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# SOLUTIONS TO CHAPTER 8

## 8.1 THE VECTOR POTENTIAL AND THE VECTOR POISSON EQUATION

**8.1.1** (a) Ampère's differential law inside the solenoid gives

$$\nabla \times \mathbf{H} = 0 \tag{1}$$

The continuity law of magnetic flux gives

$$\nabla \cdot \mu_o \mathbf{H} = 0 \tag{2}$$

Therefore, $\mathbf{H}$ is the gradient of a Laplacian potential. A uniform field is, of course, one special case of such a field. At the boundary, representing the coil as a surface current

$$\mathbf{K} = \mathbf{i}_\phi \frac{Ni}{d} \tag{3}$$

we have

$$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K} \tag{4}$$

where $\mathbf{n} = -\mathbf{i}_r$, the outside region is (b). Further we have

$$\mathbf{n} \cdot \mu_o(\mathbf{H}^a - \mathbf{H}^b) = 0 \tag{5}$$

(b) An axial $z$-directed uniform field inside, zero field outside, automatically satisfies (1), (2) and (5). On the surface we get from (4)

$$-\mathbf{i}_r \times H_z^a \mathbf{i}_z = \mathbf{i}_\phi \frac{Ni}{d}$$

and since

$$\mathbf{i}_r \times \mathbf{i}_z = -\mathbf{i}_\phi$$

$$H_z^a = \frac{Ni}{d}$$

(c) $\mathbf{A}$ is $\phi$ directed by symmetry. From the integral form of $\nabla \times \mathbf{A} = \mu_o \mathbf{H}$ we obtain

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \int_S \mu_o \mathbf{H} \cdot d\mathbf{a}$$

Taking a radius $r$ we find

$$2\pi r A_\phi(r) = \begin{cases} \pi r^2 \mu_o H_z^a & \text{for } r < a \\ \pi a^2 \mu_o H_z^a & \text{for } r > a \end{cases}$$

Therefore

$$A_\phi = \begin{cases} \frac{r}{2}\mu_o \frac{Ni}{d} & \text{for } r < a \\ \frac{a^2}{2r}\mu_o \frac{Ni}{d} & \text{for } r > a \end{cases}$$

**8.1.2** Using the coordinates defined in Fig. P4.4.3, superposition of line current vector potentials (8.1.16) gives

$$A_z = -\frac{\mu_o i}{2\pi} ln\Big[\frac{r_1 r_3}{r_2 r_4}\Big] \tag{1}$$

where

$$r_1 = \sqrt{x^2 + (y - \frac{d}{2})^2}; \quad r_3 = \sqrt{x^2 + (y + \frac{d}{2})^2}$$

$$r_2 = \sqrt{(x + \frac{d}{2})^2 + y^2}; \quad r_4 = \sqrt{(x - \frac{d}{2})^2 + y^2}$$

To linear terms in $(d/2)^2$, the numerator of this expression is

$$r_1 r_3 = \sqrt{(x^2 + y^2) + 2(x^2 - y^2)(d/2)^2} = r^2\sqrt{1 + \frac{2(x^2 - y^2)}{r^2}\Big(\frac{d}{2r}\Big)^2} \tag{2}$$

where

$$r = \sqrt{x^2 + y^2}$$

Similarly, the denominator is

$$r_2 r_4 = r^2\sqrt{1 - \frac{2(x^2 - y^2)}{r^2}\Big(\frac{d}{2r}\Big)^2} \tag{3}$$

Thus, to linear terms in $(d/2r)^2$, (1) becomes

$$\begin{aligned} A_z &\sim -\frac{\mu_o}{2\pi} i \frac{1}{2} ln\left[\frac{1 + 2\frac{(x^2 - y^2)}{r^2}\big(\frac{d}{2r}\big)^2}{1 - 2\frac{(x^2 - y^2)}{r^2}\big(\frac{d}{2r}\big)^2}\right] \\ &\simeq -\frac{\mu_o i}{4\pi} ln\Big[1 + \frac{4(x^2 - y^2)}{r^2}\Big(\frac{d}{2r}\Big)^2\Big] \end{aligned} \tag{4}$$

Observe that

$$\frac{x}{r} = \cos\phi; \quad \frac{y}{r} = \sin\phi; \quad \frac{x^2 - y^2}{r^2} = \cos^2\phi - \sin^2\phi = \cos 2\phi \tag{5}$$

and it follows that (4) is the given vector potential.

**8.1.3** We can take advantage of the analog of a solution of Poisson's equation for a two dimensional charge problem, and for a two dimensional current problem (because the structure is long, $l \gg w$ and $l \gg d$ we treat it as two dimensional). The analog charge problem is one with two charge sheets of opposite signs, producing a uniform field, and a potential $\Phi \propto y$. Thus (see Fig. S8.1.3)

$$A_z = A_o(y/d)$$

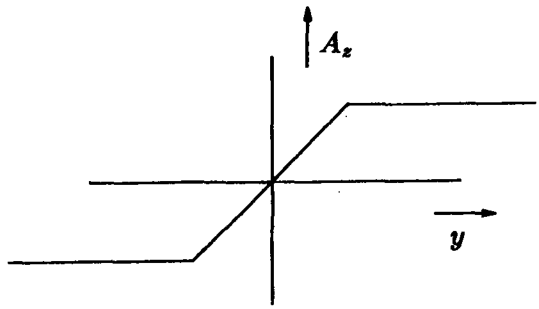


**Figure S8.1.3**

inside, $A_z =$ const outside, and we adjust $A$ so that we get the proper discontinuity of $\partial A_z/\partial y$ to account for the discontinuity of $H_x$

$$\mu_o H_x = \frac{\partial A_z}{\partial y} = \mu_o K = \mu_o \frac{Ni}{w}$$

Therefore

$$\frac{A_o}{d} = \mu_o \frac{Ni}{w}$$

and

$$\begin{aligned} A_z &= \mu_o \frac{Ni}{w} y \qquad \text{inside} \\ &= \pm \mu_o \frac{Ndi}{2w} \begin{cases} \text{top} \\ \text{bottom} \end{cases} \end{aligned}$$

## 8.2 THE BIOT-SAVART SUPERPOSITION INTEGRAL

**8.2.1** The Biot-Savart integral, (7), is evaluated recognizing that

$$(\mathbf{i}_\phi \times \mathbf{i}_{r'r})_z = \frac{r}{\sqrt{z^2 + r^2}} \tag{1}$$

Thus,

$$H_z = \frac{J_o}{4\pi}\int_0^{\Delta}\int_0^{2\pi}\int_b^a \frac{r}{\sqrt{z^2+r^2}}\frac{dzrd\phi dr}{(z^2+r^2)} \tag{2}$$

The integration on $z$ amounts to a multiplication by $\Delta$ while that on $\phi$ is simply a multiplication by $2\pi$. Thus, (2) becomes

$$H_z = \frac{\Delta J_o}{2}\int_b^a \frac{r^2 dr}{(z^2+r^2)^{3/2}} \tag{3}$$

and integration gives

$$H_z = \frac{\Delta J_o}{2}\left[\frac{-r}{\sqrt{r^2+z^2}} + ln(r+\sqrt{r^2+z^2})\right]_b^a \tag{4}$$

which is the given result.

**8.2.2** We use the Biot-Savart law,

$$\mathbf{H} = \frac{i}{4\pi}\oint \frac{d\mathbf{s}\times \mathbf{i}_{r'r}}{|\mathbf{r}-\mathbf{r}'|^2} \tag{1}$$

The field due to the turns within the width $Rd\theta$, and length $\sin\theta Rd\phi$ which produce a differential current $ids = K_o \sin^2\theta R^2 d\theta d\phi$, is (Note: $\mathbf{i}_{r'r} = -\mathbf{i}_{\mathbf{r}}$.)

$$dH_\theta = -\frac{K_o \sin^2\theta R^2 d\theta}{4\pi R^2} d\phi \tag{2}$$

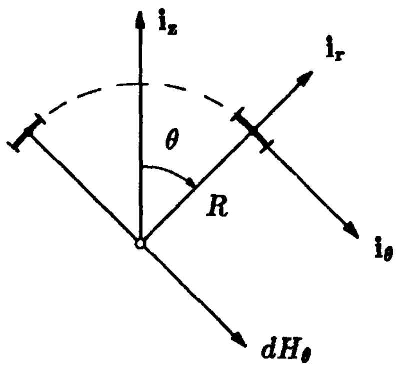


**Figure S8.2.2**

The field along the axis adds as one integrates around one turn, the components normal to the axis cancel

$$dH_z = -\sin\theta\int_0^{2\pi} dH_\theta = \frac{K_o d\theta}{2}\sin^3\theta \tag{3}$$

The total field is obtained by adding over all the currents

$$H_z = \frac{K_o}{2}\int_{\theta=0}^{\pi}\sin^3\theta d\theta = \frac{2K_o}{3} \tag{4}$$

**8.2.3** We replace $K_o \sin\theta$ by $K_o$ in Prob. 8.2.2. We can start with the integral in (4), where we drop one factor of $\sin\theta$. We get

$$H_z = \frac{K_o}{2}\int_{\theta=0}^{\pi} \sin^2\theta d\theta = \frac{\pi K_o}{4}$$

**8.2.4** We can use the result of 8.2.3 for a single shell. The total current distribution can be thought of as produced by a concentric set of shells. Each shell produces the field $\frac{\pi}{4}J_o dR$. Thus the net field at the center is

$$H_z = \frac{\pi}{4}J_o\int_0^a dR = \frac{\pi}{4}J_o R$$

**8.2.5** No matter where the vertices of the loop, (8.2.22) can be used to determine the field. However, the algebra is simplified by recognizing that the triangle not only has sides of equal length, $d$, but that the $z$ axis is at the center of the triangle. Thus, each leg makes the same contribution to the $z$ component of the field along the $z$ axis, and along that axis the $x$ and $y$ components cancel. To see that the sides are of length equal to that of the one paralleling the $x$ axis, note that the distance from the center of the leg to the vertex on the $y$ axis is $\sqrt{3/4}d$ and that based on the base $d/2$ and this distance, either of the other leg lengths must be of length $\sqrt{(d/2)^2 + (\sqrt{3/4}d)^2} = d$. Further, if the $z$ axis is at the center of the triangle, then the distance from the origin to either of the legs not parallel to the $x$ axis must be the distance to the parallel leg, $\sqrt{3/4}d/3$. Thus, we should have $2\sqrt{3/4}d/3 = \sqrt{(d/2)^2 + (\sqrt{3/4}d/3)^2}$, as indeed we do.

For the leg parallel to the $x$ axis,

$$\begin{aligned}
\mathbf{a} &= d\mathbf{i_x} \\
\mathbf{b} &= -\frac{d}{2}\mathbf{i_x} - \frac{1}{3}\sqrt{\frac{3}{4}}d\mathbf{i_y} - z\mathbf{i_z} \\
\mathbf{c} &= \frac{d}{2}\mathbf{i_x} - \frac{1}{3}\sqrt{\frac{3}{4}}d\mathbf{i_y} - z\mathbf{i_z}
\end{aligned} \tag{1}$$

Thus,

$$\begin{aligned}
\mathbf{c}\times\mathbf{a} &= -zd\mathbf{i_y} + \frac{1}{3}\sqrt{\frac{3}{4}}d^2\mathbf{i_z} \\
&\Rightarrow |\mathbf{c}\times\mathbf{a}| = d\Big(\frac{1}{12}d^2 + z^2\Big)^{1/2} \\
\mathbf{a}\cdot\mathbf{c} &= d^2/2 \quad |c| = (d^2/3 + z^2)^{1/2} \\
\mathbf{a}\cdot\mathbf{b} &= -d^2/2 \quad |b| = |c|
\end{aligned} \tag{2}$$

and the given result follows from (8.2.22), multiplied by 3 to reflect the contributions from the other two legs. This same result is obtained using either of the other legs. For example, using the back leg,

$$\begin{aligned} \mathbf{a} &= -\frac{d}{2}\mathbf{i_x} - \sqrt{\frac{3}{4}}d\mathbf{i_y} \\ \mathbf{b} &= \frac{2}{3}\sqrt{\frac{3}{4}}d\mathbf{i_y} - z\mathbf{i_z} \\ \mathbf{c} &= -\frac{d}{2}\mathbf{i_x} - \frac{1}{3}\sqrt{\frac{3}{4}}d\mathbf{i_y} - z\mathbf{i_z} \end{aligned} \tag{3}$$

**8.2.6** From (8.2.22)

$$\mathbf{H} = \frac{i}{4\pi}\frac{\mathbf{c}\times\mathbf{a}}{|\mathbf{c}\times\mathbf{a}|^2}\left(\frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{c}|} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)$$

we can find the $H$-field produced by a current stick! We look at one stick in the bottom layer of wires, extending from the position vector

$$\mathbf{b} = (x'-x)\mathbf{i_x} - \frac{d}{2}y\mathbf{i_y} - \frac{l}{2}\mathbf{i_z}$$

to the position vector

$$\mathbf{c} = (x'-x)\mathbf{i_x} - \frac{d}{2}y\mathbf{i_y} + \frac{l}{2}\mathbf{i_z}$$

with

$$\mathbf{a} \equiv \mathbf{c} - \mathbf{b} = l\mathbf{i_z}$$

Thus

$$\mathbf{c}\times\mathbf{a} = l[(x'-x)\mathbf{i_y} + \frac{d}{2}\mathbf{i_x}]$$

$$|\mathbf{c}\times\mathbf{a}|^2 = l^2[(x'-x)^2 + (d/2)^2]$$

$$|\mathbf{b}| = |\mathbf{c}| = \sqrt{(x'-x)^2 + (d/2)^2 + (l/2)^2}$$

$$\mathbf{a}\cdot\mathbf{c} = \frac{l^2}{2} \qquad \mathbf{a}\cdot\mathbf{b} = -\frac{l^2}{2}$$

Therefore, $\mathbf{H}$ due to one stick, carrying the differential current $\frac{Ni}{w}dx'$ is

$$\begin{aligned} \mathbf{H} &= \frac{Nidx'}{4\pi w}l\frac{[(x'-x)\mathbf{i_y} + \frac{d}{2}\mathbf{i_x}]}{l^2[(x'-x)^2 + (d/2)^2]}\frac{l^2}{\sqrt{(x'-x) + (d/2)^2 + (l/2)^2}} \\ &\simeq \frac{Nidx'}{2\pi w}\frac{[(x'-x)\mathbf{i_y} + \frac{d}{2}\mathbf{i_x}]}{(x'-x)^2 + (d/2)^2} \end{aligned}$$

in the limit when $l$ is very long compared with $d$ and $w$. This very same result could have been obtained from Ampère's law and symmetry considerations for an infinitely long wire (see Fig. S8.2.6)

$$\mathbf{H} = -\frac{Nidx'}{w}\frac{1}{2\pi r}\mathbf{i}_\phi = \frac{Nidx}{2\pi w}\frac{(x'-x)\mathbf{i}_y + \frac{d}{2}\mathbf{i}_x}{(x'-x)^2 + (d/2)^2}$$

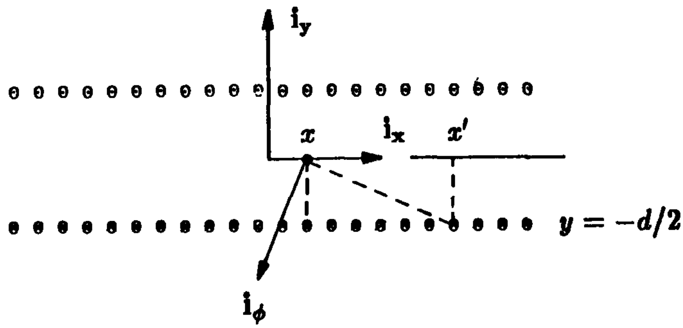


**Figure S8.2.6**

The total field is obtained by adding the contribution from a symmetrically located set of wires at the top, which cancels the $y$-component and doubles the $x$-component, and by integrating over the length of the coil

$$\begin{aligned} H_x &= \int_{-w/2}^{w/2} \frac{Nidx'}{2\pi w}\frac{d}{(x'-x)^2 + (d/2)^2} \\ &= \frac{Ni}{\pi w}\tan^{-1}\left[\frac{2}{d}\left(\frac{w}{2} - x\right)\right] + \tan^{-1}\left[\frac{2}{d}\left(\frac{w}{2} + x\right)\right] \end{aligned}$$

since

$$\int \frac{dx}{x^2 + (d/2)^2} = \frac{2}{d}\tan^{-1}(2x/d)$$

We may test this result by having $w \to \infty$. Then

$$H_x = \frac{Ni}{w}$$

QED as is correct for sheets of an infinite set.

**8.2.7** From (8.1.8) integrated over the cross-section of the stick,

$$\mathbf{A} = \frac{\mu_o}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')dv'}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_o}{4\pi}i\int_{\xi_b}^{\xi_c} \frac{\mathbf{a}}{|a|}\frac{d\xi}{|\mathbf{r}-\mathbf{r}'|} \tag{1}$$

where $\mathbf{a}/|a|$ is a unit vector in the direction of the stick and hence $[\mathbf{a}/|a|]d\xi$ is a differential length along the stick. Using the expression for $|\mathbf{r}-\mathbf{r}'|$ following (8.2.17), (1) is converted to an expression ready for integration.

$$\mathbf{A} = \frac{\mu_o}{4\pi}i\frac{\mathbf{a}}{|a|}\int_{\xi_b}^{\xi_c} \frac{d\xi}{\sqrt{\xi^2 + r_o^2}} \tag{2}$$

Integration gives

$$\mathbf{A} = \frac{\mu_o}{4\pi} i \frac{\mathbf{a}}{|a|} ln \left[ \frac{\xi_c + \sqrt{\xi_c^2 + r_o^2}}{\xi_b + \sqrt{\xi_b^2 + r_o^2}} \right] \tag{3}$$

Finally, substitution from (8.2.21) makes this expression the given result.

## 8.3 THE SCALAR MAGNETIC POTENTIAL

**8.3.1** From the Biot-Savart law

$$\mathbf{H} = \frac{i}{4\pi} \oint \frac{d\mathbf{s}' \times \mathbf{i}_{\mathbf{r}'\mathbf{r}}}{|\mathbf{r} - \mathbf{r}'|^2}$$

we find the axial field $H_z$

$$\begin{aligned} H_z &= \frac{i}{4\pi} \int_0^{2\pi} \frac{R d\phi'}{R^2/\sin^2\theta} \sin\theta = \frac{i}{4\pi} \frac{2\pi}{R} \sin^3\theta \\ &= \frac{i}{2R} \frac{R^3}{\sqrt{R^2 + z^2}^3} \end{aligned}$$

For large $z$,

$$H_z \simeq 2 \frac{i\pi R^2}{4\pi z^3}$$

which is consistent with the axial field of a dipole (see Fig. S8.3.1).

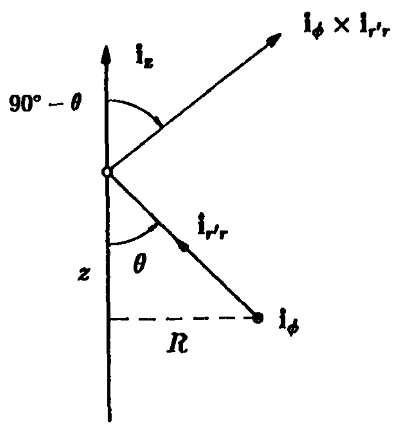


**Figure S8.3.1**

**8.3.2** The potential of one wire carrying the current $i$ in the $+z$ direction is

$$\psi = -\frac{i}{2\pi}\phi \tag{1}$$

The superposition gives (Fig. S8.3.2)

$$\Psi = -\frac{i}{2\pi}(\phi_1 - \phi_2) \tag{2}$$

The lines $\Psi = \text{const}$ are described by

$$\tan(\phi_1 - \phi_2) = \text{const} = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2} = \frac{\frac{y}{x-a} - \frac{y}{x+a}}{1 + \frac{y^2}{x^2-a^2}}$$

Therefore

$$2ya = \text{const}[x^2 - a^2 + y^2]$$

This is the equation of circles that go through the points $x = \pm a, y = 0$.

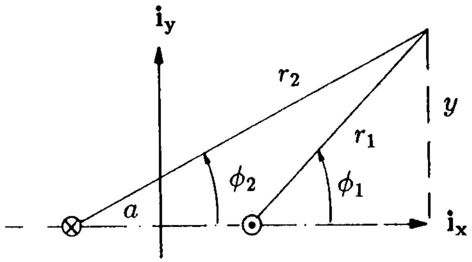


**Figure S8.3.2**

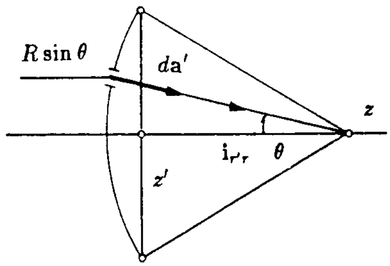


**Figure S8.3.3**

**8.3.3** Assume that the coil extends from $z = -l/2$ to $z = +l/2$. The potential of a loop is

$$\Psi(\mathbf{r}) = \frac{i}{4\pi}\Omega$$

$$\Omega = \int_0^\theta \frac{2\pi R\sin\theta R d\theta}{R^2} = 2\pi(1 - \cos\theta) = 2\pi\left(1 - \frac{(z - z')}{\sqrt{(z - z')^2 + R^2}}\right)$$

The individual differential loops of length $dz'$ carry currents $\frac{Ni}{l}dz'$. Therefore the total potential is

$$\begin{aligned}\Psi(x) &= \frac{Ni}{2l}\int_{z'=-l/2}^{z'=l/2} dz'\left(1 - \frac{(z - z')}{\sqrt{(z - z')^2 + R^2}}\right) \\ &= \frac{Ni}{2l}\left[l + \sqrt{(\frac{l}{2} - z)^2 + R^2} - \sqrt{(\frac{l}{2} + z)^2 + R^2}\right]\end{aligned}$$

We can check the result for a long coil, $l \to \infty$. Then

$$\sqrt{(\frac{l}{2} \mp z)^2 + R^2} = \frac{l}{2}\sqrt{(1 \mp \frac{2z}{l})^2 + (\frac{2R}{l})^2} \simeq \frac{l}{2}(1 \mp \frac{2z}{l})$$

and we find

$$\Psi(z) = \frac{Ni}{2l}[l - 2z]$$

giving a field

$$-\frac{\partial \Psi}{\partial z} = H_z = \frac{Ni}{l}$$

which is correct.

## 8.4 MAGNETOQUASISTATIC FIELDS IN THE PRESENCE OF PERFECT CONDUCTORS

√

**8.4.1** From (8.3.13),

$$\Psi(r \to 0) \to \frac{\pi i R^2}{4\pi}\frac{\cos\theta}{r^2} \tag{1}$$

and at $r = b$

$$\left.\frac{\partial \Psi}{\partial r}\right|_{r=b} = 0 \tag{2}$$

To meet these conditions, take the solutions to Laplace's equation

$$\Psi = \frac{\pi i R^2}{4\pi}\frac{\cos\theta}{r^2} + Ar\cos\theta \tag{3}$$

where the first automatically satisfies (1) and the coefficient $A$ of the second is determined by requiring (2). Thus,

$$\Psi = \frac{i\pi R^2}{4\pi}\left(\frac{1}{r^2} + \frac{2r}{b^3}\right)\cos\theta \tag{4}$$

The negative gradient of this magnetic potential is the given field intensity.

**8.4.2** The magnetic field of the dipole is given by (8.1.21)

$$\mathbf{H} = \frac{id}{2\pi r^2}(-\sin\phi\,\mathbf{i}_r + \cos\phi\,\mathbf{i}_\phi)$$

This corresponds to a scalar potential of

$$\Psi_d = -\frac{id}{2\pi r}\sin\phi$$

The conductor acts like a perfect conductor cancelling the normal component of $\mathbf{H}$, $H_r$. Thus we must have the total scalar potential

$$\Psi = -\frac{id}{2\pi a}\sin\phi\left(\frac{a}{r}+\frac{r}{a}\right)$$

with the field

$$\mathbf{H} = -\frac{id}{2\pi a^2}\left[\sin\phi\left(\frac{a^2}{r^2}-1\right)\mathbf{i}_r - \cos\phi\left(\frac{a^2}{r^2}+1\right)\mathbf{i}_\phi\right]$$

**8.4.3** (a) Far from the half-cylinder, the magnetic potential must become that of a uniform magnetic field in the $-z$ direction.

$$\mathbf{H}(z\to\pm\infty) = -H_o\mathbf{i}_z \Rightarrow \Psi = H_o z = H_o r\cos\phi \tag{1}$$

Thus, to satisfy the condition that there be no normal component of the field intensity at the surface of the half-cylinder, a second solution is added to this one having the same azimuthal dependence.

$$\Psi = H_o r\cos\phi + A\frac{\cos\phi}{r} \tag{2}$$

Adjusting $A$ so that

$$\frac{\partial\Psi}{\partial r}(r=R) = 0 \tag{3}$$

results in the given potential.

(b) As suggested, the field intensity shown in Fig. 8.4.2 satisfies the requirement of being tangential to the perfectly conducting surfaces. Note that the surface current density has the polarity required to exclude the magnetic field from the perfectly conducting regions, in accordance with (3).

**8.4.4** The potential $\Psi$ of the uniform field is

$$\Psi_o = H_o r\cos\theta$$

The sphere causes $H$ to be tangential. The normal component $H_r$ must be cancelled:

$$\Psi = H_o R\cos\theta\left[\frac{r}{R}+\frac{1}{2}\left(\frac{R}{r}\right)^2\right]$$

We obtain for the $H$ field

$$\mathbf{H} = -\nabla\Psi = -H_o\left\{\cos\theta\left[1-\left(\frac{R}{r}\right)^3\right]\mathbf{i}_r - \sin\theta\left[1+\frac{1}{2}\left(\frac{R}{r}\right)^3\right]\mathbf{i}_\phi\right\}$$

**8.4.5** (a) An image current is used to satisfy the condition that there be no normal component of the field intensity in the plane $y = 0$. Thus, the solution in region $y < 0$ is composed of a particular part due to the line current at $x = 0, y = -h$ and a homogeneous part equivalent to the field of a line current at $x = 0, y = h$ flowing in the opposite direction. To write these fields, first note that for a line current on the $z$ axis,

$$\begin{aligned} \mathbf{H} = \mathbf{i}_\phi\Big(\frac{i}{2\pi r}\Big) &= \frac{i}{2\pi}\Big(-\frac{\mathbf{i_x}\sin\phi + \mathbf{i_y}\cos\phi}{\sqrt{x^2+y^2}}\Big) \\ &= \frac{i}{2\pi}\Big(-\frac{\mathbf{i_x}y}{x^2+y^2} + \frac{\mathbf{i_y}x}{x^2+y^2}\Big) \end{aligned} \tag{1}$$

Translation of this field to represent first the actual and then in addition the image line current then results in the given field intensity.

(b) The surface current density that must exist at $y = 0$ if the region above sustains no field intensity is

$$\mathbf{K} = \mathbf{n}\times\mathbf{H} \Rightarrow K_z = H_x(y=0) \tag{2}$$

This is the given function.

**8.4.6** (a) The scalar potential produced by one segment of length $dx'$ is

$$d\Psi = -\frac{K_o dx'}{2\pi}\tan^{-1}\Big(\frac{y}{x-x'}\Big) = \frac{K_o dx'}{2\pi}\cot^{-1}\Big(\frac{x'-x}{y}\Big) \tag{1}$$

The integral over the strip is

$$\begin{aligned} \Psi = \int_{x'=b}^{x'=a} d\Psi = \frac{K_o}{2\pi}\Big\{&(a-x)\cot^{-1}\Big(\frac{a-x}{y}\Big) \\ &-(b-x)\cot^{-1}\Big(\frac{b-x}{y}\Big) + \frac{y}{2}\log\Big[1+\Big(\frac{a-x}{y}\Big)^2\Big] \\ &-\frac{y}{2}\log\Big[1+\Big(\frac{b-x}{y}\Big)^2\Big]\Big\} \end{aligned} \tag{2}$$

where the integral is taken from: B. O. Pearce, R. M. Foster, **A Short Table of Integrals**, 4th Ed., Ginn and Co. (1956). To this potential must be added an image potential that causes $\partial\Psi/\partial x = 0$ at $x = 0$. This is achieved by adding to (2) a potential with the replacements

$$K_o \rightarrow -K_o \quad a \rightarrow -a, \quad b \rightarrow -b$$

(b) The field $\mathbf{H} = -\nabla\Psi$ and thus from (2)

$$\begin{aligned} H_x = -\frac{K_o}{2\pi}\Big[ &- \cot^{-1}\big(\frac{a-x}{y}\big) + \cot^{-1}\big(\frac{b-x}{y}\big) \\ &+ \frac{(a-x)/y}{1+\big(\frac{a-x}{y}\big)^2} - \frac{(b-x)/y}{1+\big(\frac{b-x}{y}\big)^2} \\ &- \frac{(a-x)/y}{1+\big(\frac{a-x}{y}\big)^2} + \frac{(b-x)/y}{1+\big(\frac{b-x}{y}\big)^2}\Big] \\ = -\frac{K_o}{2\pi}\Big[ &- \cot^{-1}\big(\frac{a-x}{y}\big) + \cot^{-1}\big(\frac{b-x}{y}\big)\Big] \\ = \frac{K_o}{2\pi}\Big[ &\tan^{-1}\big(\frac{y}{a-x}\big) - \tan^{-1}\big(\frac{y}{b-x}\big)\Big] \end{aligned}$$

To this field we add

$$H_x = \frac{K_o}{2\pi}\Big[\tan^{-1}\big(\frac{y}{x+a}\big) - \tan^{-1}\big(\frac{y}{b+x}\big)\Big]$$

## 8.5 PIECE-WISE MAGNETIC FIELDS

**8.5.1** (a) The surface current density is

$$\mathbf{K} = \frac{N}{2R} i \sin\phi \mathbf{i_z} \tag{1}$$

so that the continuity conditions at the cylinder surface where $r = R$ are

$$H_\phi^a - H_\phi^b = \frac{Ni}{2R}\sin\phi \tag{2}$$

$$\mu_o H_r^a - \mu_o H_r^b = 0 \tag{3}$$

Looking forward to satisfying (2), the $\phi$ dependence of the scalar potential is taken to be $\cos\phi$. Thus, the appropriate solutions to Laplace's equation are

$$\Psi^a = A\frac{\cos\phi}{r} \tag{4}$$

$$\Psi^b = Cr\cos\phi \tag{5}$$

so that the field intensities are

$$\mathbf{H}^a = A\big(\frac{\cos\phi}{r^2}\mathbf{i_r} + \frac{\sin\phi}{r^2}\mathbf{i_\phi}\big) \tag{6}$$

$$\mathbf{H}^b = -C(\cos\phi\mathbf{i}_r - \sin\phi\mathbf{i}_\phi) \tag{7}$$

Substitution of these fields into (2) and (3) then gives

$$\frac{A}{R^2} - C = \frac{Ni}{2R} \tag{8}$$

$$\frac{A}{R^2} + C = 0 \tag{9}$$

from which it follows that

$$A = \frac{RNi}{4}; \quad C = -\frac{Ni}{4R} \tag{10}$$

Substitution of these coefficients into (4)-(7) results in the given expressions for the magnetic scalar potential and field intensity.

(b) Because the flux density is uniform over the interior of the cylinder, the flux linked by a turn in the plane $x = x' = R\cos\phi'$ is

$$\Phi_\lambda = \mu_o H_x 2R\sin\phi' = \mu_o \frac{Ni}{4R} 2R\sin\phi' \tag{11}$$

Thus, the total flux is

$$\begin{aligned}\lambda &= i\int_0^\pi \frac{\mu_o N}{2}\sin\phi'\left(\frac{N}{2R}\right)\sin\phi' R d\phi' \\ &= i\frac{\mu_o N^2}{4}\int_0^\pi \sin^2\phi' d\phi' = \left[\frac{\mu_o N^2\pi}{8}\right]i\end{aligned} \tag{12}$$

and thus the inductance is identified as that given.

**8.5.2** (a) At $r = b$, there is a jump in tangential $\mathbf{H}$:

$$\mathbf{n}\times(\mathbf{H}^{(a)} - \mathbf{H}^{(b)}) = \mathbf{K} \tag{1}$$

with region (a) outside, (b) inside the cylinder carrying the windings. Thus $\mathbf{n} = \mathbf{i}_r$ and at $r = b$

$$-\frac{1}{b}\frac{\partial\Psi^{(a)}}{\partial\phi} + \frac{1}{b}\frac{\partial\Psi^{(b)}}{\partial\phi} = K_z(\phi) \tag{2}$$

Further the normal component of $\Psi$ must be continous at $r = b$.

$$-\frac{\partial\Psi^{(a)}}{\partial r} + \frac{\partial\Psi^{(b)}}{\partial r} = 0 \tag{3}$$

At $r = a$, the normal component of $\mathbf{H}$ has to vanish:

$$\left.\frac{\partial\Psi}{\partial r}\right|_{r=a} = 0 \tag{4}$$

(b) We have a "square-wave" for the current distribution. Therefore, we need an infinite sum of terms for $\Psi$:

$$\Psi^{(b)} = \sum_{n=1}^{\infty} A_n (r/b)^n \cos n(\phi - \phi_o); \quad 0 < r < b$$

$$\Psi^{(a)} = \sum_{n=1}^{\infty} B_n (a/r)^n \cos n(\phi - \phi_o); \quad b < r < a \tag{5}$$

$$+ \sum_{n=0}^{\infty} C_n (r/a)^n \cos n(\phi - \phi_o)$$

We picked the normalization of the coefficients so that the boundary conditions are most simply stated. From (4) we have

$$-nB_n \frac{a^n}{a^{n+1}} + nC_n \frac{a^{n-1}}{a^n} = 0$$

and thus

$$C_n = B_n \tag{6}$$

From (3) we have

$$nB_n \frac{a^n}{b^{n+1}} - nC_n \frac{b^{n-1}}{a^n} + nA_n \frac{1}{b} = 0 \tag{7}$$

and using (6)

$$A_n = C_n \left[ (b/a)^n - (a/b)^n \right] \tag{8}$$

From (2) we obtain:

$$n\left[B_n (a/b)^n + C_n (b/a)^n\right] \sin n(\phi - \phi_o) - nA_n \sin n(\phi - \phi_o) = bK_z(\phi) \tag{9}$$

The expansion of the square wave $K_z(\phi)$ is

$$K_z(\phi) = K_o \sum_{\substack{n \\ n-\text{odd}}} \frac{4}{n\pi} \sin n(\phi - \phi_o) \tag{10}$$

Thus, using (6), (8) and (10) in (9) we obtain, for $n$ odd:

$$nC_n\left[(a/b)^n + (b/a)^n\right] - nC_n\left[(b/a)^n - (a/b)^n\right] = \frac{4}{n\pi} K_o$$

and $C_n = 0$ for $n$ even. Thus

$$C_n = \frac{2b}{n^2 \pi} K_o (b/a)^n = B_n \tag{11}$$

and

$$A_n = \frac{2b}{n^2\pi} K_o\big[(b/a)^{2n} - 1\big] \tag{12}$$

for $n$ odd, zero for $n$ even. We should check a few limits right away. When $a \to \infty$, we get (for $n$ odd)

$$A_n = -\frac{2b}{n^2\pi} K_o$$

and

$$\Psi^{(b)} = -\sum_{n-\text{odd}} \frac{2b}{n^2\pi} K_o (r/b)^n \cos n(\phi - \phi_o)$$

$$\Psi^{(a)} = \sum_{n-\text{odd}} \frac{2b}{n^2\pi} K_o (b/r)^n \cos n(\phi - \phi_o)$$

which gives the field due to the cylinder alone. For $a \to b$, we get $A_n = 0$

$$\Psi^{(a)} \simeq \sum_{n-\text{odd}} \frac{2b}{n^2\pi} K_o\big[(a/r)^n + (r/a)^n\big] \cos n(\phi - \phi_o)$$

$$\simeq \sum_{n-\text{odd}} \frac{4b}{n^2\pi} K_o \cos n(\phi - \phi_o)$$

There is a $\phi$ directed field in the region between the coil and the shield of magnitude

$$H_\phi \simeq -\frac{1}{a}\frac{\partial \Psi}{\partial \phi} \simeq \sum_{n-\text{odd}} \frac{4K_o}{n\pi} \sin n(\phi - \phi_o)$$

which is approximately square-wave-like. These checks confirm the correctness of the solution.

(c) The inductance of the rotor coil is computed from the flux linkage of an individual wire-loop,

$$\Phi_\lambda = l \int_{\phi=-\phi'}^{-\phi'+\pi} \mu_o H_r b d\phi \Big|_{r=b} = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} -l\mu_o \frac{nA_n}{b} \int_{\phi=-\phi'}^{-\phi'+\pi} \cos n(\phi - \phi_o) b d\phi$$

$$= \sum_{\substack{n=1\\ \text{odd}}}^{\infty} l\mu_o \frac{4K_o b}{n\pi}\Big[1 - \Big(\frac{b}{a}\Big)^{2n}\Big] \sin n(\phi' - \phi_o)$$

where $l$ is the length of the system. The flux linkage is obtained by taking the number of wires per unit circumference $N/\pi b$, multiplying them by $\Phi_\lambda$ and integrating from $\phi' = \phi_o$ to $\phi' = \phi_o + \pi$

$$\lambda = \int \frac{N}{\pi b} b d\phi' \Phi_\lambda = l\frac{N}{\pi} \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \mu_o \frac{4K_o b}{n\pi}\Big[1 - \Big(\frac{b}{a}\Big)^{2n}\Big] \int d\phi' \sin n(\phi' - \phi_o)$$

$$= l\frac{8N^2}{2\pi} \mu_o i \Bigg(\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{1}{n^2}\Big[1 - \Big(\frac{b}{a}\Big)^{2n}\Big]\Bigg)$$

where we use the fact that

$$K_o = \frac{Ni}{2b}$$

The inductance is

$$L = \frac{\lambda}{i} = \frac{8N^2}{2\pi}\mu_o l \sum_{\substack{n=1 \\ n-\text{odd}}}^{\infty} \frac{1}{n^2}\left[1 - \left(\frac{b}{a}\right)^{2n}\right]$$

The inductance is, of course, $\phi_o$ independent because the field is "tied" to the rotor and moves with $\phi_o$.

## 8.6 VECTOR POTENTIAL AND THE BOUNDARY VALUE POINT OF VIEW

**8.6.1** (a) For the two-dimensional situation under consideration, the magnetic field intensity is found from the vector potential using (8.1.17)

$$\mathbf{H} = \frac{1}{\mu_o}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi}\mathbf{i}_r - \frac{\partial A_z}{\partial r}\mathbf{i}_\phi\right) \tag{1}$$

Thus, if the vector potential were discontinuous at $r = R$, the azimuthal magnetic field intensity would be infinite there.

(b) Integration of (1) using the fields given by (1.4.7) gives

$$A_z = -\mu_o \int H_\phi dr + f(\phi) = -\mu_o J_o \begin{cases} R^3/9R + f_1; & r < R \\ \frac{R^2}{3} ln(r/R) + f_2; & R < r \end{cases} \tag{2}$$

$$A_z = \begin{cases} g_1(r); & r < R \\ g_2(r); & R < r \end{cases} \tag{3}$$

Because the integrations are performed holding $r$ and $\phi$-constant, respectively, the integration "constants" are actually functions of the "other" independent variable, as indicated. From (3) it is clear, however, that there is no dependence of $f_1$ and $f_2$ on $\phi$. Given that the vector potential is zero at $r = 0$ and that $A_z$ is continuous at $r = R$, $f_1 = 0$ and $f_2 = R^2/9$. Thus, the vector potential is as given.

(c) In terms of the vector potential, the flux is given by (8.4.12). Because there are no contributions on the radial legs and because $A_z(r = 0)$ has been defined as zero,

$$\begin{aligned} \lambda &= \oint_{C'} \mathbf{A} \cdot d\mathbf{s} = l[A_z(0) - A_z(a)] = -lA_z(a) \\ &= \frac{\mu_o l R^2 J_o}{3}\left[ln(a/R) + \frac{1}{3}\right] \end{aligned} \tag{4}$$

This illustrates how the use of $\mathbf{A}$ to represent the field makes it possible to evaluate the flux linkage without carrying out an integration.

**8.6.2** **A** must be $z$-directed and must obey Poisson's equation

$$\nabla^2 A_z = -\mu_o J_z \tag{1}$$

Now

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}\Big(r\frac{d}{dr}\Big)$$

in the special symmetry of the problem. Thus

$$\frac{1}{r}\frac{d}{dr}\Big(r\frac{d}{dr}\Big)A_z = -\mu_o J_z \tag{2}$$

and

$$A_z = -\mu_o J_z \frac{r^2}{4} \qquad r < b \tag{3}$$

Outside this region $b < r < a$, $A_z$ obeys Laplace's equation

$$A_z \propto C ln(r/b) + \text{const}$$

At $r = b$ we must have continuous $A_z$ and $dA_z/dr$ (continuous $H_\phi$). Thus,

$$\text{const} = -\mu_o J_z \frac{b^2}{4}$$

and

$$\frac{C}{b} = -\mu_o J_z \frac{b}{2}$$

Thus

$$A_z = -\mu_o J_z \frac{b^2}{4}[2ln(r/b) + 1]; \quad b < r < a$$

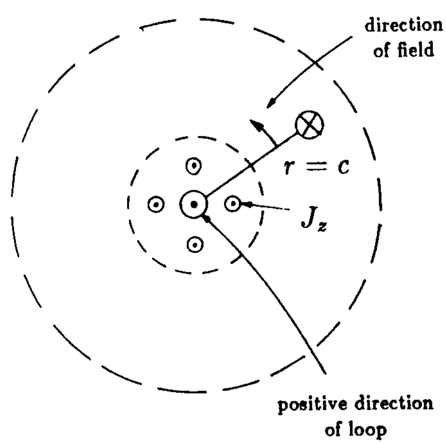


**Figure S8.6.2**

The flux is, according to (8.6.5) [see Fig. S8.6.2]

$$\lambda = l(A_z^a - A_z^b)$$

and thus

$$\lambda = -lA_z^b$$

because

$$A_z^a = 0$$

For $c < b$

$$\lambda = l\mu_o J_z \frac{c^2}{4}$$

For $c > b$

$$\lambda = l\mu_o J_z \frac{b^2}{4}[1 + 2ln(c/b)]$$

Note that $A_z \neq 0$ for $r > 0$. This should be remedied by adding a constant to $A_z$. It does not affect the flux linkage.

**8.6.3** (a) In cylindrical coordinates where there is no $\phi$ dependence, the vector potential has only a $\theta$ component

$$\mathbf{A} = A_\theta(r, z)\mathbf{i}_\theta \tag{1}$$

and the flux density is found from

$$\mu_o\mathbf{H} = \nabla \times \mathbf{A} \Rightarrow \mu_o\mathbf{H} = \mathbf{i_r}\big(-\frac{\partial A_\theta}{\partial z}\big) + \mathbf{i_z}\big[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)\big] \tag{2}$$

For reasons that are apparent in part (b), it is convenient to write $\mathbf{A}$ as

$$\mathbf{A} = \frac{\Lambda_c(r, t)}{r} \tag{3}$$

in which case, (2) becomes

$$\mu_o\mathbf{H} = \frac{1}{r}\big[-\frac{\partial \Lambda_c}{\partial z}\mathbf{i_r} + \frac{\partial \Lambda_c}{\partial r}\mathbf{i_z}\big] \tag{4}$$

(b) For any surface $S$ enclosed by the contour $C$, the net flux can be found from the vector potential by

$$\lambda = \oint_C \mathbf{A} \cdot d\mathbf{s} \tag{5}$$

In particular, consider a surface enclosed by a contour $C$ having as the first of four segments a contour spanning $0 < \phi < 2\pi$ at the radius, $a$, from the $z$ axis. The second segment connects that circular contour with a second at the radius $b$ by a segment connecting the two in a plane of constant $\phi$. The contour is closed by a second contour in an adjacent $\phi =$ constant plane joining these circular segments. Integration of (5) gives contributions only from the circular contours. The segments joining the circular contours are perpendicular to the direction of $\mathbf{A}$, and in any case make compensating contributions because they are in essentially the same $\phi =$ constant planes. Thus, the flux through the surface having outer and inner radii, $a$ and $b$ respectively, is as given.

**8.6.4** (a) The vector potential, $A_z$, satisfies Laplace's equation. The first three conditions of (8.6.18) are met by the solution

$$A_z = A_n \sinh \frac{n\pi}{a} y \sin \frac{n\pi}{a} x \tag{1}$$

The last condition is met by superimposing these solutions

$$A_z = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \sin \frac{n\pi}{a} x \tag{2}$$

and evaluating the coefficients by requiring that this function satisfy the fourth boundary condition of (8.6.18).

$$\Lambda = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} b \sin \frac{n\pi}{a} x \tag{3}$$

Multiplication by $\sin(m\pi x/a)$ and integration gives

$$-\frac{\Lambda a}{m\pi} \cos \frac{m\pi}{a} x \Big]_0^a = \frac{A_m a}{2} \sinh \frac{m\pi}{a} b \tag{4}$$

which therefore gives the coefficients as

$$A_m = \frac{2\Lambda}{m\pi \sinh \frac{m\pi}{a} b} [-\cos m\pi + 1] \tag{5}$$

so that (2) becomes the given solution.

(b) The total current in the lower plate is

$$i = \int_0^a K_z dx = -\int_0^a H_x(y=0) dx = -\int_0^a \frac{1}{\mu_o} \frac{\partial A_z}{\partial y}\Big|_{y=0} dx \tag{6}$$

Evaluation using the given vector potential gives

$$i = -\sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{8\Lambda}{\mu_o n\pi \sinh\left(\frac{n\pi b}{a}\right)} = -\sum_{n=1}^{\infty} \frac{I \sin \omega t}{2n \sinh\left(\frac{n\pi b}{a}\right)} \tag{7}$$

(c) In the limit where $b/a \gg 1$,

$$\sinh\left(\frac{n\pi b}{a}\right) \to \frac{1}{2} e^{n\pi b/a} \tag{8}$$

and (7) becomes

$$i \to -\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{I}{n} e^{-n\pi b/a} \sin \omega t \to -Ie^{-\pi b/a} \sin \omega t \tag{9}$$

Taking $ln$ of the magnitude of this expression gives

$$ln\left(\frac{|i|}{I}\right) = -\pi(b/a) \tag{10}$$

which is the straight line portion of the plotted function.

(d) In the limit $b/a \ll 1$, (7) becomes

$$i \to -\frac{1}{\mu_o} \frac{8\Lambda}{\pi^2} \frac{a}{b} \sum \frac{1}{n^2} = -\frac{1}{\mu_o} \Lambda \frac{a}{b} \tag{11}$$

This is the same as what is obtained if it is assumed that the field is uniform and simply $H_x \to \Lambda / b\mu_o$ so that

$$K_z \to -H_x \Rightarrow i \to K_z a \to -a\Lambda / b\mu_o \tag{12}$$

**8.6.5** The perfectly conducting electrodes force $H$ to be tangential to the electrodes. Thus $\partial A_z/\partial x = -\mu_o H_y$ vanishes at $y = 0, y = d$ except for the gap at $x = 0$ and $\partial A_z/\partial y = \mu_o H_x$ vanishes at $x = \pm a$. The magnetic vector potential jumps by $\Lambda$ as one goes from $x = 0_-$ to $x = 0_+$, at $y = 0$ and $y = d$. Thus $A_z$ is constant around the $\subset$ shaped contour as well as the $\supset$ shaped one. Denoting by the superscripts (a) and (b) these two regions respectively, we have for Laplacian solutions of $A_z$

$$A_z^{(a)} = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{d}(x+a) \sin \frac{n\pi}{d} y + A_o(x+a)$$

$$A_z^{(b)} = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{d}(x-a) \sin \frac{n\pi}{d} y + B_o(x-a)$$

At $x = 0$, the constants $A_o$ and $B_o$ account for the jump of $A_z, B_o = -\Lambda/2 = A_o$. The vector potential and its curl must be continuous for $0 < y < d$ at $x = 0$. We thus have $A_n = -B_n$ for all $n$ except $n = 0$. The sinusoidal series has to cancel that jump for $0 < y < d$. We must have

$$\sum_n A_n \sinh \frac{n\pi}{d} a \sin \frac{n\pi}{d} y = -\sum_{n-\text{odd}} \frac{4A_o}{n\pi} \sin \frac{n\pi}{d} y$$

and similarly for the series in region $b$. We obtain

$$A_z^{(a)} = \sum_{n-\text{odd}} \frac{2\Lambda}{n\pi} \frac{\sinh \frac{n\pi}{d}(x+a)}{\sinh \frac{n\pi}{d} a} \sin \frac{n\pi}{d} y - \frac{\Lambda}{2}(x+a)$$

$$A_z^{(b)} = \sum_{n-\text{odd}} \frac{2\Lambda}{n\pi} \frac{\sinh \frac{n\pi}{d}(x-a)}{\sinh \frac{n\pi}{d}a} \sin \frac{n\pi}{d} y - \frac{\Lambda}{2}(x-a)$$

(b) See Fig. S8.6.5.

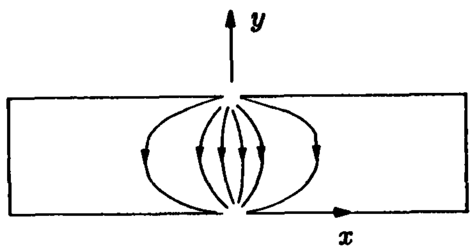


**Figure S8.6.5**

**8.6.6** (a) We must satisfy Poisson's equation for the vector potential everywhere inside the perfectly conducting boundaries

$$\nabla^2 A_z = \mu_o i n_o \sin\left(\frac{\pi x}{a}\right) \tag{1}$$

and make the normal flux density and hence $A_z$ zero on the boundaries.

$$A_z = 0 \quad \text{at} \quad x = \pm a, y = 0, y = b \tag{2}$$

A particular solution to (1) follows by looking for one that depends only on $x$.

$$\frac{\partial^2 A_{zp}}{\partial x^2} = \mu_o i n_o \sin\left(\frac{\pi x}{a}\right) \Rightarrow A_{zp} = -\mu_o i n_o \frac{a^2}{\pi^2} \sin\frac{\pi x}{a} \tag{3}$$

Then the homogeneous solution must satisfy Laplace's equation and the conditions

$$A_{zh} = 0 \quad \text{at} \quad x = \pm a; \tag{4a}$$

$$A_{zh} = \mu_o i n_o \frac{a^2}{\pi^2} \sin\frac{\pi x}{a} \quad \text{at} \quad y = 0, b \tag{4b}$$

The first of these conditions, can be met by making the $x$ dependence $\sin(\pi x/a)$. Then, the $y$ dependence must be comprised of a linear combination of $\exp(+ky)$ and $\exp(-ky)$. If the $y$ coordinate were at $y = b/2$, the second of the conditions of (4) would be even in $y$. So, make the linear combination $\cosh k(y - \frac{b}{2})]$ and for convenience adjust the coefficient so that the second of conditions (4) are met, divide this function by its value at $y = b/2$. This makes it clear that the coefficient is the value given on the boundary from (4). Thus, the desired solution, the sum of the particular and homogeneous parts, is

$$A_z = A_{zp} + A_{zh} = \frac{\mu_o i n_o a^2}{\pi^2}\left[\frac{\cosh \frac{\pi}{a}\left(y - \frac{b}{2}\right)}{\cosh\left(\frac{\pi b}{2a}\right)} - 1\right] \sin\left(\frac{\pi x}{a}\right) \tag{5}$$

(b) The flux linked by one turn is

$$\Phi_\lambda = -l[A_z(x,y) - A_z(-x,y)] = -\frac{2\mu_o i n_o a^2 l}{\pi^2}\left[\frac{\cosh\frac{\pi}{a}(y-\frac{b}{2})}{\cosh\left(\frac{\pi b}{2a}\right)} - 1\right]\sin\frac{\pi x}{a} \tag{6}$$

and the total flux of all of the windings in series is

$$\lambda = \int_0^a \int_0^b \Phi_\lambda n_o \sin\left(\frac{\pi x}{a}\right) dy dx = \frac{2\mu_o n_o^2 a^4 l}{\pi^3}\left[\frac{\pi b}{2a} - \tanh\left(\frac{\pi b}{2a}\right)\right] \tag{7}$$

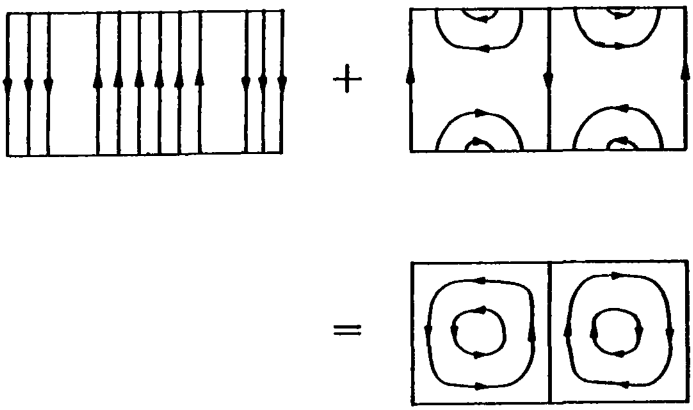

**Figure S8.6.6**

(c) A sketch of the lines of constant vector potential and thus **H** for the particular, homogeneous and total solution (the sum of these) is shown in Fig. S8.6.6. It is perhaps easiest to envision the sum by picturing the addition of contour maps of the two parts, the axes out of the paper being the height $A_z$ of the respective surfaces.

**8.6.7** (a) This is a problem involving a particular and a homogeneous solution of the vector Poisson equation. The particular solution is due to uniform current density $J_o = n_o i$

$$\mathbf{A}_p = -\mu_o n_o i \frac{x^2 - a^2}{2}\mathbf{i}_z$$

Alternatively, we may find the homogeneous solution by comparison with Prob. 8.6.6. In that problem the wire density was sinusoidal. Now it is uniform. $A_z$ was antisymmetric, now it is symmetric. We can expand the symmetric wire distribution as a square wave.

$$J_z(x,y) = n_o i = \sum_{\substack{n \\ n-\text{odd}}} \frac{4 n_o i}{n\pi}\cos\frac{n\pi}{2a}x$$

The particular solution of the vector potential is thus

$$\mathbf{A}_p = -\mathbf{i}_z \mu_o n_o i \sum_{\substack{n \\ n-\mathrm{odd}}} \frac{4}{n\pi}\Big(\frac{2a}{n\pi}\Big)^2 \cos\Big(\frac{n\pi}{2a}x\Big)$$

The complete solution is

$$\mathbf{A} = \mathbf{i}_z \mu_o n_o i \sum_{\substack{n \\ \mathrm{odd}}} \frac{4}{n\pi}\Big(\frac{2a}{n\pi}\Big)^2 \cos\Big(\frac{n\pi}{2a}x\Big)\left[\frac{\cosh \frac{n\pi}{2a}(y-\frac{b}{2})}{\cosh \frac{n\pi}{4a}b} - 1\right]$$

(b) The flux linkage of a wire at $x, y$ is

$$\lambda = lA_z(x,y)$$

and thus

$$v = \frac{d\lambda}{dt} = \mu_o n_o l \sum_{\substack{n \\ \mathrm{odd}}} \frac{4}{n\pi}\Big(\frac{2a}{n\pi}\Big)^2 \cos\Big(\frac{n\pi}{2a}x\Big)\left[\frac{\cosh \frac{n\pi}{2a}(y-\frac{b}{2})}{\cosh \frac{n\pi}{4a}b} - 1\right]\frac{di}{dt}$$

**8.6.8** (a) Here we have a solution very much like that of Prob. 8.6.6, except that the particular solution

$$\mathbf{A}_p = -\mathbf{i}_z \mu_o i n_o (a/\pi)^2 \sin\Big(\frac{\pi x}{a}\Big)$$

has to be replaced by an infinite sum whose second derivative reproduces the square wave of magnitude $in_o$. Thus

$$\mathbf{A}_b = -\mathbf{i}_z \mu_o i n_o \sum_{n-\mathrm{odd}} \frac{4}{n\pi}\Big(\frac{a}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{a}\Big)$$

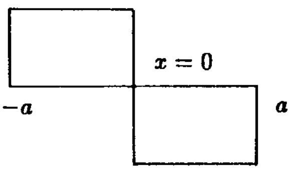


**Figure S8.6.8**

The complete solution is (compare Prob. 8.6.6)

$$\mathbf{A} = \mathbf{i}_z \mu_o i n_o \sum_{n-\mathrm{odd}} \frac{4}{n\pi}\Big(\frac{a}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{a}\Big)\left[\frac{\cosh(n\pi/a)(y-\frac{b}{2})}{\cosh\left(\frac{n\pi b}{2a}\right)} - 1\right]$$

(b) The inductance is computed from

$$Li = -\int_0^a \int_0^b n_o dx' dy' 2lA_z(x', y')$$

where $2lA_z$ is the flux linkage of one turn $n_o dx' dy'$ is the wire density. Thus integrating one typical term:

$$\int_0^u dx' \sin\left(\frac{n\pi x'}{a}\right) \int_0^b \left[\frac{\cosh \frac{n\pi}{a}\left(y - \frac{b}{2}\right)}{\cosh \frac{n\pi b}{2a}} - 1\right] dy' = 2\left(\frac{a}{n\pi}\right)\left[2\frac{a}{n\pi} \tanh \frac{n\pi b}{2a} - b\right]$$

and the inductance is

$$L = \mu_o n_o^2 l \sum_{n-\mathrm{odd}} \frac{16}{n\pi}\left(\frac{a}{n\pi}\right)^4\left[\frac{n\pi b}{2a} - \tanh\left(\frac{n\pi b}{2a}\right)\right]$$