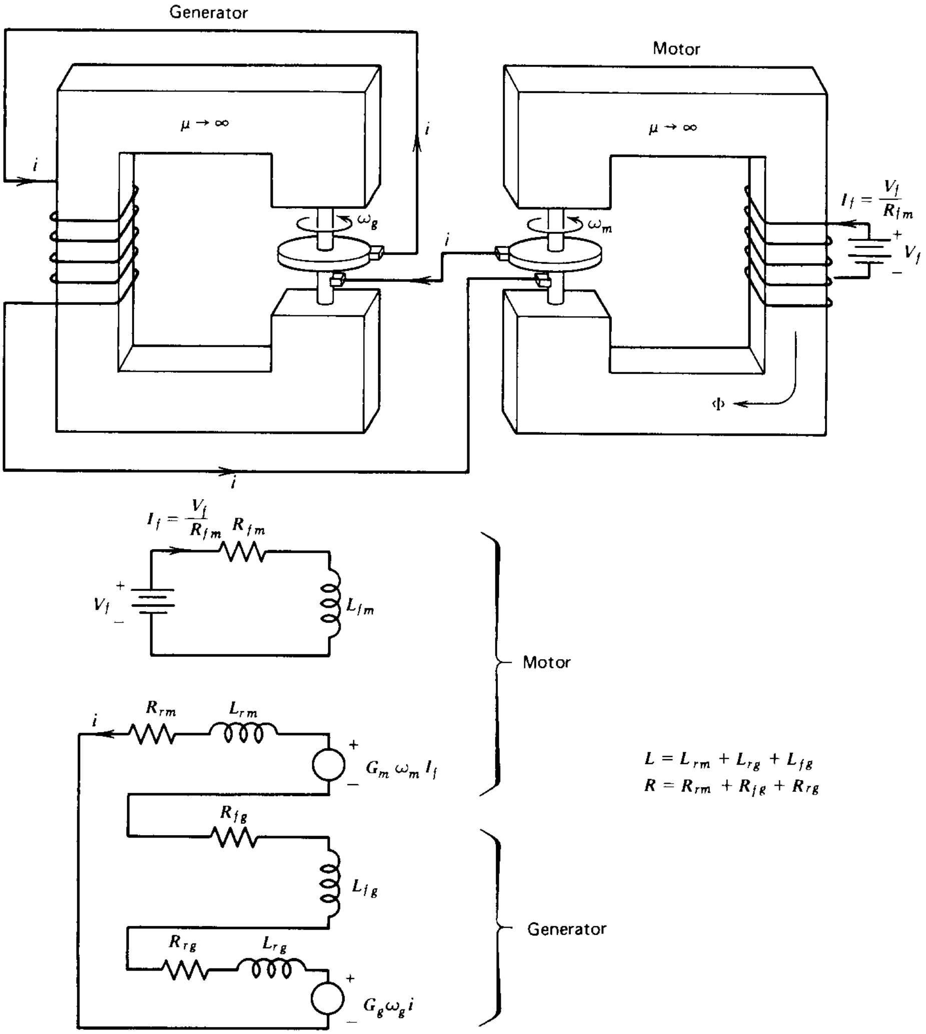


Figure 6-18 Cross connecting a homopolar generator and motor can result in spontaneous periodic speed reversals of the motor's shaft.

constrained to be a constant. The angular acceleration of the motor's shaft is equal to the torque of (20),

$$J\frac{d\omega_m}{dt} = -G_m I_f i \tag{32}$$

where J is the moment of inertia of the shaft and $I_f = V_f/R_{fm}$ is the constant motor field current.

Solutions of these coupled, linear constant coefficient differential equations are of the form

$$\begin{aligned} i &= I\,e^{st} \\ \omega &= W\,e^{st} \end{aligned} \tag{33}$$

which when substituted back into (31) and (32) yield

$$\begin{aligned} I\left(s+\frac{R-G_g\omega_g}{L}\right) - W\left(\frac{G_m I_f}{L}\right) &= 0 \\ I\left(\frac{G_m I_f}{J}\right) + Ws &= 0 \end{aligned} \tag{34}$$

Again, for nontrivial solutions the determinant of coefficients of I and W must be zero,

$$s\left(s+\frac{R-G_g\omega_g}{L}\right) + \frac{(G_m I_f)^2}{JL} = 0 \tag{35}$$

which when solved for s yields

$$s = -\frac{(R-G_g\omega_g)}{2L} \pm \left[\left(\frac{R-G_g\omega_g}{2L}\right)^2 - \frac{(G_m I_f)^2}{JL}\right]^{1/2} \tag{36}$$

For self-excitation the real part of s must be positive,

$$G_g\omega_g > R \tag{37}$$

while oscillations will occur if s has an imaginary part,

$$\frac{(G_m I_f)^2}{JL} > \left(\frac{R-G_g\omega_g}{2L}\right)^2 \tag{38}$$

Now, both the current and shaft's angular velocity spontaneously oscillate with time.

6-3-4 Basic Motors and Generators

(a) ac Machines

Alternating voltages are generated from a dc magnetic field by rotating a coil, as in Figure 6-19. An output voltage is measured via slip rings through carbon brushes. If the loop of area A is vertical at $t = 0$ linking zero flux, the imposed flux

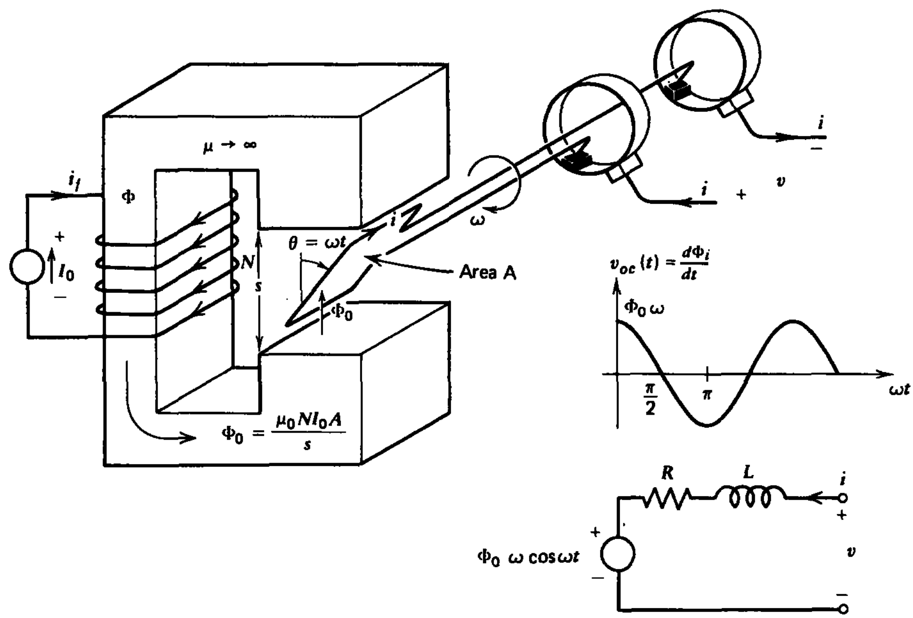


Figure 6-19 A coil rotated within a constant magnetic field generates a sinusoidal voltage.

through the loop at any time, varies sinusoidally with time due to the rotation as

$$\Phi_i = \Phi_0 \sin \omega t \tag{39}$$

Faraday's law applied to a stationary contour instantaneously passing through the wire then gives the terminal voltage as

$$v = iR + \frac{d\Phi}{dt} = iR + L\frac{di}{dt} + \Phi_0\omega \cos \omega t \tag{40}$$

where R and L are the resistance and inductance of the wire. The total flux is equal to the imposed flux of (39) as well as self-flux (accounted for by L) generated by the current i. The equivalent circuit is then similar to that of the homopolar generator, but the speed voltage term is now sinusoidal in time.

(b) dc Machines

DC machines have a similar configuration except that the slip ring is split into two sections, as in Figure 6-20*a*. Then whenever the output voltage tends to change sign, the terminals are also reversed yielding the waveform shown, which is of one polarity with periodic variations from zero to a peak value.

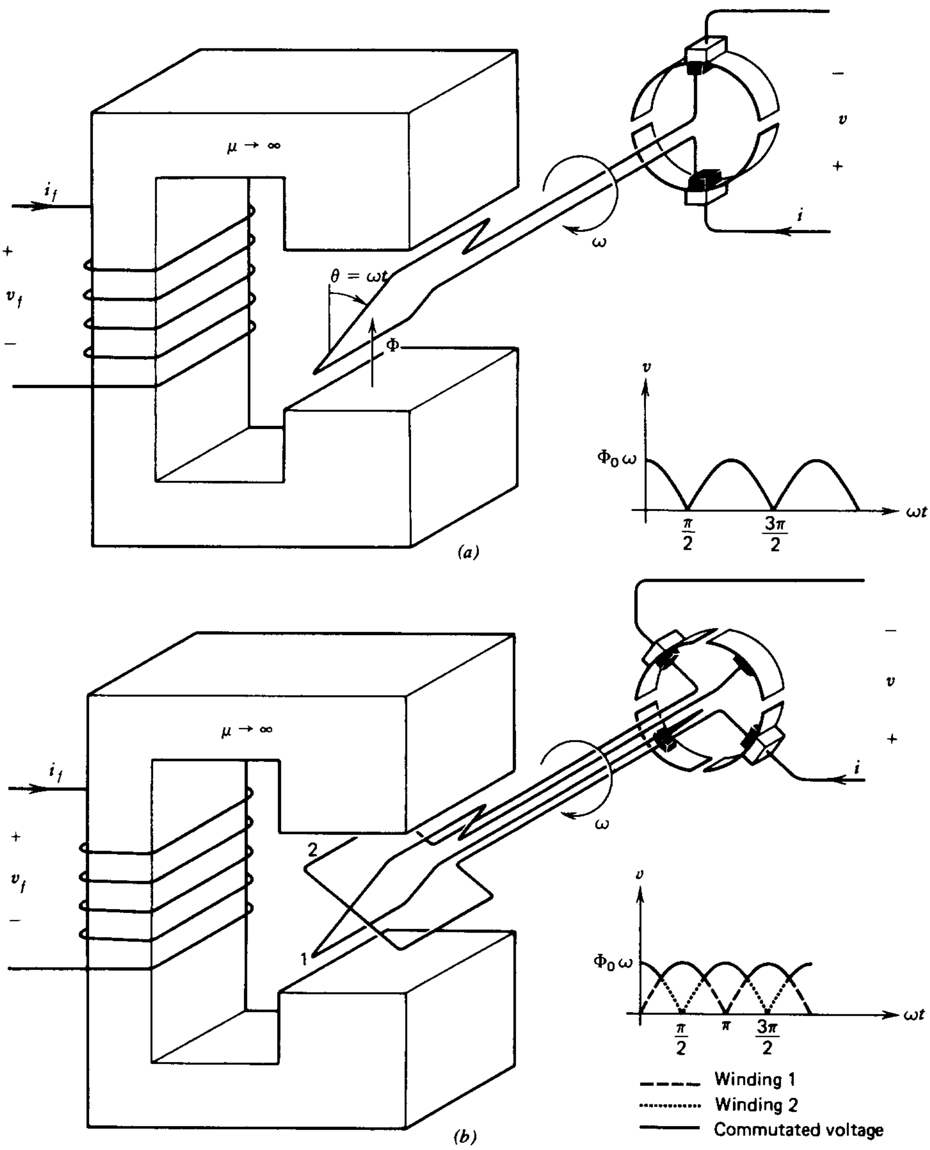


Figure 6-20 (*a*) If the slip rings are split so that when the voltage tends to change sign the terminals are also reversed, the resulting voltage is of one polarity. (*b*) The voltage waveform can be smoothed out by placing a second coil at right angles to the first and using a four-section commutator.

The voltage waveform can be smoothed out by using a four-section commutator and placing a second coil perpendicular to the first, as in Figure 6-20*b*. This second coil now generates its peak voltage when the first coil generates zero voltage. With more commutator sections and more coils, the dc voltage can be made as smooth as desired.

6-3-5 MHD Machines

Magnetohydrodynamic machines are based on the same principles as rotating machines, replacing the rigid rotor by a conducting fluid. For the linear machine in Figure 6-21, a fluid with Ohmic conductivity σ flowing with velocity v_y moves perpendicularly to an applied magnetic field $B_0\mathbf{i}_z$. The terminal voltage V is related to the electric field and current as

$$\mathbf{E}=\mathbf{i}_x\frac{V}{s},\qquad \mathbf{J}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B})=\sigma\left(\frac{V}{s}+v_yB_0\right)\mathbf{i}_x=\frac{i}{Dd}\mathbf{i}_x \tag{41}$$

which can be rewritten as

$$V=iR-v_yB_0s \tag{42}$$

which has a similar equivalent circuit as for the homopolar generator.

The force on the channel is then

$$\begin{aligned}\mathbf{f}&=\int_V \mathbf{J}\times\mathbf{B}\,dV\\&=-iB_0s\mathbf{i}_y\end{aligned} \tag{43}$$

again opposite to the fluid motion.

6-3-6 Paradoxes

Faraday's law is prone to misuse, which has led to numerous paradoxes. The confusion arises because the same

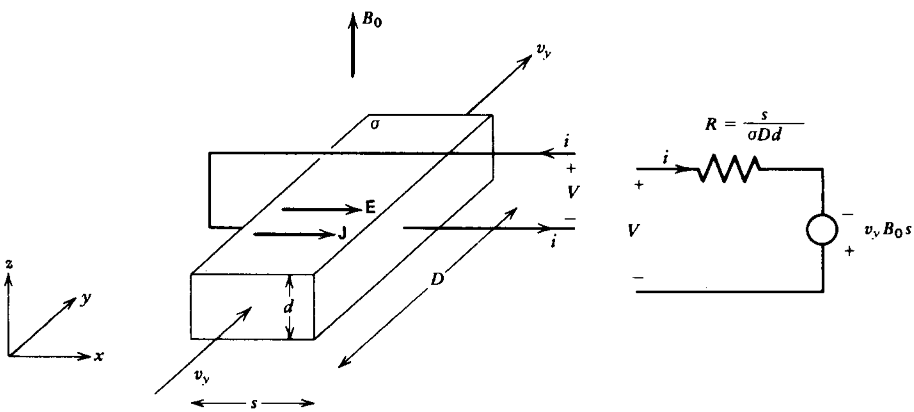


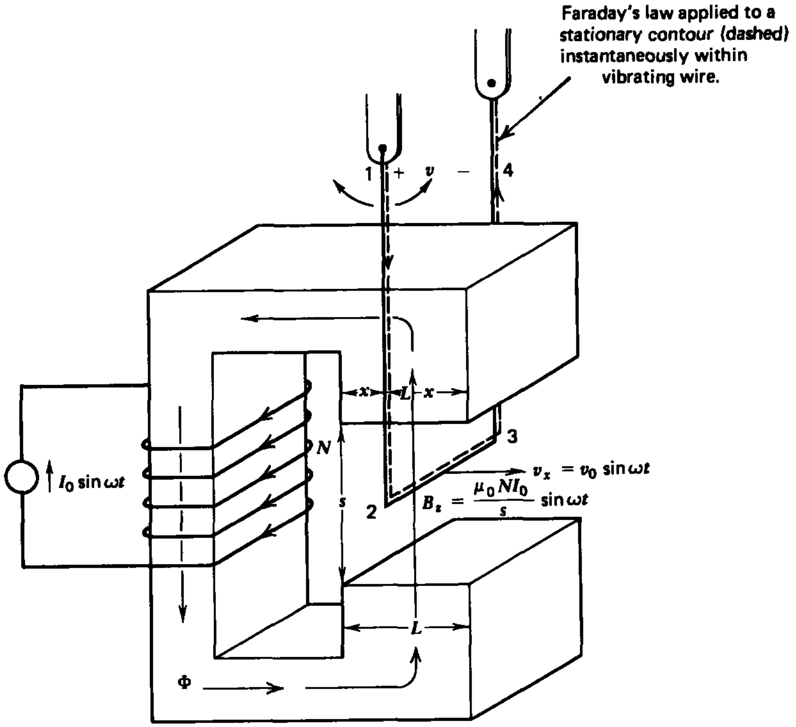
Figure 6-21 An MHD (magnetohydrodynamic) machine replaces a rotating conductor by a moving fluid.

contribution can arise from either the electromotive force side of the law, as a speed voltage when a conductor moves orthogonal to a magnetic field, or as a time rate of change of flux through the contour. This flux term itself has two contributions due to a time varying magnetic field or due to a contour that changes its shape, size, or orientation. With all these potential contributions it is often easy to miss a term or to double count.

(a) A Commutatorless dc Machine*

Many persons have tried to make a commutatorless dc machine but to no avail. One novel unsuccessful attempt is illustrated in Figure 6-22, where a highly conducting wire is vibrated within the gap of a magnetic circuit with sinusoidal velocity:

$$v_x = v_0 \sin \omega t \tag{44}$$



Fcc 6-22 It is impossible to design a commutatorless dc machine. Although the speed voltage alone can have a dc average, it will be canceled by the transformer electromotive force due to the time rate of change of magnetic flux through the loop. The total terminal voltage will always have a zero time average.

* *H. Sohon, Electrical Essays for Recreation.* Electrical Engineering, *May (1946), p. 294.*

The sinusoidal current imposes the air gap flux density at the same frequency ω:

$$B_z = B_0 \sin \omega t, \qquad B_0 = \mu_0 N I_0 / s \tag{45}$$

Applying Faraday's law to a stationary contour instantaneously within the open circuited wire yields

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \int_1^2 \cancelto{0}{\mathbf{E}'}\cdot d\mathbf{l} + \int_2^3 \underbrace{\mathbf{E}\cdot d\mathbf{l}}_{\mathbf{E}=-\mathbf{v}\times\mathbf{B}} + \int_3^4 \cancelto{0}{\mathbf{E}'}\cdot d\mathbf{l} + \int_4^1 \underbrace{\mathbf{E}\cdot d\mathbf{l}}_{-v}$$

$$= -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{46}$$

where the electric field within the highly conducting wire as measured by an observer moving with the wire is zero. The electric field on the 2–3 leg within the air gap is given by (11), where $\mathbf{E}' = 0$, while the 4–1 leg defines the terminal voltage. If we erroneously argue that the flux term on the right-hand side is zero because the magnetic field $\mathbf{B}$ is perpendicular to $\mathbf{dS}$, the terminal voltage is

$$v = v_x B_z l = v_0 B_0 l \sin^2 \omega t \tag{47}$$

which has a dc time-average value. Unfortunately, this result is not complete because we forgot to include the flux that turns the corner in the magnetic core and passes perpendicularly through our contour. Only the flux to the right of the wire passes through our contour, which is the fraction $(L-x)/L$ of the total flux. Then the correct evaluation of (46) is

$$-v + v_x B_z l = +\frac{d}{dt}[(L-x)B_z l] \tag{48}$$

where x is treated as a constant because the contour is stationary. The change in sign on the right-hand side arises because the flux passes through the contour in the direction opposite to its normal defined by the right-hand rule. The voltage is then

$$v = v_x B_z l - (L-x) l \frac{dB_z}{dt} \tag{49}$$

where the wire position is obtained by integrating (44),

$$x = \int v_x\, dt = -\frac{v_0}{\omega}(\cos \omega t - 1) + x_0 \tag{50}$$

and x_0 is the wire's position at $t=0$. Then (49) becomes

$$v = l\frac{d}{dt}(xB_z) - Ll\frac{dB_z}{dt}$$

$$= B_0 l v_0\left[\left(\frac{x_0\omega}{v_0}+1\right)\cos\omega t - \cos 2\omega t\right] - LlB_0\omega\cos\omega t \qquad (51)$$

which has a zero time average.

(b) Changes in Magnetic Flux Due to Switching

Changing the configuration of a circuit using a switch does not result in an electromotive force unless the magnetic flux itself changes.

In Figure 6-23*a*, the magnetic field through the loop is externally imposed and is independent of the switch position. Moving the switch does not induce an EMF because the magnetic flux through any surface remains unchanged.

In Figure 6-23*b*, a dc current source is connected to a circuit through a switch *S*. If the switch is instantaneously moved from contact 1 to contact 2, the magnetic field due to the source current *I* changes. The flux through any fixed area has thus changed resulting in an EMF.

(c) Time Varying Number of Turns on a Coil*

If the number of turns on a coil is changing with time, as in Figure 6-24, the voltage is equal to the time rate of change of flux through the coil. Is the voltage then

$$v \stackrel{?}{=} N\frac{d\Phi}{dt} \qquad (52)$$

or

$$v \stackrel{?}{=} \frac{d}{dt}(N\Phi) = N\frac{d\Phi}{dt} + \Phi\frac{dN}{dt} \qquad (53)$$

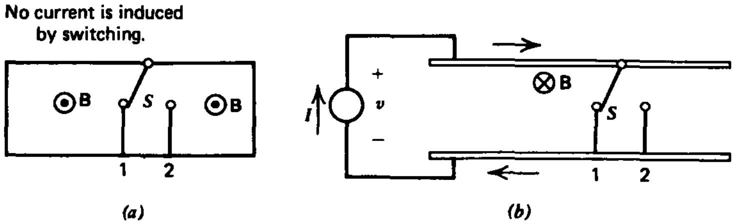


Figure 6-23 (*a*) Changes in a circuit through the use of a switch does not by itself generate an EMF. (*b*) However, an EMF can be generated if the switch changes the magnetic field.

* *L. V. Bewley.* Flux Linkages and Electromagnetic Induction. *Macmillan, New York, 1952.*

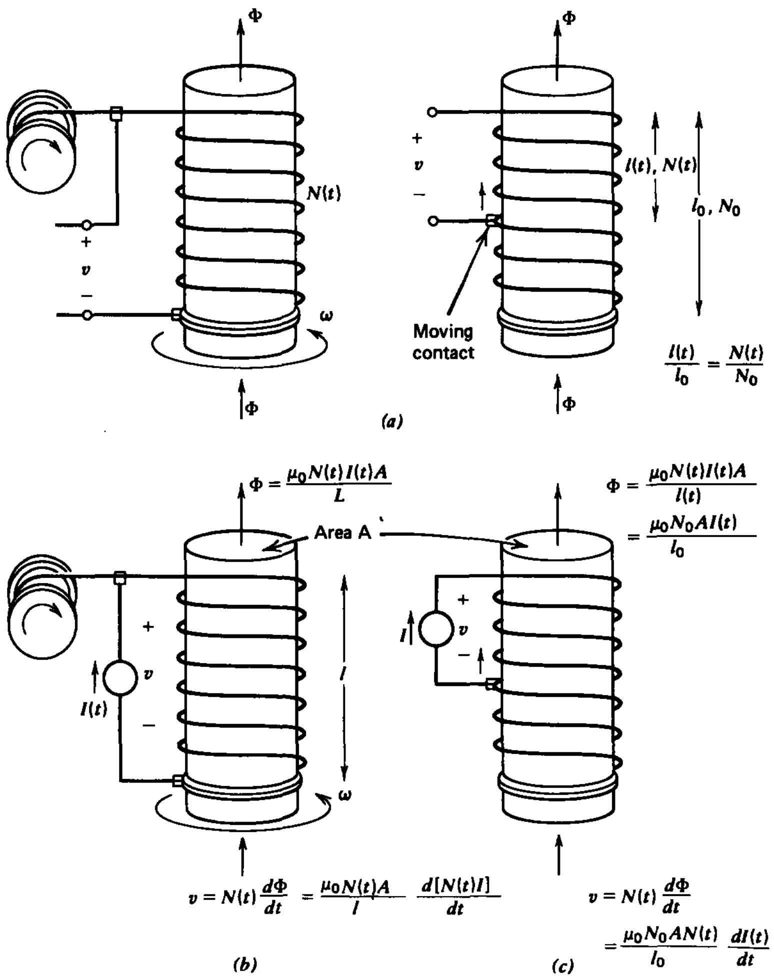


Figure 6-24 (a) If the number of turns on a coil is changing with time, the induced voltage is $v = N(t)\, d\Phi/dt$. A constant flux does not generate any voltage. (b) If the flux itself is proportional to the number of turns, a dc current can generate a voltage. (c) With the tap changing coil, the number of turns per unit length remains constant so that a dc current generates no voltage because the flux does not change with time.

For the first case a dc flux generates no voltage while the second does.

We use Faraday's law with a stationary contour instantaneously within the wire. Because the contour is stationary, its area of NA is not changing with time and so can be taken outside the time derivative in the flux term of Faraday's law so that the voltage is given by (52) and (53) is wrong. Note that there is no speed voltage contribution in the electromotive force because the velocity of the wire is in the same direction as the contour ($\mathbf{v}\times\mathbf{B}\cdot d\mathbf{l} = 0$).

If the flux Φ itself depends on the number of turns, as in Figure 6-24*b*, there may be a contribution to the voltage even if the exciting current is dc. This is true for the turns being wound onto the cylinder in Figure 6-24*b*. For the tap changing configuration in Figure 6-24*c*, with uniformly wound turns, the ratio of turns to effective length is constant so that a dc current will still not generate a voltage.

6-4 MAGNETIC DIFFUSION INTO AN OHMIC CONDUCTOR*

If the current distribution is known, the magnetic field can be directly found from the Biot–Savart or Ampere's laws. However, when the magnetic field varies with time, the generated electric field within an Ohmic conductor induces further currents that also contribute to the magnetic field.

6-4-1 Resistor–Inductor Model

A thin conducting shell of radius R_i, thickness Δ, and depth l is placed within a larger conducting cylinder, as shown in Figure 6-25. A step current I_0 is applied at $t=0$ to the larger cylinder, generating a surface current $\mathbf{K}=(I_0/l)\mathbf{i}_\phi$. If the length l is much greater than the outer radius R_0, the magnetic field is zero outside the cylinder and uniform inside for $R_i<r<R_0$. Then from the boundary condition on the discontinuity of tangential $\mathbf{H}$ given in Section 5-6-1, we have

$$\mathbf{H}_0=\frac{I_0}{l}\mathbf{i}_z, \qquad R_i<r<R_0 \tag{1}$$

The magnetic field is different inside the conducting shell because of the induced current, which from Lenz's law, flows in the opposite direction to the applied current. Because the shell is assumed to be very thin ($\Delta \ll R_i$), this induced current can be considered a surface current related to the volume current and electric field in the conductor as

$$K_\phi=J_\phi\Delta=(\sigma\Delta)E_\phi \tag{2}$$

The product $(\sigma\Delta)$ is called the surface conductivity. Then the magnetic fields on either side of the thin shell are also related by the boundary condition of Section 5-6-1:

$$H_i-H_0=K_\phi=(\sigma\Delta)E_\phi \tag{3}$$

* *Much of the treatment of this section is similar to that of H. H. Woodson and J. R. Melcher,* Electromechanical Dynamics, *Part II, Wiley, N.Y., 1968, Ch. 7.*

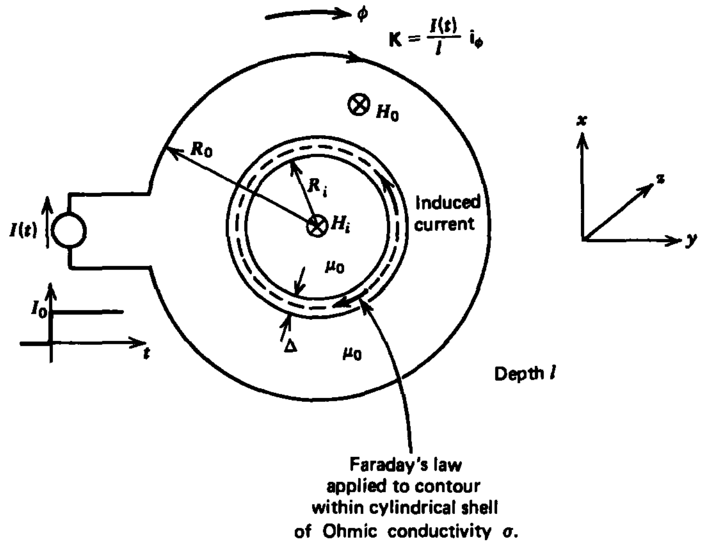


Figure 6-25 A step change in magnetic field causes the induced current within an Ohmic conductor to flow in the direction where its self-flux opposes the externally imposed flux. Ohmic dissipation causes the induced current to exponentially decay with time with a L/R time constant.

Applying Faraday's law to a contour within the conducting shell yields

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \Rightarrow E_\phi 2\pi R_i = -\mu_0 \pi R_i^2 \frac{dH_i}{dt} \tag{4}$$

where only the magnetic flux due to H_i passes through the contour. Then using (1)–(3) in (4) yields a single equation in H_i:

$$\frac{dH_i}{dt} + \frac{H_i}{\tau} = \frac{I(t)}{l\tau}, \qquad \tau = \frac{\mu_0 R_i \sigma \Delta}{2} \tag{5}$$

where we recognize the time constant τ as just being the ratio of the shell's self-inductance to resistance:

$$L = \frac{\Phi}{K_\phi l} = \frac{\mu_0 \pi R_i^2}{l}, \qquad R = \frac{2\pi R_i}{\sigma l \Delta}, \qquad \tau = \frac{L}{R} = \frac{\mu_0 R_i \sigma \Delta}{2} \tag{6}$$

The solution to (5) for a step current with zero initial magnetic field is

$$H_i = \frac{I_0}{l}(1 - e^{-t/\tau}) \tag{7}$$

Initially, the magnetic field is excluded from inside the conducting shell by the induced current. However, Ohmic dissipation causes the induced current to decay with time so that the magnetic field may penetrate through the shell with characteristic time constant τ.

6-4-2 The Magnetic Diffusion Equation

The transient solution for a thin conducting shell could be solved using the integral laws because the geometry constrained the induced current to flow azimuthally with no radial variations. If the current density is not directly known, it becomes necessary to self-consistently solve for the current density with the electric and magnetic fields:

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\quad\text{(Faraday's law)}\tag{8}$$

$$\nabla\times\mathbf{H}=\mathbf{J}_f\quad\text{(Ampere's law)}\tag{9}$$

$$\nabla\cdot\mathbf{B}=0\quad\text{(Gauss's law)}\tag{10}$$

For linear magnetic materials with constant permeability μ and constant Ohmic conductivity σ moving with velocity $\mathbf{U}$, the constitutive laws are

$$\mathbf{B}=\mu\mathbf{H},\qquad \mathbf{J}_f=\sigma(\mathbf{E}+\mathbf{U}\times\mu\mathbf{H})\tag{11}$$

We can reduce (8)–(11) to a single equation in the magnetic field by taking the curl of (9), using (8) and (11) as

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{H})&=\nabla\times\mathbf{J}_f\\&=\sigma[\nabla\times\mathbf{E}+\mu\nabla\times(\mathbf{U}\times\mathbf{H})]\\&=\mu\sigma\left(-\frac{\partial\mathbf{H}}{\partial t}+\nabla\times(\mathbf{U}\times\mathbf{H})\right)\end{aligned}\tag{12}$$

The double cross product of $\mathbf{H}$ can be simplified using the vector identity

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{H})&=\nabla(\overset{0}{\nabla\cdot\mathbf{H}})-\nabla^2\mathbf{H}\\&\Rightarrow\frac{1}{\mu\sigma}\nabla^2\mathbf{H}=\frac{\partial\mathbf{H}}{\partial t}-\nabla\times(\mathbf{U}\times\mathbf{H})\end{aligned}\tag{13}$$

where $\mathbf{H}$ has no divergence from (10). Remember that the Laplacian operator on the left-hand side of (13) also differentiates the directionally dependent unit vectors in cylindrical ($\mathbf{i}_r$ and $\mathbf{i}_\phi$) and spherical ($\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$) coordinates.

6-4-3 Transient Solution with No Motion (U = 0)

A step current is turned on at $t=0$, in the parallel plate geometry shown in Figure 6-26. By the right-hand rule and with the neglect of fringing, the magnetic field is in the z direction and only depends on the x coordinate, $B_z(x, t)$, so that (13) reduces to

$$\frac{\partial^2 H_z}{\partial x^2}-\sigma\mu\frac{\partial H_z}{\partial t}=0 \tag{14}$$

which is similar in form to the diffusion equation of a distributed resistive-capacitive cable developed in Section 3-6-4.

In the dc steady state, the second term is zero so that the solution in each region is of the form

$$\frac{\partial^2 H_z}{\partial x^2}=0 \Rightarrow H_z=ax+b \tag{15}$$

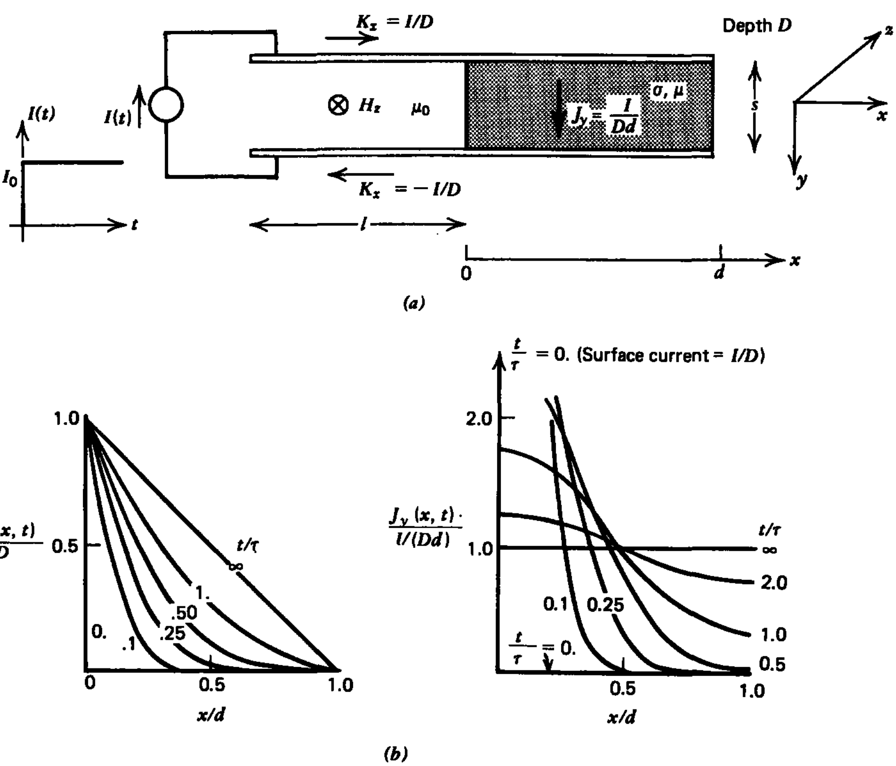


Figure 6-26 (*a*) A current source is instantaneously turned on at $t=0$. The resulting magnetic field within the Ohmic conductor remains continuous and is thus zero at $t=0$ requiring a surface current at $x=0$. (*b*) For later times the magnetic field and current diffuse into the conductor with longest time constant $\tau=\sigma\mu d^2/\pi^2$ towards a steady state of uniform current with a linear magnetic field.

where a and b are found from the boundary conditions. The current on the electrodes immediately spreads out to a uniform surface distribution $\pm(I/D)\mathbf{i}_x$ traveling from the upper to lower electrode uniformly through the Ohmic conductor. Then, the magnetic field is uniform in the free space region, decreasing linearly to zero within the Ohmic conductor being continuous across the interface at $x=0$:

$$\lim_{t\to\infty} H_z(x)=\begin{cases}\dfrac{I}{D}, & -l\le x\le 0\\ \dfrac{I}{Dd}(d-x), & 0\le x\le d\end{cases} \tag{16}$$

In the free space region where $\sigma=0$, the magnetic field remains constant for all time. Within the conducting slab, there is an initial charging transient as the magnetic field builds up to the linear steady-state distribution in (16). Because (14) is a linear equation, for the total solution of the magnetic field as a function of time and space, we use superposition and guess a solution that is the sum of the steady-state solution in (16) and a transient solution which dies off with time:

$$H_z(x,t)=\frac{I}{Dd}(d-x)+\hat{H}(x)\,e^{-\alpha t} \tag{17}$$

We follow the same procedures as for the lossy cable in Section 3-6-4. At this point we do not know the function $\hat{H}(x)$ or the parameter α. Substituting the assumed solution of (17) back into (14) yields the ordinary differential equation

$$\frac{d^2\hat{H}(x)}{dx^2}+\sigma\mu\alpha\hat{H}(x)=0 \tag{18}$$

which has the trigonometric solutions

$$\hat{H}(x)=A_1\sin\sqrt{\sigma\mu\alpha}\,x+A_2\cos\sqrt{\sigma\mu\alpha}\,x \tag{19}$$

Since the time-independent part in (17) already meets the boundary conditions of

$$\begin{aligned}H_z(x=0)&=I/D\\ H_z(x=d)&=0\end{aligned} \tag{20}$$

the transient part of the solution must be zero at the ends

$$\begin{aligned}\hat{H}(x=0)&=0\Rightarrow A_2=0\\ \hat{H}(x=d)&=0\Rightarrow A_1\sin\sqrt{\sigma\mu\alpha}\,d=0\end{aligned} \tag{21}$$

which yields the allowed values of α as

$$\sqrt{\sigma\mu\alpha}\,d=n\pi\Rightarrow\alpha_n=\frac{1}{\mu\sigma}\left(\frac{n\pi}{d}\right)^2,\quad n=1,2,3,\dots \tag{22}$$

Since there are an infinite number of allowed values of α, the most general solution is the superposition of all allowed solutions:

$$H_z(x,t)=\frac{I}{Dd}(d-x)+\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{d}e^{-\alpha_n t}\tag{23}$$

This relation satisfies the boundary conditions but not the initial conditions at $t=0$ when the current is first turned on. Before the current takes its step at $t=0$, the magnetic field is zero in the slab. Right after the current is turned on, the magnetic field must remain zero. Faraday's law would otherwise make the electric field and thus the current density infinite within the slab, which is nonphysical. Thus we impose the initial condition

$$H_z(x,t=0)=0=\frac{I}{Dd}(d-x)+\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{d}\tag{24}$$

which will allow us to solve for the amplitudes A_n by multiplying (24) through by $\sin(m\pi x/d)$ and then integrating over x from 0 to d:

$$0=\frac{I}{Dd}\int_0^d(d-x)\sin\frac{m\pi x}{d}\,dx+\sum_{n=1}^{\infty}A_n\int_0^d\sin\frac{n\pi x}{d}\sin\frac{m\pi x}{d}\,dx\tag{25}$$

The first term on the right-hand side is easily integrable* while the product of sine terms integrates to zero unless $m=n$, yielding

$$A_m=-\frac{2I}{m\pi D}\tag{26}$$

The total solution is thus

$$H_z(x,t)=\frac{I}{D}\left(1-\frac{x}{d}-2\sum_{n=1}^{\infty}\frac{\sin(n\pi x/d)}{n\pi}e^{-n^2t/\tau}\right)\tag{27}$$

where we define the fundamental continuum magnetic diffusion time constant τ as

$$\tau=\frac{1}{\alpha_1}=\frac{\mu\sigma d^2}{\pi^2}\tag{28}$$

analogous to the lumped parameter time constant of (5) and (6).

* $\int_0^d(d-x)\sin\frac{m\pi x}{d}\,dx=\frac{d^2}{m\pi}$

The magnetic field approaches the steady state in times long compared to τ. For a perfect conductor ($\sigma \to \infty$), this time is infinite and the magnetic field is forever excluded from the slab. The current then flows only along the $x=0$ surface. However, even for copper ($\sigma \approx 6 \times 10^7$ siemens/m) 10-cm thick, the time constant is $\tau \approx 80$ msec, which is fast for many applications. The current then diffuses into the conductor where the current density is easily obtained from Ampere's law as

$$\mathbf{J}_f = \nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial x}\mathbf{i}_y$$

$$= \frac{I}{Dd}\left(1 + 2\sum_{n=1}^{\infty} \cos\frac{n\pi x}{d} e^{-n^2 t/\tau}\right)\mathbf{i}_y \tag{29}$$

The diffusion of the magnetic field and current density are plotted in Figure 6-26*b* for various times

The force on the conducting slab is due to the Lorentz force tending to expand the loop and a magnetization force due to the difference of permeability of the slab and the surrounding free space as derived in Section 5-8-1:

$$\mathbf{F} = \mu_0(\mathbf{M} \cdot \nabla)\mathbf{H} + \mu_0 \mathbf{J}_f \times \mathbf{H}$$

$$= (\mu - \mu_0)(\mathbf{H} \cdot \nabla)\mathbf{H} + \mu_0 \mathbf{J}_f \times \mathbf{H} \tag{30}$$

For our case with $\mathbf{H} = H_z(x)\mathbf{i}_z$, the magnetization force density has no contribution so that (30) reduces to

$$\mathbf{F} = \mu_0 \mathbf{J}_f \times \mathbf{H}$$

$$= \mu_0(\nabla \times \mathbf{H}) \times \mathbf{H}$$

$$= \mu_0(\mathbf{H} \cdot \nabla)\mathbf{H} - \nabla(\tfrac{1}{2}\mu_0 \mathbf{H} \cdot \mathbf{H})$$

$$= -\frac{d}{dx}(\tfrac{1}{2}\mu_0 H_z^2)\mathbf{i}_x \tag{31}$$

Integrating (31) over the slab volume with the magnetic field independent of y and z,

$$f_x = -\int_0^d sD\frac{d}{dx}(\tfrac{1}{2}\mu_0 H_z^2)\,dx$$

$$= -\tfrac{1}{2}\mu_0 H_z^2 sD\big|_0^d$$

$$= \frac{1}{2}\frac{\mu_0 I^2 s}{D} \tag{32}$$

gives us a constant force with time that is independent of the permeability. Note that our approach of expressing the current density in terms of the magnetic field in (31) was easier than multiplying the infinite series of (27) and (29), as the result then only depended on the magnetic field at the boundaries that are known from the boundary conditions of (20). The resulting integration in (32) was easy because the force density in (31) was expressed as a pure derivative of x.

6-4-4 The Sinusoidal Steady State (Skin Depth)

We now place an infinitely thick conducting slab a distance d above a sinusoidally varying current sheet $K_0 \cos \omega t \mathbf{i}_y$, which lies on top of a perfect conductor, as in Figure 6-27a. The

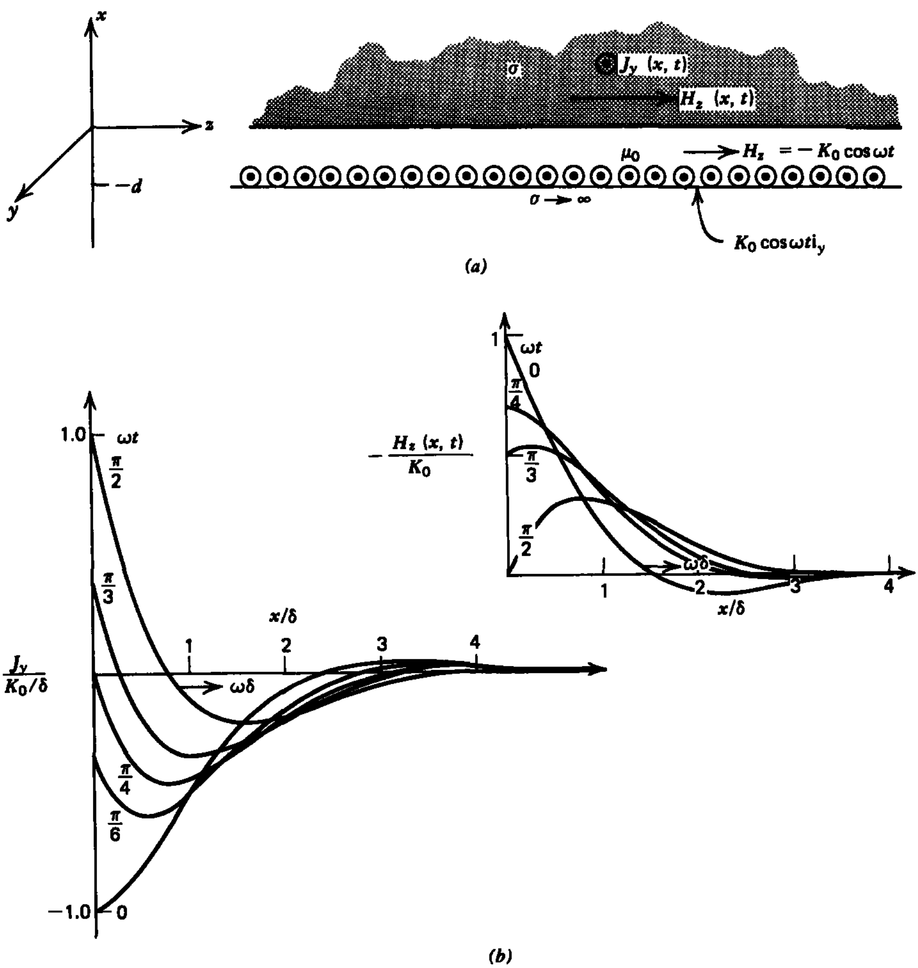


Figure 6-27 (a) A stationary conductor lies above a sinusoidal surface current placed upon a perfect conductor so that $\mathbf{H} = 0$ for $x < -d$. (b) The magnetic field and current density propagates and decays into the conductor with the same characteristic length given by the skin depth $\delta = \sqrt{2/(\omega\mu\sigma)}$. The phase speed of the wave is $\omega\delta$.

magnetic field within the conductor is then also sinusoidally varying with time:

$$H_z(x, t) = \text{Re}\,[\hat{H}_z(x)\,e^{j\omega t}] \tag{33}$$

Substituting (33) into (14) yields

$$\frac{d^2\hat{H}_z}{dx^2} - j\omega\mu\sigma\hat{H}_z = 0 \tag{34}$$

with solution

$$\hat{H}_z(x) = A_1\,e^{(1+j)x/\delta} + A_2\,e^{-(1+j)x/\delta} \tag{35}$$

where the skin depth δ is defined as

$$\delta = \sqrt{2/(\omega\mu\sigma)} \tag{36}$$

Since the magnetic field must remain finite far from the current sheet, A_1 must be zero. The magnetic field is also continuous across the $x=0$ boundary because there is no surface current, so that the solution is

$$\begin{aligned} H_z(x, t) &= \text{Re}\,[-K_0\,e^{-(1+j)x/\delta}\,e^{j\omega t}] \\ &= -K_0 \cos(\omega t - x/\delta)\,e^{-x/\delta}, \quad x \geq 0 \end{aligned} \tag{37}$$

where the magnetic field in the gap is uniform, determined by the discontinuity in tangential **H** at $x=-d$ to be $H_z = -K_y$ for $-d < x \leq 0$ since within the perfect conductor $(x < -d)\mathbf{H} = 0$. The magnetic field diffuses into the conductor as a strongly damped propagating wave with characteristic penetration depth δ. The skin depth δ is also equal to the propagating wavelength, as drawn in Figure 6-27*b*. The current density within the conductor

$$\begin{aligned} \mathbf{J}_f = \nabla \times \mathbf{H} &= -\frac{\partial H_z}{\partial x}\mathbf{i}_y \\ &= +\frac{K_0\,e^{-x/\delta}}{\delta}\left[\sin\left(\omega t - \frac{x}{\delta}\right) - \cos\left(\omega t - \frac{x}{\delta}\right)\right]\mathbf{i}_y \end{aligned} \tag{38}$$

is also drawn in Figure 6-27*b* at various times in the cycle, being confined near the interface to a depth on the order of δ. For a perfect conductor, $\delta \to 0$, and the volume current becomes a surface current.

Seawater has a conductivity of ≈ 4 siemens/m so that at a frequency of $f = 1$ MHz $(\omega = 2\pi f)$ the skin depth is $\delta \approx 0.25$ m. This is why radio communications to submarines are difficult. The conductivity of copper is $\sigma \approx 6 \times 10^7$ siemens/m so that at 60 Hz the skin depth is $\delta \approx 8$ mm. Power cables with larger radii have most of the current confined near the surface so that the center core carries very little current. This reduces the cross-sectional area through which the current flows, raising the cable resistance leading to larger power dissipation.

Again, the magnetization force density has no contribution to the force density since H_z only depends on x:

$$\begin{aligned}\mathbf{F} &= \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}+\mu_0\mathbf{J}_f\times\mathbf{H}\\ &= \mu_0(\nabla\times\mathbf{H})\times\mathbf{H}\\ &= -\nabla(\tfrac{1}{2}\mu_0\mathbf{H}\cdot\mathbf{H})\end{aligned} \tag{39}$$

The total force per unit area on the slab obtained by integrating (39) over x depends only on the magnetic field at $x=0$:

$$\begin{aligned}f_x &= -\int_0^\infty \frac{d}{dx}\left(\frac{\mu_0}{2}H_z^2\right)dx\\ &= -\tfrac{1}{2}\mu_0 H_z^2\Big|_0^\infty\\ &= \tfrac{1}{2}\mu_0 K_0^2\cos^2\omega t\end{aligned} \tag{40}$$

because again $\mathbf{H}$ is independent of y and z and the x component of the force density of (39) was written as a pure derivative with respect to x. Note that this approach was easier than integrating the cross product of (38) with (37).

This force can be used to levitate the conductor. Note that the region for $x>\delta$ is dead weight, as it contributes very little to the magnetic force.

6-4-5 Effects of Convection

A distributed dc surface current $-K_0\mathbf{i}_y$ at $x=0$ flows along parallel electrodes and returns via a conducting fluid moving to the right with constant velocity $v_0\mathbf{i}_x$, as shown in Figure 6-28*a*. The flow is not impeded by the current source at $x=0$. With the neglect of fringing, the magnetic field is purely z directed and only depends on the x coordinate, so that (13) in the dc steady state, with $\mathbf{U}=v_0\mathbf{i}_x$ being a constant, becomes*

$$\frac{d^2H_z}{dx^2}-\mu\sigma v_0\frac{dH_z}{dx}=0 \tag{41}$$

Solutions of the form

$$H_z(x)=A\,e^{px} \tag{42}$$

* $\nabla\times(\mathbf{U}\times\mathbf{H})=\mathbf{U}(\nabla\cdot\mathbf{H})^{\,0}-\mathbf{H}(\nabla\cdot\mathbf{U})^{\,0}+(\mathbf{H}\cdot\nabla)\mathbf{U}^{\,0}-(\mathbf{U}\cdot\nabla)\mathbf{H}=-v_0\dfrac{d\mathbf{H}}{dx}$

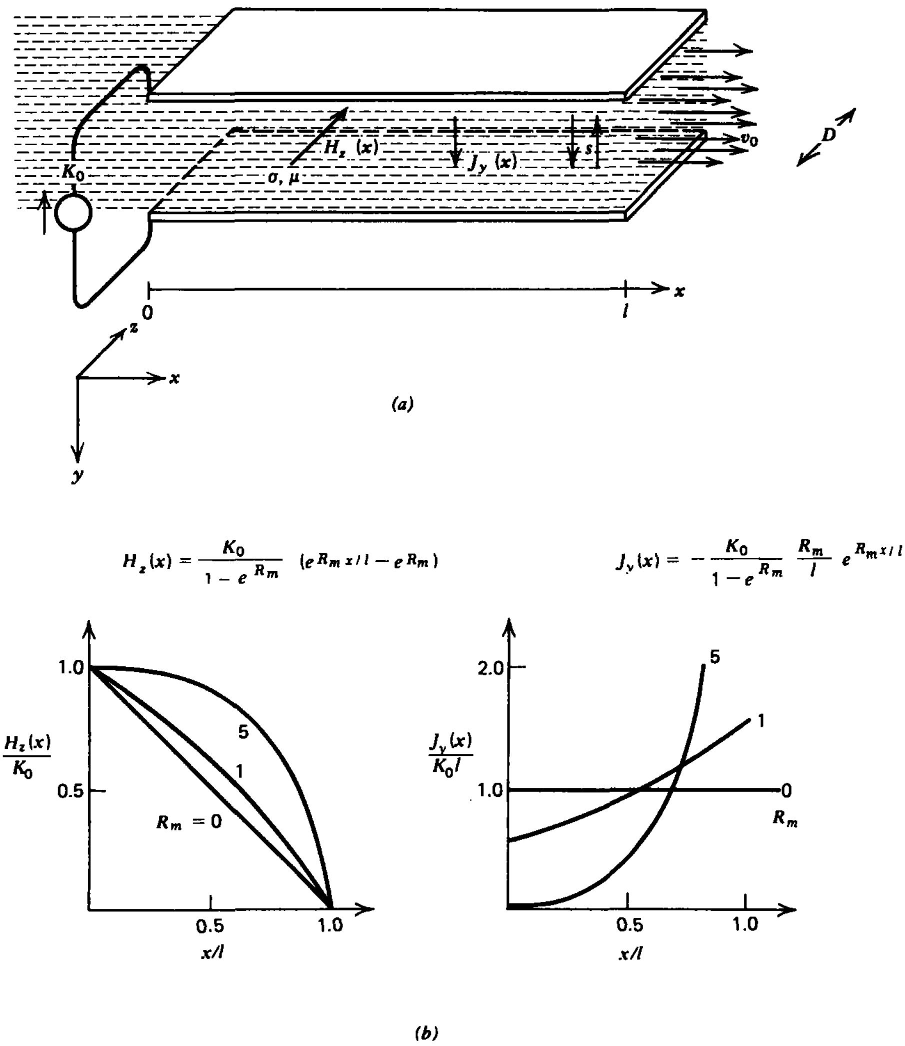


Figure 6-28 (*a*) A conducting material moving through a magnetic field tends to pull the magnetic field and current density with it. (*b*) The magnetic field and current density are greatly disturbed by the flow when the magnetic Reynolds number is large, $R_m = \sigma\mu Ul \gg 1$.

when substituted back into (41) yield two allowed values of p,

$$p^2 - \mu\sigma v_0 p = 0 \Rightarrow p = 0, \qquad p = \mu\sigma v_0 \tag{43}$$

Since (41) is linear, the most general solution is just the sum of the two allowed solutions,

$$H_z(x) = A_1 e^{R_m x/l} + A_2 \tag{44}$$

where the magnetic Reynold's number is defined as

$$R_m = \sigma\mu v_0 l = \frac{\sigma\mu l^2}{l/v_0} \tag{45}$$

and represents the ratio of a representative magnetic diffusion time given by (28) to a fluid transport time (l/v_0). The boundary conditions are

$$H_z(x=0) = K_0, \qquad H_z(x=l) = 0 \tag{46}$$

so that the solution is

$$H_z(x) = \frac{K_0}{1-e^{R_m}}(e^{R_m x/l} - e^{R_m}) \tag{47}$$

The associated current distribution is then

$$\begin{aligned} \mathbf{J}_f = \nabla \times \mathbf{H} &= -\frac{\partial H_z}{\partial x}\mathbf{i}_y \\ &= -\frac{K_0}{1-e^{R_m}}\frac{R_m}{l}e^{R_m x/l}\mathbf{i}_y \end{aligned} \tag{48}$$

The field and current distributions plotted in Figure 6-28*b* for various R_m show that the magnetic field and current are pulled along in the direction of flow. For small R_m the magnetic field is hardly disturbed from the zero flow solution of a linear field and constant current distribution. For very large $R_m \gg 1$, the magnetic field approaches a uniform distribution while the current density approaches a surface current at $x = l$.

The force on the moving fluid is independent of the flow velocity:

$$\begin{aligned} \mathbf{f} &= \int_0^l \mathbf{J} \times \mu_0 \mathbf{H} sD\, dx \\ &= -\frac{K_0^2}{(1-e^{R_m})^2}\mu_0 \frac{R_m}{l} sD \int_0^l e^{R_m x/l}(e^{R_m x/l} - e^{R_m})\, dx\, \mathbf{i}_x \\ &= -\frac{K_0^2 \mu_0 sD}{(1-e^{R_m})^2} e^{R_m x/l}\left(\frac{e^{R_m x/l}}{2} - e^{R_m}\right)\Bigg|_0^l \mathbf{i}_x \\ &= \tfrac{1}{2}\mu_0 K_0^2 sD\mathbf{i}_x \end{aligned} \tag{49}$$

6-4-6 A Linear Induction Machine

The induced currents in a conductor due to a time varying magnetic field give rise to a force that can cause the conductor to move. This describes a motor. The inverse effect is when we cause a conductor to move through a time varying magnetic field generating a current, which describes a generator.

The linear induction machine shown in Figure 6-29*a* assumes a conductor moves to the right at constant velocity $U\mathbf{i}_z$. Directly below the conductor with no gap is a surface current placed on top of an infinitely permeable medium

$$\mathbf{K}(t) = -K_0 \cos(\omega t - kz)\mathbf{i}_y = \text{Re}\,[-K_0\, e^{j(\omega t - kz)}\mathbf{i}_y] \tag{50}$$

which is a traveling wave moving to the right at speed ω/k. For $x > 0$, the magnetic field will then have x and z components of the form

$$\begin{aligned} H_z(x, z, t) &= \text{Re}\,[\hat{H}_z(x)\, e^{j(\omega t - kz)}] \\ H_x(x, z, t) &= \text{Re}\,[\hat{H}_x(x)\, e^{j(\omega t - kz)}] \end{aligned} \tag{51}$$

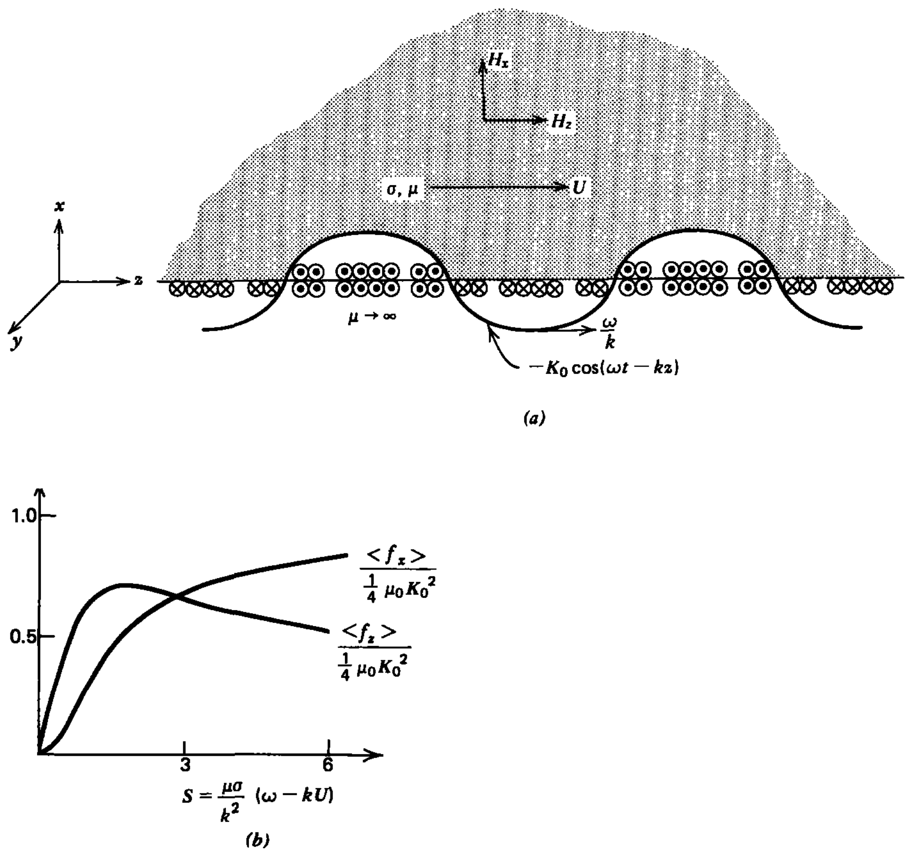


Figure 6-29 (*a*) A traveling wave of surface current induces currents in a conductor that is moving at a velocity U different from the wave speed ω/k. (*b*) The resulting forces can levitate and propel the conductor as a function of the slip S, which measures the difference in speeds of the conductor and traveling wave.

where (10) ($\nabla \cdot \mathbf{B} = 0$) requires these components to be related as

$$\frac{d\hat{H}_x}{dx} - jk\hat{H}_z = 0 \tag{52}$$

The z component of the magnetic diffusion equation of (13) is

$$\frac{d^2\hat{H}_z}{dx^2} - k^2\hat{H}_z = j\mu\sigma(\omega - kU)\hat{H}_z \tag{53}$$

which can also be written as

$$\frac{d^2\hat{H}_z}{dx^2} - \gamma^2\hat{H}_z = 0 \tag{54}$$

where

$$\gamma^2 = k^2(1 + jS), \qquad S = \frac{\mu\sigma}{k^2}(\omega - kU) \tag{55}$$

and S is known as the slip. Solutions of (54) are again exponential but complex because γ is complex:

$$\hat{H}_z = A_1 e^{\gamma x} + A_2 e^{-\gamma x} \tag{56}$$

Because $\hat{H}_z$ must remain finite far from the current sheet, $A_1 = 0$, so that using (52) the magnetic field is of the form

$$\hat{\mathbf{H}} = K_0 e^{-\gamma x}\left(\mathbf{i}_z - \frac{jk}{\gamma}\mathbf{i}_x\right) \tag{57}$$

where we use the fact that the tangential component of $\mathbf{H}$ is discontinuous in the surface current, with $\mathbf{H} = 0$ for $x<0$.

The current density in the conductor is

$$\begin{aligned} \mathbf{J}_f = \nabla \times \mathbf{H} = \mathbf{i}_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) \Rightarrow \hat{J}_y &= -jk\hat{H}_x - \frac{d\hat{H}_z}{dx} \\ &= K_0 e^{-\gamma x}\frac{(\gamma^2 - k^2)}{\gamma} \\ &= \frac{K_0 k^2 jS e^{-\gamma x}}{\gamma} \end{aligned} \tag{58}$$

If the conductor and current wave travel at the same speed ($\omega/k = U$), no current is induced as the slip is zero. Currents are only induced if the conductor and wave travel at different velocities. This is the principle of all induction machines.

The force per unit area on the conductor then has x and z components:

$$\begin{aligned}\mathbf{f} &= \int_0^\infty \mathbf{J}\times\mu_0\mathbf{H}\,dx \\ &= \int_0^\infty \mu_0 J_y(H_z\mathbf{i}_x - H_x\mathbf{i}_z)\,dx \end{aligned} \tag{59}$$

These integrations are straightforward but lengthy because first the instantaneous field and current density must be found from (51) by taking the real parts. More important is the time-average force per unit area over a period of excitation:

$$<\mathbf{f}> = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathbf{f}\,dt \tag{60}$$

Since the real part of a complex quantity is equal to half the sum of the quantity and its complex conjugate,

$$\begin{aligned} A &= \mathrm{Re}\,[\hat{A}\,e^{j\omega t}] = \tfrac{1}{2}(\hat{A}\,e^{j\omega t} + \hat{A}^*\,e^{-j\omega t}) \\ B &= \mathrm{Re}\,[\hat{B}\,e^{j\omega t}] = \tfrac{1}{2}(\hat{B}\,e^{j\omega t} + \hat{B}^*\,e^{-j\omega t}) \end{aligned} \tag{61}$$

the time-average product of two quantities is

$$\begin{aligned}\frac{\omega}{2\pi}\int_0^{2\pi/\omega} AB\,dt &= \frac{1}{4}\frac{\omega}{2\pi}\int_0^{2\pi/\omega} (\hat{A}\hat{B}\,e^{2j\omega t} + \hat{A}^*\hat{B} + \hat{A}\hat{B}^* \\ &\qquad + \hat{A}^*\hat{B}^*\,e^{-2j\omega t})\,dt \\ &= \tfrac{1}{4}(\hat{A}^*\hat{B} + \hat{A}\hat{B}^*) \\ &= \tfrac{1}{2}\,\mathrm{Re}\,(\hat{A}\hat{B}^*)\end{aligned} \tag{62}$$

which is a formula often used for the time-average power in circuits where A and B are the voltage and current.

Then using (62) in (59), the x component of the time-average force per unit area is

$$\begin{aligned}<f_x> &= \tfrac{1}{2}\,\mathrm{Re}\left(\int_0^\infty \mu_0 \hat{J}_y \hat{H}_z^*\,dx\right) \\ &= \frac{\mu_0}{2}K_0^2k^2S\,\mathrm{Re}\left(\frac{j}{\gamma}\int_0^\infty e^{-(\gamma+\gamma^*)x}\,dx\right) \\ &= \frac{\mu_0}{2}K_0^2k^2S\,\mathrm{Re}\left(\frac{j}{\gamma(\gamma+\gamma^*)}\right) \\ &= \frac{1}{4}\frac{\mu_0 K_0^2 S^2}{[1+S^2+(1+S^2)^{1/2}]} = \tfrac{1}{4}\mu_0 K_0^2\left(\frac{\sqrt{1+S^2}-1}{\sqrt{1+S^2}}\right)\end{aligned} \tag{63}$$

where the last equalities were evaluated in terms of the slip S from (55).

We similarly compute the time-average shear force per unit area as

$$\begin{aligned}<f_z> &= -\tfrac{1}{2}\,\text{Re}\left(\int_0^\infty \mu_0 J_y H_x^*\,dx\right)\\ &= \frac{\mu_0}{2}\frac{K_0^2 k^3 S}{\gamma\gamma^*}\,\text{Re}\left(\int_0^\infty e^{-(\gamma+\gamma^*)x}\,dx\right)\\ &= \frac{\mu_0}{2}\frac{k^3 K_0^2 S}{\gamma\gamma^*}\,\text{Re}\left(\frac{1}{(\gamma+\gamma^*)}\right)\\ &= \frac{\mu_0 K_0^2 S}{4\sqrt{1+S^2}\,\text{Re}\,(\sqrt{1+jS})}\end{aligned} \tag{64}$$

When the wave speed exceeds the conductor's speed ($\omega/k > U$), the force is positive as $S>0$ so that the wave pulls the conductor along. When $S<0$, the slow wave tends to pull the conductor back as $<f_z><0$. The forces of (63) and (64), plotted in Figure 6-29*b*, can be used to simultaneously lift and propel a conducting material. There is no force when the wave and conductor travel at the same speed ($\omega/k = U$) as the slip is zero ($S=0$). For large S, the levitating force $<f_x>$ approaches the constant value $\frac{1}{4}\mu_0 K_0^2$ while the shear force approaches zero. There is an optimum value of S that maximizes $<f_z>$. For smaller S, less current is induced while for larger S the phase difference between the imposed and induced currents tend to decrease the time-average force.

6-4-7 Superconductors

In the limit of infinite Ohmic conductivity ($\sigma\to\infty$), the diffusion time constant of (28) becomes infinite while the skin depth of (36) becomes zero. The magnetic field cannot penetrate a perfect conductor and currents are completely confined to the surface.

However, in this limit the Ohmic conduction law is no longer valid and we should use the superconducting constitutive law developed in Section 3-2-2*d* for a single charge carrier:

$$\frac{\partial \mathbf{J}}{\partial t} = \omega_p^2 \varepsilon \mathbf{E} \tag{65}$$

Then for a stationary medium, following the same procedure as in (12) and (13) with the constitutive law of (65), (8)–(11) reduce to

$$\nabla^2 \frac{\partial \mathbf{H}}{\partial t} - \omega_p^2 \varepsilon\mu \frac{\partial \mathbf{H}}{\partial t} = 0 \Rightarrow \nabla^2(\mathbf{H}-\mathbf{H}_0) - \omega_p^2\varepsilon\mu(\mathbf{H}-\mathbf{H}_0) = 0 \tag{66}$$

where $\mathbf{H}_0$ is the instantaneous magnetic field at $t=0$. If the superconducting material has no initial magnetic field when an excitation is first turned on, then $\mathbf{H}_0=0$.

If the conducting slab in Figure 6-27*a* becomes superconducting, (66) becomes

$$\frac{d^2H_z}{dx^2}-\frac{\omega_p^2}{c^2}H_z=0, \qquad c=\frac{1}{\sqrt{\varepsilon\mu}} \tag{67}$$

where c is the speed of light in the medium.

The solution to (67) is

$$\begin{aligned} H_z &= A_1\, e^{\omega_p x/c}+A_2\, e^{-\omega_p x/c} \\ &= -K_0 \cos \omega t\, e^{-\omega_p x/c} \end{aligned} \tag{68}$$

where we use the boundary condition of continuity of tangential $\mathbf{H}$ at $x=0$.

The current density is then

$$\begin{aligned} J_y &= -\frac{\partial H_z}{\partial x} \\ &= -\frac{K_0\omega_p}{c}\cos \omega t\, e^{-\omega_p x/c} \end{aligned} \tag{69}$$

For any frequency ω, including dc ($\omega=0$), the field and current decay with characteristic length:

$$l_c = c/\omega_p \tag{70}$$

Since the plasma frequency ω_p is typically on the order of 10^{15} radian/sec, this characteristic length is very small, $l_c \approx 3\times10^8/10^{15} \approx 3\times10^{-7}$ m. Except for this thin sheath, the magnetic field is excluded from the superconductor while the volume current is confined to this region near the interface.

There is one experimental exception to the governing equation in (66), known as the Meissner effect. If an ordinary conductor is placed within a dc magnetic field $\mathbf{H}_0$ and then cooled through the transition temperature for superconductivity, the magnetic flux is pushed out except for a thin sheath of width given by (70). This is contrary to (66), which allows the time-independent solution $\mathbf{H}=\mathbf{H}_0$, where the magnetic field remains trapped within the superconductor. Although the reason is not well understood, superconductors behave as if $\mathbf{H}_0=0$ no matter what the initial value of magnetic field.

6-5 ENERGY STORED IN THE MAGNETIC FIELD

6-5-1 A Single Current Loop

The differential amount of work necessary to overcome the electric and magnetic forces on a charge q moving an incremental distance **ds** at velocity **v** is

$$dW_q = -q(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\mathbf{ds} \tag{1}$$

(a) Electrical Work

If the charge moves solely under the action of the electrical and magnetic forces with no other forces of mechanical origin, the incremental displacement in a small time dt is related to its velocity as

$$\mathbf{ds} = \mathbf{v}\,dt \tag{2}$$

Then the magnetic field cannot contribute to any work on the charge because the magnetic force is perpendicular to the charge's displacement:

$$dW_q = -q\mathbf{v}\cdot\mathbf{E}\,dt \tag{3}$$

and the work required is entirely due to the electric field. Within a charge neutral wire, the electric field is not due to Coulombic forces but rather arises from Faraday's law. The moving charge constitutes an incremental current element,

$$q\mathbf{v} = i\,\mathbf{dl} \Rightarrow dW_q = -i\mathbf{E}\cdot\mathbf{dl}\,dt \tag{4}$$

so that the total work necessary to move all the charges in the closed wire is just the sum of the work done on each current element,

$$\begin{aligned} dW &= \oint_L dW_q = -i\,dt\oint_L \mathbf{E}\cdot\mathbf{dl} \\ &= i\,dt\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{dS} \\ &= i\,dt\frac{d\Phi}{dt} \\ &= i\,d\Phi \end{aligned} \tag{5}$$

which through Faraday's law is proportional to the change of flux through the current loop. This flux may be due to other currents and magnets (mutual flux) as well as the self-flux due to the current i. Note that the third relation in (5) is just equivalent to the circuit definition of electrical power delivered to the loop:

$$p = \frac{dW}{dt} = i\frac{d\Phi}{dt} = vi \tag{6}$$

All of this energy supplied to accelerate the charges in the wire is stored as no energy is dissipated in the lossless loop and no mechanical work is performed if the loop is held stationary.

(b) Mechanical Work

The magnetic field contributed no work in accelerating the charges. This is not true when the current-carrying wire is itself moved a small vector displacement **ds** requiring us to perform mechanical work,

$$\begin{aligned} dW &= -(i\,\mathbf{dl}\times\mathbf{B})\cdot\mathbf{ds} = i(\mathbf{B}\times\mathbf{dl})\cdot\mathbf{ds} \\ &= i\mathbf{B}\cdot(\mathbf{dl}\times\mathbf{ds}) \end{aligned} \tag{7}$$

where we were able to interchange the dot and the cross using the scalar triple product identity proved in Problem 1-10*a*. We define S_1 as the area originally bounding the loop and S_2 as the bounding area after the loop has moved the distance **ds**, as shown in Figure 6-30. The incremental area $\mathbf{dS}_3$ is then the strip joining the two positions of the loop defined by the bracketed quantity in (7):

$$\mathbf{dS}_3 = \mathbf{dl}\times\mathbf{ds} \tag{8}$$

The flux through each of the contours is

$$\Phi_1 = \int_{S_1} \mathbf{B}\cdot\mathbf{dS}, \qquad \Phi_2 = \int_{S_2} \mathbf{B}\cdot\mathbf{dS} \tag{9}$$

where their difference is just the flux that passes outward through $\mathbf{dS}_3$:

$$d\Phi = \Phi_1 - \Phi_2 = \mathbf{B}\cdot\mathbf{dS}_3 \tag{10}$$

The incremental mechanical work of (7) necessary to move the loop is then identical to (5):

$$dW = i\mathbf{B}\cdot\mathbf{dS}_3 = i\,d\Phi \tag{11}$$

Here there was no change of electrical energy input, with the increase of stored energy due entirely to mechanical work in moving the current loop.

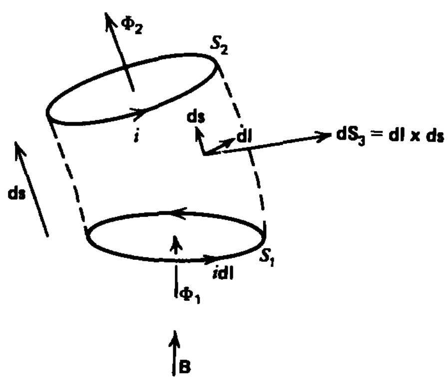


Figure 6-30 The mechanical work necessary to move a current-carrying loop is stored as potential energy in the magnetic field.

6-5-2 Energy and Inductance

If the loop is isolated and is within a linear permeable material, the flux is due entirely to the current, related through the self-inductance of the loop as

$$\Phi = Li \tag{12}$$

so that (5) or (11) can be integrated to find the total energy in a loop with final values of current I and flux Φ:

$$\begin{aligned} W &= \int_0^{\Phi} i\, d\Phi \\ &= \int_0^{\Phi} \frac{\Phi}{L}\, d\Phi \\ &= \frac{1}{2}\frac{\Phi^2}{L} = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi \end{aligned} \tag{13}$$

6-5-3 Current Distributions

The results of (13) are only true for a single current loop. For many interacting current loops or for current distributions, it is convenient to write the flux in terms of the vector potential using Stokes' theorem:

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS} = \oint_L \mathbf{A} \cdot \mathbf{dl} \tag{14}$$

Then each incremental-sized current element carrying a current I with flux $d\Phi$ has stored energy given by (13):

$$dW = \tfrac{1}{2} I\, d\Phi = \tfrac{1}{2}\mathbf{I} \cdot \mathbf{A}\, dl \tag{15}$$

For N current elements, (15) generalizes to

$$\begin{aligned} W &= \tfrac{1}{2}(\mathbf{I}_1 \cdot \mathbf{A}_1\, dl_1 + \mathbf{I}_2 \cdot \mathbf{A}_2\, dl_2 + \cdots + \mathbf{I}_N \cdot \mathbf{A}_N\, dl_N) \\ &= \tfrac{1}{2} \sum_{n=1}^{N} \mathbf{I}_n \cdot \mathbf{A}_n\, dl_n \end{aligned} \tag{16}$$

If the current is distributed over a line, surface, or volume, the summation is replaced by integration:

$$W = \begin{cases} \tfrac{1}{2} \int_L \mathbf{I}_f \cdot \mathbf{A}\, dl & \text{(line current)} \\ \tfrac{1}{2} \int_S \mathbf{K}_f \cdot \mathbf{A}\, dS & \text{(surface current)} \\ \tfrac{1}{2} \int_V \mathbf{J}_f \cdot \mathbf{A}\, dV & \text{(volume current)} \end{cases} \tag{17}$$

Remember that in (16) and (17) the currents and vector potentials are all evaluated at their final values as opposed to (11), where the current must be expressed as a function of flux.

6-5-4 Magnetic Energy Density

This stored energy can be thought of as being stored in the magnetic field. Assuming that we have a free volume distribution of current $\mathbf{J}_f$, we use (17) with Ampere's law to express $\mathbf{J}_f$ in terms of $\mathbf{H}$,

$$W=\frac{1}{2}\int_V \mathbf{J}_f\cdot\mathbf{A}\,dV=\frac{1}{2}\int_V(\nabla\times\mathbf{H})\cdot\mathbf{A}\,dV \tag{18}$$

where the volume V is just the volume occupied by the current. Larger volumes (including all space) can be used in (18), for the region outside the current has $\mathbf{J}_f=0$ so that no additional contributions arise.

Using the vector identity

$$\begin{aligned}\nabla\cdot(\mathbf{A}\times\mathbf{H})&=\mathbf{H}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{H})\\&=\mathbf{H}\cdot\mathbf{B}-\mathbf{A}\cdot(\nabla\times\mathbf{H})\end{aligned} \tag{19}$$

we rewrite (18) as

$$W=\frac{1}{2}\int_V[\mathbf{H}\cdot\mathbf{B}-\nabla\cdot(\mathbf{A}\times\mathbf{H})]\,dV \tag{20}$$

The second term on the right-hand side can be converted to a surface integral using the divergence theorem:

$$\int_V\nabla\cdot(\mathbf{A}\times\mathbf{H})\,dV=\oint_S(\mathbf{A}\times\mathbf{H})\cdot\mathbf{dS} \tag{21}$$

It now becomes convenient to let the volume extend over all space so that the surface is at infinity. If the current distribution does not extend to infinity the vector potential dies off at least as $1/r$ and the magnetic field as $1/r^2$. Then, even though the area increases as r^2, the surface integral in (21) decreases at least as $1/r$ and thus is zero when S is at infinity. Then (20) becomes simply

$$W=\frac{1}{2}\int_V\mathbf{H}\cdot\mathbf{B}\,dV=\frac{1}{2}\int_V\mu H^2\,dV=\frac{1}{2}\int_V\frac{B^2}{\mu}\,dV \tag{22}$$

where the volume V now extends over all space. The magnetic energy density is thus

$$w=\frac{1}{2}\mathbf{H}\cdot\mathbf{B}=\frac{1}{2}\mu H^2=\frac{1}{2}\frac{B^2}{\mu} \tag{23}$$

These results are only true for linear materials where μ does not depend on the magnetic field, although it can depend on position.

For a single coil, the total energy in (22) must be identical to (13), which gives us an alternate method to calculating the self-inductance from the magnetic field.

6-5-5 The Coaxial Cable

(a) External Inductance

A typical cable geometry consists of two perfectly conducting cylindrical shells of radii a and b and length l, as shown in Figure 6-31. An imposed current I flows axially as a surface current in opposite directions on each cylinder. We neglect fringing field effects near the ends so that the magnetic field is the same as if the cylinder were infinitely long. Using Ampere's law we find that

$$H_\phi = \frac{I}{2\pi r}, \quad a < r < b \tag{24}$$

The total magnetic flux between the two conductors is

$$\begin{aligned} \Phi &= \int_a^b \mu_0 H_\phi l \, dr \\ &= \frac{\mu_0 I l}{2\pi} \ln \frac{b}{a} \end{aligned} \tag{25}$$

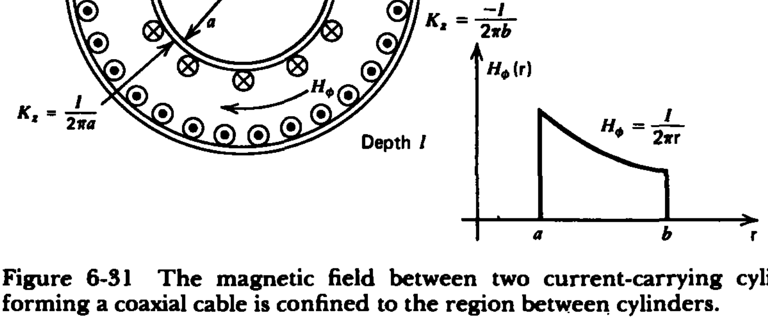


Figure 6-31 The magnetic field between two current-carrying cylindrical shells forming a coaxial cable is confined to the region between cylinders.

giving the self-inductance as

$$L=\frac{\Phi}{I}=\frac{\mu_0 l}{2\pi}\ln\frac{b}{a} \tag{26}$$

The same result can just as easily be found by computing the energy stored in the magnetic field

$$W=\tfrac{1}{2}LI^2=\tfrac{1}{2}\mu_0\int_a^b H_\phi^2 2\pi r l\,dr$$

$$=\frac{\mu_0 l I^2}{4\pi}\ln\frac{b}{a}\Rightarrow L=\frac{2W}{I^2}=\frac{\mu_0 l\ln(b/a)}{2\pi} \tag{27}$$

(b) Internal Inductance

If the inner cylinder is now solid, as in Figure 6-32, the current at low enough frequencies where the skin depth is much larger than the radius, is uniformly distributed with density

$$J_z=\frac{I}{\pi a^2} \tag{28}$$

so that a linearly increasing magnetic field is present within the inner cylinder while the outside magnetic field is

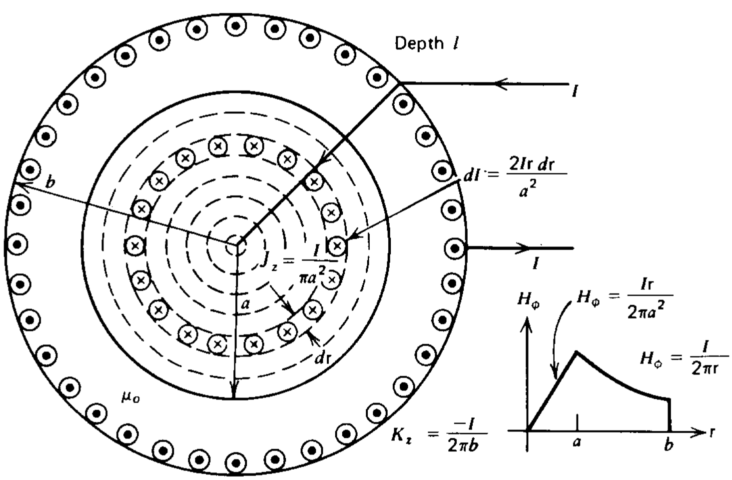


Figure 6-32 At low frequencies the current in a coaxial cable is uniformly distributed over the solid center conductor so that the internal magnetic field increases linearly with radius. The external magnetic field remains unchanged. The inner cylinder can be thought of as many incremental cylindrical shells of thickness dr carrying a fraction of the total current. Each shell links its own self-flux as well as the mutual flux of the other shells of smaller radius. The additional flux within the current-carrying conductor results in the internal inductance of the cable.

unchanged from (24):

$$H_\phi = \begin{cases} \dfrac{I\text{r}}{2\pi a^2}, & 0<\text{r}<a \\ \dfrac{I}{2\pi \text{r}}, & a<\text{r}<b \end{cases} \tag{29}$$

The self-inductance cannot be found using the flux per unit current definition for a current loop since the current is not restricted to a thin filament. The inner cylinder can be thought of as many incremental cylindrical shells, as in Figure 6-32, each linking its own self-flux as well as the mutual flux of the other shells of smaller radius. Note that each shell is at a different voltage due to the differences in enclosed flux, although the terminal wires that are in a region where the magnetic field is negligible have a well-defined unique voltage difference.

The easiest way to compute the self-inductance as seen by the terminal wires is to use the energy definition of (22):

$$\begin{aligned} W &= \tfrac{1}{2}\mu_0 \int_0^b H_\phi^2 2\pi l \text{r}\, d\text{r} \\ &= \pi l \mu_0 \left[\int_0^a \left(\frac{I\text{r}}{2\pi a^2}\right)^2 \text{r}\, d\text{r} + \int_a^b \left(\frac{I}{2\pi \text{r}}\right)^2 \text{r}\, d\text{r}\right] \\ &= \frac{\mu_0 l I^2}{4\pi}\left(\frac{1}{4} + \ln\frac{b}{a}\right) \end{aligned} \tag{30}$$

which gives the self-inductance as

$$L = \frac{2W}{I^2} = \frac{\mu_0 l}{2\pi}\left(\frac{1}{4} + \ln\frac{b}{a}\right) \tag{31}$$

The additional contribution of $\mu_0 l/8\pi$ is called the internal inductance and is due to the flux within the current-carrying conductor.

6-5-6 Self-Inductance, Capacitance, and Resistance

We can often save ourselves further calculations for the external self-inductance if we already know the capacitance or resistance for the same two-dimensional geometry composed of highly conducting electrodes with no internal inductance contribution. For the arbitrary geometry shown in Figure 6-33 of depth d, the capacitance, resistance, and inductance are defined as the ratios of line and surface integrals:

$$C = \frac{\varepsilon d \oint_S \mathbf{E} \cdot \mathbf{n}_s \, ds}{\int_L \mathbf{E} \cdot \mathbf{dl}}$$

$$R = \frac{\int_L \mathbf{E} \cdot \mathbf{dl}}{\sigma d \oint_S \mathbf{E} \cdot \mathbf{n}_s \, ds} \tag{32}$$

$$L = \frac{\mu d \int_L \mathbf{H} \cdot \mathbf{n}_l \, dl}{\oint_S \mathbf{H} \cdot \mathbf{ds}}$$

Because the homogeneous region between electrodes is charge and current free, both the electric and magnetic fields can be derived from a scalar potential that satisfies Laplace's equation. However, the electric field must be incident normally onto the electrodes while the magnetic field is incident tangentially so that **E** and **H** are perpendicular everywhere, each being along the potential lines of the other. This is accounted for in (32) and Figure 6-33 by having $\mathbf{n}_s \, ds$ perpendicular to **ds** and $\mathbf{n}_l \, dl$ perpendicular to **dl**. Then since C, R, and L are independent of the field strengths, we can take **E** and **H** to both have unit magnitude so that in the products of LC and L/R the line and surface integrals cancel:

$$LC = \varepsilon\mu d^2 = d^2/c^2, \qquad c = 1/\sqrt{\varepsilon\mu}$$
$$L/R = \mu\sigma d^2, \qquad RC = \varepsilon/\sigma \tag{33}$$

These products are then independent of the electrode geometry and depend only on the material parameters and the depth of the electrodes.

We recognize the L/R ratio to be proportional to the magnetic diffusion time of Section 6-4-3 while RC is just the charge relaxation time of Section 3-6-1. In Chapter 8 we see that the $\sqrt{LC}$ product is just equal to the time it takes an electromagnetic wave to propagate a distance d at the speed of light c in the medium.

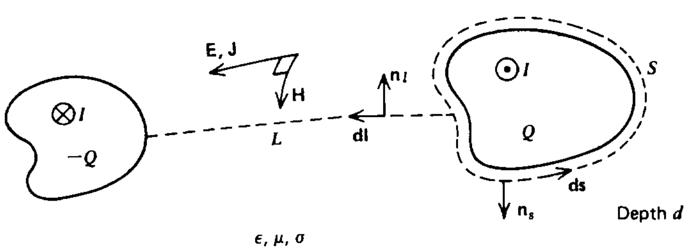


Figure 6-33 The electric and magnetic fields in the two-dimensional homogeneous charge and current-free region between hollow electrodes can be derived from a scalar potential that obeys Laplace's equation. The electric field lines are along the magnetic potential lines and vice versa so **E** and **H** are perpendicular. The inductance-capacitance product is then a constant.

6-6 THE ENERGY METHOD FOR FORCES

6-6-1 The Principle of Virtual Work

In Section 6-5-1 we calculated the energy stored in a current-carrying loop by two methods. First we calculated the electric energy input to a loop with no mechanical work done. We then obtained the same answer by computing the mechanical work necessary to move a current-carrying loop in an external field with no further electrical inputs. In the most general case, an input of electrical energy can result in stored energy dW and mechanical work by the action of a force f_x causing a small displacement dx:

$$i\,d\Phi = dW + f_x\,dx \tag{1}$$

If we knew the total energy stored in the magnetic field as a function of flux and position, the force is simply found as

$$f_x = -\left.\frac{\partial W}{\partial x}\right|_{\Phi} \tag{2}$$

We can easily compute the stored energy by realizing that no matter by what process or order the system is assembled, if the final position x and flux Φ are the same, the energy is the same. Since the energy stored is independent of the order that we apply mechanical and electrical inputs, we choose to mechanically assemble a system first to its final position x with no electrical excitations so that $\Phi = 0$. This takes no work as with zero flux there is no force of electrical origin. Once the system is mechanically assembled so that its position remains constant, we apply the electrical excitation to bring the system to its final flux value. The electrical energy required is

$$W = \int_{x\ \text{const}} i\,d\Phi \tag{3}$$

For linear materials, the flux and current are linearly related through the inductance that can now be a function of x because the inductance depends on the geometry:

$$i = \Phi/L(x) \tag{4}$$

Using (4) in (3) allows us to take the inductance outside the integral because x is held constant so that the inductance is also constant:

$$\begin{aligned} W &= \frac{1}{L(x)}\int_0^{\Phi} \Phi\,d\Phi \\ &= \frac{\Phi^2}{2L(x)} = \tfrac{1}{2}L(x)i^2 \end{aligned} \tag{5}$$

The stored energy is the same as found in Section 6-5-2 even when mechanical work is included and the inductance varies with position.

To find the force on the moveable member, we use (2) with the energy expression in (5), which depends only on flux and position:

$$\begin{aligned} f_x &= -\frac{\partial W}{\partial x}\bigg|_{\Phi} \\ &= -\frac{\Phi^2}{2}\frac{d[1/L(x)]}{dx} \\ &= \frac{1}{2}\frac{\Phi^2}{L^2(x)}\frac{dL(x)}{dx} \\ &= \tfrac{1}{2}i^2\frac{dL(x)}{dx} \end{aligned} \tag{6}$$

6-6-2 Circuit Viewpoint

This result can also be obtained using a circuit description with the linear flux-current relation of (4):

$$\begin{aligned} v &= \frac{d\Phi}{dt} \\ &= L(x)\frac{di}{dt}+i\frac{dL(x)}{dt} \\ &= L(x)\frac{di}{dt}+i\frac{dL(x)}{dx}\frac{dx}{dt} \end{aligned} \tag{7}$$

The last term, proportional to the speed of the moveable member, just adds to the usual inductive voltage term. If the geometry is fixed and does not change with time, there is no electromechanical coupling term.

The power delivered to the system is

$$p = vi = i\frac{d}{dt}[L(x)i] \tag{8}$$

which can be expanded as

$$p = \frac{d}{dt}(\tfrac{1}{2}L(x)i^2)+\tfrac{1}{2}i^2\frac{dL(x)}{dx}\frac{dx}{dt} \tag{9}$$

This is in the form

$$p = \frac{dW}{dt}+f_x\frac{dx}{dt}, \quad \begin{cases} W = \tfrac{1}{2}L(x)i^2 \\ f_x = \tfrac{1}{2}i^2\dfrac{dL(x)}{dx} \end{cases} \tag{10}$$

which states that the power delivered to the inductor is equal to the sum of the time rate of energy stored and mechanical power performed on the inductor. This agrees with the energy method approach. If the inductance does not change with time because the geometry is fixed, all the input power is stored as potential energy W.

Example 6-2 MAGNETIC FIELDS AND FORCES

(a) Relay

Find the force on the moveable slug in the magnetic circuit shown in Figure 6-34.

SOLUTION

It is necessary to find the inductance of the system as a function of the slug's position so that we can use (6). Because of the infinitely permeable core and slug, the **H** field is non-zero only in the air gap of length x. We use Ampere's law to obtain

$$H = NI/x$$

The flux through the gap

$$\Phi = \mu_0 NIA/x$$

is equal to the flux through each turn of the coil yielding the inductance as

$$L(x) = \frac{N\Phi}{I} = \frac{\mu_0 N^2 A}{x}$$

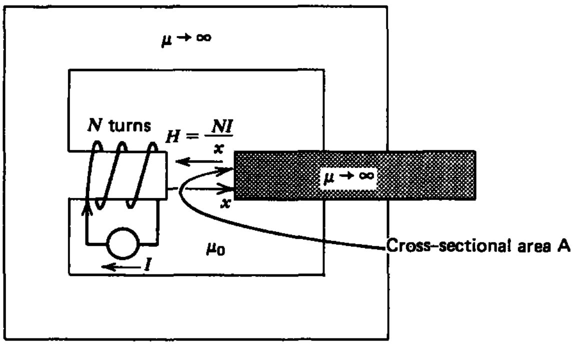


Figure 6-34 The magnetic field exerts a force on the moveable member in the relay pulling it into the magnetic circuit.

The force is then

$$f_x = \frac{1}{2}I^2 \frac{dL(x)}{dx}$$

$$= -\frac{\mu_o N^2 A I^2}{2x^2}$$

The minus sign means that the force is opposite to the direction of increasing x, so that the moveable piece is attracted to the coil.

(b) One Turn Loop

Find the force on the moveable upper plate in the one turn loop shown in Figure 6-35.

SOLUTION

The current distributes itself uniformly as a surface current $K = I/D$ on the moveable plate. If we neglect nonuniform field effects near the corners, the **H** field being tangent to the conductors just equals K:

$$H_z = I/D$$

The total flux linked by the current source is then

$$\Phi = \mu_0 H_z x l$$

$$= \frac{\mu_0 x l}{D} I$$

which gives the inductance as

$$L(x) = \frac{\Phi}{I} = \frac{\mu_0 x l}{D}$$

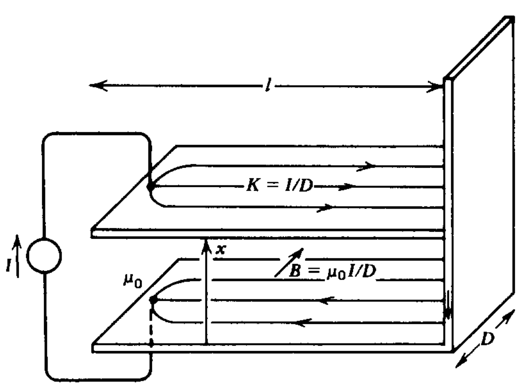


Figure 6-35 The magnetic force on a current-carrying loop tends to expand the loop.

The force is then constant

$$f_x = \frac{1}{2} I^2 \frac{dL(x)}{dx}$$
$$= \frac{1}{2} \frac{\mu_0 l I^2}{D}$$

6-6-3 Magnetization Force

A material with permeability μ is partially inserted into the magnetic circuit shown in Figure 6-36. With no free current in the moveable material, the x-directed force density from Section 5-8-1 is

$$\begin{aligned} F_x &= \mu_0 (\mathbf{M} \cdot \nabla) H_x \\ &= (\mu - \mu_0)(\mathbf{H} \cdot \nabla) H_x \\ &= (\mu - \mu_0)\left(H_x \frac{\partial H_x}{\partial x} + H_y \frac{\partial H_x}{\partial y} \right) \end{aligned} \tag{11}$$

where we neglect variations with z. This force arises in the fringing field because within the gap the magnetic field is essentially uniform:

$$H_y = NI/s \tag{12}$$

Because the magnetic field in the permeable block is curl free,

$$\nabla \times \mathbf{H} = 0 \Rightarrow \frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial x} \tag{13}$$

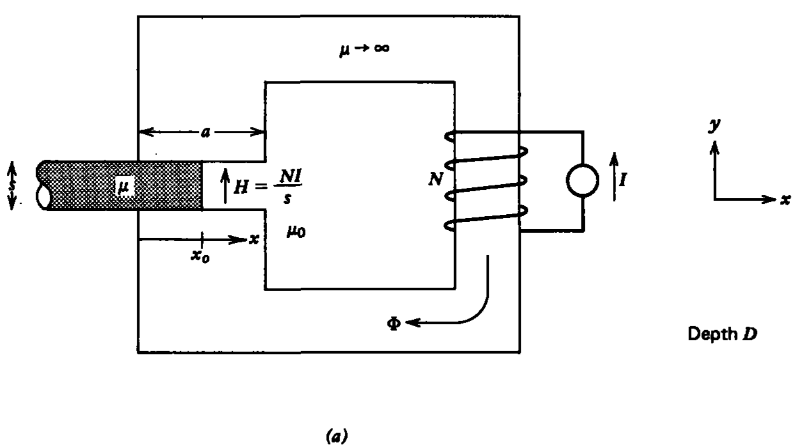


Figure 6-36 · A permeable material tends to be pulled into regions of higher magnetic field.

(11) can be rewritten as

$$F_x = \frac{(\mu - \mu_0)}{2}\frac{\partial}{\partial x}(H_x^2 + H_y^2) \tag{14}$$

The total force is then

$$\begin{aligned} f_x &= sD\int_{-\infty}^{x_0} F_x\,dx \\ &= \frac{(\mu-\mu_0)}{2} sD(H_x^2 + H_y^2)\Big|_{-\infty}^{x_0} \\ &= \frac{(\mu-\mu_0)}{2}\frac{N^2I^2D}{s} \end{aligned} \tag{15}$$

where the fields at $x = -\infty$ are zero and the field at $x = x_0$ is given by (12). High permeability material is attracted to regions of stronger magnetic field. It is this force that causes iron materials to be attracted towards a magnet. Diamagnetic materials ($\mu < \mu_0$) will be repelled.

This same result can more easily be obtained using (6) where the flux through the gap is

$$\Phi = HD[\mu x + \mu_0(a - x)] = \frac{NID}{s}[(\mu - \mu_0)x + a\mu_0] \tag{16}$$

so that the inductance is

$$L = \frac{N\Phi}{I} = \frac{N^2D}{s}[(\mu - \mu_0)x + a\mu_0] \tag{17}$$

Then the force obtained using (6) agrees with (15)

$$\begin{aligned} f_x &= \tfrac{1}{2}I^2\frac{dL(x)}{dx} \\ &= \frac{(\mu-\mu_0)}{2s}N^2I^2D \end{aligned} \tag{18}$$

PROBLEMS

Section 6-1

1. A circular loop of radius a with Ohmic conductivity σ and cross-sectional area A has its center a small distance D away from an infinitely long time varying current.