chapter 2

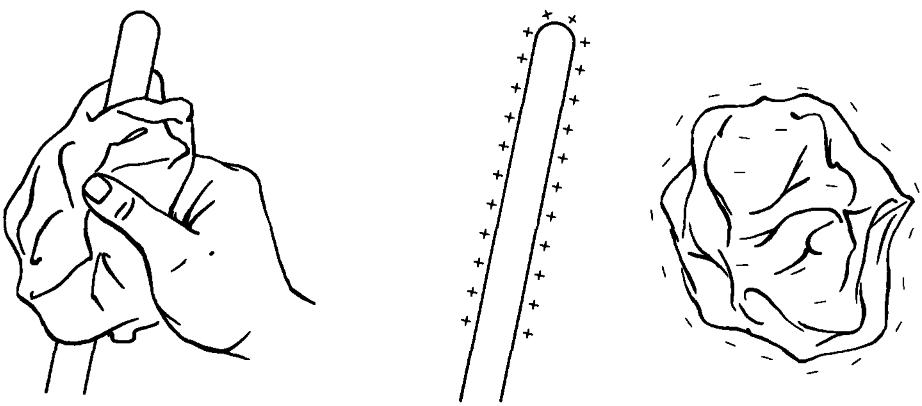
the electric field

The ancient Greeks observed that when the fossil resin amber was rubbed, small light-weight objects were attracted. Yet, upon contact with the amber, they were then repelled. No further significant advances in the understanding of this mysterious phenomenon were made until the eighteenth century when more quantitative electrification experiments showed that these effects were due to electric charges, the source of all effects we will study in this text.

2-1 ELECTRIC CHARGE

2-1-1 Charging by Contact

We now know that all matter is held together by the attractive force between equal numbers of negatively charged electrons and positively charged protons. The early researchers in the 1700s discovered the existence of these two species of charges by performing experiments like those in Figures 2-1 to 2-4. When a glass rod is rubbed by a dry cloth, as in Figure 2-1, some of the electrons in the glass are rubbed off onto the cloth. The cloth then becomes negatively charged because it now has more electrons than protons. The glass rod becomes



(a) (b)

Figure 2-1 A glass rod rubbed with a dry cloth loses some of its electrons to the cloth. The glass rod then has a net positive charge while the cloth has acquired an equal amount of negative charge. The total charge in the system remains zero.

positively charged as it has lost electrons leaving behind a surplus number of protons. If the positively charged glass rod is brought near a metal ball that is free to move as in Figure 2-2*a*, the electrons in the ball near the rod are attracted to the surface leaving uncovered positive charge on the other side of the ball. This is called electrostatic induction. There is then an attractive force of the ball to the rod. Upon contact with the rod, the negative charges are neutralized by some of the positive charges on the rod, the whole combination still retaining a net positive charge as in Figure 2-2*b*. This transfer of charge is called conduction. It is then found that the now positively charged ball is repelled from the similarly charged rod. The metal ball is said to be conducting as charges are easily induced and conducted. It is important that the supporting string not be conducting, that is, insulating, otherwise charge would also distribute itself over the whole structure and not just on the ball.

If two such positively charged balls are brought near each other, they will also repel as in Figure 2-3*a*. Similarly, these balls could be negatively charged if brought into contact with the negatively charged cloth. Then it is also found that two negatively charged balls repel each other. On the other hand, if one ball is charged positively while the other is charged negatively, they will attract. These circumstances are summarized by the simple rules:

Opposite Charges Attract. Like Charges Repel.

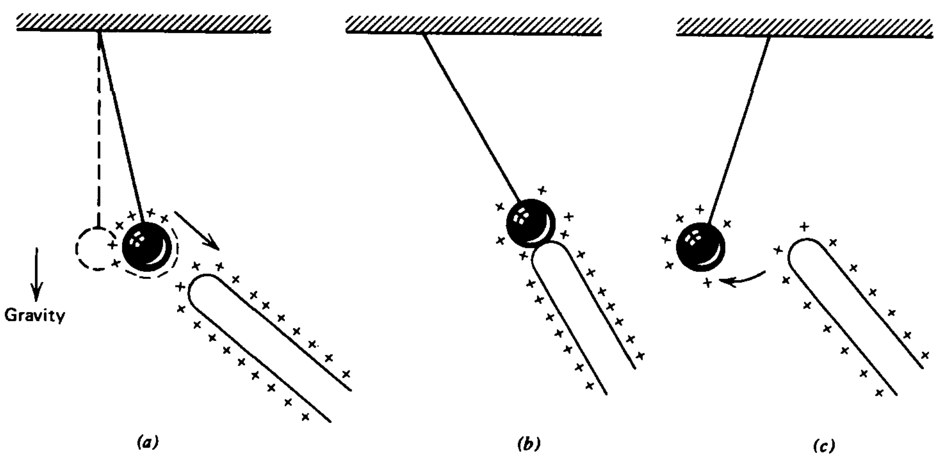


Figure 2-2 (*a*) A charged rod near a neutral ball will induce an opposite charge on the near surface. Since the ball is initially neutral, an equal amount of positive charge remains on the far surface. Because the negative charge is closer to the rod, it feels a stronger attractive force than the repelling force due to the like charges. (*b*) Upon contact with the rod the negative charge is neutralized leaving the ball positively charged. (*c*) The like charges then repel causing the ball to deflect away.

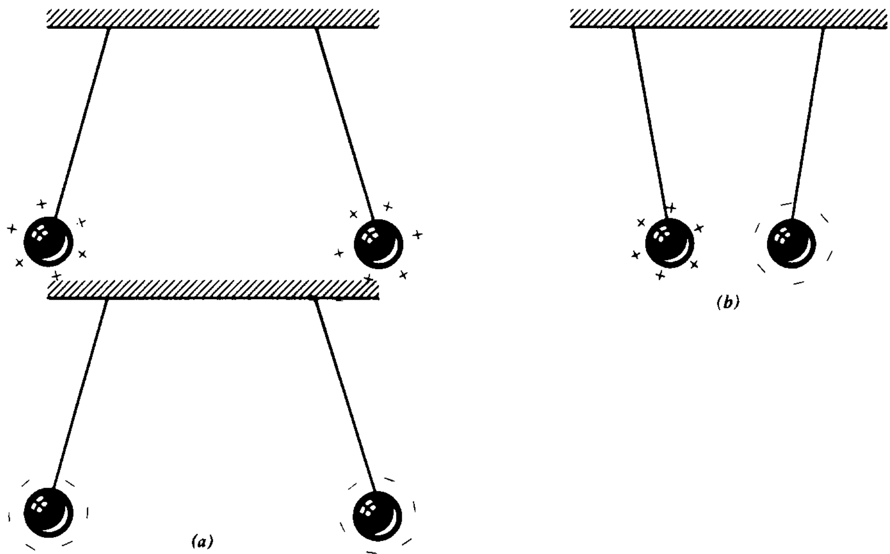


Figure 2-3 (*a*) Like charged bodies repel while (*b*) oppositely charged bodies attract.

In Figure 2-2*a*, the positively charged rod attracts the negative induced charge but repels the uncovered positive charge on the far end of the ball. The net force is attractive because the positive charge on the ball is farther away from the glass rod so that the repulsive force is less than the attractive force.

We often experience nuisance frictional electrification when we walk across a carpet or pull clothes out of a dryer. When we comb our hair with a plastic comb, our hair often becomes charged. When the comb is removed our hair still stands up, as like charged hairs repel one another. Often these effects result in sparks because the presence of large amounts of charge actually pulls electrons from air molecules.

2-1-2 Electrostatic Induction

Even without direct contact net charge can also be placed on a body by electrostatic induction. In Figure 2-4*a* we see two initially neutral suspended balls in contact acquiring opposite charges on each end because of the presence of a charged rod. If the balls are now separated, each half retains its net charge even if the inducing rod is removed. The net charge on the two balls is zero, but we have been able to isolate net positive and negative charges on each ball.

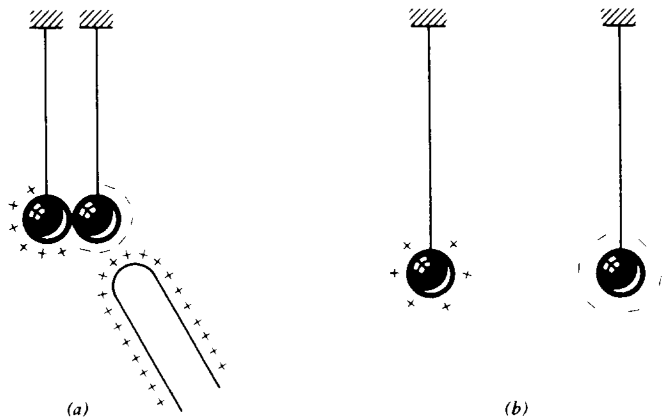


Figure 2-4 A net charge can be placed on a body without contact by electrostatic induction. (*a*) When a charged body is brought near a neutral body, the near side acquires the opposite charge. Being neutral, the far side takes on an equal but opposite charge. (*b*) If the initially neutral body is separated, each half retains its charge.

2-1-3 Faraday's "Ice-Pail" Experiment

These experiments showed that when a charged conductor contacted another conductor, whether charged or not, the total charge on both bodies was shared. The presence of charge was first qualitatively measured by an electroscope that consisted of two attached metal foil leaves. When charged, the mutual repulsion caused the leaves to diverge.

In 1843 Michael Faraday used an electroscope to perform the simple but illuminating "ice-pail" experiment illustrated in Figure 2-5. When a charged body is inside a closed isolated conductor, an equal amount of charge appears on the outside of the conductor as evidenced by the divergence of the electroscope leaves. This is true whether or not the charged body has contacted the inside walls of the surrounding conductor. If it has not, opposite charges are induced on the inside wall leaving unbalanced charge on the outside. If the charged body is removed, the charge on the inside and outside of the conductor drops to zero. However, if the charged body does contact an inside wall, as in Figure 2-5*c*, all the charge on the inside wall and ball is neutralized leaving the outside charged. Removing the initially charged body as in Figure 2-5*d* will find it uncharged, while the ice-pail now holds the original charge.

If the process shown in Figure 2-5 is repeated, the charge on the pail can be built up indefinitely. This is the principle of electrostatic generators where large amounts of charge are stored by continuous deposition of small amounts of charge.

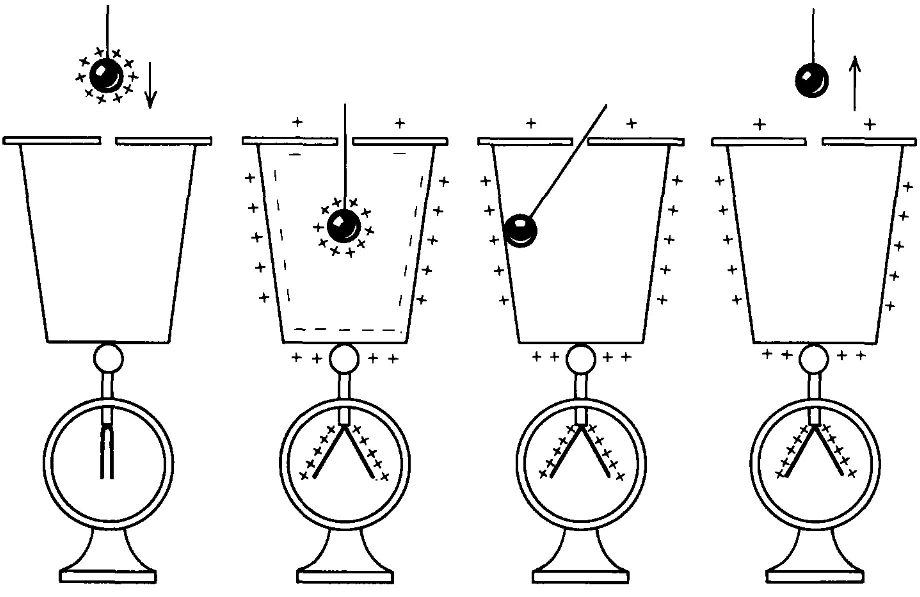


Figure 2-5 Faraday first demonstrated the principles of charge conservation by attaching an electroscope to an initially uncharged metal ice pail. (*a*) When all charges are far away from the pail, there is no charge on the pail nor on the flexible gold leaves of the electroscope attached to the outside of the can, which thus hang limply. (*b*) As a charged ball comes within the pail, opposite charges are induced on the inner surface. Since the pail and electroscope were originally neutral, unbalanced charge appears on the outside of which some is on the electroscope leaves. The leaves being like charged repel each other and thus diverge. (*c*) Once the charged ball is within a closed conducting body, the charge on the outside of the pail is independent of the position of the charged ball. If the charged ball contacts the inner surface of the pail, the inner charges neutralize each other. The outside charges remain unchanged. (*d*) As the now uncharged ball leaves the pail, the distributed charge on the outside of the pail and electroscope remains unchanged.

This large accumulation of charge gives rise to a large force on any other nearby charge, which is why electrostatic generators have been used to accelerate charged particles to very high speeds in atomic studies.

2-2 THE COULOMB FORCE LAW BETWEEN STATIONARY CHARGES

2-2-1 Coulomb's Law

It remained for Charles Coulomb in 1785 to express these experimental observations in a quantitative form. He used a very sensitive torsional balance to measure the force between two stationary charged balls as a function of their distance apart. He discovered that the force between two small charges q_1 and q_2 (idealized as point charges of zero size) is proportional to their magnitudes and inversely proportional to the square of the distance r_{12} between them, as illustrated in Figure 2-6. The force acts along the line joining the charges in the same or opposite direction of the unit vector $\mathbf{i}_{12}$ and is attractive if the charges are of opposite sign and repulsive if like charged. The force $\mathbf{F}_2$ on charge q_2 due to charge q_1 is equal in magnitude but opposite in direction to the force $\mathbf{F}_1$ on q_1, the net force on the pair of charges being zero.

$$\mathbf{F}_2 = -\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\mathbf{i}_{12}\ \text{nt}\ [\text{kg}-\text{m}-\text{s}^{-2}] \tag{1}$$

2-2-2 Units

The value of the proportionality constant $1/4\pi\varepsilon_0$ depends on the system of units used. Throughout this book we use SI units (Système International d'Unités) for which the base units are taken from the rationalized MKSA system of units where distances are measured in meters (m), mass in kilograms (kg), time in seconds (s), and electric current in amperes (A). The unit of charge is a coulomb where 1 coulomb = 1 ampere-second. The adjective "rationalized" is used because the factor of 4π is arbitrarily introduced into the proportionality factor in Coulomb's law of (1). It is done this way so as to cancel a 4π that will arise from other more often used laws we will introduce shortly. Other derived units are formed by combining base units.

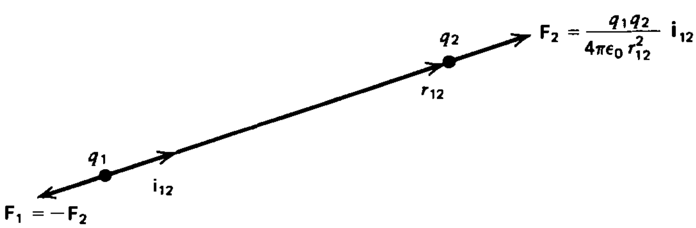


Figure 2-6 The Coulomb force between two point charges is proportional to the magnitude of the charges and inversely proportional to the square of the distance between them. The force on each charge is equal in magnitude but opposite in direction. The force vectors are drawn as if q_1 and q_2 are of the same sign so that the charges repel. If q_1 and q_2 are of opposite sign, both force vectors would point in the opposite directions, as opposite charges attract.

The parameter ε_0 is called the permittivity of free space and has a value

$$\varepsilon_0=(4\pi\times 10^{-7}c^2)^{-1}$$
$$\approx\frac{10^{-9}}{36\pi}\approx 8.8542\times 10^{-12}\text{ farad/m }[\text{A}^2-\text{s}^4-\text{kg}^{-1}-\text{m}^{-3}] \quad (2)$$

where c is the speed of light in vacuum ($c\approx 3\times 10^8$ m/sec).

This relationship between the speed of light and a physical constant was an important result of the early electromagnetic theory in the late nineteenth century, and showed that light is an electromagnetic wave; see the discussion in Chapter 7.

To obtain a feel of how large the force in (1) is, we compare it with the gravitational force that is also an inverse square law with distance. The smallest unit of charge known is that of an electron with charge e and mass m_e

$$e\approx 1.60\times 10^{-19}\text{ Coul},\quad m_e\approx 9.11\times 10^{-31}\text{ kg}$$

Then, the ratio of electric to gravitational force magnitudes for two electrons is independent of their separation:

$$\frac{\mathbf{F}_e}{\mathbf{F}_g}=-\frac{e^2/(4\pi\varepsilon_0 r^2)}{Gm_e^2/r^2}=-\frac{e^2}{m_e^2}\frac{1}{4\pi\varepsilon_0 G}\approx -4.16\times 10^{42} \quad (3)$$

where $G=6.67\times 10^{-11}$ [m^3-s^{-2}-kg^{-1}] is the gravitational constant. This ratio is so huge that it exemplifies why electrical forces often dominate physical phenomena. The minus sign is used in (3) because the gravitational force between two masses is always attractive while for two like charges the electrical force is repulsive.

2-2-3 The Electric Field

If the charge q_1 exists alone, it feels no force. If we now bring charge q_2 within the vicinity of q_1, then q_2 feels a force that varies in magnitude and direction as it is moved about in space and is thus a way of mapping out the vector force field due to q_1. A charge other than q_2 would feel a different force from q_2 proportional to its own magnitude and sign. It becomes convenient to work with the quantity of force per unit charge that is called the electric field, because this quantity is independent of the particular value of charge used in mapping the force field. Considering q_2 as the test charge, the electric field due to q_1 at the position of q_2 is defined as

$$\mathbf{E}_2=\lim_{q_2\to 0}\frac{\mathbf{F}_2}{q_2}=\frac{q_1}{4\pi\varepsilon_0 r_{12}^2}\mathbf{i}_{12}\text{ volts/m }[\text{kg-m-s}^{-3}-\text{A}^{-1}] \quad (4)$$

In the definition of (4) the charge q_1 must remain stationary. This requires that the test charge q_2 be negligibly small so that its force on q_1 does not cause q_1 to move. In the presence of nearby materials, the test charge q_2 could also induce or cause redistribution of the charges in the material. To avoid these effects in our definition of the electric field, we make the test charge infinitely small so its effects on nearby materials and charges are also negligibly small. Then (4) will also be a valid definition of the electric field when we consider the effects of materials. To correctly map the electric field, the test charge must not alter the charge distribution from what it is in the absence of the test charge.

2-2-4 Superposition

If our system only consists of two charges, Coulomb's law (1) completely describes their interaction and the definition of an electric field is unnecessary. The electric field concept is only useful when there are large numbers of charge present as each charge exerts a force on all the others. Since the forces on a particular charge are linear, we can use superposition, whereby if a charge q_1 alone sets up an electric field $\mathbf{E}_1$, and another charge q_2 alone gives rise to an electric field $\mathbf{E}_2$, then the resultant electric field with both charges present is the vector sum $\mathbf{E}_1+\mathbf{E}_2$. This means that if a test charge q_p is placed at point P in Figure 2-7, in the vicinity of N charges it will feel a force

$$\mathbf{F}_p = q_p\mathbf{E}_P \tag{5}$$

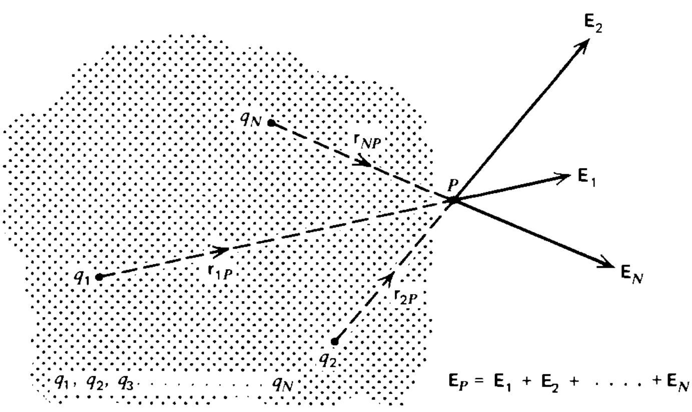


Figure 2-7 The electric field due to a collection of point charges is equal to the vector sum of electric fields from each charge alone.

where $\mathbf{E}_P$ is the vector sum of the electric fields due to all the N-point charges,

$$\mathbf{E}_P=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{1P}^2}\mathbf{i}_{1P}+\frac{q_2}{r_{2P}^2}\mathbf{i}_{2P}+\frac{q_3}{r_{3P}^2}\mathbf{i}_{3P}+\cdots+\frac{q_N}{r_{NP}^2}\mathbf{i}_{NP}\right)$$

$$=\frac{1}{4\pi\varepsilon_0}\sum_{n=1}^{N}\frac{q_n}{r_{nP}^2}\mathbf{i}_{nP} \qquad (6)$$

Note that $\mathbf{E}_P$ has no contribution due to q_p since a charge cannot exert a force upon itself.

EXAMPLE 2-1 TWO-POINT CHARGES

Two-point charges are a distance a apart along the z axis as shown in Figure 2-8. Find the electric field at any point in the $z=0$ plane when the charges are:

(a) both equal to q
(b) of opposite polarity but equal magnitude $\pm q$. This configuration is called an electric dipole.

SOLUTION

(a) In the $z=0$ plane, each point charge alone gives rise to field components in the $\mathbf{i}_r$ and $\mathbf{i}_z$ directions. When both charges are equal, the superposition of field components due to both charges cancel in the z direction but add radially:

$$E_r(z=0)=\frac{q}{4\pi\varepsilon_0}\frac{2r}{[r^2+(a/2)^2]^{3/2}}$$

As a check, note that far away from the point charges ($r\gg a$) the field approaches that of a point charge of value $2q$:

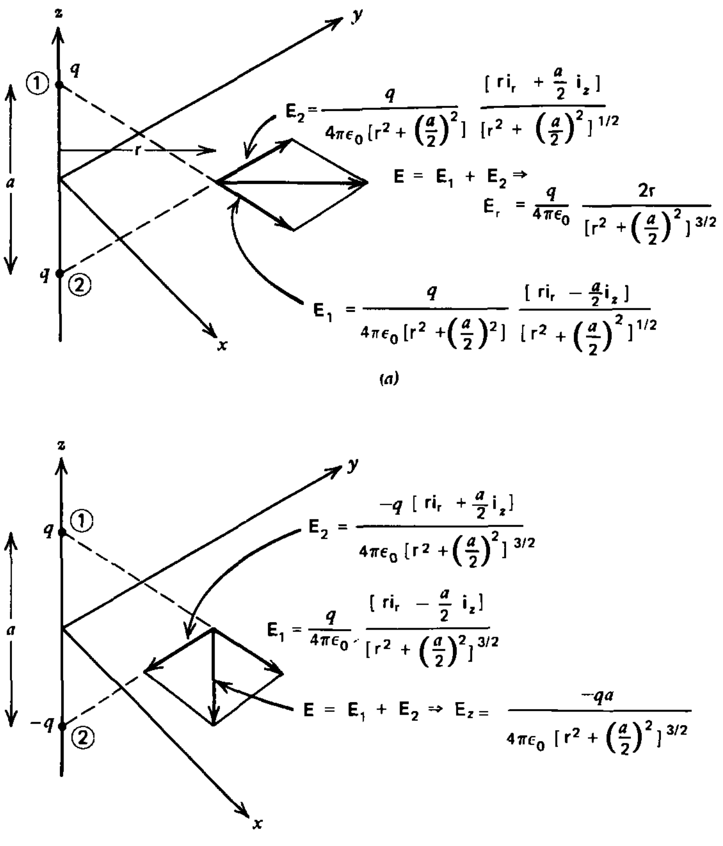
$$\lim_{r\gg a} E_r(z=0)=\frac{2q}{4\pi\varepsilon_0 r^2}$$

(b) When the charges have opposite polarity, the total electric field due to both charges now cancel in the radial direction but add in the z direction:

$$E_z(z=0)=\frac{-q}{4\pi\varepsilon_0}\frac{a}{[r^2+(a/2)^2]^{3/2}}$$

Far away from the point charges the electric field dies off as the inverse cube of distance:

$$\lim_{r\gg a} E_z(z=0)=\frac{-qa}{4\pi\varepsilon_0 r^3}$$



(b)

Figure 2-8 Two equal magnitude point charges are a distance a apart along the z axis. (a) When the charges are of the same polarity, the electric field due to each is radially directed away. In the $z = 0$ symmetry plane, the net field component is radial. (b) When the charges are of opposite polarity, the electric field due to the negative charge is directed radially inwards. In the $z = 0$ symmetry plane, the net field is now $-z$ directed.

The faster rate of decay of a dipole field is because the net charge is zero so that the fields due to each charge tend to cancel each other out.

2-3 CHARGE DISTRIBUTIONS

The method of superposition used in Section 2.2.4 will be used throughout the text in relating fields to their sources. We first find the field due to a single-point source. Because the field equations are linear, the net field due to many point sources is just the superposition of the fields from each source alone. Thus, knowing the electric field for a single-point charge at an arbitrary position immediately gives us the total field for any distribution of point charges.

In typical situations, one coulomb of total charge may be present requiring 6.25×10^{18} elementary charges ($e\approx1.60\times10^{-19}$ coul). When dealing with such a large number of particles, the discrete nature of the charges is often not important and we can consider them as a continuum. We can then describe the charge distribution by its density. The same model is used in the classical treatment of matter. When we talk about mass we do not go to the molecular scale and count the number of molecules, but describe the material by its mass density that is the product of the local average number of molecules in a unit volume and the mass per molecule.

2-3-1 Line, Surface, and Volume Charge Distributions

We similarly speak of charge densities. Charges can distribute themselves on a line with line charge density λ (coul/m), on a surface with surface charge density σ (coul/m^2) or throughout a volume with volume charge density ρ (coul/m^3).

Consider a distribution of free charge dq of differential size within a macroscopic distribution of line, surface, or volume charge as shown in Figure 2-9. Then, the total charge q within each distribution is obtained by summing up all the differential elements. This requires an integration over the line, surface, or volume occupied by the charge.

$$dq=\begin{cases}\lambda\,dl\\ \sigma\,dS\\ \rho\,dV\end{cases}\Rightarrow q=\begin{cases}\displaystyle\int_L\lambda\,dl & \text{(line charge)}\\ \displaystyle\int_S\sigma\,dS & \text{(surface charge)}\\ \displaystyle\int_V\rho\,dV & \text{(volume charge)}\end{cases}\qquad(1)$$

EXAMPLE 2-2 CHARGE DISTRIBUTIONS

Find the total charge within each of the following distributions illustrated in Figure 2-10.

(a) Line charge λ_0 uniformly distributed in a circular hoop of radius a.

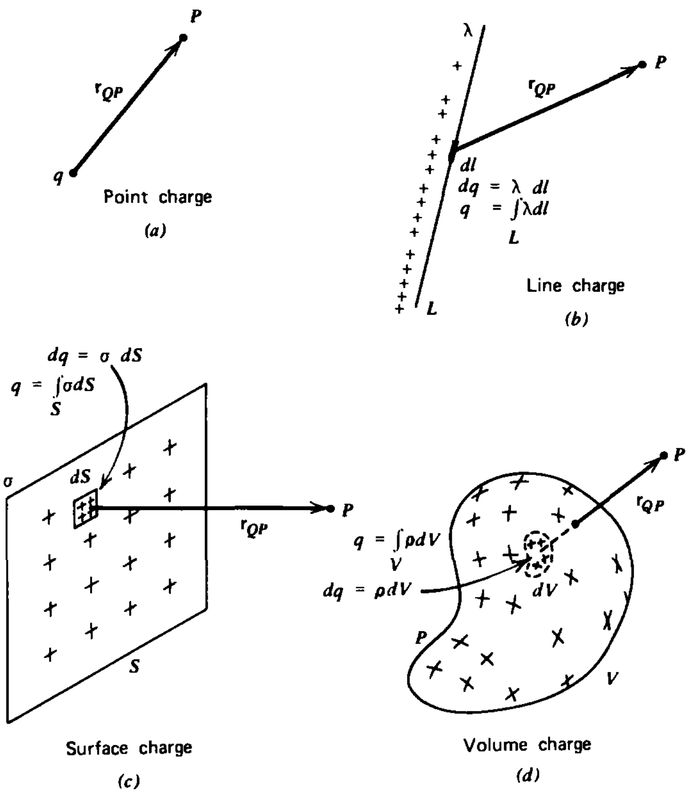


Figure 2-9 Charge distributions. (*a*) Point charge; (*b*) Line charge; (*c*) Surface charge; (*d*) Volume charge.

SOLUTION

$$q = \int_L \lambda\, dl = \int_0^{2\pi} \lambda_0 a\, d\phi = 2\pi a\lambda_0$$

(b) Surface charge σ_0 uniformly distributed on a circular disk of radius a.

SOLUTION

$$q = \int_S \sigma\, dS = \int_{r=0}^{a}\int_{\phi=0}^{2\pi} \sigma_0 r\, dr\, d\phi = \pi a^2\sigma_0$$

(c) Volume charge ρ_0 uniformly distributed throughout a sphere of radius R.

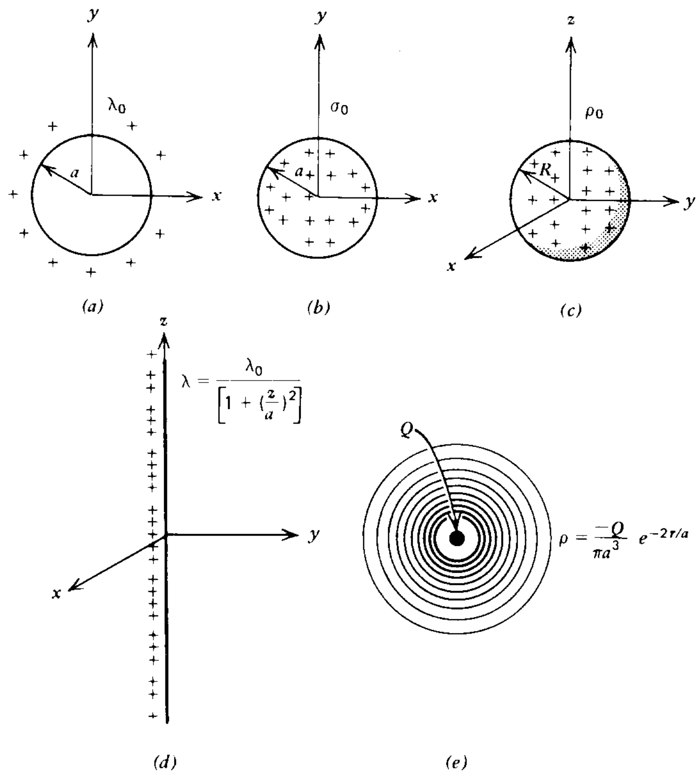


Figure 2-10 Charge distributions of Example 2-2. (*a*) Uniformly distributed line charge on a circular hoop. (*b*) Uniformly distributed surface charge on a circular disk. (*c*) Uniformly distributed volume charge throughout a sphere. (*d*) Nonuniform line charge distribution. (*e*) Smooth radially dependent volume charge distribution throughout all space, as a simple model of the electron cloud around the positively charged nucleus of the hydrogen atom.

SOLUTION

$$q=\int_V \rho\, dV=\int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\rho_0 r^2 \sin\theta\, dr\, d\theta\, d\phi=\tfrac{4}{3}\pi R^3\rho_0$$

(d) A line charge of infinite extent in the z direction with charge density distribution

$$\lambda=\frac{\lambda_0}{[1+(z/a)^2]}$$

SOLUTION

$$q=\int_L \lambda\, dl=\int_{-\infty}^{+\infty}\frac{\lambda_0\, dz}{[1+(z/a)^2]}=\lambda_0 a \tan^{-1}\frac{z}{a}\bigg|_{-\infty}^{+\infty}=\lambda_0\pi a$$

(e) The electron cloud around the positively charged nucleus Q in the hydrogen atom is simply modeled as the spherically symmetric distribution

$$\rho(r) = -\frac{Q}{\pi a^3} e^{-2r/a}$$

where a is called the Bohr radius.

SOLUTION

The total charge in the cloud is

$$\begin{aligned} q &= \int_V \rho\, dV \\ &= -\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{Q}{\pi a^3} e^{-2r/a} r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= -\int_{r=0}^{\infty} \frac{4Q}{a^3} e^{-2r/a} r^2\, dr \\ &= -\frac{4Q}{a^3}\left(-\frac{a}{2}\right) e^{-2r/a}\left[r^2 - \frac{a^2}{2}\left(-\frac{2r}{a} - 1\right)\right]\Bigg|_{r=0}^{\infty} \\ &= -Q \end{aligned}$$

2-3-2 The Electric Field Due to a Charge Distribution

Each differential charge element dq as a source at point Q contributes to the electric field at a point P as

$$d\mathbf{E} = \frac{dq}{4\pi\varepsilon_0 r_{QP}^2}\mathbf{i}_{QP} \tag{2}$$

where r_{QP} is the distance between Q and P with $\mathbf{i}_{QP}$ the unit vector directed from Q to P. To find the total electric field, it is necessary to sum up the contributions from each charge element. This is equivalent to integrating (2) over the entire charge distribution, remembering that both the distance r_{QP} and direction $\mathbf{i}_{QP}$ vary for each differential element throughout the distribution

$$\mathbf{E} = \int_{\text{all } q} \frac{dq}{4\pi\varepsilon_0 r_{QP}^2}\mathbf{i}_{QP} \tag{3}$$

where (3) is a line integral for line charges ($dq = \lambda\, dl$), a surface integral for surface charges ($dq = \sigma\, dS$), a volume integral for a volume charge distribution ($dq = \rho\, dV$), or in general, a combination of all three.

If the total charge distribution is known, the electric field is obtained by performing the integration of (3). Some general rules and hints in using (3) are:

1. It is necessary to distinguish between the coordinates of the field points and the charge source points. Always integrate over the coordinates of the charges.
2. Equation (3) is a vector equation and so generally has three components requiring three integrations. Symmetry arguments can often be used to show that particular field components are zero.
3. The distance r_{QP} is always positive. In taking square roots, always make sure that the positive square root is taken.
4. The solution to a particular problem can often be obtained by integrating the contributions from simpler differential size structures.

2-3-3 Field Due to an Infinitely Long Line Charge

An infinitely long uniformly distributed line charge λ_0 along the z axis is shown in Figure 2-11. Consider the two symmetrically located charge elements dq_1 and dq_2 a distance z above and below the point P, a radial distance r away. Each charge element alone contributes radial and z components to the electric field. However, just as we found in Example 2-1a, the two charge elements together cause equal magnitude but oppositely directed z field components that thus cancel leaving only additive radial components:

$$dE_r = \frac{\lambda_0\, dz}{4\pi\varepsilon_0(z^2+r^2)}\cos\theta = \frac{\lambda_0 r\, dz}{4\pi\varepsilon_0(z^2+r^2)^{3/2}} \tag{4}$$

To find the total electric field we integrate over the length of the line charge:

$$\begin{aligned} E_r &= \frac{\lambda_0 r}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dz}{(z^2+r^2)^{3/2}} \\ &= \frac{\lambda_0 r}{4\pi\varepsilon_0}\frac{z}{r^2(z^2+r^2)^{1/2}}\bigg|_{z=-\infty}^{+\infty} \\ &= \frac{\lambda_0}{2\pi\varepsilon_0 r} \end{aligned} \tag{5}$$

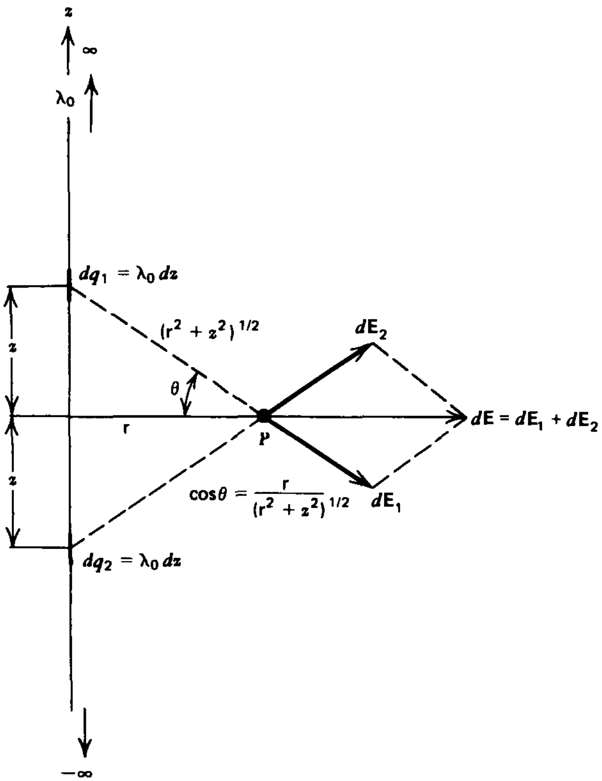
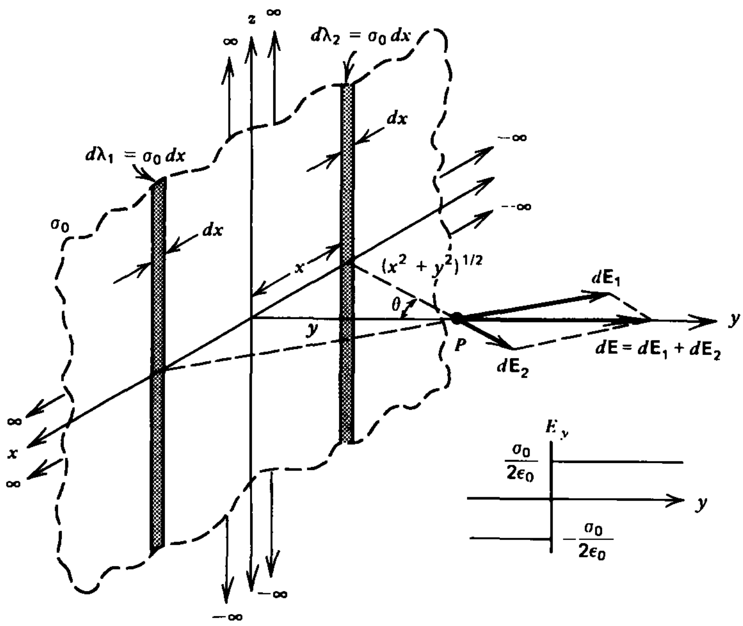


Figure 2-11 An infinitely long uniform distribution of line charge only has a radially directed electric field because the z components of the electric field are canceled out by symmetrically located incremental charge elements as also shown in Figure 2-8*a*.

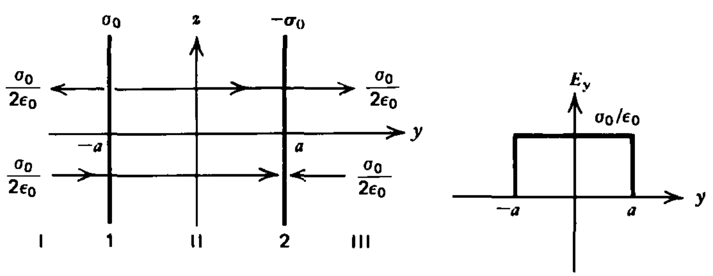
2-3-4 Field Due to Infinite Sheets of Surface Charge

(a) Single Sheet

A surface charge sheet of infinite extent in the $y=0$ plane has a uniform surface charge density σ_0 as in Figure 2-12*a*. We break the sheet into many incremental line charges of thickness dx with $d\lambda=\sigma_0\,dx$. We could equivalently break the surface into incremental horizontal line charges of thickness dz. Each incremental line charge alone has a radial field component as given by (5) that in Cartesian coordinates results in x and y components. Consider the line charge $d\lambda_1$, a distance x to the left of P, and the symmetrically placed line charge $d\lambda_2$ the same distance x to the right of P. The x components of the resultant fields cancel while the y



(a)



(b)

Figure 2-12 (a) The electric field from a uniformly surface charged sheet of infinite extent is found by summing the contributions from each incremental line charge element. Symmetrically placed line charge elements have x field components that cancel, but y field components that add. (b) Two parallel but oppositely charged sheets of surface charge have fields that add in the region between the sheets but cancel outside. (c) The electric field from a volume charge distribution is obtained by summing the contributions from each incremental surface charge element.

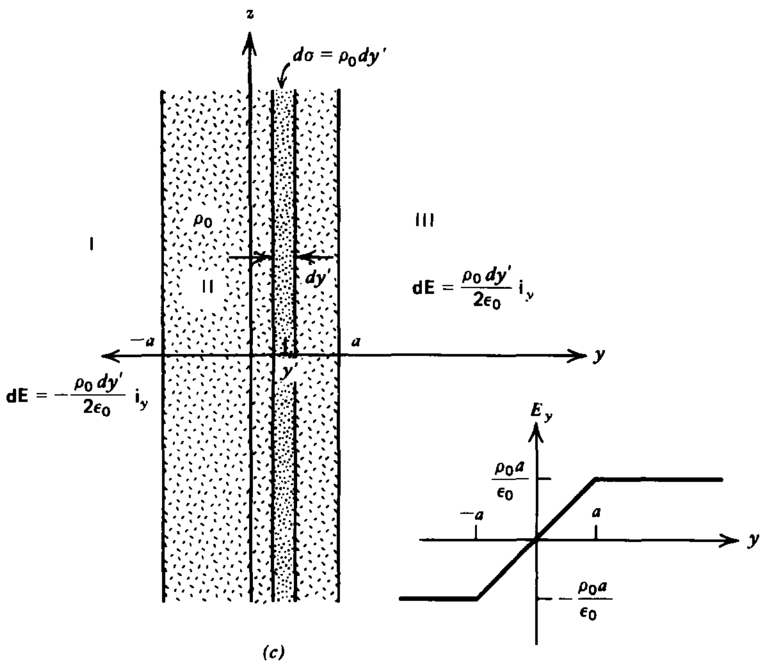


Fig. 2-12(c)

components add:

$$dE_y = \frac{\sigma_0\,dx}{2\pi\varepsilon_0(x^2+y^2)^{1/2}}\cos\theta = \frac{\sigma_0 y\,dx}{2\pi\varepsilon_0(x^2+y^2)} \tag{6}$$

The total field is then obtained by integration over all line charge elements:

$$\begin{aligned} E_y &= \frac{\sigma_0 y}{2\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dx}{x^2+y^2} \\ &= \frac{\sigma_0 y}{2\pi\varepsilon_0}\frac{1}{y}\tan^{-1}\frac{x}{y}\bigg|_{x=-\infty}^{+\infty} \\ &= \begin{cases} \sigma_0/2\varepsilon_0, & y>0 \\ -\sigma_0/2\varepsilon_0, & y<0 \end{cases} \end{aligned} \tag{7}$$

where we realized that the inverse tangent term takes the sign of the ratio x/y so that the field reverses direction on each side of the sheet. The field strength does not decrease with distance from the infinite sheet.

(b) Parallel Sheets of Opposite Sign

A capacitor is formed by two oppositely charged sheets of surface charge a distance $2a$ apart as shown in Figure 2-12b.

The fields due to each charged sheet alone are obtained from (7) as

$$\mathbf{E}_1=\begin{cases}\dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y>-a\\[2ex] -\dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y<-a\end{cases}\qquad \mathbf{E}_2=\begin{cases}-\dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y>a\\[2ex] \dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y<a\end{cases}\tag{8}$$

Thus, outside the sheets in regions I and III the fields cancel while they add in the enclosed region II. The nonzero field is confined to the region between the charged sheets and is independent of the spacing:

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2=\begin{cases}(\sigma_0/\varepsilon_0)\mathbf{i}_y, & |y|<a\\ 0 & |y|>a\end{cases}\tag{9}$$

(c) Uniformly Charged Volume

A uniformly charged volume with charge density ρ_0 of infinite extent in the x and z directions and of width $2a$ is centered about the y axis, as shown in Figure 2-12c. We break the volume distribution into incremental sheets of surface charge of width dy' with differential surface charge density $d\sigma=\rho_0\,dy'$. It is necessary to distinguish the position y' of the differential sheet of surface charge from the field point y. The total electric field is the sum of all the fields due to each differentially charged sheet. The problem breaks up into three regions. In region I, where $y\leq -a$, each surface charge element causes a field in the negative y direction:

$$E_y=\int_{-a}^{a}-\frac{\rho_0}{2\varepsilon_0}\,dy'=-\frac{\rho_0 a}{\varepsilon_0},\quad y\leq -a\tag{10}$$

Similarly, in region III, where $y\geq a$, each charged sheet gives rise to a field in the positive y direction:

$$E_y=\int_{-a}^{a}\frac{\rho_0\,dy'}{2\varepsilon_0}=\frac{\rho_0 a}{\varepsilon_0},\quad y\geq a\tag{11}$$

For any position y in region II, where $-a\leq y\leq a$, the charge to the right of y gives rise to a negatively directed field while the charge to the left of y causes a positively directed field:

$$E_y=\int_{-a}^{y}\frac{\rho_0\,dy'}{2\varepsilon_0}+\int_{y}^{a}(-)\frac{\rho_0}{2\varepsilon_0}\,dy'=\frac{\rho_0 y}{\varepsilon_0},\quad -a\leq y\leq a\tag{12}$$

The field is thus constant outside of the volume of charge and in opposite directions on either side being the same as for a surface charged sheet with the same total charge per unit area, $\sigma_0 = \rho_0 2a$. At the boundaries $y = \pm a$, the field is continuous, changing linearly with position between the boundaries:

$$E_y = \begin{cases} -\dfrac{\rho_0 a}{\varepsilon_0}, & y \le -a \\ \dfrac{\rho_0 y}{\varepsilon_0}, & -a \le y \le a \\ \dfrac{\rho_0 a}{\varepsilon_0}, & y \ge a \end{cases} \qquad (13)$$

2-3-5 Superposition of Hoops of Line Charge

(a) Single Hoop

Using superposition, we can similarly build up solutions starting from a circular hoop of radius a with uniform line charge density λ_0 centered about the origin in the $z = 0$ plane as shown in Figure 2-13a. Along the z axis, the distance to the hoop perimeter $(a^2 + z^2)^{1/2}$ is the same for all incremental point charge elements $dq = \lambda_0 a\, d\phi$. Each charge element alone contributes z- and r-directed electric field components. However, along the z axis symmetrically placed elements 180° apart have z components that add but radial components that cancel. The z-directed electric field along the z axis is then

$$E_z = \int_0^{2\pi} \frac{\lambda_0 a\, d\phi \cos\theta}{4\pi\varepsilon_0(z^2 + a^2)} = \frac{\lambda_0 a z}{2\varepsilon_0(a^2 + z^2)^{3/2}} \qquad (14)$$

The electric field is in the —z direction along the z axis below the hoop.

The total charge on the hoop is $q = 2\pi a\lambda_0$ so that (14) can also be written as

$$E_z = \frac{qz}{4\pi\varepsilon_0(a^2 + z^2)^{3/2}} \qquad (15)$$

When we get far away from the hoop ($|z| \gg a$), the field approaches that of a point charge:

$$\lim_{|z| \gg a} E_z = \pm\frac{q}{4\pi\varepsilon_0 z^2} \begin{cases} z > 0 \\ z < 0 \end{cases} \qquad (16)$$

(b) Disk of Surface Charge

The solution for a circular disk of uniformly distributed surface charge σ_0 is obtained by breaking the disk into incremental hoops of radius r with line charge $d\lambda = \sigma_0\, dr$ as in

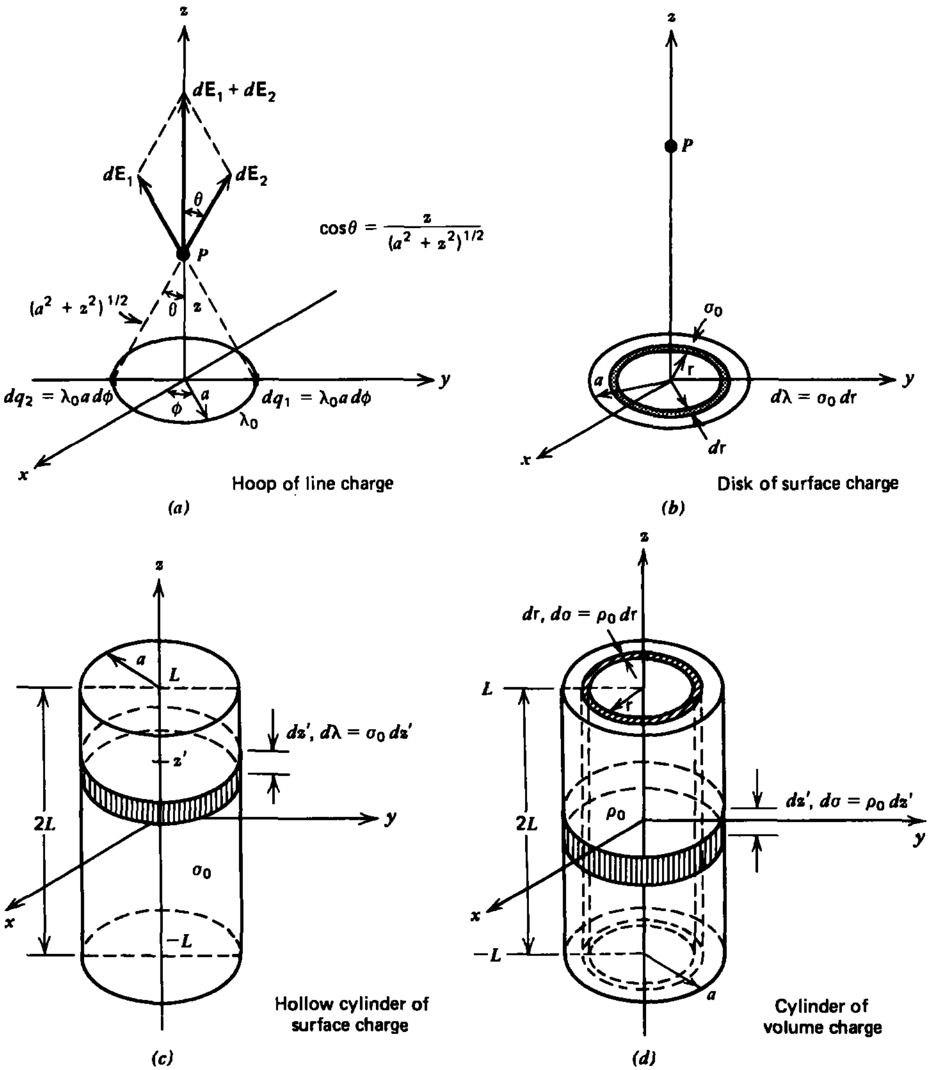


Figure 2-13 (*a*) The electric field along the symmetry z axis of a uniformly distributed hoop of line charge is z directed. (*b*) The axial field from a circular disk of surface charge is obtained by radially summing the contributions of incremental hoops of line charge. (*c*) The axial field from a hollow cylinder of surface charge is obtained by axially summing the contributions of incremental hoops of line charge. (*d*) The axial field from a cylinder of volume charge is found by summing the contributions of axial incremental disks or of radial hollow cylinders of surface charge.

Figure 2-13*b*. Then the incremental z-directed electric field along the z axis due to a hoop of radius r is found from (14) as

$$dE_z = \frac{\sigma_0 r z\, dr}{2\varepsilon_0 (r^2 + z^2)^{3/2}} \tag{17}$$

where we replace a with r, the radius of the incremental hoop. The total electric field is then

$$\begin{aligned} E_z &= \frac{\sigma_0 z}{2\varepsilon_0}\int_0^a \frac{\mathrm{r}\,d\mathrm{r}}{(\mathrm{r}^2+z^2)^{3/2}} \\ &= -\frac{\sigma_0 z}{2\varepsilon_0(\mathrm{r}^2+z^2)^{1/2}}\bigg|_0^a \\ &= -\frac{\sigma_0}{2\varepsilon_0}\left(\frac{z}{(a^2+z^2)^{1/2}}-\frac{z}{|z|}\right) \\ &= \pm\frac{\sigma_0}{2\varepsilon_0}-\frac{\sigma_0 z}{2\varepsilon_0(a^2+z^2)^{1/2}}\begin{cases} z>0 \\ z<0 \end{cases} \end{aligned} \tag{18}$$

where care was taken at the lower limit ($\mathrm{r}=0$), as the magnitude of the square root must always be used.

As the radius of the disk gets very large, this result approaches that of the uniform field due to an infinite sheet of surface charge:

$$\lim_{a\to\infty} E_z = \pm\frac{\sigma_0}{2\varepsilon_0}\begin{cases} z>0 \\ z<0 \end{cases} \tag{19}$$

(c) Hollow Cylinder of Surface Charge

A hollow cylinder of length $2L$ and radius a has its axis along the z direction and is centered about the $z=0$ plane as in Figure 2-13c. Its outer surface at $\mathrm{r}=a$ has a uniform distribution of surface charge σ_0. It is necessary to distinguish between the coordinate of the field point z and the source point at z' ($-L\le z'\le L$). The hollow cylinder is broken up into incremental hoops of line charge $d\lambda=\sigma_0\,dz'$. Then, the axial distance from the field point at z to any incremental hoop of line charge is $(z-z')$. The contribution to the axial electric field at z due to the incremental hoop at z' is found from (14) as

$$dE_z = \frac{\sigma_0 a(z-z')\,dz'}{2\varepsilon_0[a^2+(z-z')^2]^{3/2}} \tag{20}$$

which when integrated over the length of the cylinder yields

$$\begin{aligned} E_z &= \frac{\sigma_0 a}{2\varepsilon_0}\int_{-L}^{+L}\frac{(z-z')\,dz'}{[a^2+(z-z')^2]^{3/2}} \\ &= \frac{\sigma_0 a}{2\varepsilon_0}\frac{1}{[a^2+(z-z')^2]^{1/2}}\bigg|_{z'=-L}^{+L} \\ &= \frac{\sigma_0 a}{2\varepsilon_0}\left(\frac{1}{[a^2+(z-L)^2]^{1/2}}-\frac{1}{[a^2+(z+L)^2]^{1/2}}\right) \end{aligned} \tag{21}$$

(d) Cylinder of Volume Charge

If this same cylinder is uniformly charged throughout the volume with charge density ρ_0, we break the volume into differential-size hollow cylinders of thickness dr with incremental surface charge $d\sigma = \rho_0\, dr$ as in Figure 2-13d. Then, the z-directed electric field along the z axis is obtained by integration of (21) replacing a by r:

$$\begin{aligned}
E_z &= \frac{\rho_0}{2\varepsilon_0}\int_0^a r\left(\frac{1}{[r^2+(z-L)^2]^{1/2}} - \frac{1}{[r^2+(z+L)^2]^{1/2}}\right) dr \\
&= \frac{\rho_0}{2\varepsilon_0}\{[r^2+(z-L)^2]^{1/2} - [r^2+(z+L)^2]^{1/2}\}\Big|_0^a \\
&= \frac{\rho_0}{2\varepsilon_0}\{[a^2+(z-L)^2]^{1/2} - |z-L| - [a^2+(z+L)^2]^{1/2} \\
&\quad + |z+L|\}
\end{aligned} \tag{22}$$

where at the lower $r=0$ limit we always take the positive square root.

This problem could have equally well been solved by breaking the volume charge distribution into many differential-sized surface charged disks at position z' $(-L \le z' \le L)$, thickness dz', and effective surface charge density $d\sigma = \rho_0\, dz'$. The field is then obtained by integrating (18).

2-4 GAUSS'S LAW

We could continue to build up solutions for given charge distributions using the coulomb superposition integral of Section 2.3.2. However, for geometries with spatial symmetry, there is often a simpler way using some vector properties of the inverse square law dependence of the electric field.

2-4-1 Properties of the Vector Distance Between Two Points, $\mathbf{r}_{QP}$

(a) $\mathbf{r}_{QP}$

In Cartesian coordinates the vector distance $\mathbf{r}_{QP}$ between a source point at Q and a field point at P directed from Q to P as illustrated in Figure 2-14 is

$$\mathbf{r}_{QP} = (x - x_Q)\mathbf{i}_x + (y - y_Q)\mathbf{i}_y + (z - z_Q)\mathbf{i}_z \tag{1}$$

with magnitude

$$r_{QP} = [(x - x_Q)^2 + (y - y_Q)^2 + (z - z_Q)^2]^{1/2} \tag{2}$$

The unit vector in the direction of $\mathbf{r}_{QP}$ is

$$\mathbf{i}_{QP} = \frac{\mathbf{r}_{QP}}{r_{QP}} \tag{3}$$

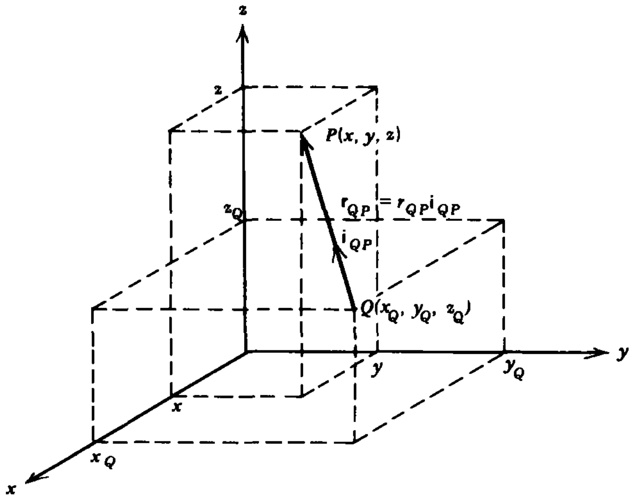


Figure 2-14 The vector distance $\mathbf{r}_{QP}$ between two points Q and P.

(b) Gradient of the Reciprocal Distance, $\nabla(1/r_{QP})$

Taking the gradient of the reciprocal of (2) yields

$$\begin{aligned}\nabla\left(\frac{1}{r_{QP}}\right) &= \mathbf{i}_x\frac{\partial}{\partial x}\left(\frac{1}{r_{QP}}\right)+\mathbf{i}_y\frac{\partial}{\partial y}\left(\frac{1}{r_{QP}}\right)+\mathbf{i}_z\frac{\partial}{\partial z}\left(\frac{1}{r_{QP}}\right)\\ &= -\frac{1}{r_{QP}^3}[(x-x_Q)\mathbf{i}_x+(y-y_Q)\mathbf{i}_y+(z-z_Q)\mathbf{i}_z]\\ &= -\mathbf{i}_{QP}/r_{QP}^2 \end{aligned} \tag{4}$$

which is the negative of the spatially dependent term that we integrate to find the electric field in Section 2.3.2.

(c) Laplacian of the Reciprocal Distance

Another useful identity is obtained by taking the divergence of the gradient of the reciprocal distance. This operation is called the Laplacian of the reciprocal distance. Taking the divergence of (4) yields

$$\begin{aligned}\nabla^2\left(\frac{1}{r_{QP}}\right) &= \nabla\cdot\left[\nabla\left(\frac{1}{r_{QP}}\right)\right]\\ &= \nabla\cdot\left(\frac{-\mathbf{i}_{QP}}{r_{QP}^2}\right)\\ &= -\frac{\partial}{\partial x}\left(\frac{x-x_Q}{r_{QP}^3}\right)-\frac{\partial}{\partial y}\left(\frac{y-y_Q}{r_{QP}^3}\right)-\frac{\partial}{\partial z}\left(\frac{z-z_Q}{r_{QP}^3}\right)\\ &= -\frac{3}{r_{QP}^3}+\frac{3}{r_{QP}^5}[(x-x_Q)^2+(y-y_Q)^2+(z-z_Q)^2] \end{aligned} \tag{5}$$

Using (2) we see that (5) reduces to

$$\nabla^2\left(\frac{1}{r_{QP}}\right)=\begin{cases}0, & r_{QP}\neq 0\\ \text{undefined} & r_{QP}=0\end{cases}\tag{6}$$

Thus, the Laplacian of the inverse distance is zero for all nonzero distances but is undefined when the field point is coincident with the source point.

2-4-2 Gauss's Law In Integral Form

(a) Point Charge Inside or Outside a Closed Volume

Now consider the two cases illustrated in Figure 2-15 where an arbitrarily shaped closed volume V either surrounds a point charge q or is near a point charge q outside the surface S. For either case the electric field emanates radially from the point charge with the spatial inverse square law. We wish to calculate the flux of electric field through the surface S surrounding the volume V:

$$\begin{aligned}\Phi&=\oint_S \mathbf{E}\cdot d\mathbf{S}\\ &=\oint_S \frac{q}{4\pi\varepsilon_0 r_{QP}^2}\mathbf{i}_{QP}\cdot d\mathbf{S}\\ &=\oint_S \frac{-q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r_{QP}}\right)\cdot d\mathbf{S}\end{aligned}\tag{7}$$

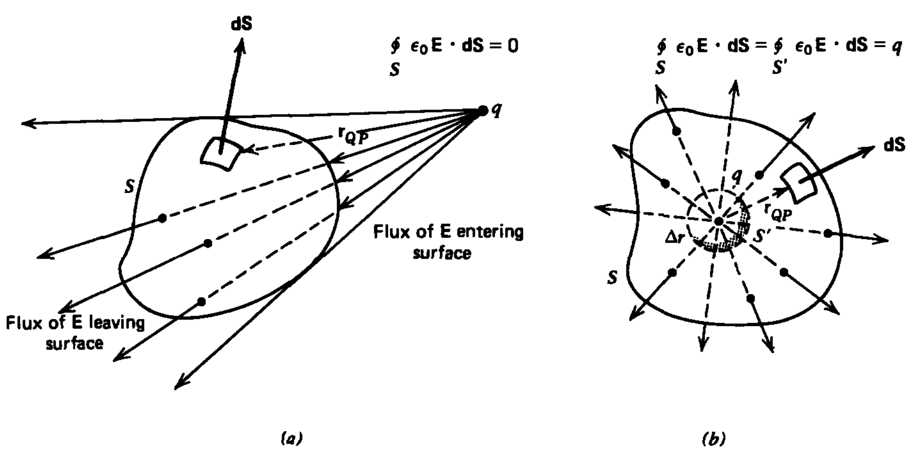


Figure 2-15 (*a*) The net flux of electric field through a closed surface S due to an outside point charge is zero because as much flux enters the near side of the surface as leaves on the far side. (*b*) All the flux of electric field emanating from an enclosed point charge passes through the surface.

where we used (4). We can now use the divergence theorem to convert the surface integral to a volume integral:

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{-q}{4\pi\varepsilon_0}\int_V \nabla\cdot\left[\nabla\left(\frac{1}{r_{QP}}\right)\right]dV \tag{8}$$

When the point charge q is outside the surface every point in the volume has a nonzero value of r_{QP}. Then, using (6) with $r_{QP}\neq 0$, we see that the net flux of $\mathbf{E}$ through the surface is zero.

This result can be understood by examining Figure 2-15*a*. The electric field emanating from q on that part of the surface S nearest q has its normal component oppositely directed to $\mathbf{dS}$ giving a negative contribution to the flux. However, on the opposite side of S the electric field exits with its normal component in the same direction as $\mathbf{dS}$ giving a positive contribution to the flux. We have shown that these flux contributions are equal in magnitude but opposite in sign so that the net flux is zero.

As illustrated in Figure 2-15*b*, assuming q to be positive, we see that when S surrounds the charge the electric field points outwards with normal component in the direction of $\mathbf{dS}$ everywhere on S so that the flux must be positive. If q were negative, $\mathbf{E}$ and $\mathbf{dS}$ would be oppositely directed everywhere so that the flux is also negative. For either polarity with nonzero q, the flux cannot be zero. To evaluate the value of this flux we realize that (8) is zero everywhere except where $r_{QP}=0$ so that the surface S in (8) can be shrunk down to a small spherical surface S' of infinitesimal radius Δr surrounding the point charge; the rest of the volume has $r_{QP}\neq 0$ so that $\nabla\cdot\nabla(1/r_{QP})=0$. On this incremental surface we know the electric field is purely radial in the same direction as $\mathbf{d}S'$ with the field due to a point charge:

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \oint_{S'} \mathbf{E}\cdot d\mathbf{S}' = \frac{q}{4\pi\varepsilon_0(\Delta r)^2}4\pi(\Delta r)^2 = \frac{q}{\varepsilon_0} \tag{9}$$

If we had many point charges within the surface S, each charge q_i gives rise to a flux q_i/ε_0 so that Gauss's law states that the net flux of $\varepsilon_0\mathbf{E}$ through a closed surface is equal to the net charge enclosed by the surface:

$$\oint_S \varepsilon_0\mathbf{E}\cdot d\mathbf{S} = \sum_{\substack{\text{all } q_i\\ \text{inside } S}} q_i. \tag{10}$$

Any charges outside S do not contribute to the flux.

(b) Charge Distributions

For continuous charge distributions, the right-hand side of (10) includes the sum of all enclosed incremental charge elements so that the total charge enclosed may be a line, surface, and/or volume integral in addition to the sum of point charges:

$$\oint_S \varepsilon_0 \mathbf{E}\cdot d\mathbf{S} = \sum_{\substack{\text{all } q_i \\ \text{inside } S}} q_i + \int_{\substack{\text{all } q \\ \text{inside } S}} dq$$

$$= \left(\sum q_i + \int_L \lambda\, dl + \int_S \sigma\, dS + \int_V \rho\, dV\right)\Bigg|_{\substack{\text{all charge} \\ \text{inside } S}} \tag{11}$$

Charges outside the volume give no contribution to the total flux through the enclosing surface.

Gauss's law of (11) can be used to great advantage in simplifying computations for those charges distributed with spatial symmetry. The trick is to find a surface S that has sections tangent to the electric field so that the dot product is zero, or has surfaces perpendicular to the electric field and upon which the field is constant so that the dot product and integration become pure multiplications. If the appropriate surface is found, the surface integral becomes very simple to evaluate.

Coulomb's superposition integral derived in Section 2.3.2 is often used with symmetric charge distributions to determine if any field components are zero. Knowing the direction of the electric field often suggests the appropriate Gaussian surface upon which to integrate (11). This integration is usually much simpler than using Coulomb's law for each charge element.

2-4-3 Spherical Symmetry

(a) Surface Charge

A sphere of radius R has a uniform distribution of surface charge σ_0 as in Figure 2-16a. Measure the angle θ from the line joining any point P at radial distance r to the sphere center. Then, the distance from P to any surface charge element on the sphere is independent of the angle ϕ. Each differential surface charge element at angle θ contributes field components in the radial and θ directions, but symmetrically located charge elements at $-\phi$ have equal field magnitude components that add radially but cancel in the θ direction.

Realizing from the symmetry that the electric field is purely radial and only depends on r and not on θ or ϕ, we draw Gaussian spheres of radius r as in Figure 2-16b both inside ($r<R$) and outside ($r>R$) the charged sphere. The Gaussian sphere inside encloses no charge while the outside sphere

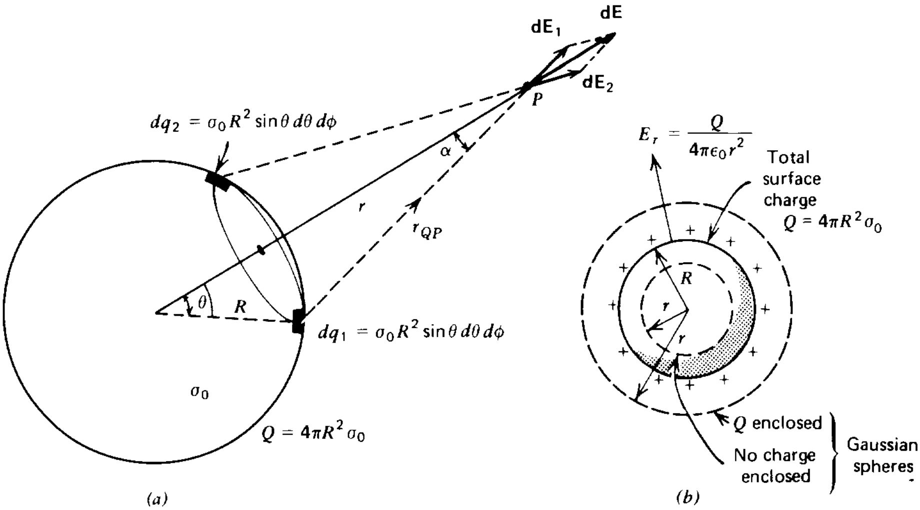


Figure 2-16 A sphere of radius R with uniformly distributed surface charge σ_0. (a) Symmetrically located charge elements show that the electric field is purely radial. (b) Gauss's law, applied to concentric spherical surfaces inside ($r<R$) and outside ($r>R$) the charged sphere, easily shows that the electric field within the sphere is zero and outside is the same as if all the charge $Q=4\pi R^2\sigma_0$ were concentrated as a point charge at the origin.

encloses all the charge $Q=\sigma_0 4\pi R^2$:

$$\oint_S \varepsilon_0 \mathbf{E}\cdot \mathbf{dS} = \varepsilon_0 E_r 4\pi r^2 = \begin{cases} \sigma_0 4\pi R^2 = Q, & r>R \\ 0, & r<R \end{cases} \tag{12}$$

so that the electric field is

$$E_r = \begin{cases} \dfrac{\sigma_0 R^2}{\varepsilon_0 r^2} = \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r>R \\ 0, & r<R \end{cases} \tag{13}$$

The integration in (12) amounts to just a multiplication of $\varepsilon_0 E_r$ and the surface area of the Gaussian sphere because on the sphere the electric field is constant and in the same direction as the normal $\mathbf{i}_r$. The electric field outside the sphere is the same as if all the surface charge were concentrated as a point charge at the origin.

The zero field solution for $r<R$ is what really proved Coulomb's law. After all, Coulomb's small spheres were not really point charges and his measurements did have small sources of errors. Perhaps the electric force only varied inversely with distance by some power close to two, $r^{-2+\delta}$, where δ is very small. However, only the inverse square law

gives a zero electric field within a uniformly surface charged sphere. This zero field result is true for any closed conducting body of arbitrary shape charged on its surface with no enclosed charge. Extremely precise measurements were made inside such conducting surface charged bodies and the electric field was always found to be zero. Such a closed conducting body is used for shielding so that a zero field environment can be isolated and is often called a Faraday cage, after Faraday's measurements of actually climbing into a closed hollow conducting body charged on its surface to verify the zero field results.

To appreciate the ease of solution using Gauss's law, let us redo the problem using the superposition integral of Section 2.3.2. From Figure 2-16*a* the incremental radial component of electric field due to a differential charge element is

$$dE_r = \frac{\sigma_0 R^2 \sin\theta\, d\theta\, d\phi}{4\pi\varepsilon_0 r_{QP}^2} \cos\alpha \tag{14}$$

From the law of cosines the angles and distances are related as

$$\begin{aligned} r_{QP}^2 &= r^2 + R^2 - 2rR\cos\theta \\ R^2 &= r^2 + r_{QP}^2 - 2rr_{QP}\cos\alpha \end{aligned} \tag{15}$$

so that α is related to θ as

$$\cos\alpha = \frac{r - R\cos\theta}{[r^2 + R^2 - 2rR\cos\theta]^{1/2}} \tag{16}$$

Then the superposition integral of Section 2.3.2 requires us to integrate (14) as

$$E_r = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\sigma_0 R^2 \sin\theta (r - R\cos\theta)\, d\theta\, d\phi}{4\pi\varepsilon_0 [r^2 + R^2 - 2rR\cos\theta]^{3/2}} \tag{17}$$

After performing the easy integration over ϕ that yields the factor of 2π, we introduce the change of variable:

$$\begin{aligned} u &= r^2 + R^2 - 2rR\cos\theta \\ du &= 2rR\sin\theta\, d\theta \end{aligned} \tag{18}$$

which allows us to rewrite the electric field integral as

$$\begin{aligned} E_r &= \int_{u=(r-R)^2}^{(r+R)^2} \frac{\sigma_0 R[u + r^2 - R^2]\, du}{8\varepsilon_0 r^2 u^{3/2}} \\ &= \frac{\sigma_0 R}{4\varepsilon_0 r^2} \left(u^{1/2} - \frac{(r^2 - R^2)}{u^{1/2}} \right) \Bigg|_{(r-R)^2}^{(r+R)^2} \\ &= \frac{\sigma_0 R}{4\varepsilon_0 r^2} \left[(r+R) - |r-R| - (r^2 - R^2)\left(\frac{1}{r+R} - \frac{1}{|r-R|} \right) \right] \end{aligned} \tag{19}$$

where we must be very careful to take the positive square root in evaluating the lower limit of the integral for $r<R$. Evaluating (19) for r greater and less than R gives us (13), but with a lot more effort.

(b) Volume Charge Distribution

If the sphere is uniformly charged throughout with density ρ_0, then the Gaussian surface in Figure 2-17a for $r>R$ still encloses the total charge $Q=\frac{4}{3}\pi R^3\rho_0$. However, now the smaller Gaussian surface with $r<R$ encloses a fraction of the total charge:

$$\oint_S \varepsilon_0\mathbf{E}\cdot\mathbf{dS}=\varepsilon_0 E_r 4\pi r^2=\begin{cases}\rho_0\frac{4}{3}\pi r^3=Q(r/R)^3, & r<R\\[2ex] \rho_0\frac{4}{3}\pi R^3=Q, & r>R\end{cases} \tag{20}$$

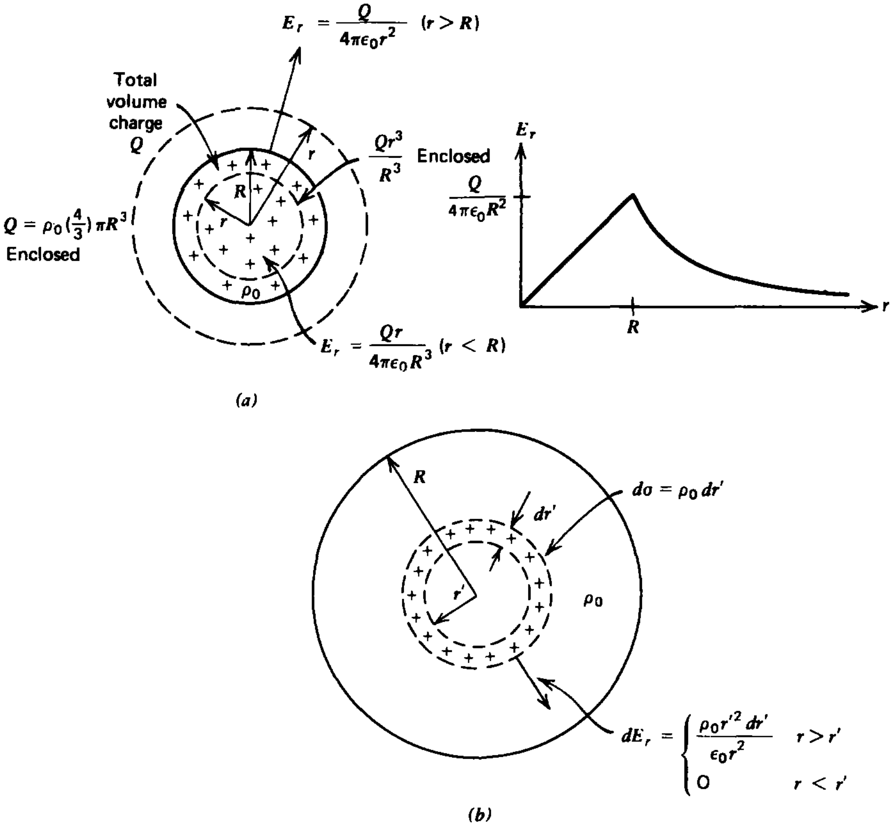


Figure 2-17 (a) Gaussian spheres for a uniformly charged sphere show that the electric field outside the sphere is again the same as if all the charge $Q=\frac{4}{3}\pi R^3\rho_0$ were concentrated as a point charge at $r=0$. (b) The solution is also obtained by summing the contributions from incremental spherical shells of surface charge.

so that the electric field is

$$E_r = \begin{cases} \dfrac{\rho_0 r}{3\varepsilon_0} = \dfrac{Qr}{4\pi\varepsilon_0 R^3}, & r<R \\ \dfrac{\rho_0 R^3}{3\varepsilon_0 r^2} = \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r>R \end{cases} \tag{21}$$

This result could also have been obtained using the results of (13) by breaking the spherical volume into incremental shells of radius r', thickness dr', carrying differential surface charge $d\sigma = \rho_0\, dr'$ as in Figure 2-17*b*. Then the contribution to the field is zero inside each shell but nonzero outside:

$$dE_r = \begin{cases} 0, & r<r' \\ \dfrac{\rho_0 r'^2\, dr'}{\varepsilon_0 r^2}, & r>r' \end{cases} \tag{22}$$

The total field outside the sphere is due to all the differential shells, while the field inside is due only to the enclosed shells:

$$E_r = \begin{cases} \displaystyle\int_0^r \frac{r'^2 \rho_0\, dr'}{\varepsilon_0 r^2} = \frac{\rho_0 r}{3\varepsilon_0} = \frac{Qr}{4\pi\varepsilon_0 R^3}, & r<R \\ \displaystyle\int_0^R \frac{r'^2 \rho_0\, dr'}{\varepsilon_0 r^2} = \frac{\rho_0 R^3}{3\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}, & r>R \end{cases} \tag{23}$$

which agrees with (21).

2-4-4 Cylindrical Symmetry

(a) Hollow Cylinder of Surface Charge

An infinitely long cylinder of radius a has a uniform distribution of surface charge σ_0, as shown in Figure 2-18*a*. The angle ϕ is measured from the line joining the field point P to the center of the cylinder. Each incremental line charge element $d\lambda = \sigma_0 a\, d\phi$ contributes to the electric field at P as given by the solution for an infinitely long line charge in Section 2.3.3. However, the symmetrically located element at $-\phi$ gives rise to equal magnitude field components that add radially as measured from the cylinder center but cancel in the ϕ direction.

Because of the symmetry, the electric field is purely radial so that we use Gauss's law with a concentric cylinder of radius r and height L, as in Figure 2-18*b* where L is arbitrary. There is no contribution to Gauss's law from the upper and lower surfaces because the electric field is purely tangential. Along the cylindrical wall at radius r, the electric field is constant and

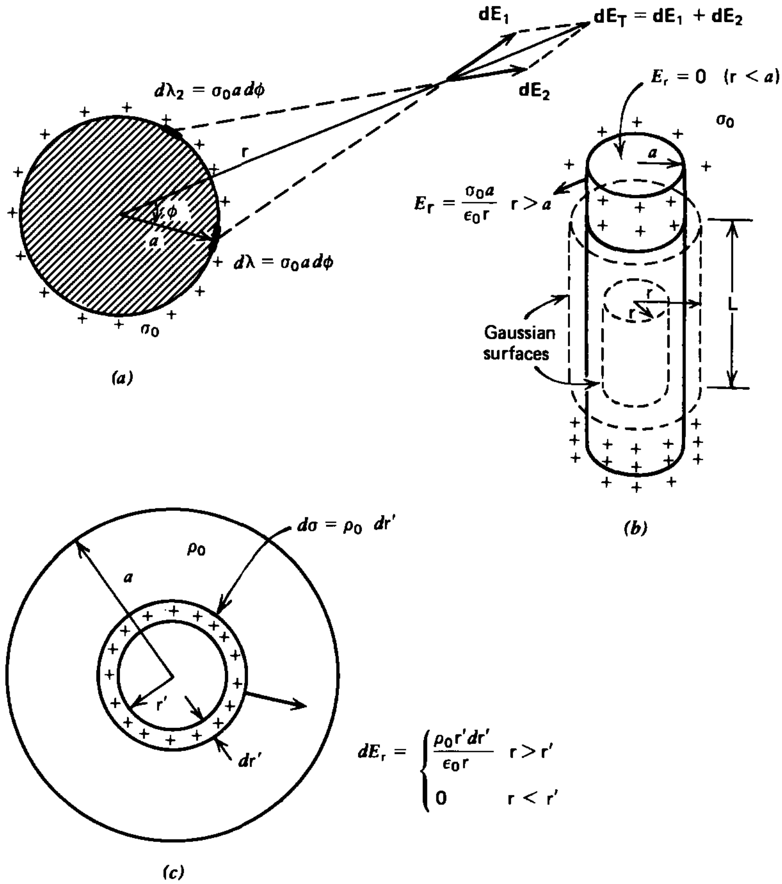


Figure 2-18 (a) Symmetrically located line charge elements on a cylinder with uniformly distributed surface charge show that the electric field is purely radial. (b) Gauss's law applied to concentric cylindrical surfaces shows that the field inside the surface charged cylinder is zero while outside it is the same as if all the charge per unit length $\sigma_0\, 2\pi a$ were concentrated at the origin as a line charge. (c) In addition to using the surfaces of (b) with Gauss's law for a cylinder of volume charge, we can also sum the contributions from incremental hollow cylinders of surface charge.

purely normal so that Gauss's law simply yields

$$\oint_S \varepsilon_0 \mathbf{E}\cdot d\mathbf{S} = \varepsilon_0 E_r 2\pi r L = \begin{cases} \sigma_0 2\pi a L, & r > a \\ 0 & r < a \end{cases} \tag{24}$$

where for $r < a$ no charge is enclosed, while for $r > a$ all the charge within a height L is enclosed. The electric field outside the cylinder is then the same as if all the charge per unit length $\lambda = \sigma_0 2\pi a$ were concentrated along the axis of the cylinder:

$$E_r = \begin{cases} \dfrac{\sigma_0 a}{\varepsilon_0 r} = \dfrac{\lambda}{2\pi\varepsilon_0 r} & r > a \\ 0, & r < a \end{cases} \tag{25}$$

Note in (24) that the arbitrary height L canceled out.

(b) Cylinder of Volume Charge

If the cylinder is uniformly charged with density ρ_0, both Gaussian surfaces in Figure 2-18*b* enclose charge

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \varepsilon_0 E_r 2\pi r L = \begin{cases} \rho_0 \pi a^2 L, & r > a \\ \rho_0 \pi r^2 L, & r < a \end{cases} \tag{26}$$

so that the electric field is

$$E_r = \begin{cases} \dfrac{\rho_0 a^2}{2\varepsilon_0 r} = \dfrac{\lambda}{2\pi\varepsilon_0 r}, & r > a \\[2ex] \dfrac{\rho_0 r}{2\varepsilon_0} = \dfrac{\lambda r}{2\pi\varepsilon_0 a^2} & r < a \end{cases} \tag{27}$$

where $\lambda = \rho_0 \pi a^2$ is the total charge per unit length on the cylinder.

Of course, this result could also have been obtained by integrating (25) for all differential cylindrical shells of radius r′ with thickness dr' carrying incremental surface charge $d\sigma = \rho_0\, dr'$, as in Figure 2-18*c*.

$$E_r = \begin{cases} \displaystyle\int_0^a \frac{\rho_0 r'}{\varepsilon_0 r}\, dr' = \frac{\rho_0 a^2}{2\varepsilon_0 r} = \frac{\lambda}{2\pi\varepsilon_0 r}, & r > a \\[2ex] \displaystyle\int_0^r \frac{\rho_0 r'}{\varepsilon_0 r}\, dr' = \frac{\rho_0 r}{2\varepsilon_0} = \frac{\lambda r}{2\pi\varepsilon_0 a^2}, & r < a \end{cases} \tag{28}$$

2-4-5 Gauss's Law and the Divergence Theorem

If a volume distribution of charge ρ is completely surrounded by a closed Gaussian surface S, Gauss's law of (11) is

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \int_V \rho\, dV \tag{29}$$

The left-hand side of (29) can be changed to a volume integral using the divergence theorem:

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \int_V \nabla \cdot (\varepsilon_0 \mathbf{E})\, dV = \int_V \rho\, dV \tag{30}$$

Since (30) must hold for any volume, the volume integrands in (30) must be equal, yielding the point form of Gauss's law:

$$\nabla \cdot (\varepsilon_0 \mathbf{E}) = \rho \qquad (31)$$

Since the permittivity of free space ε_0 is a constant, it can freely move outside the divergence operator.

2-4-6 Electric Field Discontinuity Across a Sheet of Surface Charge

In Section 2.3.4*a* we found that the electric field changes direction discontinuously on either side of a straight sheet of surface charge. We can be more general by applying the surface integral form of Gauss's law in (30) to the differential-sized pill-box surface shown in Figure 2-19 surrounding a small area dS of surface charge:

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \int_S \sigma\, dS \Rightarrow \varepsilon_0(E_{2n} - E_{1n})\, dS = \sigma\, dS \qquad (32)$$

where E_{2n} and E_{1n} are the perpendicular components of electric field on each side of the interface. Only the upper and lower surfaces of the pill-box contribute in (32) because the surface charge is assumed to have zero thickness so that the short cylindrical surface has zero area. We thus see that the surface charge density is proportional to the discontinuity in the normal component of electric field across the sheet:

$$\varepsilon_0(E_{2n} - E_{1n}) = \sigma \Rightarrow \mathbf{n} \cdot \varepsilon_0(\mathbf{E}_2 - \mathbf{E}_1) = \sigma \qquad (33)$$

where $\mathbf{n}$ is perpendicular to the interface directed from region 1 to region 2.

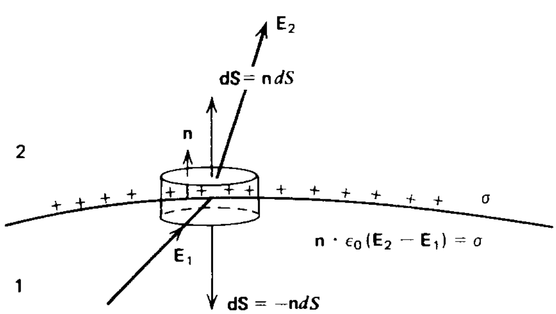


Figure 2-19 Gauss's law applied to a differential sized pill-box surface enclosing some surface charge shows that the normal component of $\varepsilon_0\mathbf{E}$ is discontinuous in the surface charge density.

2-5 THE ELECTRIC POTENTIAL

If we have two charges of opposite sign, work must be done to separate them in opposition to the attractive coulomb force. This work can be regained if the charges are allowed to come together. Similarly, if the charges have the same sign, work must be done to push them together; this work can be regained if the charges are allowed to separate. A charge gains energy when moved in a direction opposite to a force. This is called potential energy because the amount of energy depends on the position of the charge in a force field.

2-5-1 Work Required to Move a Point Charge

The work W required to move a test charge q_t along any path from the radial distance r_a to the distance r_b with a force that just overcomes the coulombic force from a point charge q, as shown in Figure 2-20, is

$$\begin{aligned} W &= -\int_{r_a}^{r_b} \mathbf{F}\cdot d\mathbf{l} \\ &= -\frac{qq_t}{4\pi\varepsilon_0}\int_{r_a}^{r_b} \frac{\mathbf{i}_r \cdot d\mathbf{l}}{r^2} \end{aligned} \tag{1}$$

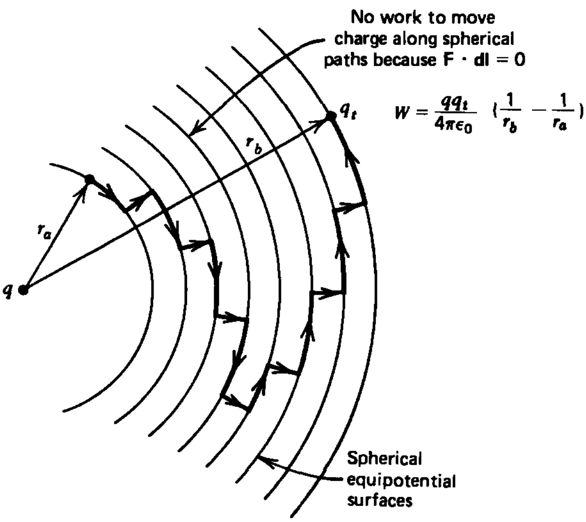


Figure 2-20 It takes no work to move a test charge q_t along the spherical surfaces perpendicular to the electric field due to a point charge q. Such surfaces are called equipotential surfaces.

The minus sign in front of the integral is necessary because the quantity W represents the work we must exert on the test charge in opposition to the coulombic force between charges. The dot product in (1) tells us that it takes no work to move the test charge perpendicular to the electric field, which in this case is along spheres of constant radius. Such surfaces are called equipotential surfaces. Nonzero work is necessary to move q to a different radius for which $\mathbf{dl} = dr\,\mathbf{i}_r$. Then, the work of (1) depends only on the starting and ending positions (r_a and r_b) of the path and not on the shape of the path itself:

$$\begin{aligned} W &= -\frac{qq_t}{4\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{dr}{r^2} \\ &= \frac{qq_t}{4\pi\varepsilon_0}\left(\frac{1}{r_b}-\frac{1}{r_a}\right) \end{aligned} \tag{2}$$

We can convince ourselves that the sign is correct by examining the case when r_b is bigger than r_a and the charges q and q_t are of opposite sign and so attract each other. To separate the charges further requires us to do work on q_t so that W is positive in (2). If q and q_t are the same sign, the repulsive coulomb force would tend to separate the charges further and perform work on q_t. For force equilibrium, we would have to exert a force opposite to the direction of motion so that W is negative.

If the path is closed so that we begin and end at the same point with $r_a = r_b$, the net work required for the motion is zero. If the charges are of the opposite sign, it requires positive work to separate them, but on the return, equal but opposite work is performed on us as the charges attract each other.

If there was a distribution of charges with net field $\mathbf{E}$, the work in moving the test charge against the total field $\mathbf{E}$ is just the sum of the works necessary to move the test charge against the field from each charge alone. Over a closed path this work remains zero:

$$W = \oint_L -q_t\mathbf{E}\cdot\mathbf{dl} = 0 \Rightarrow \oint_L \mathbf{E}\cdot\mathbf{dl} = 0 \tag{3}$$

which requires that the line integral of the electric field around the closed path also be zero.

2-5-2 The Electric Field and Stokes' Theorem

Using Stokes' theorem of Section 1.5.3, we can convert the line integral of the electric field to a surface integral of the curl of the electric field:

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{E})\cdot d\mathbf{S} \tag{4}$$

From Section 1.3.3, we remember that the gradient of a scalar function also has the property that its line integral around a closed path is zero. This means that the electric field can be determined from the gradient of a scalar function V called the potential having units of volts [kg-m^2-s^{-3}-A^{-1}]:

$$\mathbf{E} = -\nabla V \tag{5}$$

The minus sign is introduced by convention so that the electric field points in the direction of decreasing potential. From the properties of the gradient discussed in Section 1.3.1 we see that the electric field is always perpendicular to surfaces of constant potential.

By applying the right-hand side of (4) to an area of differential size or by simply taking the curl of (5) and using the vector identity of Section 1.5.4a that the curl of the gradient is zero, we reach the conclusion that the electric field has zero curl:

$$\nabla\times\mathbf{E} = 0 \tag{6}$$

2-5-3 The Potential and the Electric Field

The potential difference between the two points at r_a and r_b is the work per unit charge necessary to move from r_a to r_b:

$$\begin{aligned} V(r_b) - V(r_a) &= \frac{W}{q_t} \\ &= -\int_{r_a}^{r_b} \mathbf{E}\cdot d\mathbf{l} = +\int_{r_b}^{r_a} \mathbf{E}\cdot d\mathbf{l} \end{aligned} \tag{7}$$

Note that (3), (6), and (7) are the fields version of Kirchoff's circuit voltage law that the algebraic sum of voltage drops around a closed loop is zero.

The advantage to introducing the potential is that it is a scalar from which the electric field can be easily calculated. The electric field must be specified by its three components, while if the single potential function V is known, taking its negative gradient immediately yields the three field components. This is often a simpler task than solving for each field component separately. Note in (5) that adding a constant to the potential does not change the electric field, so the potential is only uniquely defined to within a constant. It is necessary to specify a reference zero potential that is often taken at infinity. In actual practice zero potential is often assigned to the earth's surface so that common usage calls the reference point "ground."

The potential due to a single point charge q is

$$V(r_b)-V(r_a)=-\int_{r_a}^{r_b}\frac{q\,dr}{4\pi\varepsilon_0 r^2}=\frac{q}{4\pi\varepsilon_0 r}\bigg|_{r_a}^{r_b}$$

$$=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_b}-\frac{1}{r_a}\right) \tag{8}$$

If we pick our reference zero potential at $r_a=\infty$, $V(r_a)=0$ so that $r_b=r$ is just the radial distance from the point charge. The scalar potential V is then interpreted as the work per unit charge necessary to bring a charge from infinity to some distance r from the point charge q:

$$V(r)=\frac{q}{4\pi\varepsilon_0 r} \tag{9}$$

The net potential from many point charges is obtained by the sum of the potentials from each charge alone. If there is a continuous distribution of charge, the summation becomes an integration over all the differential charge elements dq:

$$V=\int_{\text{all } q}\frac{dq}{4\pi\varepsilon_0 r_{QP}} \tag{10}$$

where the integration is a line integral for line charges, a surface integral for surface charges, and a volume integral for volume charges.

The electric field formula of Section 2.3.2 obtained by superposition of coulomb's law is easily re-obtained by taking the negative gradient of (10), recognizing that derivatives are to be taken with respect to field positions (x, y, z) while the integration is over source positions (x_Q, y_Q, z_Q). The del operator can thus be brought inside the integral and operates only on the quantity r_{QP}:

$$\mathbf{E}=-\nabla V=-\int_{\text{all } q}\frac{dq}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r_{QP}}\right)$$

$$=\int_{\text{all } q}\frac{dq}{4\pi\varepsilon_0 r_{QP}^2}\mathbf{i}_{QP} \tag{11}$$

where we use the results of Section 2.4.1b for the gradient of the reciprocal distance.

2-5-4 Finite Length Line Charge

To demonstrate the usefulness of the potential function, consider the uniform distribution of line charge λ_0 of finite length $2L$ centered on the z axis in Figure 2-21. Distinguishing between the position of the charge element $dq = \lambda_0\, dz'$ at z' and the field point at coordinate z, the distance between source and field point is

$$r_{QP} = [r^2 + (z - z')^2]^{1/2} \tag{12}$$

Substituting into (10) yields

$$\begin{aligned} V &= \int_{-L}^{L} \frac{\lambda_0\, dz'}{4\pi\varepsilon_0[r^2 + (z - z')^2]^{1/2}} \\ &= -\frac{\lambda_0}{4\pi\varepsilon_0} \ln\left(\frac{z - L + [r^2 + (z - L)^2]^{1/2}}{z + L + [r^2 + (z + L)^2]^{1/2}}\right) \\ &= -\frac{\lambda_0}{4\pi\varepsilon_0}\left(\sinh^{-1}\frac{z - L}{r} - \sinh^{-1}\frac{z + L}{r}\right) \end{aligned} \tag{13}$$

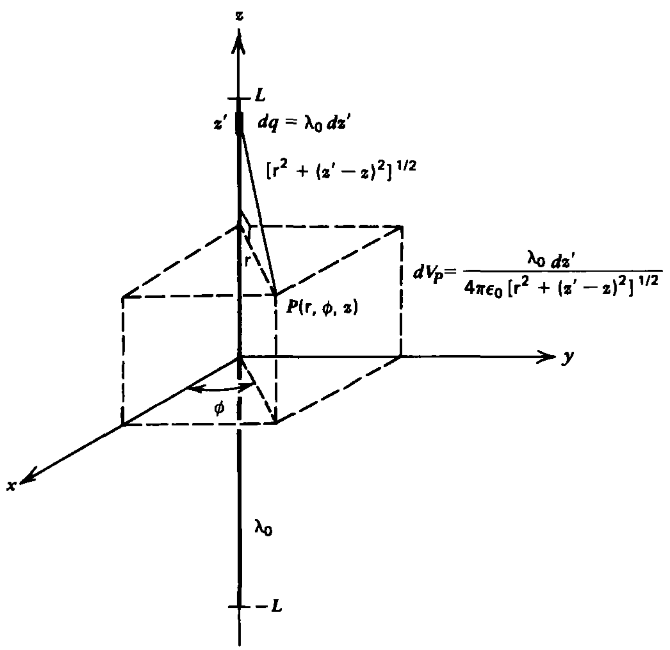


Figure 2-21 The potential from a finite length of line charge is obtained by adding the potentials due to each incremental line charge element.

The field components are obtained from (13) by taking the negative gradient of the potential:

$$E_z = -\frac{\partial V}{\partial z} = \frac{\lambda_0}{4\pi\varepsilon_0}\left(\frac{1}{[r^2+(z-L)^2]^{1/2}} - \frac{1}{[r^2+(z+L)^2]^{1/2}}\right)$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{\lambda_0 r}{4\pi\varepsilon_0}\left(\frac{1}{[r^2+(z-L)^2]^{1/2}[z-L+[r^2+(z-L)^2]^{1/2}]}\right.$$

$$\left. - \frac{1}{[r^2+(z+L)^2]^{1/2}[z+L+[r^2+(z+L)^2]^{1/2}]}\right)$$

$$= -\frac{\lambda_0}{4\pi\varepsilon_0 r}\left(\frac{z-L}{[r^2+(z-L)^2]^{1/2}} - \frac{z+L}{[r^2+(z+L)^2]^{1/2}}\right) \tag{14}$$

As L becomes large, the field and potential approaches that of an infinitely long line charge:

$$\lim_{L\to\infty}\begin{cases} E_z = 0 \\ E_r = \dfrac{\lambda_0}{2\pi\varepsilon_0 r} \\ V = -\dfrac{\lambda_0}{2\pi\varepsilon_0}(\ln r - \ln 2L) \end{cases} \tag{15}$$

The potential has a constant term that becomes infinite when L is infinite. This is because the zero potential reference of (10) is at infinity, but when the line charge is infinitely long the charge at infinity is nonzero. However, this infinite constant is of no concern because it offers no contribution to the electric field.

Far from the line charge the potential of (13) approaches that of a point charge $2\lambda_0 L$:

$$\lim_{r^2=r^2+z^2\gg L^2} V = \frac{\lambda_0(2L)}{4\pi\varepsilon_0 r} \tag{16}$$

Other interesting limits of (14) are

$$\lim_{z=0}\begin{cases} E_z = 0 \\ E_r = \dfrac{\lambda_0 L}{2\pi\varepsilon_0 r(r^2+L^2)^{1/2}} \end{cases}$$

$$\lim_{r=0}\begin{cases} E_z = \dfrac{\lambda_0}{4\pi\varepsilon_0}\left(\dfrac{1}{|z-L|} - \dfrac{1}{|z+L|}\right) = \begin{cases} \dfrac{\pm\lambda_0 L}{2\pi\varepsilon_0(z^2-L^2)}, & z>L \\ & z<-L \\ \dfrac{\lambda_0 z}{2\pi\varepsilon_0(L^2-z^2)}, & -L\le z\le L \end{cases} \\ E_r = 0 \end{cases} \tag{17}$$

2-5-5 Charged Spheres

(a) Surface Charge

A sphere of radius R supports a uniform distribution of surface charge σ_0 with total charge $Q=\sigma_0 4\pi R^2$, as shown in Figure 2-22*a*. Each incremental surface charge element contributes to the potential as

$$dV=\frac{\sigma_0 R^2 \sin\theta\, d\theta\, d\phi}{4\pi\varepsilon_0 r_{QP}} \tag{18}$$

where from the law of cosines

$$r_{QP}^2=R^2+r^2-2rR\cos\theta \tag{19}$$

so that the differential change in r_{QP} about the sphere is

$$2r_{QP}\, dr_{QP}=2rR\sin\theta\, d\theta \tag{20}$$

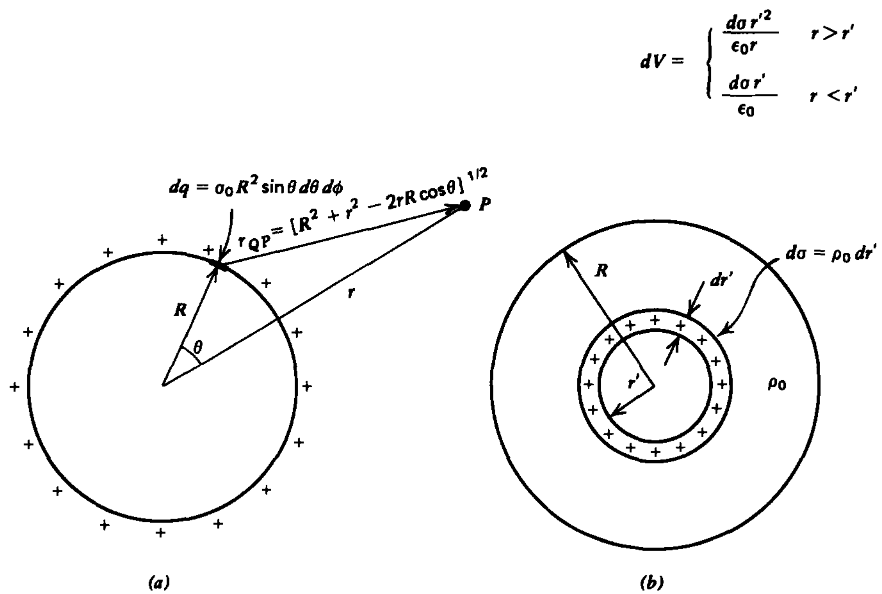


Figure 2-22 (*a*) A sphere of radius R supports a uniform distribution of surface charge σ_0. (*b*) The potential due to a uniformly volume charged sphere is found by summing the potentials due to differential sized shells.

Therefore, the total potential due to the whole charged sphere is

$$V=\int_{r_{QP}=|r-R|}^{r+R}\int_{\phi=0}^{2\pi}\frac{\sigma_0 R}{4\pi\varepsilon_0 r}\,dr_{QP}\,d\phi$$
$$=\frac{\sigma_0 R}{2\varepsilon_0 r}r_{QP}\bigg|_{|r-R|}^{r+R}$$
$$=\begin{cases}\dfrac{\sigma_0 R^2}{\varepsilon_0 r}=\dfrac{Q}{4\pi\varepsilon_0 r}, & r>R\\[2ex] \dfrac{\sigma_0 R}{\varepsilon_0}=\dfrac{Q}{4\pi\varepsilon_0 R}, & r<R\end{cases}\qquad(21)$$

Then, as found in Section 2.4.3*a* the electric field is

$$E_r=-\frac{\partial V}{\partial r}=\begin{cases}\dfrac{\sigma_0 R^2}{\varepsilon_0 r^2}=\dfrac{Q}{4\pi\varepsilon_0 r^2}, & r>R\\[2ex] 0 & r<R\end{cases}\qquad(22)$$

Outside the sphere, the potential of (21) is the same as if all the charge Q were concentrated at the origin as a point charge, while inside the sphere the potential is constant and equal to the surface potential.

(b) Volume Charge

If the sphere is uniformly charged with density ρ_0 and total charge $Q=\frac{4}{3}\pi R^3\rho_0$, the potential can be found by breaking the sphere into differential size shells of thickness dr' and incremental surface charge $d\sigma=\rho_0\,dr'$. Then, integrating (21) yields

$$V=\begin{cases}\displaystyle\int_0^R\frac{\rho_0 r'^2}{\varepsilon_0 r}dr'=\frac{\rho_0 R^3}{3\varepsilon_0 r}=\frac{Q}{4\pi\varepsilon_0 r}, & r>R\\[2ex] \displaystyle\int_0^r\frac{\rho_0 r'^2}{\varepsilon_0 r}dr'+\int_r^R\frac{\rho_0 r'}{\varepsilon_0}dr'=\frac{\rho_0}{2\varepsilon_0}\left(R^2-\frac{r^2}{3}\right)\\[2ex] \qquad=\dfrac{3Q}{8\pi\varepsilon_0 R^3}\left(R^2-\dfrac{r^2}{3}\right) & r<R\end{cases}\qquad(23)$$

where we realized from (21) that for $r<R$ the interior shells have a different potential contribution than exterior shells.

Then, the electric field again agrees with Section 2.4.3*b*:

$$E_r=-\frac{\partial V}{\partial r}=\begin{cases}\dfrac{\rho_0 R^3}{3\varepsilon_0 r^2}=\dfrac{Q}{4\pi\varepsilon_0 r^2}, & r>R\\[2ex] \dfrac{\rho_0 r}{3\varepsilon_0}=\dfrac{Qr}{4\pi\varepsilon_0 R^3}, & r<R\end{cases}\qquad(24)$$

(c) Two Spheres

Two conducting spheres with respective radii R_1 and R_2 have their centers a long distance D apart as shown in Figure 2-23. Different charges Q_1 and Q_2 are put on each sphere. Because $D \gg R_1 + R_2$, each sphere can be treated as isolated. The potential on each sphere is then

$$V_1 = \frac{Q_1}{4\pi\varepsilon_0 R_1}, \qquad V_2 = \frac{Q_2}{4\pi\varepsilon_0 R_2} \tag{25}$$

If a wire is connected between the spheres, they are forced to be at the same potential:

$$V_0 = \frac{q_1}{4\pi\varepsilon_0 R_1} = \frac{q_2}{4\pi\varepsilon_0 R_2} \tag{26}$$

causing a redistribution of charge. Since the total charge in the system must be conserved,

$$q_1 + q_2 = Q_1 + Q_2 \tag{27}$$

Eq. (26) requires that the charges on each sphere be

$$q_1 = \frac{R_1(Q_1+Q_2)}{R_1+R_2}, \qquad q_2 = \frac{R_2(Q_1+Q_2)}{R_1+R_2} \tag{28}$$

so that the system potential is

$$V_0 = \frac{Q_1+Q_2}{4\pi\varepsilon_0(R_1+R_2)} \tag{29}$$

Even though the smaller sphere carries less total charge, from (22) at $r=R$, where $E_r(R) = \sigma_0/\varepsilon_0$, we see that the surface electric field is stronger as the surface charge density is larger:

$$\begin{aligned} E_1(r=R_1) &= \frac{q_1}{4\pi\varepsilon_0 R_1^2} = \frac{Q_1+Q_2}{4\pi\varepsilon_0 R_1(R_1+R_2)} = \frac{V_0}{R_1} \\ E_2(r=R_2) &= \frac{q_2}{4\pi\varepsilon_0 R_2^2} = \frac{Q_1+Q_2}{4\pi\varepsilon_0 R_2(R_1+R_2)} = \frac{V_0}{R_2} \end{aligned} \tag{30}$$

For this reason, the electric field is always largest near corners and edges of equipotential surfaces, which is why

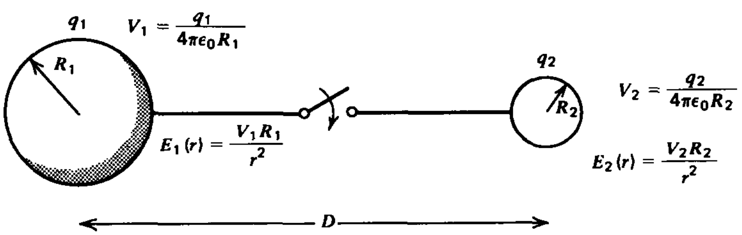


Figure 2-23 The charges on two spheres a long distance apart ($D \gg R_1 + R_2$) must redistribute themselves when connected by a wire so that each sphere is at the same potential. The surface electric field is then larger at the smaller sphere.

sharp points must be avoided in high-voltage equipment. When the electric field exceeds a critical amount E_b, called the breakdown strength, spark discharges occur as electrons are pulled out of the surrounding medium. Air has a breakdown strength of $E_b \approx 3\times 10^6$ volts/m. If the two spheres had the same radius of 1 cm (10^{-2} m), the breakdown strength is reached when $V_0 \approx 30{,}000$ volts. This corresponds to a total system charge of $Q_1 + Q_2 \approx 6.7\times 10^{-8}$ coul.

2-5-6 Poisson's and Laplace's Equations

The general governing equations for the free space electric field in integral and differential form are thus summarized as

$$\oint_S \varepsilon_0 \mathbf{E}\cdot \mathbf{dS} = \int_V \rho\, dV \Rightarrow \nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \tag{31}$$

$$\oint_L \mathbf{E}\cdot\mathbf{dl} = 0 \Rightarrow \nabla\times\mathbf{E} = 0 \Rightarrow \mathbf{E} = -\nabla V \tag{32}$$

The integral laws are particularly useful for geometries with great symmetry and with one-dimensional fields where the charge distribution is known. Often, the electrical potential of conducting surfaces are constrained by external sources so that the surface charge distributions, themselves sources of electric field are not directly known and are in part due to other charges by induction and conduction. Because of the coulombic force between charges, the charge distribution throughout space itself depends on the electric field and it is necessary to self-consistently solve for the equilibrium between the electric field and the charge distribution. These complications often make the integral laws difficult to use, and it becomes easier to use the differential form of the field equations. Using the last relation of (32) in Gauss's law of (31) yields a single equation relating the Laplacian of the potential to the charge density:

$$\nabla\cdot(\nabla V) = \nabla^2 V = -\rho/\varepsilon_0 \tag{33}$$

which is called Poisson's equation. In regions of zero charge ($\rho = 0$) this equation reduces to Laplace's equation, $\nabla^2 V = 0$.

2-6 THE METHOD OF IMAGES WITH LINE CHARGES AND CYLINDERS

2-6-1 Two Parallel Line Charges

The potential of an infinitely long line charge λ is given in Section 2.5.4 when the length of the line L is made very large. More directly, knowing the electric field of an infinitely long line charge from Section 2.3.3 allows us to obtain the potential by direct integration:

$$E_r = -\frac{\partial V}{\partial r} = \frac{\lambda}{2\pi\varepsilon_0 r} \Rightarrow V = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r}{r_0} \tag{1}$$

where r_0 is the arbitrary reference position of zero potential.

If we have two line charges of opposite polarity $\pm\lambda$ a distance $2a$ apart, we choose our origin halfway between, as in Figure 2-24*a*, so that the potential due to both charges is just the superposition of potentials of (1):

$$V = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{y^2+(x+a)^2}{y^2+(x-a)^2}\right)^{1/2} \tag{2}$$

where the reference potential point r_0 cancels out and we use Cartesian coordinates. Equipotential lines are then

$$\frac{y^2+(x+a)^2}{y^2+(x-a)^2} = e^{-4\pi\varepsilon_0 V/\lambda} = K_1 \tag{3}$$

where K_1 is a constant on an equipotential line. This relation is rewritten by completing the squares as

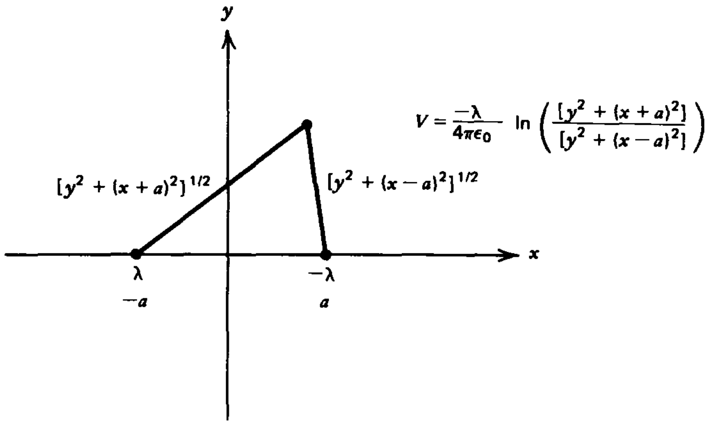
$$\left(x - \frac{a(1+K_1)}{K_1-1}\right)^2 + y^2 = \frac{4K_1a^2}{(1-K_1)^2} \tag{4}$$

which we recognize as circles of radius $r = 2a\sqrt{K_1}/|1-K_1|$ with centers at $y=0, x=a(1+K_1)/(K_1-1)$, as drawn by dashed lines in Figure 2-24*b*. The value of $K_1 = 1$ is a circle of infinite radius with center at $x = \pm\infty$ and thus represents the $x=0$ plane. For values of K_1 in the interval $0 \le K_1 \le 1$ the equipotential circles are in the left half-plane, while for $1 \le K_1 \le \infty$ the circles are in the right half-plane.

The electric field is found from (2) as

$$\mathbf{E} = -\nabla V = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{-4axy\mathbf{i}_y + 2a(y^2+a^2-x^2)\mathbf{i}_x}{[y^2+(x+a)^2][y^2+(x-a)^2]}\right) \tag{5}$$

One way to plot the electric field distribution graphically is by drawing lines that are everywhere tangent to the electric field, called field lines or lines of force. These lines are everywhere perpendicular to the equipotential surfaces and tell us the direction of the electric field. The magnitude is proportional to the density of lines. For a single line charge, the field lines emanate radially. The situation is more complicated for the two line charges of opposite polarity in Figure 2-24 with the field lines always starting on the positive charge and terminating on the negative charge.



(a)

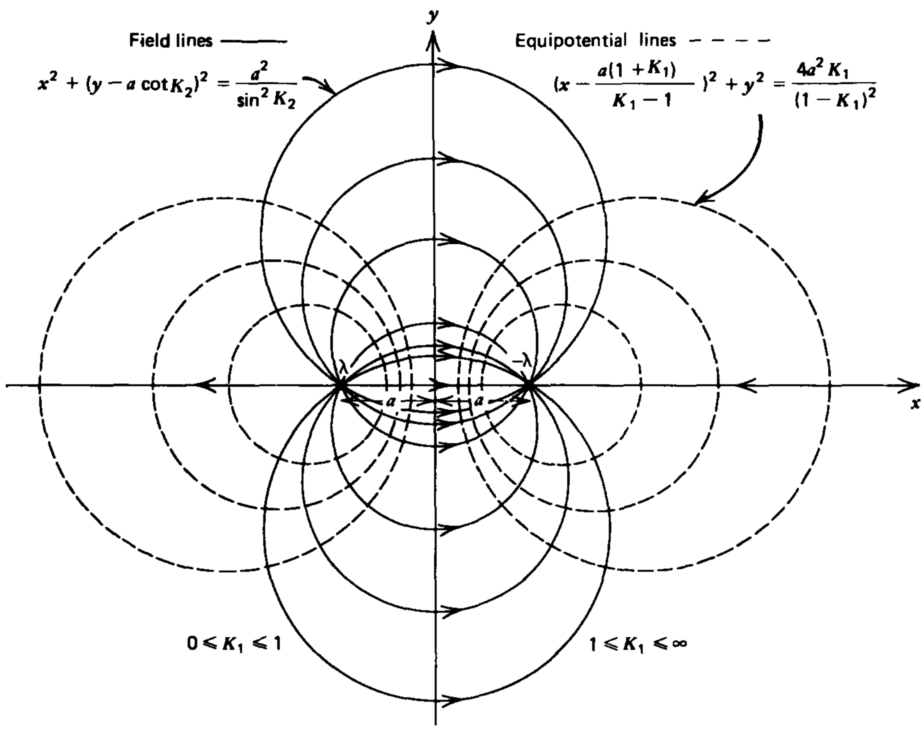


Figure 2-24 (*a*) Two parallel line charges of opposite polarity a distance $2a$ apart. (*b*) The equipotential (dashed) and field (solid) lines form a set of orthogonal circles.

For the field given by (5), the equation for the lines tangent to the electric field is

$$\frac{dy}{dx}=\frac{E_y}{E_x}=-\frac{2xy}{y^2+a^2-x^2}\Rightarrow\frac{d(x^2+y^2)}{a^2-(x^2+y^2)}+d(\ln y)=0 \qquad (6)$$

where the last equality is written this way so the expression can be directly integrated to

$$x^2+(y-a\cot K_2)^2=\frac{a^2}{\sin^2 K_2} \qquad (7)$$

where K_2 is a constant determined by specifying a single coordinate (x_0, y_0) along the field line of interest. The field lines are also circles of radius $a/\sin K_2$ with centers at $x=0, y=a\cot K_2$ as drawn by the solid lines in Figure 2-24*b*.

2-6-2 The Method of Images

(a) General properties

When a conductor is in the vicinity of some charge, a surface charge distribution is induced on the conductor in order to terminate the electric field, as the field within the equipotential surface is zero. This induced charge distribution itself then contributes to the external electric field subject to the boundary condition that the conductor is an equipotential surface so that the electric field terminates perpendicularly to the surface. In general, the solution is difficult to obtain because the surface charge distribution cannot be known until the field is known so that we can use the boundary condition of Section 2.4.6. However, the field solution cannot be found until the surface charge distribution is known.

However, for a few simple geometries, the field solution can be found by replacing the conducting surface by equivalent charges within the conducting body, called images, that guarantee that all boundary conditions are satisfied. Once the image charges are known, the problem is solved as if the conductor were not present but with a charge distribution composed of the original charges plus the image charges.

(b) Line Charge Near a Conducting Plane

The method of images can adapt a known solution to a new problem by replacing conducting bodies with an equivalent charge. For instance, we see in Figure 2-24*b* that the field lines are all perpendicular to the $x=0$ plane. If a conductor were placed along the $x=0$ plane with a single line charge λ at $x=-a$, the potential and electric field for $x<0$ is the same as given by (2) and (5).

A surface charge distribution is induced on the conducting plane in order to terminate the incident electric field as the field must be zero inside the conductor. This induced surface charge distribution itself then contributes to the external electric field for $x<0$ in exactly the same way as for a single image line charge $-\lambda$ at $x=+a$.

The force per unit length on the line charge λ is due only to the field from the image charge $-\lambda$;

$$\mathbf{f}=\lambda\mathbf{E}(-a,0)=\frac{\lambda^2}{2\pi\varepsilon_0(2a)}\mathbf{i}_x=\frac{\lambda^2}{4\pi\varepsilon_0 a}\mathbf{i}_x \tag{8}$$

From Section 2.4.6 we know that the surface charge distribution on the plane is given by the discontinuity in normal component of electric field:

$$\sigma(x=0)=-\varepsilon_0 E_x(x=0)=\frac{-\lambda a}{\pi(y^2+a^2)} \tag{9}$$

where we recognize that the field within the conductor is zero. The total charge per unit length on the plane is obtained by integrating (9) over the whole plane:

$$\begin{aligned}\lambda_T&=\int_{-\infty}^{+\infty}\sigma(x=0)\,dy\\&=-\frac{\lambda a}{\pi}\int_{-\infty}^{+\infty}\frac{dy}{y^2+a^2}\\&=-\frac{\lambda a}{\pi}\frac{1}{a}\tan^{-1}\frac{y}{a}\Big|_{-\infty}^{+\infty}\\&=-\lambda\end{aligned} \tag{10}$$

and just equals the image charge.

2-6-3 Line Charge and Cylinder

Because the equipotential surfaces of (4) are cylinders, the method of images also works with a line charge λ a distance D from the center of a conducting cylinder of radius R as in Figure 2-25. Then the radius R and distance a must fit (4) as

$$R=\frac{2a\sqrt{K_1}}{|1-K_1|},\qquad \pm a+\frac{a(1+K_1)}{K_1-1}=D \tag{11}$$

where the upper positive sign is used when the line charge is outside the cylinder, as in Figure 2-25*a*, while the lower negative sign is used when the line charge is within the cylinder, as in Figure 2-25*b*. Because the cylinder is chosen to be in the right half-plane, $1\le K_1\le\infty$, the unknown parameters K_1

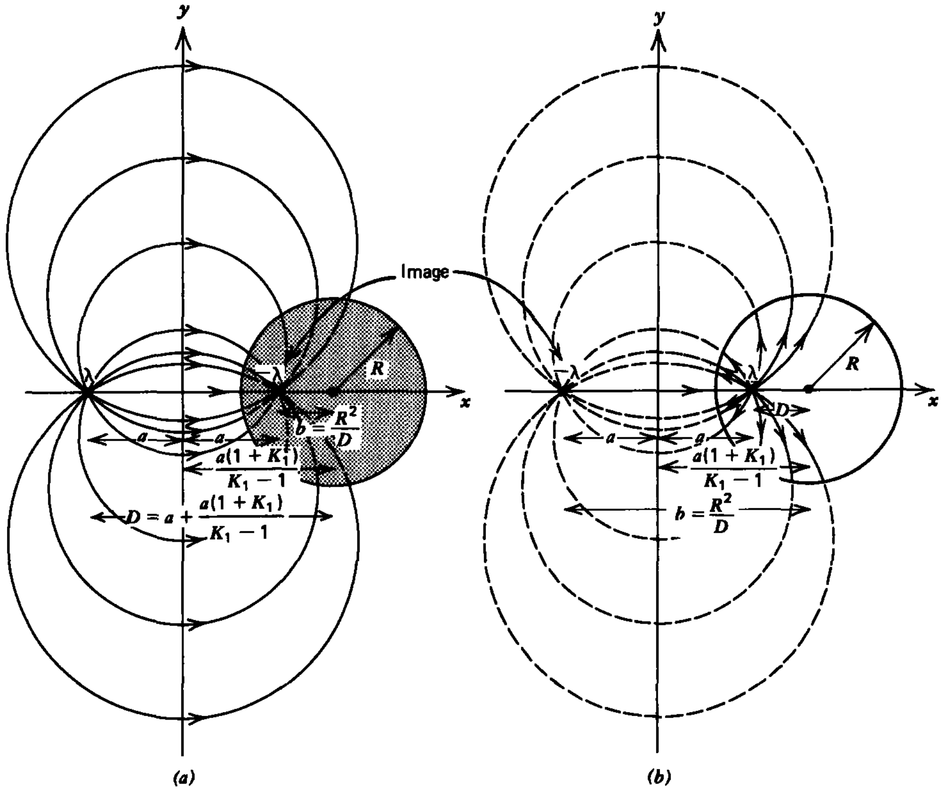


Figure 2-25 The electric field surrounding a line charge λ a distance D from the center of a conducting cylinder of radius R is the same as if the cylinder were replaced by an image charge $-\lambda$, a distance $b = R^2/D$ from the center. (a) Line charge outside cylinder. (b) Line charge inside cylinder.

and a are expressed in terms of the given values R and D from (11) as

$$K_1 = \left(\frac{D^2}{R^2}\right)^{\pm 1}, \qquad a = \pm\frac{D^2 - R^2}{2D} \tag{12}$$

For either case, the image line charge then lies a distance b from the center of the cylinder:

$$b = \frac{a(1+K_1)}{K_1 - 1} \mp a = \frac{R^2}{D} \tag{13}$$

being inside the cylinder when the inducing charge is outside ($R < D$), and vice versa, being outside the cylinder when the inducing charge is inside ($R > D$).

The force per unit length on the cylinder is then just due to the force on the image charge:

$$f_x = -\frac{\lambda^2}{2\pi\varepsilon_0(D-b)} = -\frac{\lambda^2 D}{2\pi\varepsilon_0(D^2-R^2)} \tag{14}$$

2-6-4 Two Wire Line

(a) Image Charges

We can continue to use the method of images for the case of two parallel equipotential cylinders of differing radii R_1 and R_2 having their centers a distance D apart as in Figure 2-26. We place a line charge λ a distance b_1 from the center of cylinder 1 and a line charge $-\lambda$ a distance b_2 from the center of cylinder 2, both line charges along the line joining the centers of the cylinders. We simultaneously treat the cases where the cylinders are adjacent, as in Figure 2-26*a*, or where the smaller cylinder is inside the larger one, as in Figure 2-26*b*.

The position of the image charges can be found using (13) realizing that the distance from each image charge to the center of the opposite cylinder is $D-b$ so that

$$b_1 = \frac{R_1^2}{D\mp b_2}, \qquad b_2 = \pm\frac{R_2^2}{D-b_1} \tag{15}$$

where the upper signs are used when the cylinders are adjacent and lower signs are used when the smaller cylinder is inside the larger one. We separate the two coupled equations in (15) into two quadratic equations in b_1 and b_2:

$$\begin{aligned} b_1^2 - \frac{[D^2-R_2^2+R_1^2]}{D}b_1 + R_1^2 = 0 \\ b_2^2 \mp \frac{[D^2-R_1^2+R_2^2]}{D}b_2 + R_2^2 = 0 \end{aligned} \tag{16}$$

with resulting solutions

$$\begin{aligned} b_2 &= \pm\frac{[D^2-R_1^2+R_2^2]}{2D} - \left[\left(\frac{D^2-R_1^2+R_2^2}{2D}\right)^2 - R_2^2\right]^{1/2} \\ b_1 &= \frac{[D^2+R_1^2-R_2^2]}{2D} \mp \left[\left(\frac{D^2+R_1^2-R_2^2}{2D}\right)^2 - R_1^2\right]^{1/2} \end{aligned} \tag{17}$$

We were careful to pick the roots that lay outside the region between cylinders. If the equal magnitude but opposite polarity image line charges are located at these positions, the cylindrical surfaces are at a constant potential.

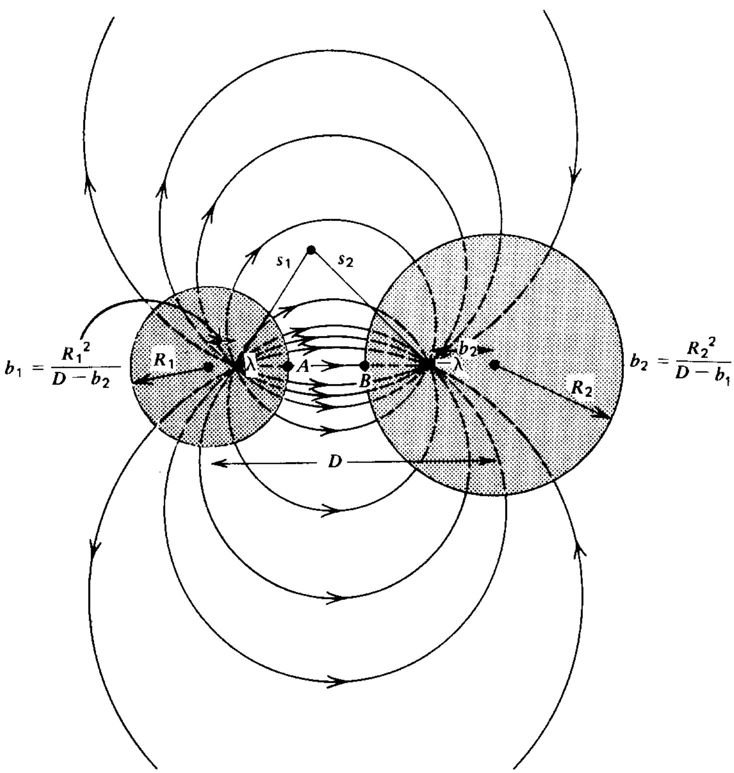


Figure 2-26 The solution for the electric field between two parallel conducting cylinders is found by replacing the cylinders by their image charges. The surface charge density is largest where the cylinder surfaces are closest together. This is called the proximity effect. (*a*) Adjacent cylinders. (*b*) Smaller cylinder inside the larger one.

(b) Force of Attraction

The attractive force per unit length on cylinder 1 is the force on the image charge λ due to the field from the opposite image charge $-\lambda$:

$$f_x = \frac{\lambda^2}{2\pi\varepsilon_0[\pm(D-b_1)-b_2]}$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0\left[\left(\frac{D^2-R_1^2+R_2^2}{2D}\right)^2-R_2^2\right]^{1/2}}$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0\left[\left(\frac{D^2-R_2^2+R_1^2}{2D}\right)^2-R_1^2\right]^{1/2}} \tag{18}$$

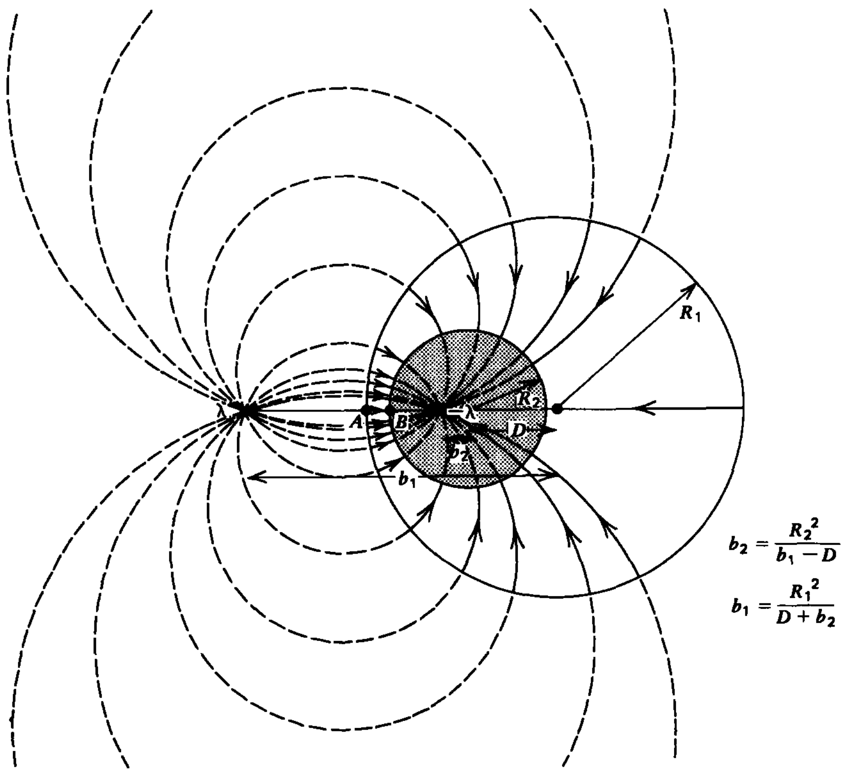


Fig. 2-26(b)

(c) Capacitance Per Unit Length

The potential of (2) in the region between the two cylinders depends on the distances from any point to the line charges:

$$V = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{s_1}{s_2} \tag{19}$$

To find the voltage difference between the cylinders we pick the most convenient points labeled A and B in Figure 2-26:

$$\begin{array}{ll} \qquad A & \qquad B \\ s_1 = \pm(R_1 - b_1) & s_1 = \pm(D - b_1 \mp R_2) \\ s_2 = \pm(D \mp b_2 - R_1) & s_2 = R_2 - b_2 \end{array} \tag{20}$$

although any two points on the surfaces could have been used. The voltage difference is then

$$V_1 - V_2 = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\pm\frac{(R_1 - b_1)(R_2 - b_2)}{(D \mp b_2 - R_1)(D - b_1 \mp R_2)}\right) \tag{21}$$

This expression can be greatly reduced using the relations

$$D \mp b_2 = \frac{R_1^2}{b_1}, \qquad D - b_1 = \pm\frac{R_2^2}{b_2} \tag{22}$$

to

$$\begin{aligned} V_1 - V_2 &= -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b_1 b_2}{R_1 R_2} \\ &= \frac{\lambda}{2\pi\varepsilon_0}\ln\left\{\pm\frac{[D^2 - R_1^2 - R_2^2]}{2R_1R_2}\right. \\ &\quad \left. + \left[\left(\frac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right)^2 - 1\right]^{1/2}\right\} \end{aligned} \tag{23}$$

The potential difference $V_1 - V_2$ is linearly related to the line charge λ through a factor that only depends on the geometry of the conductors. This factor is defined as the capacitance per unit length and is the ratio of charge per unit length to potential difference:

$$\begin{aligned} C = \frac{\lambda}{V_1 - V_2} &= \frac{2\pi\varepsilon_0}{\ln\left\{\pm\dfrac{[D^2 - R_1^2 - R_2^2]}{2R_1R_2} + \left[\left(\dfrac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right)^2 - 1\right]^{1/2}\right\}} \\ &= \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\pm\dfrac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right)} \end{aligned} \tag{24}$$

where we use the identity*

$$\ln[y + (y^2 - 1)^{1/2}] = \cosh^{-1} y \tag{25}$$

We can examine this result in various simple limits. Consider first the case for adjacent cylinders ($D > R_1 + R_2$).

1. If the distance D is much larger than the radii,

$$\lim_{D \gg (R_1 + R_2)} C \approx \frac{2\pi\varepsilon_0}{\ln[D^2/(R_1R_2)]} = \frac{2\pi\varepsilon_0}{\cosh^{-1}[D^2/(2R_1R_2)]} \tag{26}$$

2. The capacitance between a cylinder and an infinite plane can be obtained by letting one cylinder have infinite radius but keeping finite the closest distance $s =$

* $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$(e^x)^2 - 2ye^x + 1 = 0$

$e^x = y \pm (y^2 - 1)^{1/2}$

$x = \cosh^{-1} y = \ln[y \pm (y^2 - 1)^{1/2}]$

$D-R_1-R_2$ between cylinders. If we let R_1 become infinite, the capacitance becomes

$$\lim_{\substack{R_1\to\infty \\ D-R_1-R_2=s\ \text{(finite)}}} C = \frac{2\pi\varepsilon_0}{\ln\left\{\frac{s+R_2}{R_2}+\left[\left(\frac{s+R_2}{R_2}\right)^2-1\right]^{1/2}\right\}}$$
$$= \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{s+R_2}{R_2}\right)} \tag{27}$$

3. If the cylinders are identical so that $R_1=R_2\equiv R$, the capacitance per unit length reduces to

$$\lim_{R_1=R_2\equiv R} C = \frac{\pi\varepsilon_0}{\ln\left\{\frac{D}{2R}+\left[\left(\frac{D}{2R}\right)^2-1\right]^{1/2}\right\}} = \frac{\pi\varepsilon_0}{\cosh^{-1}\frac{D}{2R}} \tag{28}$$

4. When the cylinders are concentric so that $D=0$, the capacitance per unit length is

$$\lim_{D=0} C = \frac{2\pi\varepsilon_0}{\ln(R_1/R_2)} = \frac{2\pi\varepsilon_0}{\cosh^{-1}[(R_1^2+R_2^2)/(2R_1R_2)]} \tag{29}$$

2-7 THE METHOD OF IMAGES WITH POINT CHARGES AND SPHERES

2-7-1 Point Charge and a Grounded Sphere

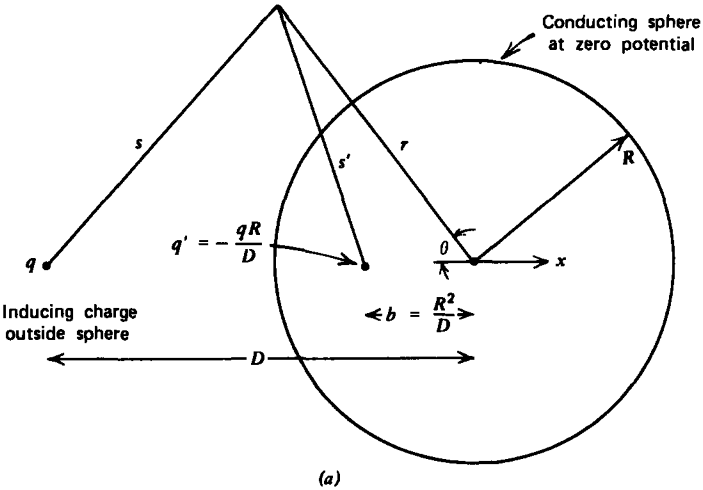
A point charge q is a distance D from the center of the conducting sphere of radius R at zero potential as shown in Figure 2-27a. We try to use the method of images by placing a single image charge q' a distance b from the sphere center along the line joining the center to the point charge q.

We need to find values of q' and b that satisfy the zero potential boundary condition at $r=R$. The potential at any point P outside the sphere is

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{s}+\frac{q'}{s'}\right) \tag{1}$$

where the distance from P to the point charges are obtained from the law of cosines:

$$\begin{aligned} s &= [r^2+D^2-2rD\cos\theta]^{1/2} \\ s' &= [b^2+r^2-2rb\cos\theta]^{1/2} \end{aligned} \tag{2}$$



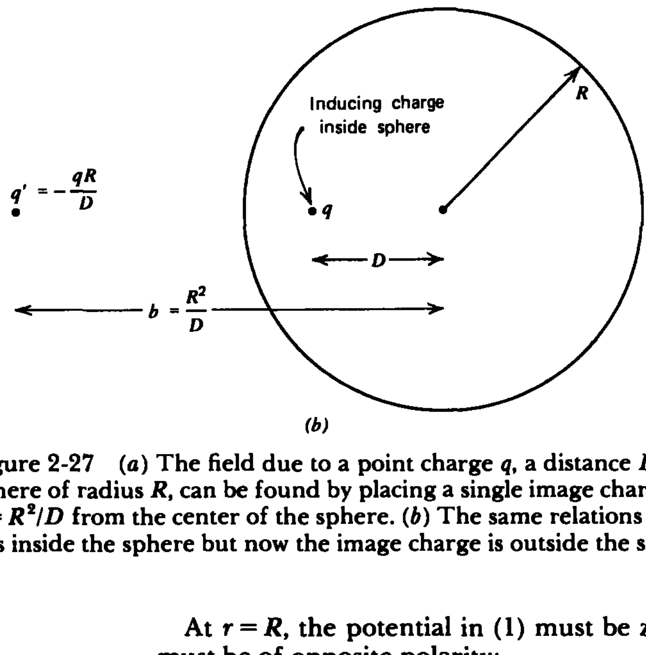


Figure 2-27 (*a*) The field due to a point charge q, a distance D outside a conducting sphere of radius R, can be found by placing a single image charge $-qR/D$ at a distance $b = R^2/D$ from the center of the sphere. (*b*) The same relations hold true if the charge q is inside the sphere but now the image charge is outside the sphere, since $D < R$.

At $r = R$, the potential in (1) must be zero so that q and q' must be of opposite polarity:

$$\left(\frac{q}{s}+\frac{q'}{s'}\right)\bigg|_{r=R}=0 \Rightarrow \left(\frac{q}{s}\right)^2=\left(\frac{q'}{s'}\right)^2\bigg|_{r=R} \tag{3}$$

where we square the equalities in (3) to remove the square roots when substituting (2),

$$q^2[b^2+R^2-2Rb\cos\theta]=q'^2[R^2+D^2-2RD\cos\theta] \tag{4}$$

Since (4) must be true for all values of θ, we obtain the following two equalities:

$$\begin{aligned} q^2(b^2+R^2) &= q'^2(R^2+D^2) \\ q^2 b &= q'^2 D \end{aligned} \tag{5}$$

Eliminating q and q' yields a quadratic equation in b:

$$b^2 - bD\left[1+\left(\frac{R}{D}\right)^2\right] + R^2 = 0 \tag{6}$$

with solution

$$\begin{aligned} b &= \frac{D}{2}\left[1+\left(\frac{R}{D}\right)^2\right] \pm \sqrt{\left\{\frac{D}{2}\left[1+\left(\frac{R}{D}\right)^2\right]\right\}^2 - R^2} \\ &= \frac{D}{2}\left[1+\left(\frac{R}{D}\right)^2\right] \pm \sqrt{\left\{\frac{D}{2}\left[1-\left(\frac{R}{D}\right)^2\right]\right\}^2} \\ &= \frac{D}{2}\left\{\left[1+\left(\frac{R}{D}\right)^2\right] \pm \left[1-\left(\frac{R}{D}\right)^2\right]\right\} \end{aligned} \tag{7}$$

We take the lower negative root so that the image charge is inside the sphere with value obtained from using (7) in (5):

$$b = \frac{R^2}{D}, \qquad q' = -q\frac{R}{D} \tag{8}$$

remembering from (3) that q and q' have opposite sign. We ignore the $b = D$ solution with $q' = -q$ since the image charge must always be outside the region of interest. If we allowed this solution, the net charge at the position of the inducing charge is zero, contrary to our statement that the net charge is q.

The image charge distance b obeys a similar relation as was found for line charges and cylinders in Section 2.6.3. Now, however, the image charge magnitude does not equal the magnitude of the inducing charge because not all the lines of force terminate on the sphere. Some of the field lines emanating from q go around the sphere and terminate at infinity.

The force on the grounded sphere is then just the force on the image charge $-q'$ due to the field from q:

$$f_x = \frac{qq'}{4\pi\varepsilon_0(D-b)^2} = -\frac{q^2R}{4\pi\varepsilon_0 D(D-b)^2} = -\frac{q^2RD}{4\pi\varepsilon_0(D^2-R^2)^2} \tag{9}$$

The electric field outside the sphere is found from (1) using (2) as

$$\mathbf{E}=-\nabla V=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{s^3}[(r-D\cos\theta)\mathbf{i}_r+D\sin\theta\mathbf{i}_\theta]\right.$$
$$\left.+\frac{q'}{s'^3}[(r-b\cos\theta)\mathbf{i}_r+b\sin\theta\mathbf{i}_\theta]\right) \tag{10}$$

On the sphere where $s'=(R/D)s$, the surface charge distribution is found from the discontinuity in normal electric field as given in Section 2.4.6:

$$\sigma(r=R)=\varepsilon_0 E_r(r=R)=-\frac{q(D^2-R^2)}{4\pi R[R^2+D^2-2RD\cos\theta]^{3/2}} \tag{11}$$

The total charge on the sphere

$$q_T=\int_0^\pi \sigma(r=R)2\pi R^2\sin\theta\,d\theta$$
$$=-\frac{q}{2}R(D^2-R^2)\int_0^\pi \frac{\sin\theta\,d\theta}{[R^2+D^2-2RD\cos\theta]^{3/2}} \tag{12}$$

can be evaluated by introducing the change of variable

$$u=R^2+D^2-2RD\cos\theta,\qquad du=2RD\sin\theta\,d\theta \tag{13}$$

so that (12) integrates to

$$q_T=-\frac{q(D^2-R^2)}{4D}\int_{(D-R)^2}^{(D+R)^2}\frac{du}{u^{3/2}}$$
$$=-\frac{q(D^2-R^2)}{4D}\left(-\frac{2}{u^{1/2}}\right)\Bigg|_{(D-R)^2}^{(D+R)^2}=-\frac{qR}{D} \tag{14}$$

which just equals the image charge q'.

If the point charge q is inside the grounded sphere, the image charge and its position are still given by (8), as illustrated in Figure 2-27b. Since $D<R$, the image charge is now outside the sphere.

2-7-2 Point Charge Near a Grounded Plane

If the point charge is a distance a from a grounded plane, as in Figure 2-28a, we consider the plane to be a sphere of infinite radius R so that $D=R+a$. In the limit as R becomes infinite, (8) becomes

$$\lim_{\substack{R\to\infty\\ D=R+a}} q'=-q,\qquad b=\frac{R}{(1+a/R)}=R-a \tag{15}$$

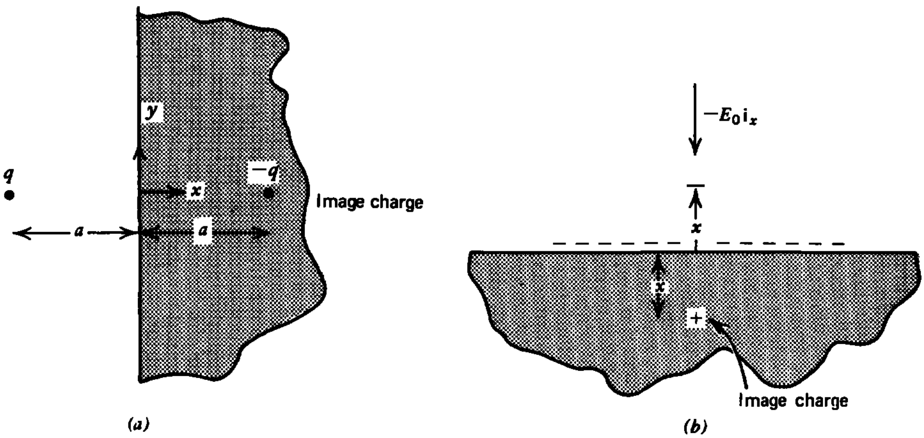


Figure 2-28 (*a*) A point charge q near a conducting plane has its image charge $-q$ symmetrically located behind the plane. (*b*) An applied uniform electric field causes a uniform surface charge distribution on the conducting plane. Any injected charge must overcome the restoring force due to its image in order to leave the electrode.

so that the image charge is of equal magnitude but opposite polarity and symmetrically located on the opposite side of the plane.

The potential at any point (x, y, z) outside the conductor is given in Cartesian coordinates as

$$V=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{[(x+a)^2+y^2+z^2]^{1/2}}-\frac{1}{[(x-a)^2+y^2+z^2]^{1/2}}\right) \tag{16}$$

with associated electric field

$$\mathbf{E}=-\nabla V=\frac{q}{4\pi\varepsilon_0}\left(\frac{(x+a)\mathbf{i}_x+y\mathbf{i}_y+z\mathbf{i}_z}{[(x+a)^2+y^2+z^2]^{3/2}}-\frac{(x-a)\mathbf{i}_x+y\mathbf{i}_y+z\mathbf{i}_z}{[(x-a)^2+y^2+z^2]^{3/2}}\right) \tag{17}$$

Note that as required the field is purely normal to the grounded plane

$$E_y(x=0)=0, \qquad E_z(x=0)=0 \tag{18}$$

The surface charge density on the conductor is given by the discontinuity of normal **E**:

$$\begin{aligned}\sigma(x=0)&=-\varepsilon_0E_x(x=0)\\&=-\frac{q}{4\pi}\frac{2a}{[y^2+z^2+a^2]^{3/2}}\\&=-\frac{qa}{2\pi(r^2+a^2)^{3/2}};\quad r^2=y^2+z^2\end{aligned} \tag{19}$$

where the minus sign arises because the surface normal points in the negative x direction.

The total charge on the conducting surface is obtained by integrating (19) over the whole surface:

$$\begin{aligned} q_T &= \int_0^\infty \sigma(x=0) 2\pi \mathrm{r}\, d\mathrm{r} \\ &= -qa \int_0^\infty \frac{\mathrm{r}\, d\mathrm{r}}{(\mathrm{r}^2+a^2)^{3/2}} \\ &= \frac{qa}{(\mathrm{r}^2+a^2)^{1/2}}\bigg|_0^\infty = -q \end{aligned} \tag{20}$$

As is always the case, the total charge on a conducting surface must equal the image charge.

The force on the conductor is then due only to the field from the image charge:

$$\mathbf{f} = -\frac{q^2}{16\pi\varepsilon_0 a^2}\mathbf{i}_x \tag{21}$$

This attractive force prevents charges from escaping from an electrode surface when an electric field is applied. Assume that an electric field $-E_0\mathbf{i}_x$ is applied perpendicular to the electrode shown in Figure (2-28*b*). A uniform negative surface charge distribution $\sigma = -\varepsilon_0 E_0$ as given in (2.4.6) arises to terminate the electric field as there is no electric field within the conductor. There is then an upwards Coulombic force on the surface charge, so why aren't the electrons pulled out of the electrode? Imagine an ejected charge $-q$ a distance x from the conductor. From (15) we know that an image charge $+q$ then appears at $-x$ which tends to pull the charge $-q$ back to the electrode with a force given by (21) with $a = x$ in opposition to the imposed field that tends to pull the charge away from the electrode. The total force on the charge $-q$ is then

$$f_x = qE_0 - \frac{q^2}{4\pi\varepsilon_0(2x)^2} \tag{22}$$

The force is zero at position x_c

$$f_x = 0 \Rightarrow x_c = \left[\frac{q}{16\pi\varepsilon_0 E_0}\right]^{1/2} \tag{23}$$

For an electron ($q = 1.6\times 10^{-19}$ coulombs) in a field of $E_0 = 10^6$ v/m, $x_c \approx 1.9\times 10^{-8}$ m. For smaller values of x the net force is negative tending to pull the charge back to the electrode. If the charge can be propelled past x_c by external forces, the imposed field will then carry the charge away from the electrode. If this external force is due to heating of the electrode, the process is called thermionic emission. High field emission even with a cold electrode occurs when the electric field E_0 becomes sufficiently large (on the order of 10^{10} v/m) that the coulombic force overcomes the quantum mechanical binding forces holding the electrons within the electrode.

2-7-3 Sphere With Constant Charge

If the point charge q is outside a conducting sphere $(D>R)$ that now carries a constant total charge Q_0, the induced charge is still $q'=-qR/D$. Since the total charge on the sphere is Q_0, we must find another image charge that keeps the sphere an equipotential surface and has value Q_0+qR/D. This other image charge must be placed at the center of the sphere, as in Figure 2-29*a*. The original charge q plus the image charge $q'=-qR/D$ puts the sphere at zero potential. The additional image charge at the center of the sphere raises the potential of the sphere to

$$V=\frac{Q_0+qR/D}{4\pi\varepsilon_0 R} \tag{24}$$

The force on the sphere is now due to the field from the point charge q acting on the two image charges:

$$\begin{aligned} f_x &= \frac{q}{4\pi\varepsilon_0}\left(-\frac{qR}{D(D-b)^2}+\frac{(Q_0+qR/D)}{D^2}\right) \\ &= \frac{q}{4\pi\varepsilon_0}\left(-\frac{qRD}{(D^2-R^2)^2}+\frac{(Q_0+qR/D)}{D^2}\right) \end{aligned} \tag{25}$$

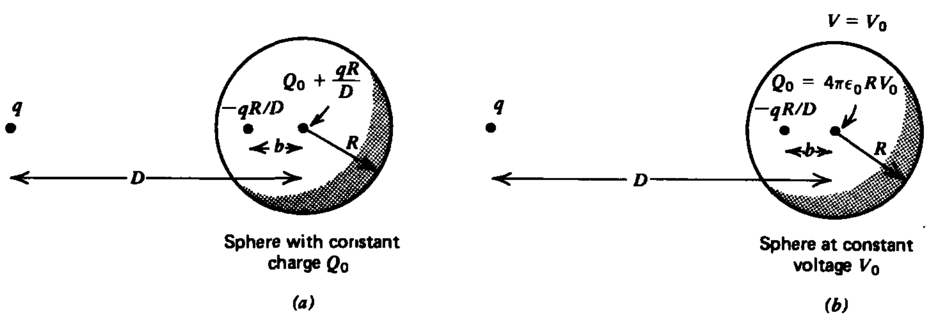


Figure 2-29 (*a*) If a conducting sphere carries a constant charge Q_0 or (*b*) is at a constant voltage V_0, an additional image charge is needed at the sphere center when a charge q is nearby.

2-7-4 Constant Voltage Sphere

If the sphere is kept at constant voltage V_0, the image charge $q' = -qR/D$ at distance $b = R^2/D$ from the sphere center still keeps the sphere at zero potential. To raise the potential of the sphere to V_0, another image charge,

$$Q_0 = 4\pi\varepsilon_0 R V_0 \tag{26}$$

must be placed at the sphere center, as in Figure 2-29*b*. The force on the sphere is then

$$f_x = \frac{q}{4\pi\varepsilon_0}\left(-\frac{qR}{D(D-b)^2} + \frac{Q_0}{D^2}\right) \tag{27}$$

PROBLEMS

Section 2.1

1. Faraday's "ice-pail" experiment is repeated with the following sequence of steps:

   (i) A ball with total charge Q is brought inside an insulated metal ice-pail without touching.
   (ii) The outside of the pail is momentarily connected to the ground and then disconnected so that once again the pail is insulated.
   (iii) Without touching the pail, the charged ball is removed.

   (a) Sketch the charge distribution on the inside and outside of the pail during each step.
   (b) What is the net charge on the pail after the charged ball is removed?

2. A sphere initially carrying a total charge Q is brought into momentary contact with an uncharged identical sphere.
   (a) How much charge is on each sphere?
   (b) This process is repeated for N identical initially uncharged spheres. How much charge is on each of the spheres including the original charged sphere?
   (c) What is the total charge in the system after the N contacts?

Section 2.2

3. The charge of an electron was first measured by Robert A. Millikan in 1909 by measuring the electric field necessary to levitate a small charged oil drop against its weight. The oil droplets were sprayed and became charged by frictional electrification.