chapter 4

electric field boundary value problems

The electric field distribution due to external sources is disturbed by the addition of a conducting or dielectric body because the resulting induced charges also contribute to the field. The complete solution must now also satisfy boundary conditions imposed by the materials.

4-1 THE UNIQUENESS THEOREM

Consider a linear dielectric material where the permittivity may vary with position:

$$\mathbf{D}=\varepsilon(r)\mathbf{E}=-\varepsilon(r)\nabla V \tag{1}$$

The special case of different constant permittivity media separated by an interface has $\varepsilon(r)$ as a step function. Using (1) in Gauss's law yields

$$\nabla\cdot[\varepsilon(r)\nabla V]=-\rho_f \tag{2}$$

which reduces to Poisson's equation in regions where $\varepsilon(r)$ is a constant. Let us call V_p a solution to (2).

The solution V_L to the homogeneous equation

$$\nabla\cdot[\varepsilon(r)\nabla V]=0 \tag{3}$$

which reduces to Laplace's equation when $\varepsilon(r)$ is constant, can be added to V_p and still satisfy (2) because (2) is linear in the potential:

$$\nabla\cdot[\varepsilon(r)\nabla(V_p+V_L)]=\nabla\cdot[\varepsilon(r)\nabla V_p]+\underbrace{\nabla\cdot[\varepsilon(r)\nabla V_L]}_{0}=-\rho_f \tag{4}$$

Any linear physical problem must only have one solution yet (3) and thus (2) have many solutions. We need to find what boundary conditions are necessary to uniquely specify this solution. Our method is to consider two different solutions V_1 and V_2 for the same charge distribution

$$\nabla\cdot(\varepsilon\nabla V_1)=-\rho_f,\qquad \nabla\cdot(\varepsilon\nabla V_2)=-\rho_f \tag{5}$$

so that we can determine what boundary conditions force these solutions to be identical, $V_1=V_2$.

The difference of these two solutions $V_T = V_1 - V_2$ obeys the homogeneous equation

$$\nabla \cdot (\varepsilon \nabla V_T) = 0 \tag{6}$$

We examine the vector expansion

$$\nabla \cdot (\varepsilon V_T \nabla V_T) = V_T \underbrace{\nabla \cdot (\varepsilon \nabla V_T)}_{0} + \varepsilon \nabla V_T \cdot \nabla V_T = \varepsilon |\nabla V_T|^2 \tag{7}$$

noting that the first term in the expansion is zero from (6) and that the second term is never negative.

We now integrate (7) over the volume of interest V, which may be of infinite extent and thus include all space

$$\int_V \nabla \cdot (\varepsilon V_T \nabla V_T)\, dV = \oint_S \varepsilon V_T \nabla V_T \cdot \mathbf{dS} = \int_V \varepsilon |\nabla V_T|^2\, dV \tag{8}$$

The volume integral is converted to a surface integral over the surface bounding the region using the divergence theorem. Since the integrand in the last volume integral of (8) is never negative, the integral itself can only be zero if V_T is zero at every point in the volume making the solution unique ($V_T = 0 \Rightarrow V_1 = V_2$). To force the volume integral to be zero, the surface integral term in (8) must be zero. This requires that on the surface S the two solutions must have the same value ($V_1 = V_2$) or their normal derivatives must be equal $[\nabla V_1 \cdot \mathbf{n} = \nabla V_2 \cdot \mathbf{n}]$. This last condition is equivalent to requiring that the normal components of the electric fields be equal ($\mathbf{E} = -\nabla V$).

Thus, a problem is uniquely posed when in addition to giving the charge distribution, the potential or the normal component of the electric field on the bounding surface surrounding the volume is specified. The bounding surface can be taken in sections with some sections having the potential specified and other sections having the normal field component specified.

If a particular solution satisfies (2) but it does not satisfy the boundary conditions, additional homogeneous solutions where $\rho_f = 0$, must be added so that the boundary conditions are met. No matter how a solution is obtained, even if guessed, if it satisfies (2) and all the boundary conditions, it is the only solution.

4-2 BOUNDARY VALUE PROBLEMS IN CARTESIAN GEOMETRIES

For most of the problems treated in Chapters 2 and 3 we restricted ourselves to one-dimensional problems where the electric field points in a single direction and only depends on that coördinate. For many cases, the volume is free of charge so that the system is described by Laplace's equation. Surface charge is present only on interfacial boundaries separating dissimilar conducting materials. We now consider such volume charge-free problems with two- and three dimensional variations.

4-2-1 Separation of Variables

Let us assume that within a region of space of constant permittivity with no volume charge, that solutions do not depend on the z coordinate. Then Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0 \tag{1}$$

We try a solution that is a product of a function only of the x coordinate and a function only of y:

$$V(x, y)=X(x)\, Y(y) \tag{2}$$

This assumed solution is often convenient to use if the system boundaries lay in constant x or constant y planes. Then along a boundary, one of the functions in (2) is constant. When (2) is substituted into (1) we have

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0\Rightarrow\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0 \tag{3}$$

where the partial derivatives become total derivatives because each function only depends on a single coordinate. The second relation is obtained by dividing through by XY so that the first term is only a function of x while the second is only a function of y.

The only way the sum of these two terms can be zero for all values of x and y is if each term is separately equal to a constant so that (3) separates into two equations,

$$\frac{1}{X}\frac{d^2X}{dx^2}=k^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2}=-k^2 \tag{4}$$

where k^2 is called the separation constant and in general can be a complex number. These equations can then be rewritten as the ordinary differential equations:

$$\frac{d^2X}{dx^2}-k^2X=0, \qquad \frac{d^2Y}{dy^2}+k^2Y=0 \tag{5}$$

4-2-2 Zero Separation Constant Solutions

When the separation constant is zero ($k^2 = 0$) the solutions to (5) are

$$X = a_1 x + b_1, \qquad Y = c_1 y + d_1 \tag{6}$$

where a_1, b_1, c_1, and d_1 are constants. The potential is given by the product of these terms which is of the form

$$V = a_2 + b_2 x + c_2 y + d_2 xy \tag{7}$$

The linear and constant terms we have seen before, as the potential distribution within a parallel plate capacitor with no fringing, so that the electric field is uniform. The last term we have not seen previously.

(a) Hyperbolic Electrodes

A hyperbolically shaped electrode whose surface shape obeys the equation $xy = ab$ is at potential V_0 and is placed above a grounded right-angle corner as in Figure 4-1. The

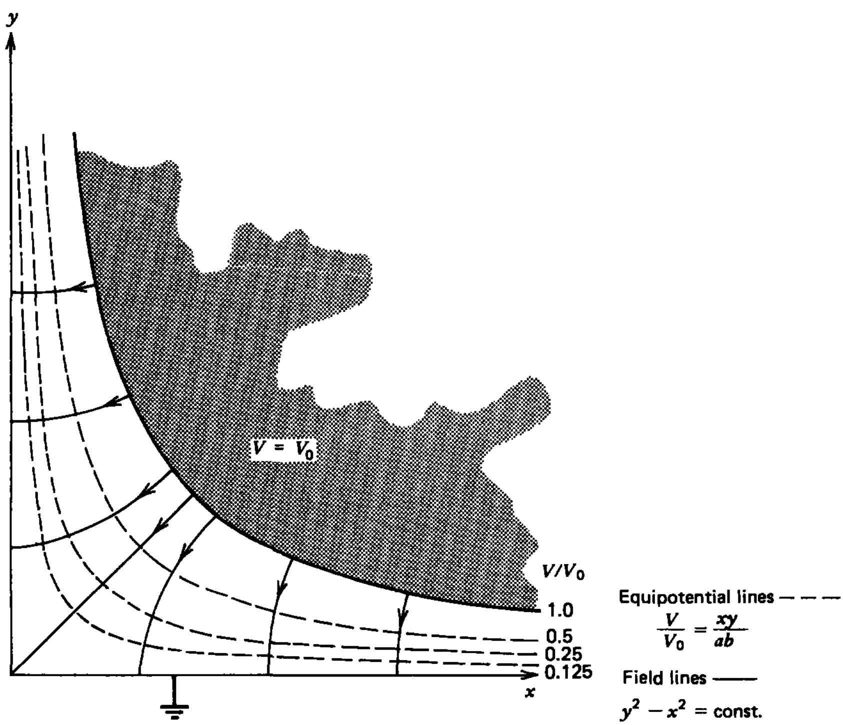


Figure 4-1 The equipotential and field lines for a hyperbolically shaped electrode at potential V_0 above a right-angle conducting corner are orthogonal hyperbolas.

boundary conditions are

$$V(x=0)=0, \qquad V(y=0)=0, \qquad V(xy=ab)=V_0 \tag{8}$$

so that the solution can be obtained from (7) as

$$V(x, y)=V_0xy/(ab) \tag{9}$$

The electric field is then

$$\mathbf{E}=-\nabla V=-\frac{V_0}{ab}[y\mathbf{i}_x+x\mathbf{i}_y] \tag{10}$$

The field lines drawn in Figure 4-1 are the perpendicular family of hyperbolas to the equipotential hyperbolas in (9):

$$\frac{dy}{dx}=\frac{E_y}{E_x}=\frac{x}{y}\Rightarrow y^2-x^2=\text{const} \tag{11}$$

(b) Resistor in an Open Box

A resistive medium is contained between two electrodes, one of which extends above and is bent through a right-angle corner as in Figure 4-2. We try zero separation constant

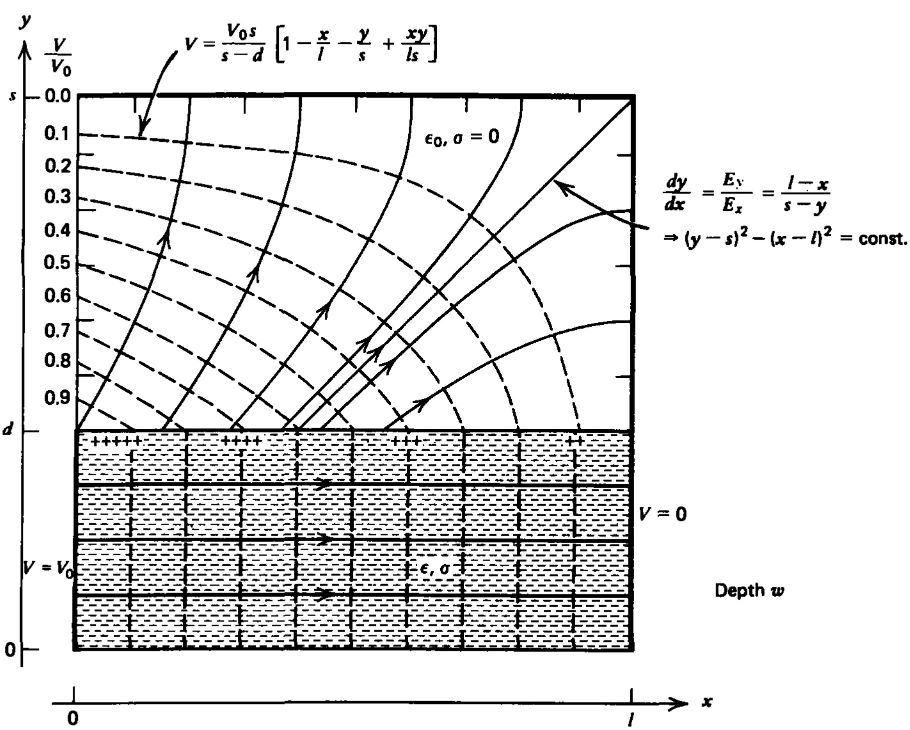


Figure 4-2 A resistive medium partially fills an open conducting box.

solutions given by (7) in each region enclosed by the electrodes:

$$V=\begin{cases} a_1+b_1x+c_1y+d_1xy, & o\le y\le d \\ a_2+b_2x+c_2y+d_2xy, & d\le y\le s \end{cases} \tag{12}$$

With the potential constrained on the electrodes and being continuous across the interface, the boundary conditions are

$$V(x=0)=V_0=a_1+c_1y\Rightarrow a_1=V_0,\quad c_1=0 \qquad (0\le y\le d)$$

$$V(x=l)=0=\begin{cases} \underbrace{a_1}_{V_0}+b_1l+\cancelto{0}{c_1}y+d_1ly\Rightarrow b_1=-V_0/l, \quad d_1=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (0\le y\le d) \\ a_2+b_2l+c_2y+d_2ly \Rightarrow a_2+b_2l=0, \quad c_2+d_2l=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (d\le y\le s) \end{cases}$$

$$V(y=s)=0=a_2+b_2x+c_2s+d_2xs \Rightarrow a_2+c_2s=0, \quad b_2+d_2s=0$$

$$\begin{aligned} V(y=d_+)=V(y=d_-)&=a_1+b_1x+\cancelto{0}{c_1}\,d+\cancelto{0}{d_1}\,xd \\ &=a_2+b_2x+c_2d+d_2xd \end{aligned} \tag{13}$$

$$\Rightarrow a_1=V_0=a_2+c_2d, \quad b_1=-V_0/l=b_2+d_2d$$

so that the constants in (12) are

$$\begin{aligned} &a_1=V_0, \quad b_1=-V_0/l, \quad c_1=0, \quad d_1=0 \\ &a_2=\frac{V_0}{(1-d/s)}, \quad b_2=-\frac{V_0}{l(1-d/s)}, \\ &c_2=-\frac{V_0}{s(1-d/s)}, \quad d_2=\frac{V_0}{ls(1-d/s)} \end{aligned} \tag{14}$$

The potential of (12) is then

$$V=\begin{cases} V_0(1-x/l), & 0\le y\le d \\ \dfrac{V_0s}{s-d}\left(1-\dfrac{x}{l}-\dfrac{y}{s}+\dfrac{xy}{ls}\right), & d\le y\le s \end{cases} \tag{15}$$

with associated electric field

$$\mathbf{E}=-\nabla V=\begin{cases} \dfrac{V_0}{l}\mathbf{i}_x, & 0\le y\le d \\ \dfrac{V_0s}{s-d}\left[\dfrac{\mathbf{i}_x}{l}\left(1-\dfrac{y}{s}\right)+\dfrac{\mathbf{i}_y}{s}\left(1-\dfrac{x}{l}\right)\right], & d<y<s \end{cases} \tag{16}$$

Note that in the dc steady state, the conservation of charge boundary condition of Section 3-3-5 requires that no current cross the interfaces at $y=0$ and $y=d$ because of the surrounding zero conductivity regions. The current and, thus, the electric field within the resistive medium must be purely tangential to the interfaces, $E_y(y=d_-)=E_y(y=0_+)=0$. The surface charge density on the interface at $y=d$ is then due only to the normal electric field above, as below, the field is purely tangential:

$$\sigma_f(y=d)=\varepsilon_0 E_y(y=d_+)-\cancelto{0}{\varepsilon E_y}(y=d_-)=\frac{\varepsilon_0 V_0}{s-d}\left(1-\frac{x}{l}\right) \tag{17}$$

The interfacial shear force is then

$$F_x=\int_0^l \sigma_f E_x(y=d)w\,dx=\frac{\varepsilon_0 V_0^2}{2(s-d)}w \tag{18}$$

If the resistive material is liquid, this shear force can be used to pump the fluid.*

4-2-3 Nonzero Separation Constant Solutions

Further solutions to (5) with nonzero separation constant ($k^2\neq 0$) are

$$\begin{aligned} X&=A_1\sinh kx+A_2\cosh kx=B_1\,e^{kx}+B_2\,e^{-kx}\\ Y&=C_1\sin ky+C_2\cos ky=D_1\,e^{jky}+D_2\,e^{-jky}\end{aligned} \tag{19}$$

When k is real, the solutions of X are hyperbolic or equivalently exponential, as drawn in Figure 4-3, while those of Y are trigonometric. If k is pure imaginary, then X becomes trigonometric and Y is hyperbolic (or exponential).

The solution to the potential is then given by the product of X and Y:

$$\begin{aligned} V=&E_1\sin ky\sinh kx+E_2\sin ky\cosh kx\\ &+E_3\cos ky\sinh kx+E_4\cos ky\cosh kx\end{aligned} \tag{20}$$

or equivalently

$$\mathrm{V}=F_1\sin ky\,e^{kx}+F_2\sin ky\,e^{-kx}+F_3\cos ky\,e^{kx}+F_4\cos ky\,e^{-kx} \tag{21}$$

We can always add the solutions of (7) or any other Laplacian solutions to (20) and (21) to obtain a more general

* *See J. R. Melcher and G. I. Taylor, Electrohydrodynamics: A Review of the Role of Interfacial Shear Stresses,* Annual Rev. Fluid Mech., *Vol. 1, Annual Reviews, Inc., Palo Alto, Calif., 1969, ed. by Sears and Van Dyke, pp. 111–146. See also J. R. Melcher, "Electric Fields and Moving Media", film produced for the National Committee on Electrical Engineering Films by the Educational Development Center, 39 Chapel St., Newton, Mass. 02160. This film is described in* IEEE Trans. Education ***E-17***, *(1974) pp. 100–110.*

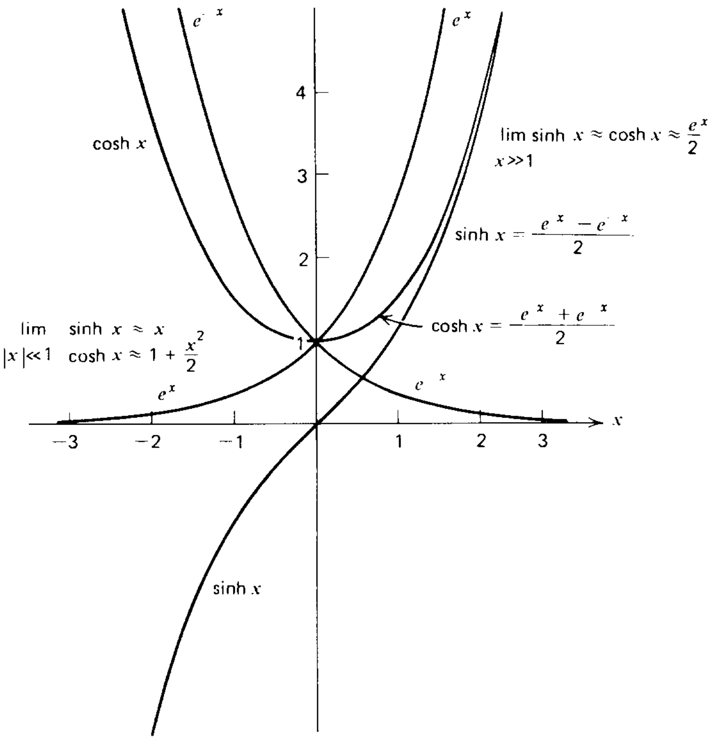


Figure 4-3 The exponential and hyperbolic functions for positive and negative arguments.

solution because Laplace's equation is linear. The values of the coefficients and of k are determined by boundary conditions.

When regions of space are of infinite extent in the x direction, it is often convenient to use the exponential solutions in (21) as it is obvious which solutions decay as x approaches $\pm\infty$. For regions of finite extent, it is usually more convenient to use the hyperbolic expressions of (20). A general property of Laplace solutions are that they are oscillatory in one direction and decay in the perpendicular direction.

4-2-4 Spatially Periodic Excitation

A sheet in the $x = 0$ plane has the imposed periodic potential, $V = V_0 \sin ay$ shown in Figure 4-4. In order to meet this boundary condition we use the solution of (21) with $k = a$. The potential must remain finite far away from the source so

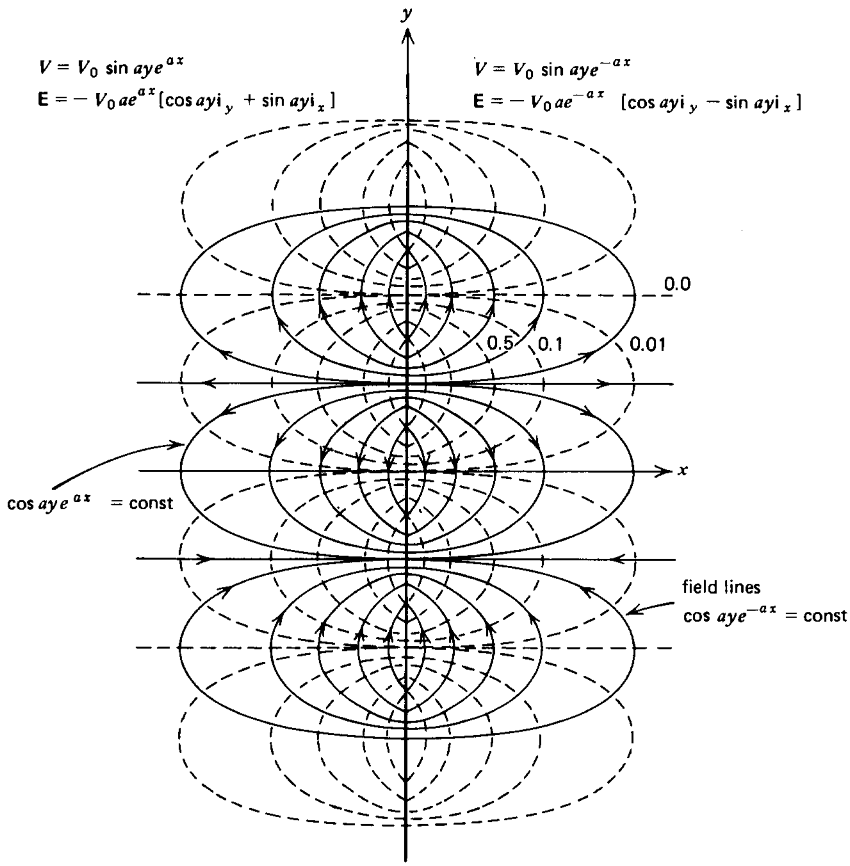


Figure 4-4 The potential and electric field decay away from an infinite sheet with imposed spatially periodic voltage. The field lines emanate from positive surface charge on the sheet and terminate on negative surface charge.

we write the solution separately for positive and negative x as

$$V=\begin{cases} V_0 \sin ay\, e^{-ax}, & x \ge 0 \\ V_0 \sin ay\, e^{ax}, & x \le 0 \end{cases} \tag{22}$$

where we picked the amplitude coefficients to be continuous and match the excitation at $x=0$. The electric field is then

$$\mathbf{E}=-\nabla V=\begin{cases} -V_0 a\, e^{-ax}[\cos ay\mathbf{i}_y-\sin ay\mathbf{i}_x], & x>0 \\ -V_0 a\, e^{ax}[\cos ay\mathbf{i}_y+\sin ay\mathbf{i}_x], & x<0 \end{cases} \tag{23}$$

The surface charge density on the sheet is given by the discontinuity in normal component of $\mathbf{D}$ across the sheet:

$$\begin{aligned}\sigma_f(x=0)&=\varepsilon[E_x(x=0_+)-E_x(x=0_-)]\\ &=2\varepsilon V_0 a \sin ay\end{aligned} \tag{24}$$

The field lines drawn in Figure 4-4 obey the equation

$$\frac{dy}{dx}=\frac{E_y}{E_x}=\mp \cot ay \Rightarrow \cos ay\, e^{\mp ax}=\text{const} \quad \begin{cases} x>0 \\ x<0 \end{cases} \tag{25}$$

4-2-5 Rectangular Harmonics

When excitations are not sinusoidally periodic in space, they can be made so by expressing them in terms of a trigonometric Fourier series. Any periodic function of y can be expressed as an infinite sum of sinusoidal terms as

$$f(y)=\tfrac{1}{2}b_0+\sum_{n=1}^{\infty}\left(a_n \sin\frac{2n\pi y}{\lambda}+b_n\cos\frac{2n\pi y}{\lambda}\right) \tag{26}$$

where λ is the fundamental period of $f(y)$.

The Fourier coefficients a_n are obtained by multiplying both sides of the equation by $\sin(2p\pi y/\lambda)$ and integrating over a period. Since the parameter p is independent of the index n, we may bring the term inside the summation on the right hand side. Because the trigonometric functions are orthogonal to one another, they integrate to zero except when the function multiplies itself:

$$\begin{aligned} &\int_0^{\lambda}\sin\frac{2p\pi y}{\lambda}\sin\frac{2n\pi y}{\lambda}\,dy=\begin{cases}0, & p\neq n\\ \lambda/2, & p=n\end{cases}\\ &\int_0^{\lambda}\sin\frac{2p\pi y}{\lambda}\cos\frac{2n\pi y}{\lambda}\,dy=0\end{aligned} \tag{27}$$

Every term in the series for $n\neq p$ integrates to zero. Only the term for $n=p$ is nonzero so that

$$a_p=\frac{2}{\lambda}\int_0^{\lambda} f(y)\sin\frac{2p\pi y}{\lambda}\,dy \tag{28}$$

To obtain the coefficients b_n, we similarly multiply by $\cos(2p\pi y/\lambda)$ and integrate over a period:

$$b_p=\frac{2}{\lambda}\int_0^{\lambda} f(y)\cos\frac{2p\pi y}{\lambda}\,dy \tag{29}$$

Consider the conducting rectangular box of infinite extent in the x and z directions and of width d in the y direction shown in Figure 4-5. The potential along the $x=0$ edge is V_0 while all other surfaces are grounded at zero potential. Any periodic function can be used for $f(y)$ if over the interval $0\leq y\leq d, f(y)$ has the properties

$$f(y)=V_0,\ 0<y<d;\ f(y=0)=f(y=d)=0 \tag{30}$$

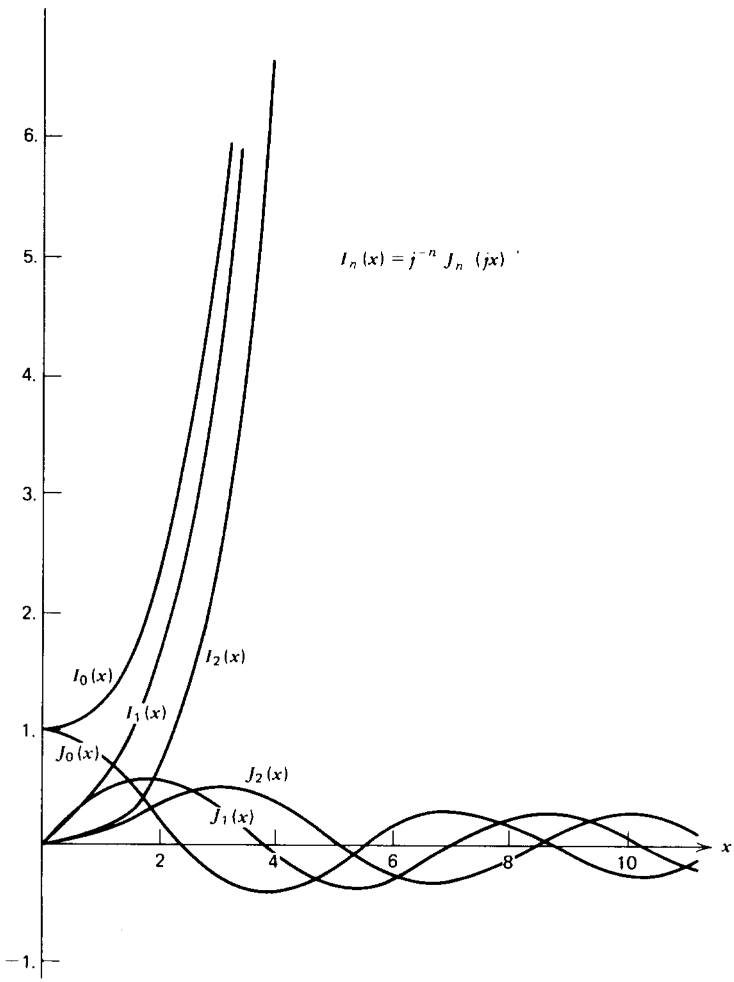
where n^2 is the second separation constant. The angular dependence thus has the same solutions as for the two-dimensional case

$$\Phi = \begin{cases} B_1 \sin n\phi + B_2 \cos n\phi, & n \neq 0 \\ B_3\phi + B_4, & n = 0 \end{cases} \tag{36}$$

The resulting differential equation for the radial dependence

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (k^2 r^2 - n^2)R = 0 \tag{37}$$

is Bessel's equation and for nonzero k has solutions in terms



(a)

Figure 4-9 The Bessel functions (a) $J_n(x)$ and $I_n(x)$, and (b) $Y_n(x)$ and $K_n(x)$.

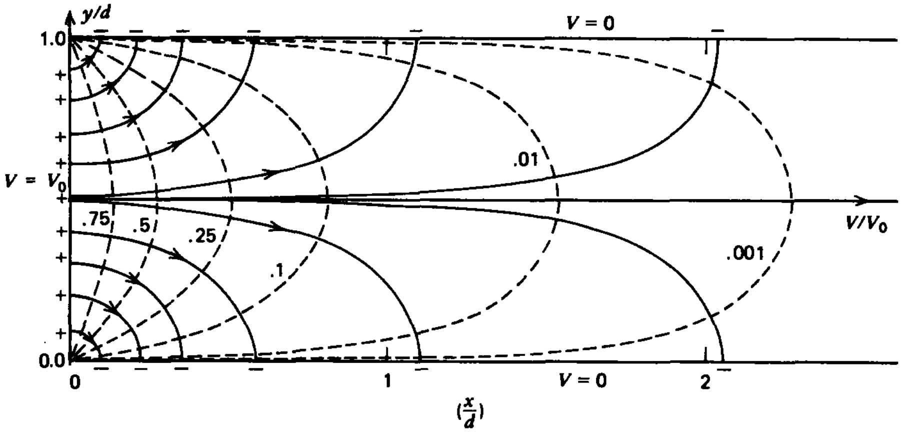


Figure 4-5 An open conducting box of infinite extent in the x and z directions and of finite width d in the y direction, has zero potential on all surfaces except the closed end at $x = 0$, where $V = V_0$.

In particular, we choose the periodic square wave function with $\lambda = 2d$ shown in Figure 4-6 so that performing the integrations in (28) and (29) yields

$$a_p = -\frac{2V_0}{p\pi}(\cos p\pi - 1)$$

$$= \begin{cases} 0, & p \text{ even} \\ 4V_0/p\pi, & p \text{ odd} \end{cases} \tag{31}$$

$$b_p = 0$$

Thus the constant potential at $x = 0$ can be written as the Fourier sine series

$$V(x=0) = V_0 = \frac{4V_0}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin(n\pi y/d)}{n} \tag{32}$$

In Figure 4-6 we plot various partial sums of the Fourier series to show that as the number of terms taken becomes large, the series approaches the constant value V_0 except for the Gibbs overshoot of about 18% at $y = 0$ and $y = d$ where the function is discontinuous.

The advantage in writing V_0 in a Fourier sine series is that each term in the series has a similar solution as found in (22) where the separation constant for each term is $k_n = n\pi/d$ with associated amplitude $4V_0/(n\pi)$.

The solution is only nonzero for $x > 0$ so we immediately write down the total potential solution as

$$V(x, y) = \frac{4V_0}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \sin\frac{n\pi y}{d} e^{-n\pi x/d} \tag{33}$$

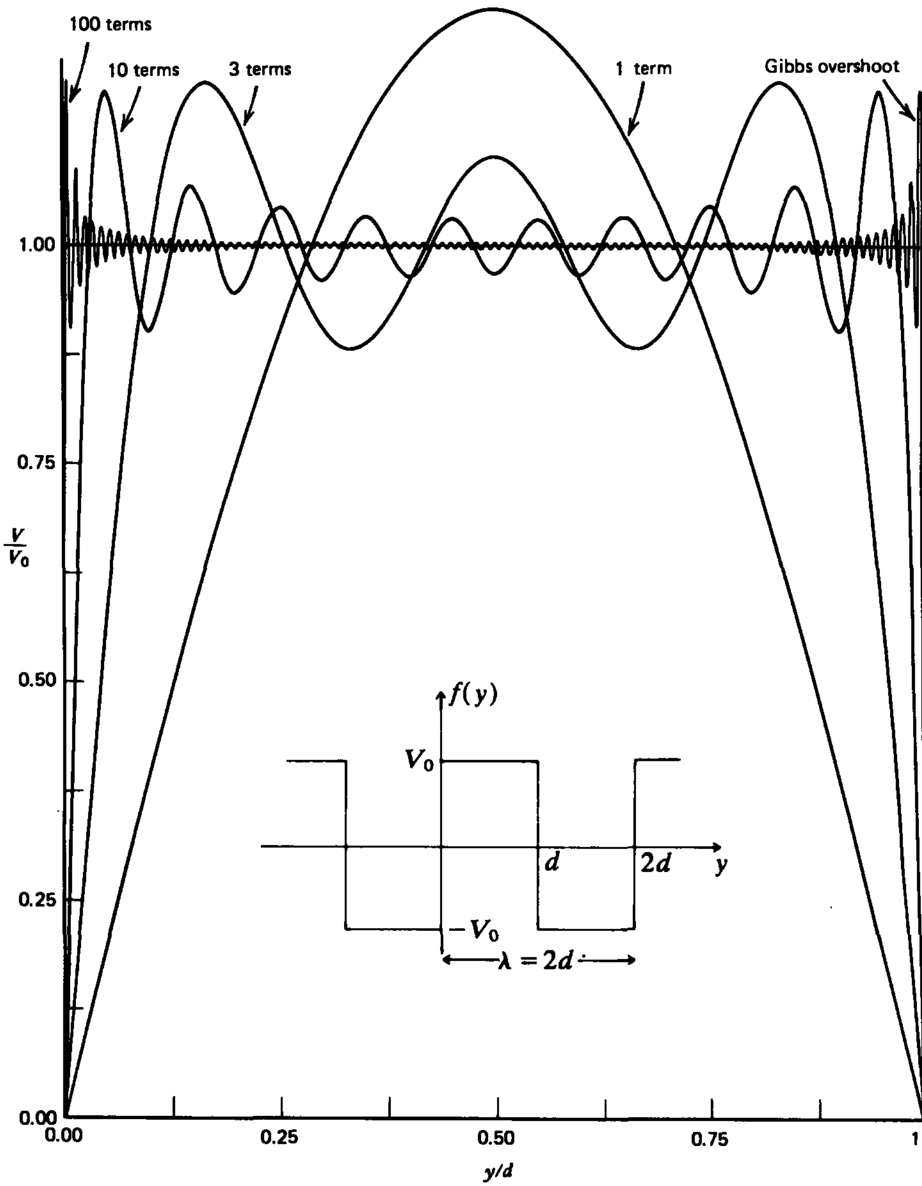


Figure 4-6 Fourier series expansion of the imposed constant potential along the $x = 0$ edge in Figure 4-5 for various partial sums. As the number of terms increases, the series approaches a constant except at the boundaries where the discontinuity in potential gives rise to the Gibbs phenomenon of an 18% overshoot with narrow width.

The electric field is then

$$\mathbf{E} = -\nabla V = -\frac{4V_0}{d} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(-\sin\frac{n\pi y}{d}\mathbf{i}_x + \cos\frac{n\pi y}{d}\mathbf{i}_y\right) e^{-n\pi x/d} \tag{34}$$

The field and equipotential lines are sketched in Figure 4-5. Note that for $x \gg d$, the solution is dominated by the first harmonic. Far from a source, Laplacian solutions are insensitive to the details of the source geometry.

4-2-6 Three-Dimensional Solutions

If the potential depends on the three coordinates (x, y, z), we generalize our approach by trying a product solution of the form

$$V(x, y, z) = X(x)\, Y(y)\, Z(z) \tag{35}$$

which, when substituted into Laplace's equation, yields after division through by XYZ

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0 \tag{36}$$

three terms each wholly a function of a single coordinate so that each term again must separately equal a constant:

$$\frac{1}{X}\frac{d^2X}{dx^2}=-k_x^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2}=-k_y^2, \qquad \frac{1}{Z}\frac{d^2Z}{dz^2}=k_z^2=k_x^2+k_y^2 \tag{37}$$

We change the sign of the separation constant for the z dependence as the sum of separation constants must be zero. The solutions for nonzero separation constants are

$$\begin{aligned} X &= A_1 \sin k_x x + A_2 \cos k_x x \\ Y &= B_1 \sin k_y y + B_2 \cos k_y y \\ Z &= C_1 \sinh k_z z + C_2 \cosh k_z z = D_1\, e^{k_z z} + D_2\, e^{-k_z z} \end{aligned} \tag{38}$$

The solutions are written as if k_x, k_y, and k_z are real so that the x and y dependence is trigonometric while the z dependence is hyperbolic or equivalently exponential. However, k_x, k_y, or k_z may be imaginary converting hyperbolic functions to trigonometric and vice versa. Because the squares of the separation constants must sum to zero at least one of the solutions in (38) must be trigonometric and one must be hyperbolic. The remaining solution may be either trigonometric or hyperbolic depending on the boundary conditions. If the separation constants are all zero, in addition to the solutions of (6) we have the similar addition

$$Z = e_1 z + f_1 \tag{39}$$

4-3 SEPARATION OF VARIABLES IN CYLINDRICAL GEOMETRY

Product solutions to Laplace's equation in cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}=0 \tag{1}$$

also separate into solvable ordinary differential equations.

4-3-1 Polar Solutions

If the system geometry does not vary with z, we try a solution that is a product of functions which only depend on the radius r and angle ϕ:

$$V(r,\phi)=R(r)\Phi(\phi) \tag{2}$$

which when substituted into (1) yields

$$\frac{\Phi}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{R}{r^2}\frac{d^2\Phi}{d\phi^2}=0 \tag{3}$$

This assumed solution is convenient when boundaries lay at a constant angle of ϕ or have a constant radius, as one of the functions in (2) is then constant along the boundary.

For (3) to separate, each term must only be a function of a single variable, so we multiply through by $r^2/R\Phi$ and set each term equal to a constant, which we write as n^2:

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=n^2, \qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-n^2 \tag{4}$$

The solution for Φ is easily solved as

$$\Phi=\begin{cases}A_1\sin n\phi+A_2\cos n\phi, & n\neq 0\\ B_1\phi+B_2, & n=0\end{cases} \tag{5}$$

The solution for the radial dependence is not as obvious. However, if we can find two independent solutions by any means, including guessing, the total solution is uniquely given as a linear combination of the two solutions. So, let us try a power-law solution of the form

$$R=Ar^p \tag{6}$$

which when substituted into (4) yields

$$p^2=n^2 \Rightarrow p=\pm n \tag{7}$$

For $n \neq 0$, (7) gives us two independent solutions. When $n = 0$ we refer back to (4) to solve

$$r\frac{dR}{dr} = \text{const} \Rightarrow R = D_1 \ln r + D_2 \tag{8}$$

so that the solutions are

$$R = \begin{cases} C_1 r^n + C_2 r^{-n}, & n \neq 0 \\ D_1 \ln r + D_2, & n = 0 \end{cases} \tag{9}$$

We recognize the $n = 0$ solution for the radial dependence as the potential due to a line charge. The $n = 0$ solution for the ϕ dependence shows that the potential increases linearly with angle. Generally n can be any complex number, although in usual situations where the domain is periodic and extends over the whole range $0 \leq \phi \leq 2\pi$, the potential at $\phi = 2\pi$ must equal that at $\phi = 0$ since they are the same point. This requires that n be an integer.

EXAMPLE 4-1 SLANTED CONDUCTING PLANES

Two planes of infinite extent in the z direction at an angle α to one another, as shown in Figure 4-7, are at a potential difference v. The planes do not intersect but come sufficiently close to one another that fringing fields at the electrode ends may be neglected. The electrodes extend from $r = a$ to $r = b$. What is the approximate capacitance per unit length of the structure?

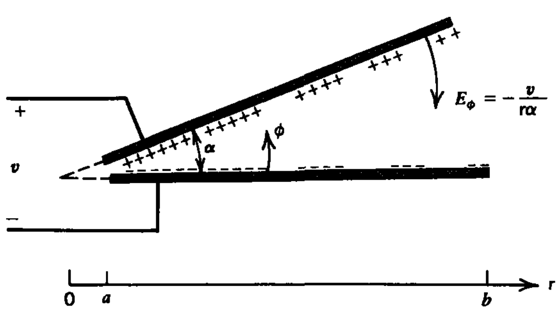


Figure 4-7 Two conducting planes at angle α stressed by a voltage v have a ϕ-directed electric field.

SOLUTION

We try the $n=0$ solution of (5) with no radial dependence as

$$V = B_1\phi + B_2$$

The boundary conditions impose the constraints

$$V(\phi=0)=0, \qquad V(\phi=\alpha)=v \Rightarrow V = v\phi/\alpha$$

The electric field is

$$E_\phi = -\frac{1}{r}\frac{dV}{d\phi} = -\frac{v}{r\alpha}$$

The surface charge density on the upper electrode is then

$$\sigma_f(\phi=\alpha) = -\varepsilon E_\phi(\phi=\alpha) = \frac{\varepsilon v}{r\alpha}$$

with total charge per unit length

$$\lambda(\phi=\alpha) = \int_{r=a}^{b} \sigma_f(\phi=\alpha)\, dr = \frac{\varepsilon v}{\alpha}\ln\frac{b}{a}$$

so that the capacitance per unit length is

$$C = \frac{\lambda}{v} = \frac{\varepsilon \ln(b/a)}{\alpha}$$

4-3-2 Cylinder in a Uniform Electric Field

(a) Field Solutions

An infinitely long cylinder of radius a, permittivity ε_2, and Ohmic conductivity σ_2 is placed within an infinite medium of permittivity ε_1 and conductivity σ_1. A uniform electric field at infinity $\mathbf{E} = E_0\mathbf{i}_x$ is suddenly turned on at $t=0$. This problem is analogous to the series lossy capacitor treated in Section 3-6-3. As there, we will similarly find that:

(i) At $t=0$ the solution is the same as for two lossless dielectrics, independent of the conductivities, with no interfacial surface charge, described by the boundary condition

$$\sigma_f(r=a) = D_r(r=a_+) - D_r(r=a_-) = 0$$
$$\Rightarrow \varepsilon_1 E_r(r=a_+) = \varepsilon_2 E_r(r=a_-) \quad (10)$$

(ii) As $t\to\infty$, the steady-state solution depends only on the conductivities, with continuity of normal current at the cylinder interface,

$$J_r(r = a_+) = J_r(r = a_-) \Rightarrow \sigma_1 E_r(r = a_+) = \sigma_2 E_r(r = a_-) \tag{11}$$

(iii) The time constant describing the transition from the initial to steady-state solutions will depend on some weighted average of the ratio of permittivities to conductivities.

To solve the general transient problem we must find the potential both inside and outside the cylinder, joining the solutions in each region via the boundary conditions at $r = a$.

Trying the nonzero n solutions of (5) and (9), n must be an integer as the potential at $\phi = 0$ and $\phi = 2\pi$ must be equal, since they are the same point. For the most general case, an infinite series of terms is necessary, superposing solutions with $n = 1, 2, 3, 4, \cdots$. However, because of the form of the uniform electric field applied at infinity, expressed in cylindrical coordinates as

$$\mathbf{E}(r \to \infty) = E_0 \mathbf{i}_x = E_0[\mathbf{i}_r \cos\phi - \mathbf{i}_\phi \sin\phi] \tag{12}$$

we can meet all the boundary conditions using only the $n = 1$ solution.

Keeping the solution finite at $r = 0$, we try solutions of the form

$$V(r, \phi) = \begin{cases} A(t) r \cos\phi, & r \le a \\ [B(t) r + C(t)/r] \cos\phi, & r \ge a \end{cases} \tag{13}$$

with associated electric field

$$\mathbf{E} = -\nabla V = \begin{cases} -A(t)[\cos\phi\, \mathbf{i}_r - \sin\phi\, \mathbf{i}_\phi] = -A(t)\mathbf{i}_x, & r < a \\ -[B(t) - C(t)/r^2] \cos\phi\, \mathbf{i}_r & \\ \quad +[B(t) + C(t)/r^2] \sin\phi\, \mathbf{i}_\phi, & r > a \end{cases} \tag{14}$$

We do not consider the $\sin\phi$ solution of (5) in (13) because at infinity the electric field would have to be y directed:

$$V = Dr \sin\phi \Rightarrow \mathbf{E} = -\nabla V = -D[\mathbf{i}_r \sin\phi + \mathbf{i}_\phi \cos\phi] = -D\mathbf{i}_y \tag{15}$$

The electric field within the cylinder is x directed. The solution outside is in part due to the imposed x-directed uniform field, so that as $r \to \infty$ the field of (14) must approach (12), requiring that $B(t) = -E_0$. The remaining contribution to the external field is equivalent to a two-dimensional line dipole (see Problem 3.1), with dipole moment per unit length:

$$p_x = \lambda d = 2\pi\varepsilon C(t) \tag{16}$$

The other time-dependent amplitudes $A(t)$ and $C(t)$ are found from the following additional boundary conditions:

(i) the potential is continuous at $\mathrm{r}=a$, which is the same as requiring continuity of the tangential component of **E**:

$$V(\mathrm{r}=a_+)=V(\mathrm{r}=a_-)\Rightarrow E_\phi(\mathrm{r}=a_-)=E_\phi(\mathrm{r}=a_+)$$
$$\Rightarrow Aa=Ba+C/a \tag{17}$$

(ii) charge must be conserved on the interface:

$$J_\mathrm{r}(\mathrm{r}=a_+)-J_\mathrm{r}(\mathrm{r}=a_-)+\frac{\partial\sigma_f}{\partial t}=0$$
$$\Rightarrow \sigma_1E_\mathrm{r}(\mathrm{r}=a_+)-\sigma_2E_\mathrm{r}(\mathrm{r}=a_-)$$
$$+\frac{\partial}{\partial t}[\varepsilon_1E_\mathrm{r}(\mathrm{r}=a_+)-\varepsilon_2E_\mathrm{r}(\mathrm{r}=a_-)]=0 \tag{18}$$

In the steady state, (18) reduces to (11) for the continuity of normal current, while for $t=0$ the time derivative must be noninfinite so σ_f is continuous and thus zero as given by (10).

Using (17) in (18) we obtain a single equation in $C(t)$:

$$\frac{dC}{dt}+\left(\frac{\sigma_1+\sigma_2}{\varepsilon_1+\varepsilon_2}\right)C=\frac{-a^2}{\varepsilon_1+\varepsilon_2}\left(E_0(\sigma_1-\sigma_2)+(\varepsilon_1-\varepsilon_2)\frac{dE_0}{dt}\right) \tag{19}$$

Since E_0 is a step function in time, the last term on the right-hand side is an impulse function, which imposes the initial condition

$$C(t=0)=-a^2\frac{(\varepsilon_1-\varepsilon_2)}{\varepsilon_1+\varepsilon_2}E_0 \tag{20}$$

so that the total solution to (19) is

$$C(t)=a^2E_0\left(\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}+\frac{2(\sigma_1\varepsilon_2-\sigma_2\varepsilon_1)}{(\sigma_1+\sigma_2)(\varepsilon_1+\varepsilon_2)}e^{-t/\tau}\right),\quad \tau=\frac{\varepsilon_1+\varepsilon_2}{\sigma_1+\sigma_2} \tag{21}$$

The interfacial surface charge is

$$\begin{aligned}\sigma_f(\mathrm{r}=\mathrm{a},t)&=\varepsilon_1E_\mathrm{r}(\mathrm{r}=\mathrm{a}_+)-\varepsilon_2E_\mathrm{r}(\mathrm{r}=a_-)\\&=\left[-\varepsilon_1\left(B-\frac{C}{a^2}\right)+\varepsilon_2A\right]\cos\phi\\&=\left[(\varepsilon_1-\varepsilon_2)E_0+(\varepsilon_1+\varepsilon_2)\frac{C}{a^2}\right]\cos\phi\\&=\frac{2(\sigma_2\varepsilon_1-\sigma_1\varepsilon_2)}{\sigma_1+\sigma_2}E_0[1-e^{-t/\tau}]\cos\phi\end{aligned} \tag{22}$$

The upper part of the cylinder $(-\pi/2 \le \phi \le \pi/2)$ is charged of one sign while the lower half $(\pi/2 \le \phi \le \frac{3}{2}\pi)$ is charged with the opposite sign, the net charge on the cylinder being zero. The cylinder is uncharged at each point on its surface if the relaxation times in each medium are the same, $\varepsilon_1/\sigma_1 = \varepsilon_2/\sigma_2$

The solution for the electric field at $t = 0$ is

$$\mathbf{E}(t=0) = \begin{cases} \dfrac{2\varepsilon_1 E_0}{\varepsilon_1+\varepsilon_2}[\cos\phi\,\mathbf{i}_r - \sin\phi\,\mathbf{i}_\phi] = \dfrac{2\varepsilon_1 E_0}{\varepsilon_1+\varepsilon_2}\mathbf{i}_x, & r<a \\ E_0\left[\left(1+\dfrac{a^2}{r^2}\dfrac{\varepsilon_2-\varepsilon_1}{\varepsilon_1+\varepsilon_2}\right)\cos\phi\,\mathbf{i}_r \right. & \\ \left. \quad -\left(1-\dfrac{a^2}{r^2}\dfrac{\varepsilon_2-\varepsilon_1}{\varepsilon_1+\varepsilon_2}\right)\sin\phi\,\mathbf{i}_\phi\right], & r>a \end{cases} \tag{23}$$

The field inside the cylinder is in the same direction as the applied field, and is reduced in amplitude if $\varepsilon_2 > \varepsilon_1$ and increased in amplitude if $\varepsilon_2 < \varepsilon_1$, up to a limiting factor of two as ε_1 becomes large compared to ε_2. If $\varepsilon_2 = \varepsilon_1$, the solution reduces to the uniform applied field everywhere.

The dc steady-state solution is identical in form to (23) if we replace the permittivities in each region by their conductivities;

$$\mathbf{E}(t\to\infty) = \begin{cases} \dfrac{2\sigma_1 E_0}{\sigma_1+\sigma_2}[\cos\phi\,\mathbf{i}_r - \sin\phi\,\mathbf{i}_\phi] = \dfrac{2\sigma_1 E_0}{\sigma_1+\sigma_2}\mathbf{i}_x, & r<a \\ E_0\left[\left(1+\dfrac{a^2}{r^2}\dfrac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\cos\phi\,\mathbf{i}_r \right. & \\ \left. \quad -\left(1-\dfrac{a^2}{r^2}\dfrac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\sin\phi\,\mathbf{i}_\phi\right], & r>a \end{cases} \tag{24}$$

(b) Field Line Plotting

Because the region outside the cylinder is charge free, we know that $\nabla \cdot \mathbf{E} = 0$. From the identity derived in Section 1-5-4*b*, that the divergence of the curl of a vector is zero, we thus know that the polar electric field with no z component can be expressed in the form

$$\begin{aligned} \mathbf{E}(r, \phi) &= \nabla \times \Sigma(r, \phi)\mathbf{i}_z \\ &= \frac{1}{r}\frac{\partial \Sigma}{\partial \phi}\mathbf{i}_r - \frac{\partial \Sigma}{\partial r}\mathbf{i}_\phi \end{aligned} \tag{25}$$

where Σ is called the stream function. Note that the stream function vector is in the direction perpendicular to the electric field so that its curl has components in the same direction as the field.

Along a field line, which is always perpendicular to the equipotential lines,

$$\frac{dr}{r\,d\phi}=\frac{E_r}{E_\phi}=-\frac{1}{r}\frac{\partial\Sigma/\partial\phi}{\partial\Sigma/\partial r} \tag{26}$$

By cross multiplying and grouping terms on one side of the equation, (26) reduces to

$$d\Sigma=\frac{\partial\Sigma}{\partial r}dr+\frac{\partial\Sigma}{\partial\phi}d\phi=0\Rightarrow\Sigma=\text{const} \tag{27}$$

Field lines are thus lines of constant Σ.

For the steady-state solution of (24), outside the cylinder

$$\begin{aligned}\frac{1}{r}\frac{\partial\Sigma}{\partial\phi}&=E_r=E_0\left(1+\frac{a^2}{r^2}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\cos\phi\\-\frac{\partial\Sigma}{\partial r}&=E_\phi=-E_0\left(1-\frac{a^2}{r^2}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\sin\phi\end{aligned} \tag{28}$$

we find by integration that

$$\Sigma=E_0\left(r+\frac{a^2}{r}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\sin\phi \tag{29}$$

The steady-state field and equipotential lines are drawn in Figure 4-8 when the cylinder is perfectly conducting ($\sigma_2\to\infty$) or perfectly insulating ($\sigma_2=0$).

If the cylinder is highly conducting, the internal electric field is zero with the external electric field incident radially, as drawn in Figure 4-8*a*. In contrast, when the cylinder is perfectly insulating, the external field lines must be purely tangential to the cylinder as the incident normal current is zero, and the internal electric field has double the strength of the applied field, as drawn in Figure 4-8*b*.

4-3-3 Three-Dimensional Solutions

If the electric potential depends on all three coordinates, we try a product solution of the form

$$V(r,\phi,z)=R(r)\Phi(\phi)Z(z) \tag{30}$$

which when substituted into Laplace's equation yields

$$\frac{Z\Phi}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{RZ}{r^2}\frac{d^2\Phi}{d\phi^2}+R\Phi\frac{d^2Z}{dz^2}=0 \tag{31}$$

We now have a difficulty, as we cannot divide through by a factor to make each term a function only of a single variable.

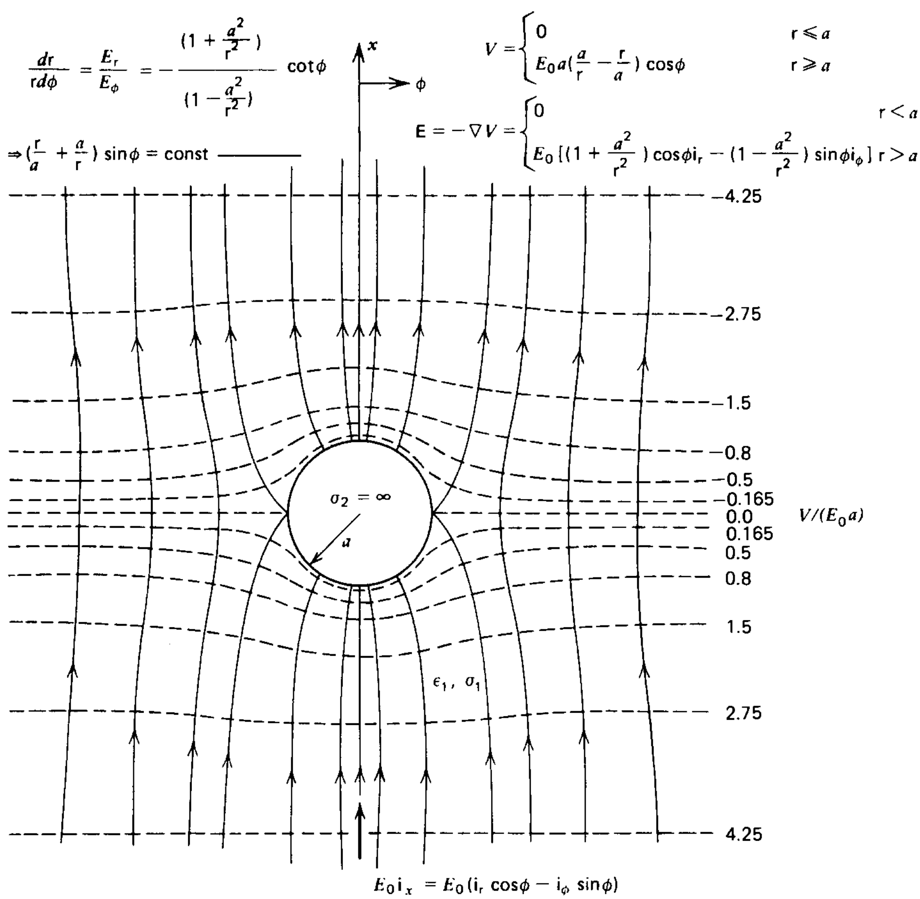


Figure 4-8 Steady-state field and equipotential lines about a (*a*) perfectly conducting or (*b*) perfectly insulating cylinder in a uniform electric field.

However, by dividing through by $V = R\Phi Z$,

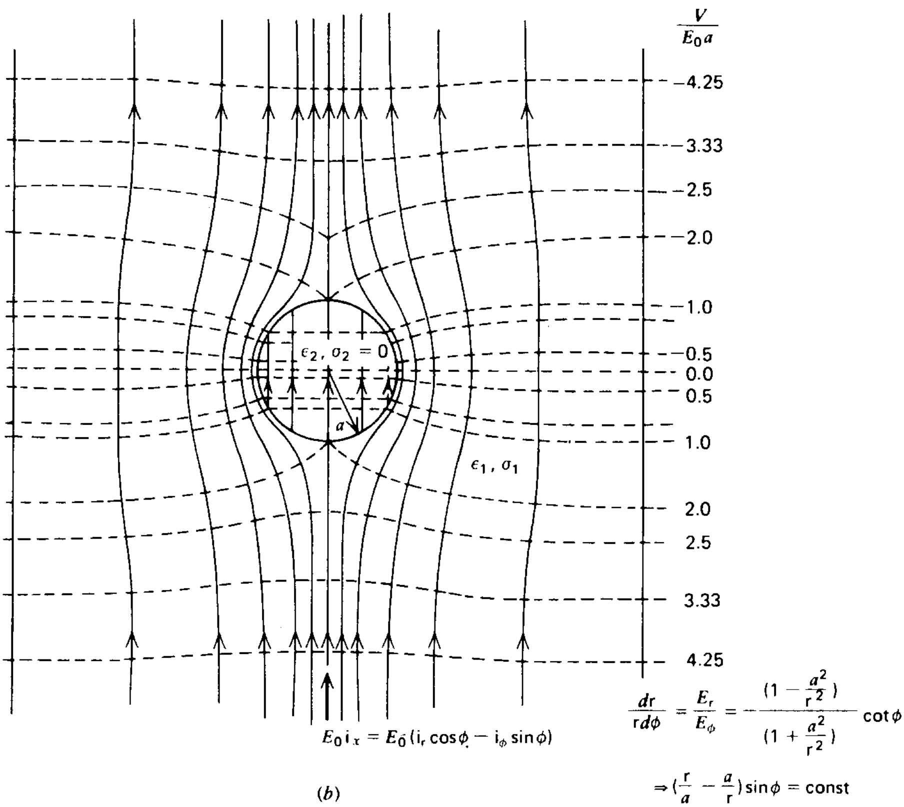
$$\underbrace{\frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2}}_{-k^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{k^2} = 0 \tag{32}$$

we see that the first two terms are functions of r and ϕ while the last term is only a function of z. This last term must therefore equal a constant:

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2 \Rightarrow Z = \begin{cases} A_1 \sinh kz + A_2 \cosh kz, & k \neq 0 \\ A_3 z + A_4, & k = 0 \end{cases} \tag{33}$$

$$V = \begin{cases} -2E_0 r\cos\phi & r \le a \\ -E_0 a\left(\dfrac{a}{r} + \dfrac{r}{a}\right)\cos\phi & r \ge a \end{cases}$$

$$\mathbf{E} = -\nabla V = \begin{cases} 2E_0(\cos\phi\,\mathbf{i}_r - \sin\phi\,\mathbf{i}_\phi) = 2E_0\,\mathbf{i}_x & r < a \\ E_0\left[\left(1 - \dfrac{a^2}{r^2}\right)\cos\phi\,\mathbf{i}_r - \left(1 + \dfrac{a^2}{r^2}\right)\sin\phi\,\mathbf{i}_\phi\right] & r > a \end{cases}$$



(*b*)

Figure 4-8*b*

The first two terms in (32) must now sum to $-k^2$ so that after multiplying through by r^2 we have

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + k^2 r^2 + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \tag{34}$$

Now again the first two terms are only a function of r, while the last term is only a function of ϕ so that (34) again separates:

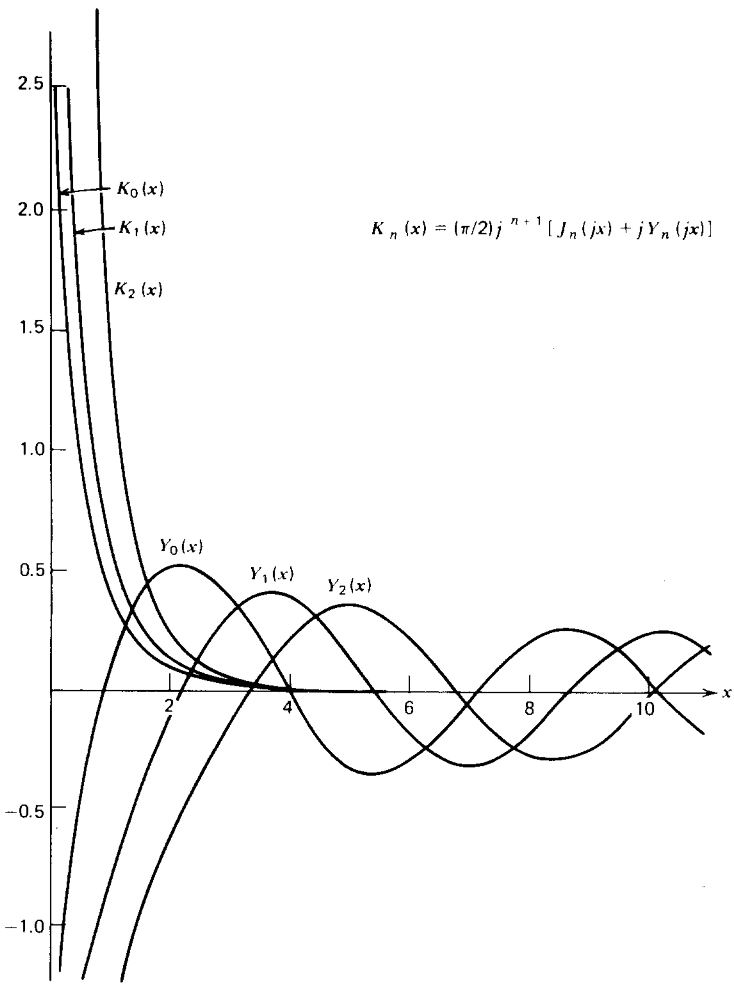
$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + k^2 r^2 = n^2, \qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -n^2 \tag{35}$$

of tabulated functions:

$$R=\begin{cases} C_1J_n(kr)+C_2Y_n(kr), & k\neq 0 \\ C_3r^n+C_4r^{-n}, & k=0, \quad n\neq 0 \\ C_5\ln r+C_6, & k=0, \quad n=0 \end{cases} \tag{38}$$

where J_n is called a Bessel function of the first kind of order n and Y_n is called the nth-order Bessel function of the second kind. When $n=0$, the Bessel functions are of zero order while if $k=0$ the solutions reduce to the two-dimensional solutions of (9).

Some of the properties and limiting values of the Bessel functions are illustrated in Figure 4-9. Remember that k



(*b*)

Figure 4-9*b*

can also be purely imaginary as well as real. When k is real so that the z dependence is hyperbolic or equivalently exponential, the Bessel functions are oscillatory while if k is imaginary so that the axial dependence on z is trigonometric, it is convenient to define the nonoscillatory modified Bessel functions as

$$\begin{aligned} I_n(k\mathrm{r}) &= j^{-n} J_n(jk\mathrm{r}) \\ K_n(k\mathrm{r}) &= \frac{\pi}{2} j^{n+1}[J_n(jk\mathrm{r}) + jY_n(jk\mathrm{r})] \end{aligned} \tag{39}$$

As in rectangular coordinates, if the solution to Laplace's equation decays in one direction, it is oscillatory in the perpendicular direction.

4-3-4 High Voltage Insulator Bushing

The high voltage insulator shown in Figure 4-10 consists of a cylindrical disk with Ohmic conductivity σ supported by a perfectly conducting cylindrical post above a ground plane.*

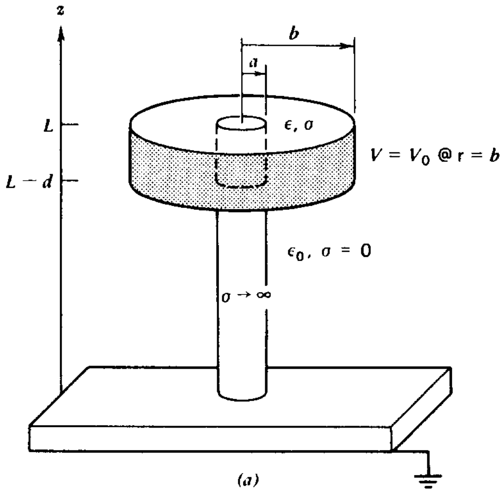
The plane at $z=0$ and the post at $\mathrm{r}=a$ are at zero potential, while a constant potential is imposed along the circumference of the disk at $\mathrm{r}=b$. The region below the disk is free space so that no current can cross the surfaces at $z=L$ and $z=L-d$. Because the boundaries lie along surfaces at constant z or constant r we try the simple zero separation constant solutions in (33) and (38), which are independent of angle ϕ:

$$V(\mathrm{r}, z) = \begin{cases} A_1 z + B_1 z \ln \mathrm{r} + C_1 \ln \mathrm{r} + D_1, & L-d < z < L \\ A_2 z + B_2 z \ln \mathrm{r} + C_2 \ln \mathrm{r} + D_2, & 0 \le z \le L-d \end{cases} \tag{40}$$

Applying the boundary conditions we relate the coefficients as

$$\begin{aligned} &V(z=0) = 0 \Rightarrow C_2 = D_2 = 0 \\ &V(\mathrm{r}=a) = 0 \Rightarrow \begin{cases} A_2 + B_2 \ln a = 0 \\ A_1 + B_1 \ln a = 0 \\ C_1 \ln a + D_1 = 0 \end{cases} \\ &V(\mathrm{r}=b, z > L-d) = V_0 \Rightarrow \begin{cases} A_1 + B_1 \ln b = 0 \\ C_1 \ln b + D_1 = V_0 \end{cases} \\ &V(z=(L-d)_-) = V(z=(L-d)_+) \Rightarrow (L-d)(A_2 + B_2 \ln \mathrm{r}) \\ &\qquad = (L-d)(A_1 + B_1 \ln \mathrm{r}) + C_1 \ln \mathrm{r} + D_1 \end{aligned} \tag{41}$$

* *M. N. Horenstein, "Particle Contamination of High Voltage DC Insulators," PhD thesis, Massachusetts Institute of Technology, 1978.*



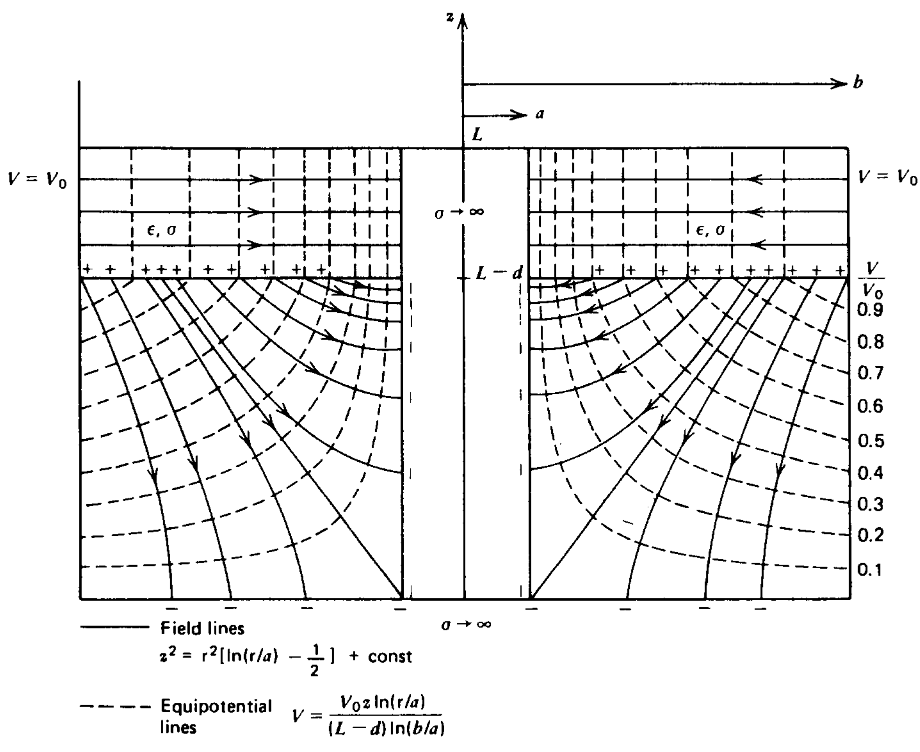


Figure 4-10 (a) A finitely conducting disk is mounted upon a perfectly conducting cylindrical post and is placed on a perfectly conducting ground plane. (b) Field and equipotential lines.

which yields the values

$$A_1 = B_1 = 0, \qquad C_1 = \frac{V_0}{\ln(b/a)}, \qquad D_1 = -\frac{V_0 \ln a}{\ln(b/a)}$$

$$A_2 = -\frac{V_0 \ln a}{(L-d)\ln(b/a)}, \quad B_2 = \frac{V_0}{(L-d)\ln(b/a)}, \qquad C_2 = D_2 = 0 \tag{42}$$

The potential of (40) is then

$$V(\mathrm{r}, z) = \begin{cases} \dfrac{V_0 \ln(\mathrm{r}/a)}{\ln(b/a)}, & L-d \le z \le L \\ \dfrac{V_0 z \ln(\mathrm{r}/a)}{(L-d)\ln(b/a)}, & 0 \le z \le L-d \end{cases} \tag{43}$$

with associated electric field

$$\mathbf{E} = -\nabla V = \begin{cases} -\dfrac{V_0}{\mathrm{r}\ln(b/a)}\mathbf{i}_\mathrm{r}, & L-d < z < L \\ -\dfrac{V_0}{(L-d)\ln(b/a)}\left(\ln\dfrac{\mathrm{r}}{a}\mathbf{i}_z + \dfrac{z}{\mathrm{r}}\mathbf{i}_\mathrm{r}\right), & 0 < z < L-d \end{cases} \tag{44}$$

The field lines in the free space region are

$$\frac{d\mathrm{r}}{dz} = \frac{E_\mathrm{r}}{E_z} = \frac{z}{\mathrm{r}\ln(\mathrm{r}/a)} \Rightarrow z^2 = \mathrm{r}^2\left[\ln\frac{\mathrm{r}}{a} - \frac{1}{2}\right] + \text{const} \tag{45}$$

and are plotted with the equipotential lines in Figure 4-10b.

4-4 PRODUCT SOLUTIONS IN SPHERICAL GEOMETRY

In spherical coordinates, Laplace's equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \tag{1}$$

4-4-1 One-Dimensional Solutions

If the solution only depends on a single spatial coordinate, the governing equations and solutions for each of the three coordinates are

$$\text{(i)} \quad \frac{d}{dr}\left(r^2\frac{dV(r)}{dr}\right) = 0 \Rightarrow V(r) = \frac{A_1}{r} + A_2 \tag{2}$$

$$\text{(ii)}\quad \frac{d}{d\theta}\left(\sin\theta\frac{dV(\theta)}{d\theta}\right)=0\Rightarrow V(\theta)=B_1\ln\left(\tan\frac{\theta}{2}\right)+B_2 \tag{3}$$

$$\text{(iii)}\quad \frac{d^2V(\phi)}{d\phi^2}=0\Rightarrow V(\phi)=C_1\phi+C_2 \tag{4}$$

We recognize the radially dependent solution as the potential due to a point charge. The new solutions are those which only depend on θ or ϕ.

EXAMPLE 4-2 TWO CONES

Two identical cones with surfaces at angles $\theta=\alpha$ and $\theta=\pi-\alpha$ and with vertices meeting at the origin, are at a potential difference v, as shown in Figure 4-11. Find the potential and electric field.

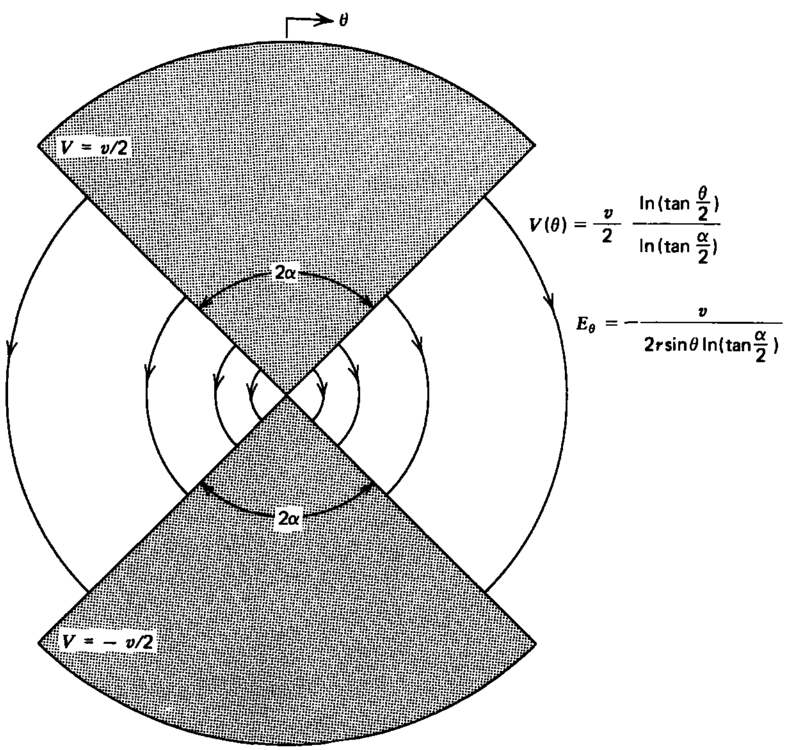


Figure 4-11 Two cones with vertices meeting at the origin are at a potential difference v.

SOLUTION

Because the boundaries are at constant values of θ, we try (3) as a solution:

$$V(\theta)=B_1 \ln[\tan(\theta/2)]+B_2$$

From the boundary conditions we have

$$V(\theta=\alpha)=\frac{v}{2}$$

$$V(\theta=\pi-\alpha)=\frac{-v}{2} \Rightarrow B_1=\frac{v}{2\ln[\tan(\alpha/2)]}, \qquad B_2=0$$

so that the potential is

$$V(\theta)=\frac{v\ln[\tan(\theta/2)]}{2\ln[\tan(\alpha/2)]}$$

with electric field

$$\mathbf{E}=-\nabla V=\frac{-v}{2r\sin\theta\ln[\tan(\alpha/2)]}\mathbf{i}_\theta$$

4-4-2 Axisymmetric Solutions

If the solution has no dependence on the coordinate ϕ, we try a product solution

$$V(r,\theta)=R(r)\Theta(\theta) \tag{5}$$

which when substituted into (1), after multiplying through by $r^2/R\Theta$, yields

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0 \tag{6}$$

Because each term is again only a function of a single variable, each term is equal to a constant. Anticipating the form of the solution, we choose the separation constant as $n(n+1)$ so that (6) separates to

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-n(n+1)R=0 \tag{7}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+n(n+1)\Theta\sin\theta=0 \tag{8}$$

For the radial dependence we try a power-law solution

$$R = Ar^p \tag{9}$$

which when substituted back into (7) requires

$$p(p+1) = n(n+1) \tag{10}$$

which has the two solutions

$$p = n, \qquad p = -(n+1) \tag{11}$$

When $n = 0$ we re-obtain the $1/r$ dependence due to a point charge.

To solve (8) for the θ dependence it is convenient to introduce the change of variable

$$\beta = \cos\theta \tag{12}$$

so that

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{d\beta}\frac{d\beta}{d\theta} = -\sin\theta\frac{d\Theta}{d\beta} = -(1-\beta^2)^{1/2}\frac{d\Theta}{d\beta} \tag{13}$$

Then (8) becomes

$$\frac{d}{d\beta}\left((1-\beta^2)\frac{d\Theta}{d\beta}\right) + n(n+1)\Theta = 0 \tag{14}$$

which is known as Legendre's equation. When n is an integer, the solutions are written in terms of new functions:

$$\Theta = B_n P_n(\beta) + C_n Q_n(\beta) \tag{15}$$

where the $P_n(\beta)$ are called Legendre polynomials of the first kind and are tabulated in Table 4-1. The Q_n solutions are called the Legendre functions of the second kind for which the first few are also tabulated in Table 4-1. Since all the Q_n are singular at $\theta = 0$ and $\theta = \pi$, where $\beta = \pm 1$, for all problems which include these values of angle, the coefficients C_n in (15) must be zero, so that many problems only involve the Legendre polynomials of first kind, $P_n(\cos\theta)$. Then using (9)–(11) and (15) in (5), the general solution for the potential with no ϕ dependence can be written as

$$V(r,\theta) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-(n+1)}) P_n(\cos\theta) \tag{16}$$

Table 4-1 Legendre polynomials of first and second kind

n	$P_n(\beta=\cos\theta)$	$Q_n(\beta=\cos\theta)$
0	1	$\frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$
1	$\beta=\cos\theta$	$\frac{1}{2}\beta\ln\left(\frac{1+\beta}{1-\beta}\right)-1$
2	$\frac{1}{2}(3\beta^2-1)$ $=\frac{1}{2}(3\cos^2\theta-1)$	$\frac{1}{4}(3\beta^2-1)\ln\left(\frac{1+\beta}{1-\beta}\right)-\frac{3\beta}{2}$
3	$\frac{1}{2}(5\beta^3-3\beta)$ $=\frac{1}{2}(5\cos^3\theta-3\cos\theta)$	$\frac{1}{4}(5\beta^3-3\beta)\ln\left(\frac{1+\beta}{1-\beta}\right)-\frac{5}{2}\beta^2+\frac{2}{3}$
⋮		
m	$\frac{1}{2^m m!}\frac{d^m}{d\beta^m}(\beta^2-1)^m$	

4-4-3 Conducting Sphere in a Uniform Field

(a) Field Solution

A sphere of radius R, permittivity ε_2, and Ohmic conductivity σ_2 is placed within a medium of permittivity ε_1 and conductivity σ_1. A uniform dc electric field $E_0\mathbf{i}_z$ is applied at infinity. Although the general solution of (16) requires an infinite number of terms, the form of the uniform field at infinity in spherical coordinates,

$$\mathbf{E}(r\to\infty)=E_0\mathbf{i}_z=E_0(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta) \tag{17}$$

suggests that all the boundary conditions can be met with just the $n=1$ solution:

$$V(r,\theta)=\begin{cases}Ar\cos\theta, & r\le R\\ (Br+C/r^2)\cos\theta, & r\ge R\end{cases} \tag{18}$$

We do not include the $1/r^2$ solution within the sphere ($r<R$) as the potential must remain finite at $r=0$. The associated

electric field is

$$\mathbf{E}=-\nabla V=\begin{cases}-A(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)=-A\mathbf{i}_z, & r<R\\ -(B-2C/r^3)\cos\theta\mathbf{i}_r+(B+C/r^3)\sin\theta\mathbf{i}_\theta, & r>R\end{cases}\tag{19}$$

The electric field within the sphere is uniform and z directed while the solution outside is composed of the uniform z-directed field, for as $r\to\infty$ the field must approach (17) so that $B=-E_0$, plus the field due to a point dipole at the origin, with dipole moment

$$p_z=4\pi\varepsilon_1 C\tag{20}$$

Additional steady-state boundary conditions are the continuity of the potential at $r=R$ [equivalent to continuity of tangential $\mathbf{E}(r=R)$], and continuity of normal current at $r=R$,

$$\begin{aligned}&V(r=R_+)=V(r=R_-)\Rightarrow E_\theta(r=R_+)=E_\theta(r=R_-)\\ &\qquad\Rightarrow AR=BR+C/R^2\\ &J_r(r=R_+)=J_r(r=R_-)\Rightarrow\sigma_1E_r(r=R_+)=\sigma_2E_r(r=R_-)\\ &\qquad\Rightarrow\sigma_1(B-2C/R^3)=\sigma_2A\end{aligned}\tag{21}$$

for which solutions are

$$A=-\frac{3\sigma_1}{2\sigma_1+\sigma_2}E_0,\qquad B=-E_0,\qquad C=\frac{(\sigma_2-\sigma_1)R^3}{2\sigma_1+\sigma_2}E_0\tag{22}$$

The electric field of (19) is then

$$\mathbf{E}=\begin{cases}\dfrac{3\sigma_1E_0}{2\sigma_1+\sigma_2}(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)=\dfrac{3\sigma_1E_0}{2\sigma_1+\sigma_2}\mathbf{i}_z, & r<R\\ E_0\left[\left(1+\dfrac{2R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\cos\theta\mathbf{i}_r\right.\\ \qquad\left.-\left(1-\dfrac{R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\sin\theta\mathbf{i}_\theta\right], & r>R\end{cases}\tag{23}$$

The interfacial surface charge is

$$\begin{aligned}\sigma_f(r=R)&=\varepsilon_1E_r(r=R_+)-\varepsilon_2E_r(r=R_-)\\ &=\frac{3(\sigma_2\varepsilon_1-\sigma_1\varepsilon_2)E_0}{2\sigma_1+\sigma_2}\cos\theta\end{aligned}\tag{24}$$

which is of one sign on the upper part of the sphere and of opposite sign on the lower half of the sphere. The total charge on the entire sphere is zero. The charge is zero at every point on the sphere if the relaxation times in each region are equal:

$$\frac{\varepsilon_1}{\sigma_1} = \frac{\varepsilon_2}{\sigma_2} \tag{25}$$

The solution if both regions were lossless dielectrics with no interfacial surface charge, is similar in form to (23) if we replace the conductivities by their respective permittivities.

(b) Field Line Plotting

As we saw in Section 4-3-2*b* for a cylindrical geometry, the electric field in a volume charge-free region has no divergence, so that it can be expressed as the curl of a vector. For an axisymmetric field in spherical coordinates we write the electric field as

$$\begin{aligned}\mathbf{E}(r,\theta) &= \nabla \times \left(\frac{\Sigma(r,\theta)}{r\sin\theta}\mathbf{i}_\phi\right)\\ &= \frac{1}{r^2\sin\theta}\frac{\partial\Sigma}{\partial\theta}\mathbf{i}_r - \frac{1}{r\sin\theta}\frac{\partial\Sigma}{\partial r}\mathbf{i}_\theta\end{aligned} \tag{26}$$

Note again, that for a two-dimensional electric field, the stream function vector points in the direction orthogonal to both field components so that its curl has components in the same direction as the field. The stream function Σ is divided by $r\sin\theta$ so that the partial derivatives in (26) only operate on Σ.

The field lines are tangent to the electric field

$$\frac{dr}{r\,d\theta} = \frac{E_r}{E_\theta} = -\frac{1}{r}\frac{\partial\Sigma/\partial\theta}{\partial\Sigma/\partial r} \tag{27}$$

which after cross multiplication yields

$$d\Sigma = \frac{\partial\Sigma}{\partial r}dr + \frac{\partial\Sigma}{\partial\theta}d\theta = 0 \Rightarrow \Sigma = \text{const} \tag{28}$$

so that again Σ is constant along a field line.

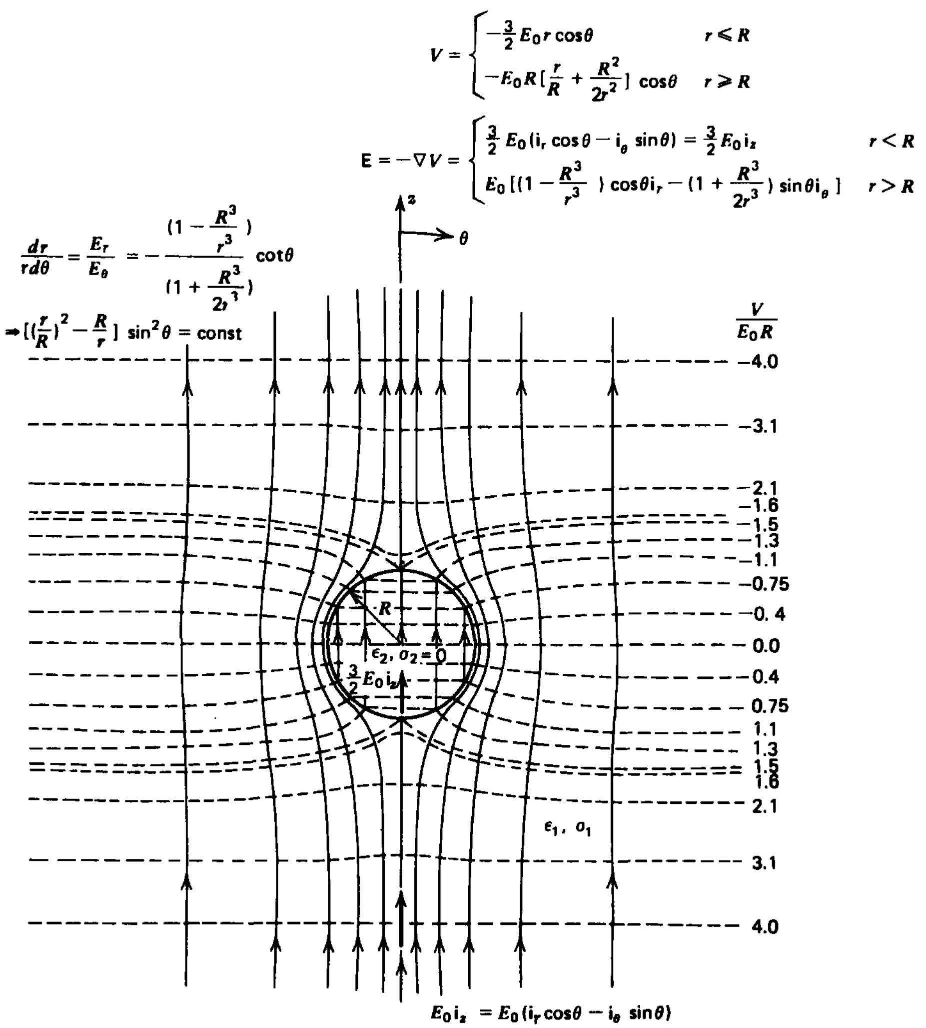
For the solution of (23) outside the sphere, we relate the field components to the stream function using (26) as

$$\begin{aligned}E_r &= \frac{1}{r^2\sin\theta}\frac{\partial\Sigma}{\partial\theta} = E_0\left(1 + \frac{2R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\cos\theta\\ E_\theta &= -\frac{1}{r\sin\theta}\frac{\partial\Sigma}{\partial r} = -E_0\left(1 - \frac{R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\sin\theta\end{aligned} \tag{29}$$

so that by integration the stream function is

$$\Sigma = E_0\left(\frac{r^2}{2}+\frac{R^3(\sigma_2-\sigma_1)}{r(2\sigma_1+\sigma_2)}\right)\sin^2\theta \tag{30}$$

The steady-state field and equipotential lines are drawn in Figure 4-12 when the sphere is perfectly insulating ($\sigma_2 = 0$) or perfectly conducting ($\sigma_2 \to \infty$).



(a)

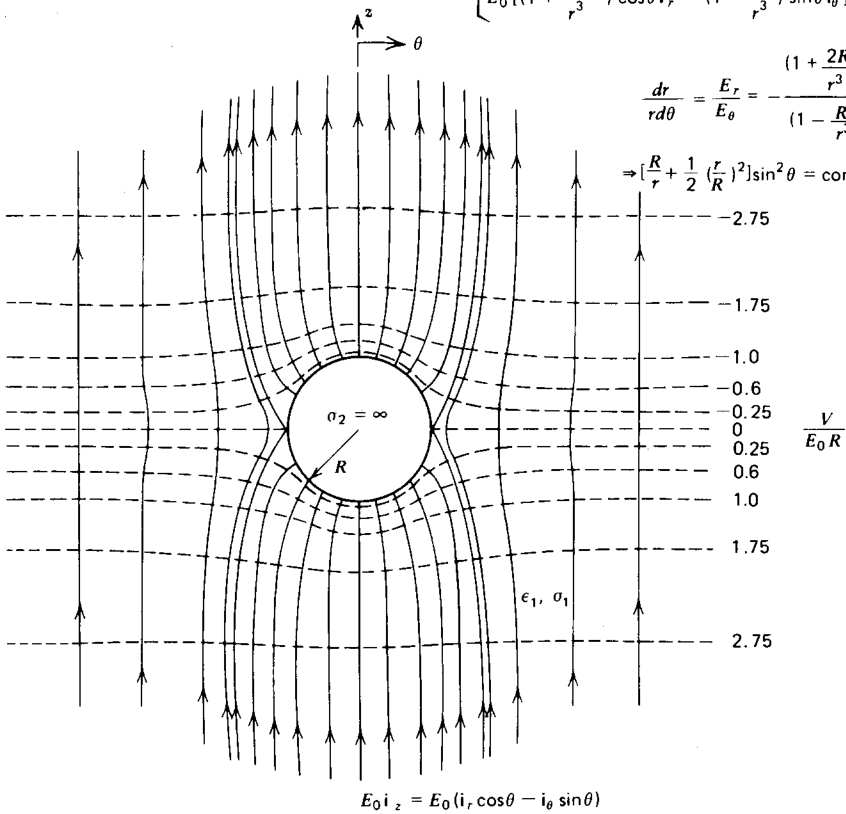
Figure 4-12 Steady-state field and equipotential lines about a (a) perfectly insulating or (b) perfectly conducting sphere in a uniform electric field.

$$V=\begin{cases}0 & r\leqslant R\\ -E_0R\left(\dfrac{r}{R}-\dfrac{R^2}{r^2}\right)\cos\theta & r\geqslant R\end{cases}$$

$$\mathbf{E}=-\nabla V=\begin{cases}0 & r<R\\ E_0\left[\left(1+\dfrac{2R^3}{r^3}\right)\cos\theta\,\mathbf{i}_r-\left(1-\dfrac{R^3}{r^3}\right)\sin\theta\,\mathbf{i}_\theta\right] & r>R\end{cases}$$

$$\frac{dr}{r\,d\theta}=\frac{E_r}{E_\theta}=-\frac{\left(1+\dfrac{2R^3}{r^3}\right)}{\left(1-\dfrac{R^3}{r^3}\right)}\cot\theta$$

$$\Rightarrow\left[\frac{R}{r}+\frac{1}{2}\left(\frac{r}{R}\right)^2\right]\sin^2\theta=\text{const}$$



(b)

Figure 4-12*b*

If the conductivity of the sphere is less than that of the surrounding medium ($\sigma_2<\sigma_1$), the electric field within the sphere is larger than the applied field. The opposite is true for ($\sigma_2>\sigma_1$). For the insulating sphere in Figure 4-12*a*, the field lines go around the sphere as no current can pass through.

For the conducting sphere in Figure 4-12*b*, the electric field lines must be incident perpendicularly. This case is used as a polarization model, for as we see from (23) with $\sigma_2\to\infty$, the external field is the imposed field plus the field of a point dipole with moment,

$$p_z = 4\pi\varepsilon_1 R^3 E_0 \tag{31}$$

If a dielectric is modeled as a dilute suspension of noninteracting, perfectly conducting spheres in free space with number density N, the dielectric constant is

$$\varepsilon = \frac{\varepsilon_0 E_0 + P}{E_0} = \frac{\varepsilon_0 E_0 + N p_z}{E_0} = \varepsilon_0(1 + 4\pi R^3 N) \tag{32}$$

4-4-4 Charged Particle Precipitation Onto a Sphere

The solution for a perfectly conducting sphere surrounded by free space in a uniform electric field has been used as a model for the charging of rain drops.* This same model has also been applied to a new type of electrostatic precipitator where small charged particulates are collected on larger spheres.†

Then, in addition to the uniform field $E_0\mathbf{i}_z$ applied at infinity, a uniform flux of charged particulate with charge density ρ_0, which we take to be positive, is also injected, which travels along the field lines with mobility μ. Those field lines that start at infinity where the charge is injected and that approach the sphere with negative radial electric field, deposit charged particulate, as in Figure 4-13. The charge then redistributes itself uniformly on the equipotential surface so that the total charge on the sphere increases with time. Those field lines that do not intersect the sphere or those that start on the sphere do not deposit any charge.

We assume that the self-field due to the injected charge is very much less than the applied field E_0. Then the solution of (23) with $\sigma_2 = \infty$ is correct here, with the addition of the radial field of a uniformly charged sphere with total charge $Q(t)$:

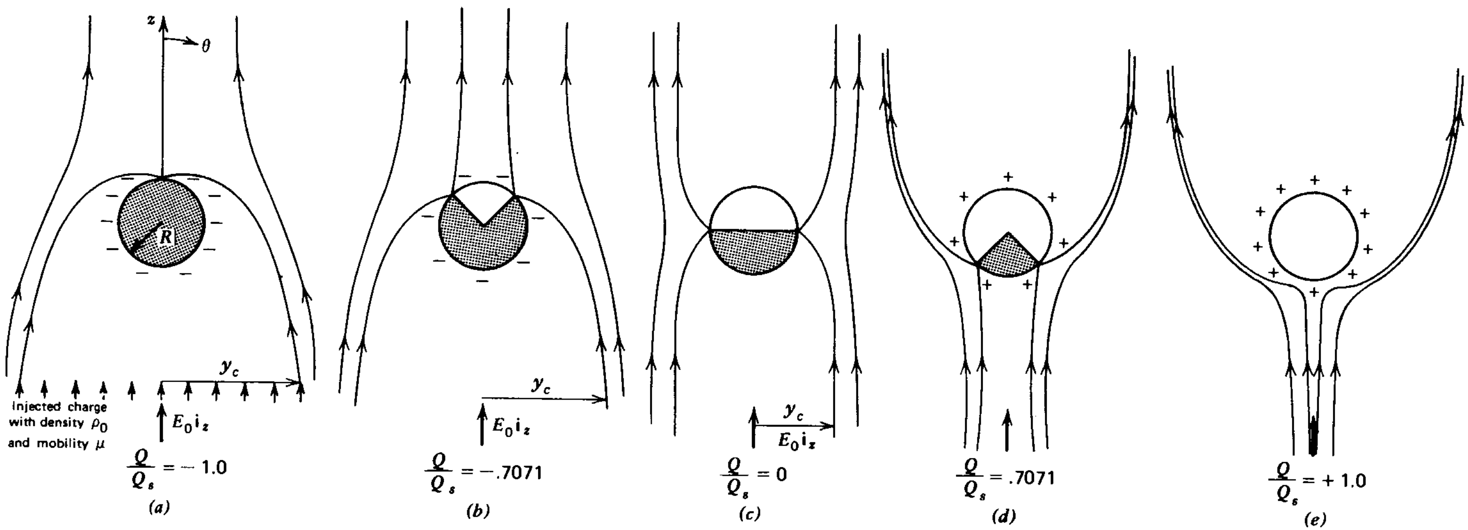
$$\mathbf{E} = \left[E_0\left(1 + \frac{2R^3}{r^3}\right)\cos\theta + \frac{Q}{4\pi\varepsilon r^2}\right]\mathbf{i}_r - E_0\left(1 - \frac{R^3}{r^3}\right)\sin\theta\,\mathbf{i}_\theta, \qquad r > R \tag{33}$$

Charge only impacts the sphere where $E_r(r = R)$ is negative:

$$E_r(r = R) = 3E_0\cos\theta + \frac{Q}{4\pi\varepsilon R^2} < 0 \tag{34}$$

* *See*: *F. J. W. Whipple and J. A. Chalmers, On Wilson's Theory of the Collection of Charge by Falling Drops*, Quart. J. Roy. Met. Soc. ***70***, *(1944), p. 103.*

† *See*: *H. J. White,* Industrial Electrostatic Precipitation *Addison-Wesley, Reading. Mass. 1963, pp. 126–137.*



$$E_0 R^2 \left[\frac{R}{r} + \frac{1}{2}\left(\frac{r}{R}\right)^2\right] \sin^2\theta - \frac{Q\cos\theta}{4\pi\epsilon} = \text{constant}$$

Figure 4-13 Electric field lines around a uniformly charged perfectly conducting sphere in a uniform electric field with continuous positive charge injection from $z = -\infty$. Only those field lines that impact on the sphere with the electric field radially inward $[E_r(R) < 0]$ deposit charge. (a) If the total charge on the sphere starts out as negative charge with magnitude greater or equal to the critical charge, the field lines within the distance y_c of the z axis impact over the entire sphere. (b)–(d) As the sphere charges up it tends to repel some of the incident charge and only part of the sphere collects charge. With increasing charge the angular window for charge collection decreases as does y_c. (e) For $Q \geq Q_s$ no further charge collects on the sphere so that the charge remains constant thereafter. The angular window and y_c have shrunk to zero.

which gives us a window for charge collection over the range of angle, where

$$\cos\theta \le -\frac{Q}{12\pi\varepsilon E_0 R^2} \tag{35}$$

Since the magnitude of the cosine must be less than unity, the maximum amount of charge that can be collected on the sphere is

$$Q_s = 12\pi\varepsilon E_0 R^2 \tag{36}$$

As soon as this saturation charge is reached, all field lines emanate radially outward from the sphere so that no more charge can be collected. We define the critical angle θ_c as the angle where the radial electric field is zero, defined when (35) is an equality $\cos\theta_c = -Q/Q_s$. The current density charging the sphere is

$$\begin{aligned} J_r &= \rho_0 \mu E_r(r = R) \\ &= 3\rho_0\mu E_0(\cos\theta + Q/Q_s), \qquad \theta_c < \theta < \pi \end{aligned} \tag{37}$$

The total charging current is then

$$\begin{aligned} \frac{dQ}{dt} &= -\int_{\theta=\theta_c}^{\pi} J_r 2\pi R^2 \sin\theta\, d\theta \\ &= -6\pi\rho_0\mu E_0 R^2 \int_{\theta=\theta_c}^{\pi} (\cos\theta + Q/Q_s)\sin\theta\, d\theta \\ &= -6\pi\rho_0\mu E_0 R^2 (-\tfrac{1}{4}\cos 2\theta - (Q/Q_s)\cos\theta)\big|_{\theta=\theta_c}^{\pi} \\ &= -6\pi\rho_0\mu E_0 R^2 (-\tfrac{1}{4}(1 - \cos 2\theta_c) + (Q/Q_s)(1 + \cos\theta_c)) \end{aligned} \tag{38}$$

As long as $|Q| < Q_s$, θ_c is defined by the equality in (35). If Q exceeds Q_s, which can only occur if the sphere is intentionally overcharged, then $\theta_c = \pi$ and no further charging can occur as dQ/dt in (38) is zero. If Q is negative and exceeds Q_s in magnitude, $Q < -Q_s$, then the whole sphere collects charge as $\theta_c = 0$. Then for these conditions we have

$$\cos\theta_c = \begin{cases} -1, & Q > Q_s \\ -Q/Q_s & -Q_s < Q < Q_s \\ 1, & Q < -Q_s \end{cases} \tag{39}$$

$$\cos 2\theta_c = 2\cos^2\theta_c - 1 = \begin{cases} 1, & |Q| > Q_s \\ 2(Q/Q_s)^2 - 1, & |Q| < Q_s \end{cases} \tag{40}$$

so that (38) becomes

$$\frac{d\frac{Q}{Q_s}}{dt}=\begin{cases}0, & Q>Q_s\\ \frac{\rho_0\mu}{4\varepsilon}\left(1-\frac{Q}{Q_s}\right)^2, & -Q_s<Q<Q_s\\ -\frac{\rho_0\mu}{\varepsilon}\frac{Q}{Q_s}, & Q<-Q_s\end{cases} \tag{41}$$

with integrated solutions

$$\frac{Q}{Q_s}=\begin{cases}\frac{Q_0}{Q_s}, & Q>Q_s\\ \dfrac{\frac{Q_0}{Q_s}+\frac{(t-t_0)}{4\tau}\left(1-\frac{Q_0}{Q_s}\right)}{1+\frac{(t-t_0)}{4\tau}\left(1-\frac{Q_0}{Q_s}\right)}, & -Q_s<Q<Q_s\\ \frac{Q_0}{Q_s}e^{-t/\tau}, & Q<-Q_s\end{cases} \tag{42}$$

where Q_0 is the initial charge at $t=0$ and the characteristic charging time is

$$\tau=\varepsilon/(\rho_0\mu) \tag{43}$$

If the initial charge Q_0 is less than $-Q_s$, the charge magnitude decreases with the exponential law in (42) until the total charge reaches $-Q_s$ at $t=t_0$. Then the charging law switches to the next regime with $Q_0=-Q_s$, where the charge passes through zero and asymptotically slowly approaches $Q=Q_s$. The charge can never exceed Q_s unless externally charged. It then remains constant at this value repelling any additional charge. If the initial charge Q_0 has magnitude less than Q_s, then $t_0=0$. The time dependence of the charge is plotted in Figure 4-14 for various initial charge values Q_0. No matter the initial value of Q_0 for $Q<Q_s$, it takes many time constants for the charge to closely approach the saturation value Q_s. The force of repulsion on the injected charge increases as the charge on the sphere increases so that the charging current decreases.

The field lines sketched in Figure 4-13 show how the fields change as the sphere charges up. The window for charge collection decreases with increasing charge. The field lines are found by adding the stream function of a uniformly charged sphere with total charge Q to the solution of (30)

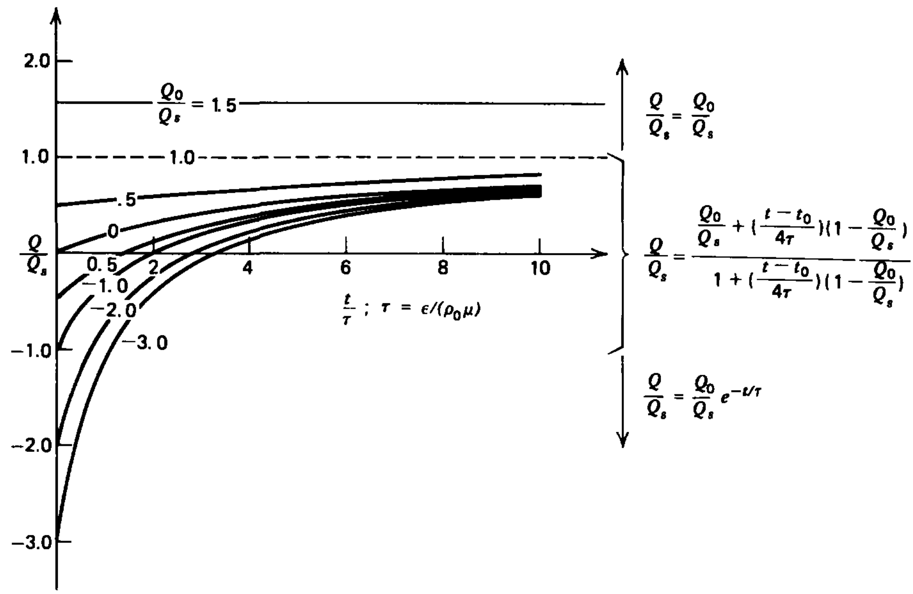


Figure 4-14 There are three regimes describing the charge build-up on the sphere. It takes many time constants $[\tau = \varepsilon/(\rho_0\mu)]$ for the charge to approach the saturation value Q_s, because as the sphere charges up the Coulombic repulsive force increases so that most of the charge goes around the sphere. If the sphere is externally charged to a valuė in excess of the saturation charge, it remains constant as all additional charge is completely repelled.

with $\sigma_2 \to \infty$:

$$\Sigma = E_0 R^2 \left[\frac{R}{r} + \frac{1}{2}\left(\frac{r}{R}\right)^2\right] \sin^2\theta - \frac{Q\cos\theta}{4\pi\varepsilon} \tag{44}$$

The streamline intersecting the sphere at $r = R, \theta = \theta_c$ separates those streamlines that deposit charge onto the sphere from those that travel past.

4-5 A NUMERICAL METHOD—SUCCESSIVE RELAXATION

In many cases, the geometry and boundary conditions are irregular so that closed form solutions are not possible. It then becomes necessary to solve Poisson's equation by a computational procedure. In this section we limit ourselves to dependence on only two Cartesian coordinates.

4-5-1 Finite Difference Expansions

The Taylor series expansion to second order of the potential V, at points a distance Δx on either side of the coordinate

(x, y), is

$$
\begin{aligned}
V(x+\Delta x, y) &\approx V(x, y)+\left.\frac{\partial V}{\partial x}\right|_{x,y} \Delta x+\left.\frac{1}{2} \frac{\partial^2 V}{\partial x^2}\right|_{x,y} (\Delta x)^2 \\
V(x-\Delta x, y) &\approx V(x, y)-\left.\frac{\partial V}{\partial x}\right|_{x,y} \Delta x+\left.\frac{1}{2} \frac{\partial^2 V}{\partial x^2}\right|_{x,y} (\Delta x)^2
\end{aligned} \tag{1}
$$

If we add these two equations and solve for the second derivative, we have

$$
\frac{\partial^2 V}{\partial x^2} \approx \frac{V(x+\Delta x, y)+V(x-\Delta x, y)-2V(x, y)}{(\Delta x)^2} \tag{2}
$$

Performing similar operations for small variations from y yields

$$
\frac{\partial^2 V}{\partial y^2} \approx \frac{V(x, y+\Delta y)+V(x, y-\Delta y)-2V(x, y)}{(\Delta y)^2} \tag{3}
$$

If we add (2) and (3) and furthermore let $\Delta x=\Delta y$, Poisson's equation can be approximated as

$$
\begin{aligned}
\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2} \approx \frac{1}{(\Delta x)^2} &[V(x+\Delta x, y)+V(x-\Delta x, y) \\
&+V(x, y+\Delta y)+V(x, y-\Delta y)-4V(x, y)]=-\frac{\rho_f(x, y)}{\varepsilon}
\end{aligned} \tag{4}
$$

so that the potential at (x, y) is equal to the average potential of its four nearest neighbors plus a contribution due to any volume charge located at (x, y):

$$
\begin{aligned}
V(x, y)=\tfrac{1}{4}&[V(x+\Delta x, y)+V(x-\Delta x, y) \\
&+V(x, y+\Delta y)+V(x, y-\Delta y)]+\frac{\rho_f(x, y)(\Delta x)^2}{4\varepsilon}
\end{aligned} \tag{5}
$$

The components of the electric field are obtained by taking the difference of the two expressions in (1)

$$
\begin{aligned}
E_x(x, y) &= -\left.\frac{\partial V}{\partial x}\right|_{x,y} \approx -\frac{1}{2\Delta x}[V(x+\Delta x, y)-V(x-\Delta x, y)] \\
E_y(x, y) &= -\left.\frac{\partial V}{\partial y}\right|_{x,y} \approx -\frac{1}{2\Delta y}[V(x, y+\Delta y)-V(x, y-\Delta y)]
\end{aligned} \tag{6}
$$

4-5-2 Potential Inside a Square Box

Consider the square conducting box whose sides are constrained to different potentials, as shown in Figure (4-15). We discretize the system by drawing a square grid with four

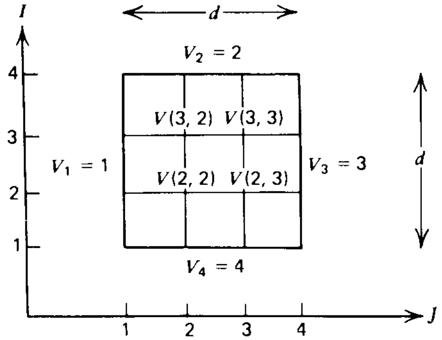


Figure 4-15 The potentials at the four interior points of a square conducting box with imposed potentials on its surfaces are found by successive numerical relaxation. The potential at any charge free interior grid point is equal to the average potential of the four adjacent points.

interior points. We must supply the potentials along the boundaries as proved in Section 4-1:

$$\begin{aligned} V_1 &= \sum_{I=1}^{4} V(I, J=1) = 1, \qquad & V_3 &= \sum_{I=1}^{4} V(I, J=4) = 3 \\ V_2 &= \sum_{J=1}^{4} V(I=4, J) = 2, \qquad & V_4 &= \sum_{J=1}^{4} V(I=1, J) = 4 \end{aligned} \tag{7}$$

Note the discontinuity in the potential at the corners.

We can write the charge-free discretized version of (5) as

$$V(I, J) = \tfrac{1}{4}[V(I+1, J) + V(I-1, J) + V(I, J+1) + V(I, J-1)] \tag{8}$$

We then guess any initial value of potential for all interior grid points not on the boundary. The boundary potentials must remain unchanged. Taking the interior points one at a time, we then improve our initial guess by computing the average potential of the four surrounding points.

We take our initial guess for all interior points to be zero inside the box:

$$\begin{aligned} V(2, 2) &= 0, \qquad & V(3, 3) &= 0 \\ V(3, 2) &= 0, \qquad & V(2, 3) &= 0 \end{aligned} \tag{9}$$

Then our first improved estimate for $V(2, 2)$ is

$$\begin{aligned} V(2, 2) &= \tfrac{1}{4}[V(2, 1) + V(2, 3) + V(1, 2) + V(3, 2)] \\ &= \tfrac{1}{4}[1 + 0 + 4 + 0] = 1.25 \end{aligned} \tag{10}$$

Using this value of $V(2,2)$ we improve our estimate for $V(3,2)$ as

$$\begin{aligned} V(3,2) &= \tfrac{1}{4}[V(2,2)+V(4,2)+V(3,1)+V(3,3)] \\ &= \tfrac{1}{4}[1.25+2+1+0] = 1.0625 \end{aligned} \tag{11}$$

Similarly for $V(3,3)$,

$$\begin{aligned} V(3,3) &= \tfrac{1}{4}[V(3,2)+V(3,4)+V(2,3)+V(4,3)] \\ &= \tfrac{1}{4}[1.0625+3+0+2] = 1.5156 \end{aligned} \tag{12}$$

and $V(2,3)$

$$\begin{aligned} V(2,3) &= \tfrac{1}{4}[V(2,2)+V(2,4)+V(1,3)+V(3,3)] \\ &= \tfrac{1}{4}[1.25+3+4+1.5156] = 2.4414 \end{aligned} \tag{13}$$

We then continue and repeat the procedure for the four interior points, always using the latest values of potential. As the number of iterations increase, the interior potential values approach the correct solutions. Table 4-2 shows the first ten iterations and should be compared to the exact solution to four decimal places, obtained by superposition of the rectangular harmonic solution in Section 4-2-5 (see problem 4-4):

$$\begin{aligned} V(x,y) = \sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} \frac{4}{n\pi \sinh n\pi} &\left[\sin\frac{n\pi y}{d}\left(V_3 \sinh\frac{n\pi x}{d}\right.\right. \\ &\left. - V_1 \sinh\frac{n\pi(x-d)}{d}\right) \\ &\left. + \sin\frac{n\pi x}{d}\left(V_2 \sinh\frac{n\pi y}{d} - V_4 \sinh\frac{n\pi(y-d)}{d}\right)\right] \end{aligned} \tag{14}$$

where V_1, V_2, V_3 and V_4 are the boundary potentials that for this case are

$$V_1 = 1, \qquad V_2 = 2, \qquad V_3 = 3, \qquad V_4 = 4 \tag{15}$$

To four decimal places the numerical solutions remain unchanged for further iterations past ten.

Table 4-2 Potential values for the four interior points in Figure 4-15 obtained by successive relaxation for the first ten iterations

	0	1	2	3	4	5
V_1	0	1.2500	2.1260	2.3777	2.4670	2.4911
V_2	0	1.0625	1.6604	1.9133	1.9770	1.9935
V_3	0	1.5156	2.2755	2.4409	2.4829	2.4952
V_4	0	2.4414	2.8504	2.9546	2.9875	2.9966

	6	7	8	9	10	Exact
V_1	2.4975	2.4993	2.4998	2.4999	2.5000	2.5000
V_2	1.9982	1.9995	1.9999	2.0000	2.0000	1.9771
V_3	2.4987	2.4996	2.4999	2.5000	2.5000	2.5000
V_4	2.9991	2.9997	2.9999	3.0000	3.0000	3.0229

The results are surprisingly good considering the coarse grid of only four interior points. This relaxation procedure can be used for any values of boundary potentials, for any number of interior grid points, and can be applied to other boundary shapes. The more points used, the greater the accuracy. The method is easily implemented as a computer algorithm to do the repetitive operations.

PROBLEMS

Section 4.2

1. The hyperbolic electrode system of Section 4-2-2*a* only extends over the range $0 \leq x \leq x_0$, $0 \leq y \leq y_0$ and has a depth D.

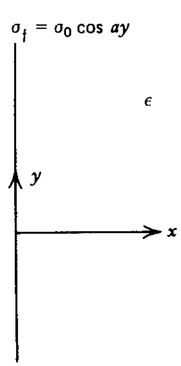
(a) Neglecting fringing field effects what is the approximate capacitance?

(b) A small positive test charge q (image charge effects are negligible) with mass m is released from rest from the surface of the hyperbolic electrode at $x = x_0$, $y = ab/x_0$. What is the velocity of the charge as a function of its position?

(c) What is the velocity of the charge when it hits the opposite electrode?

2. A sheet of free surface charge at $x = 0$ has charge distribution

$$\sigma_f = \sigma_0 \cos ay$$



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