

MIT OpenCourseWare  
<http://ocw.mit.edu>

*Continuum Electromechanics*

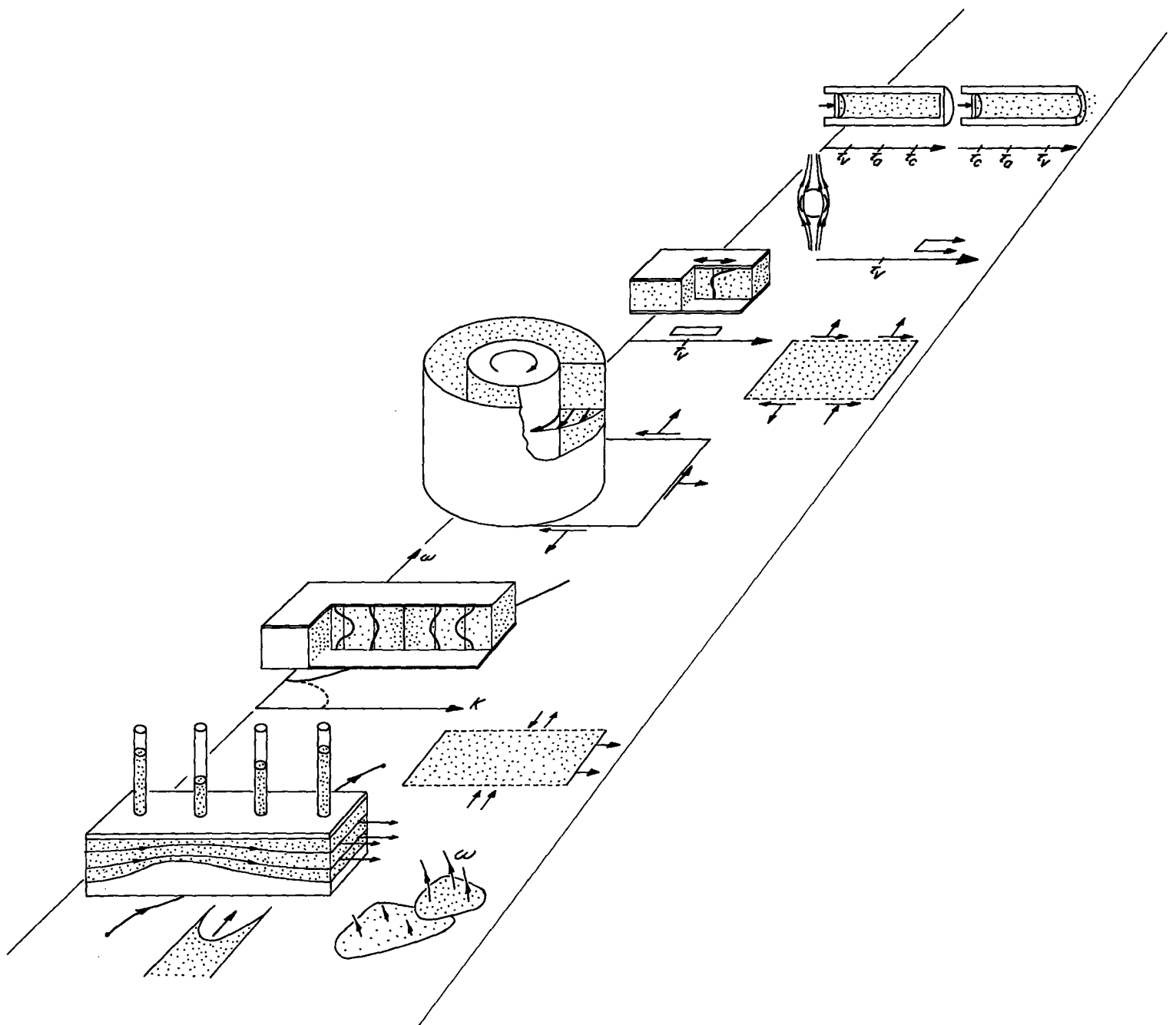
For any use or distribution of this textbook, please cite as follows:

Melcher, James R. *Continuum Electromechanics*. Cambridge, MA: MIT Press, 1981. Copyright Massachusetts Institute of Technology. ISBN: 9780262131650. Also available online from MIT OpenCourseWare at <http://ocw.mit.edu> (accessed MM DD, YYYY) under Creative Commons license Attribution-NonCommercial-Share Alike.

For more information about citing these materials or our Terms of Use, visit:  
<http://ocw.mit.edu/terms>.

---

## Laws, Approximations and Relations of Fluid Mechanics



## 7.1 Introduction

The following chapters carry the subject of continuum electromechanics to its third level. Not only do the field sources assume distributions consistent with deformations of the support medium, the medium is itself free to respond to the associated electromagnetic forces. For gases and liquids, as well as fluid-like continua such as certain plasma models and electron beams, this response must be consistent with the mechanical laws and relations now derived. The role of this chapter is the mechanical analogue of the electromagnetic one played by Chap. 2.

The chapter is organized so that Secs. 7.2-7.9 are sufficient background in incompressible inviscid fluid mechanics to proceed directly with related electromechanical studies. An even wider range of electromechanical coupling mechanisms than might be imagined at this point are tied to fluid interfaces. This makes fluid interfaces (Sec. 7.5), surface tension (Sec. 7.6) and jump conditions (Sec. 7.7) appropriate for early discussion.

Compressibility and related acoustic phenomena are taken up in Secs. 7.10-7.12. Then, contributions of fluid friction, the consequence of fluid viscosity, are taken up in Secs. 7.13-7.17. The resulting Navier-Stokes's equations are summarized in Sec. 7.16.

Overlaying the derivation of the laws of fluid mechanics is the development of relations that play a role in the following chapters for describing the continuum mechanics that is analogous to that for the electric and magnetic transfer relations in the preceding chapters. Transfer relations describing an incompressible and inviscid inertial continuum (Sec. 7.9) will be used many times. Also for future reference are the relations of Sec. 7.11, which embody the acoustic phenomena associated with compressibility, those of Sec. 7.19, which establish the interplay between viscous and inertial effects, and of Sec. 7.20, which describe "creep flow," in which fluid friction overwhelms inertia.

Viscous diffusion, the diffusion of vorticity, has considerable analogy to magnetic diffusion. Thus, the studies of Chap. 6 are a useful background for understanding the interplay of inertia and fluid friction.

This chapter is largely concerned with general laws and relations. The chapters which follow make extensive use of these results in specific case studies.

Chapter 2 begins with a discussion of the two quasistatic limits of the general laws of electrodynamics, identifying rate processes brought in by electrical dissipation in each of these approximations. This chapter ends with a similar discussion.

## 7.2 Conservation of Mass

With the mass per unit volume of a continuous medium defined as  $\rho$ , a statement of mass conservation for a volume  $V$  of fixed identity is

$$\frac{d}{dt} \int_V \rho dV = 0 \quad (1)$$

Here, the volume  $V$  is defined such that it always encloses the same material. The surface  $S$  enclosing the materials therefore moves with the material, and the velocity  $\vec{v}$  is the velocity of surface and material alike.

With the integral theorem of Eq. 2.6.5, it is possible to express Eq. 1 as the integral form of mass conservation:

$$\int_V \frac{\partial \rho}{\partial t} dV + \oint_S \rho \vec{v} \cdot \vec{n} da = 0 \quad (2)$$

Written in this form, the law applies for  $V$  and  $S$  either fixed or enclosing material of fixed identity. Using Gauss' theorem, the surface integral can again be expressed as a volume integral, so that the equation involves one integral over the volume,  $V$ . Because  $V$  is arbitrary, it follows that the integrand must vanish:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0 \quad (3)$$

This is the required differential law of mass conservation.

Incompressibility: If fluid motions are typified by times that are long compared to the transit time of an acoustic wave through a length typifying the system, for important classes of flows the mass density in the vicinity of a given fluid particle remains constant. In view of the definition of the

convection derivative, Sec. 2.4, this means that

$$\frac{D\rho}{Dt} = 0 \quad (4)$$

For incompressible motions, the mass density evolves much as the free charge density in an insulating fluid (Sec. 5.10). If fluid particles of interest originate where the mass density is uniform, it follows that the mass density in the region occupied by this same fluid at a later time is also uniform. Thus, the solution to Eq. 4,  $\rho = \text{constant}$ , is a special "homogeneous" or "uniform" density case.

From Eqs. 3 and 4, it follows from conservation of mass that for an incompressible fluid

$$\nabla \cdot \vec{v} = 0 \quad (5)$$

whether the fluid is homogeneous or not.

The quasistatic nature of the incompressible model is investigated in Secs. 7.12 and 7.22.

### 7.3 Conservation of Momentum

Because momentum is a vector field, rather than a scalar one, it is convenient to deal with its individual components in Cartesian coordinates. Of course, this in no way restricts the validity of the resulting equation of motion.

Again, with the understanding that the volume  $V$  always encloses the same material, and hence that its surface deforms with the local velocity of the material, conservation of momentum for the  $i$ th component is

$$\frac{d}{dt} \int_V \rho v_i dV = \int_V F_i dV \quad (1)$$

The integral on the right represents contributions to the total force acting on the volume that come from the surrounding material (viscous and pressure forces) and from "external" sources, such as gravity and electromagnetic fields.

Use of the integral theorem, Eq. 2.6.5, gives the integral law for conservation of momentum:

$$\int_V \frac{\partial \rho v_i}{\partial t} dV + \oint_S \rho v_i \vec{v} \cdot \vec{n} da = \int_V F_i dV \quad (2)$$

Gauss' theorem, Eq. 2.6.2, makes possible a conversion of the surface integral to a volume integral:

$$\int_V \left( \frac{\partial \rho v_i}{\partial t} + \nabla \cdot \rho v_i \vec{v} \right) dV = \int_V F_i dV \quad (3)$$

Expansion of terms on the left gives

$$\int_V \left\{ v_i \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} \right] + \rho \left[ \frac{\partial v_i}{\partial t} + \vec{v} \cdot \nabla v_i \right] \right\} dV = \int_V F_i dV \quad (4)$$

Again, the integrand of the volume integrations collected together must vanish, but note that conservation of mass, Eq. 7.2.3, requires that the first term in brackets vanish. Thus, the differential law representing conservation of momentum is

$$\rho \left[ \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right] = \vec{F} \quad (5)$$

On the left is the time-rate of change of  $\vec{v}$  for an observer moving with the fluid, the convective derivative as discussed in Sec. 2.4. Even though the mass density appears "outside" the convective derivative, this equation is valid even if  $\rho$  is a function of space and time.

### 7.4 Equations of Motion for an Inviscid Fluid

To complete the integral or differential force laws, Eqs. 7.3.2 and 7.3.5, it is necessary to take account of how the surrounding fluid exerts a force on the element of interest. This is naturally done by considering the associated traction exerted on the surface  $S$  that encloses the fluid volume  $V$ .

In an inviscid (frictionless) fluid, this traction acts normal to the surface and is of the same magnitude regardless of the local surface orientation. With  $\vec{n}$  the local normal vector to the surface, the traction due to the surrounding fluid is

$$\vec{T} = -p\vec{n} \quad (1)$$

where the minus sign is introduced so that the pressure,  $p$ , will be a positive quantity. That this traction is consistent with a stress

$$T_{ij} = -p\delta_{ij} \quad (2)$$

can be seen by substituting Eq. 2 into the relation between stress and traction, Eq. 3.9.5.

The force density associated with this stress is found by taking the tensor divergence of Eq. 2 (Eq. 3.15.1),

$$F_i^P = -\frac{\partial}{\partial x_j} (p\delta_{ij}) = -\frac{\partial p}{\partial x_i} \quad (3)$$

With forces such as due to gravity and of electric or magnetic origin represented by the external force density  $\vec{F}_{ex}$ , the force equation, Eq. 7.3.5, becomes

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right] + \nabla p = \vec{F}_{ex} \quad (4)$$

By way of discussing what is required to complete the formulation of the fluid mechanics, suppose that  $\vec{F}_{ex}$  is a given driving function. Then, the dependent variables are  $\vec{v}$ ,  $\rho$  and  $p$ . For incompressible fluid, Eqs. 7.2.4 and 7.2.5 are the two additional scalar laws required to describe the fluid mechanics. Constitutive laws for compressible flows are introduced in Sec. 7.10. Contributions of viscosity to the stress are taken up in Secs. 7.13-7.16.

### 7.5 Eulerian Description of the Fluid Interface

In electromagnetic theory, the boundary and the field are easily distinguished. In fluid mechanics, the boundary of a given fluid region may be the interface between two fluids. Then, the boundary is in fact a part of the fluid and flow is intrinsically linked to a deformation of the interface.

An interface can be represented analytically by

$$F(x,y,z,t) = 0 \quad (1)$$

That is, of all possible spatial coordinates  $(x,y,z)$ , at some time,  $t$ , only those that make  $F = 0$  comprise an interface. Figure 7.5.1 illustrates a particular case where it is convenient to denote the surface elevation above the  $y$ - $z$  plane as  $\xi(y,z,t)$ , and

$$F = \xi - x = 0 \quad (2)$$

In the language of electrostatics,  $F$  could be regarded as a surface of zero potential. This observation is useful, because it is a reminder that the normal vector  $\vec{n}$  to the interface is given by the geometry of the interface alone, and is

$$\vec{n} = \frac{\nabla F}{|\nabla F|} \quad (3)$$

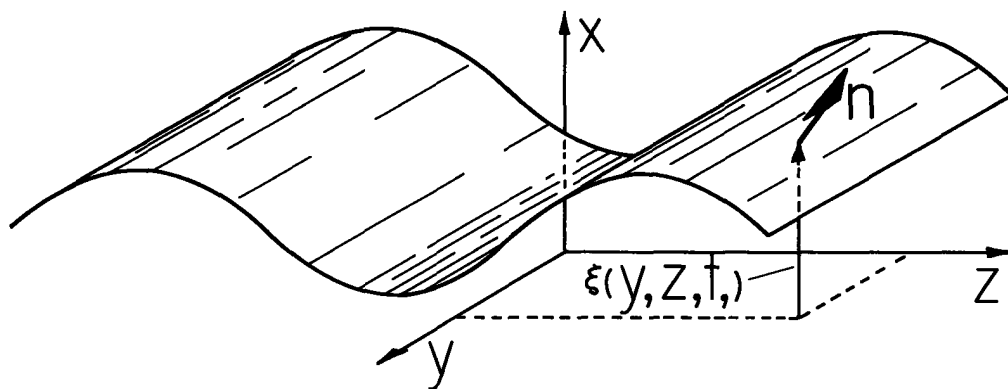


Fig. 7.5.1. Fluid interface.

The geometric relation between  $F$  and  $\vec{n}$  is the same as that between the electric potential  $\Phi$  and the electric field intensity  $\vec{E}$ . The normal to the interface is the gradient of  $F$  normalized to ensure unit magnitude.

What is the relationship between the interface geometry and the velocity  $\vec{v}$  of the fluid adjacent to the interface? The interface is presumed to be a surface cut from the total fluid volume and always composed of the same material particles. Thus, the interface could be distinguished from the remainder of the fluid by dye markers. As the fluid deforms, it is presumed that the surface remains contiguous. Dyed particles always have adjacent dyed neighbors within the plane of the interface, and undyed neighbors in the adjacent regions of fluid bulk.

By definition, the convective derivative of Sec. 2.4 is the rate of change with respect to time for an observer moving with a particle of fluid. By the definition of what is meant by the "interface," the rate of change of  $F$  for an observer on the interface must be zero. Hence, the required relationship between the surface geometry and the fluid velocity is

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \vec{v} \cdot \nabla F = 0 \quad (4)$$

on  $F = 0$ .

For the particular case illustrated by Eq. 2 and Fig. 7.5.1, this condition requires that on the surface,

$$v_x = \frac{\partial \xi}{\partial t} + v_y \frac{\partial \xi}{\partial y} + v_z \frac{\partial \xi}{\partial z} \quad (5)$$

The relation is seen to be physically reasonable by considering limiting situations such as: (a) a flat interface that moves in the  $x$  direction in a time-varying fashion,  $\xi = \xi(t)$ ; (b) an interface that is stationary but deformed,  $\xi = \xi(y,z)$ .

## 7.6 Surface Tension Surface Force Density

If viewed on a millimeter scale, a liquid can take on many of the appearances of an elastic solid. As if enclosed by an elastic "skin," drops of water suffer oscillations and capillary ripples have the appearance of a liquid surface covered by an elastic membrane. Although similar in effect to a membrane under tension, these attributes of the interface are a consequence of the difference between forces on a molecule deep within the bulk of a fluid and near an interface. Because of this difference, energy is required to make an interface between two fluids.

Energy Constitutive Law for a Clean Interface: A clean interface is one made up of molecules from one or the other of the bulk phases. Thus, there are no molecules attributable to the interface itself (as for example there are when an interface between water and air is covered by a film of oil). Because the nature of the interface is therefore completely determined by the bulk phases, it follows that increasing the interfacial area by the increment  $\delta A$  results in a proportionate increase in the energy  $W_s$  associated with the interface,

$$\delta W_s = \gamma \delta A \quad (1)$$

For a given pair of fluids, the surface tension is a constant physical property having the same units as for the tension of a membrane, newton/m. Typical values are given in Table 7.6.1.

Table 7.6.1. Illustrative values of surface tension.<sup>1</sup>

Substances	Temperature (°C)	Surface tension (newton/m)
Water/air	18	$7.30 \times 10^{-2}$
Acetone/air	20	$2.37 \times 10^{-2}$
Nitrobenzene/air	20	$4.39 \times 10^{-2}$
Water/Carbon tetrachloride	20	$4.5 \times 10^{-2}$
Water/mercury	20	$3.75 \times 10^{-1}$

1. Values taken from Handbook of Chemistry and Physics, College Edition, 49th ed., Robert C. West, ed., The Chemical Rubber Co., Cleveland, Ohio, pp. F-30-32.

**Surface Energy Conservation:** With the objective a relationship between the geometry of an interface and an effective force per unit area  $\vec{T}_s$  acting on the interface, the procedure is now analogous to that followed in Chap. 3. Instead of an electric or magnetic energy subsystem, energy conservation is now written for the "surface subsystem." Some external agent used to put an increment of energy into this system will either increase its stored energy by  $\delta W_s$ , or do work on the external mechanical subsystem through the agent of a force per unit area  $T_s$  displacing an area  $A$  of the interface by an amount  $\delta\xi$ . Thus,

$$\text{incremental input of energy} = \delta W_s + T_s A \delta\xi \quad (2)$$

Inputs on the left might come from changing the chemical nature of the bulk fluids. For interfaces of interest here, there are no such inputs of energy, and Eq. 2 is set equal to zero. The only way in which  $W_s$  can be altered is through the mechanical work done by displacing the interface. Thus for a clean interface,  $\delta W_s$  is given by Eq. 1,

$$\gamma \delta A + T_s A \delta\xi = 0 \quad (3)$$

To deduce  $T_s$  from this expression,  $\delta A$  must be related to the surface geometry and hence to  $\delta\xi$ .

**Surface Force Density Related to Interfacial Curvature:** In the geometric construction of Fig. 7.6.1, the local curvature of the elemental area  $A$  is represented by radii of curvature  $R_1$  and  $R_2$ , defined for orthogonal directions within the local plane of the interface. To find the change in area  $\delta A$ , caused by the displacement  $\delta\xi$ , note that

$$A + \delta A = (x + \delta x)(y + \delta y) \approx xy + y\delta x + x\delta y \quad (4)$$

In addition, the similarity of triangles requires that

$$\frac{x + \delta x}{R_1 + \delta\xi} = \frac{x}{R_1}; \quad \frac{y + \delta y}{R_2 + \delta\xi} = \frac{y}{R_2} \quad (5)$$

which shows that

$$\delta x = \frac{x}{R_1} \delta\xi; \quad \delta y = \frac{y}{R_2} \delta\xi \quad (6)$$

From Eqs. 4 and 6, it follows that because  $xy = A$

$$\delta A = y\delta x + x\delta y = A(\delta\xi) \left[ \frac{1}{R_1} + \frac{1}{R_2} \right] \quad (7)$$

In turn, this result can be substituted into Eq. 3 to give

$$\left[ \gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + T_s \right] A \delta\xi = 0 \quad (8)$$

Because  $\delta\xi$  is arbitrary

$$\vec{T}_s = -\gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \vec{n} \quad (9)$$

This surface force density of Young and Laplace<sup>2</sup> has been written as a vector which, if positive, acts in the direction of the normal  $\vec{n}$ . A radius of curvature has a sign that is positive if the associated center of curvature is in the region from which  $\vec{n}$  is directed. If the center of curvature is in the region into which  $\vec{n}$  is directed, the associated radius is taken as negative.

The implications of Eq. 9 for the static equilibrium of a liquid are illustrated in Fig. 7.6.2. The pair of glass plates are wetted by the liquid so that the radius of curvature of the interface is essentially equal to half the local distance between the plates. Thus, where the plates are closest together the radius of curvature is least and the surface force density is accordingly largest. Note that the radius of curvature is also negative, so that  $T_s$  acts from liquid to air with a net effect of making the interface rise between the plates. The height of rise is greatest to the right, where the plates are closest together. The height of rise,  $\xi(r)$ , is found in Sec. 7.8.

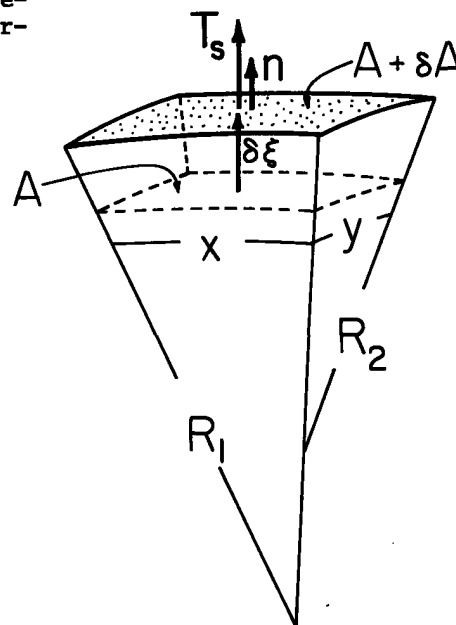
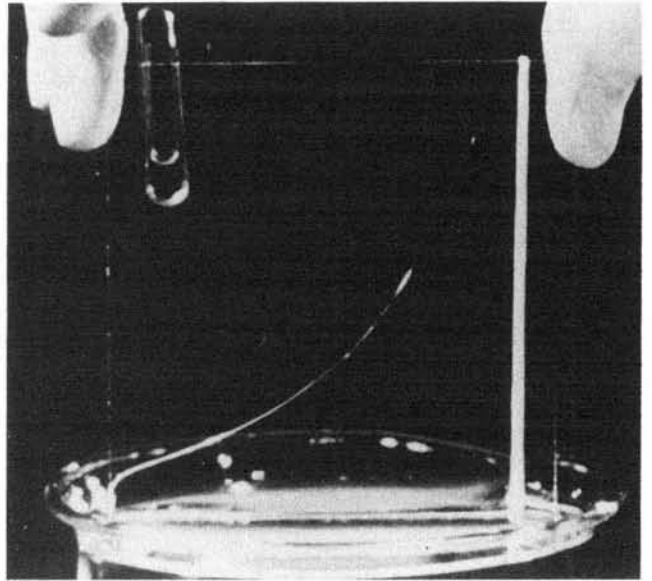
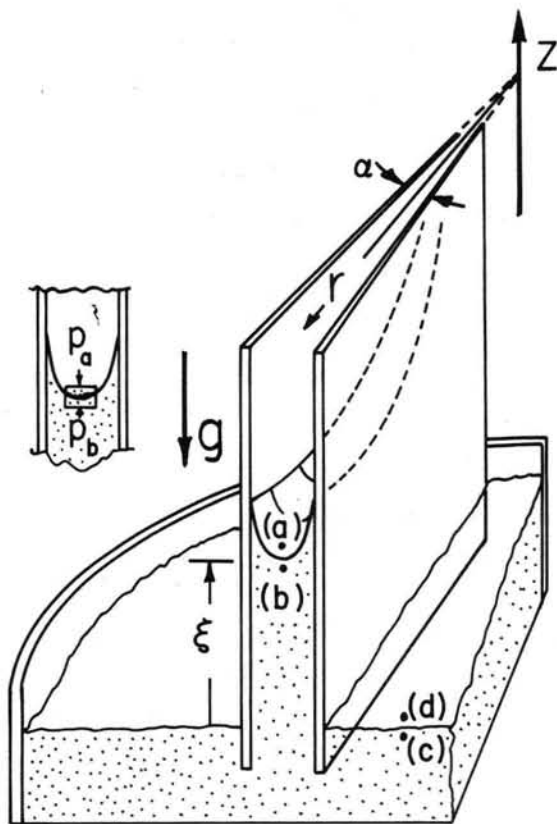


Fig. 7.6.1. Section of interface that suffers perpendicular displacement  $\delta\xi$  to make new surface  $\delta A$ .

2. A. W. Adamson, Physical Chemistry of Surfaces, Interscience, New York, 1960, pp. 4-6.



Courtesy of Education Development Center, Inc. Used with permission.

Fig. 7.6.2. Because of surface tension, fluid wetting pair of glass plates rises to a height  $\xi(r)$  determined by the surface tension  $\gamma$  and local distance between plates. Experiment from film "Surface Tension in Fluid Mechanics" (Reference 9, Appendix C).

Surface Force Density Related to Interfacial Deformation: Three commonly encountered interfacial configurations are shown in Table 7.6.2. In "equilibrium," these are respectively planar, circular cylindrical and spherical in shape. To describe the dynamics of the interface, the surface force density due to surface tension must be expressed in terms of the perturbation  $\xi$  from these equilibria. This could be done by evaluating Eq. 9, but is more easily accomplished by returning to Eq. 3.

Consider the volume, shown in Fig. 7.6.3, that is "cut out" by the surface segment A as it displaces an amount  $\delta\xi$ . For this volume V, enclosed by the surface S having the outward normal vector  $\vec{n}_s$ , Gauss' theorem states that

$$\int_V \nabla \cdot \vec{C} dV = \oint_S \vec{C} \cdot \vec{n}_s da \quad (10)$$

The vector  $\vec{C}$  is arbitrary, and now chosen to be the vector  $\vec{n}$  normal to the interface (not to the surface S enclosing the volume element). Thus,  $\vec{n} = \vec{n}_s$  on the upper surface but  $\vec{n} = -\vec{n}_s$  on the lower surface. On the remaining sides,  $\vec{n}$  is perpendicular to  $\vec{n}_s$ . It follows that the right-hand side of Eq. 10 is the required change in area,  $\delta A$ . Because the area A is itself elemental, the left-hand side of Eq. 10 is  $\nabla \cdot \vec{n} \delta\xi$  and Eq. 10 becomes

$$\delta A = \nabla \cdot \vec{n} \delta\xi \quad (11)$$

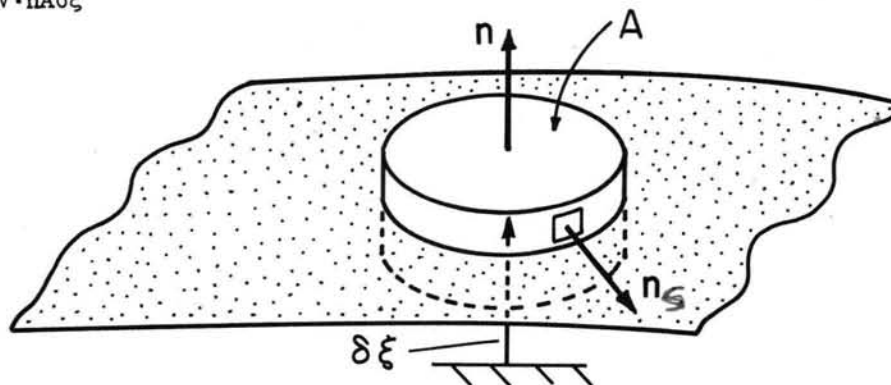
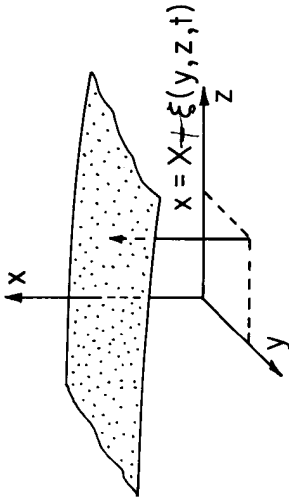
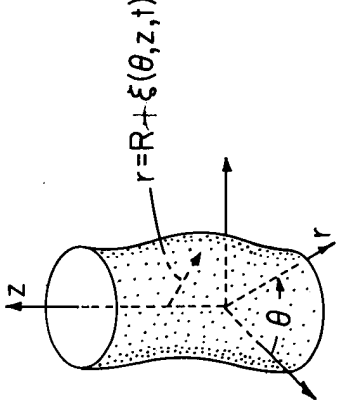
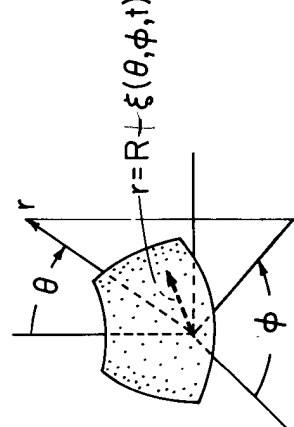


Fig. 7.6.3

Elemental volume V enclosed by surface S intersecting interface between fluids.



Table 7.6.2. Summary of normal vector and surface tension surface force density for small perturbations from planar, circular cylindrical and spherical equilibria.

	<p>(a) <math>\vec{n} = \vec{i}_x - \frac{\partial \xi}{\partial y} \vec{i}_y - \frac{\partial \xi}{\partial z} \vec{i}_z</math></p> <p>(b) <math>(\vec{T}_s)_x = \gamma \left( \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} \right)</math></p> <p>(c) <math>\xi = \text{Re } \tilde{\xi} \exp -j(k_y y + k_z z)</math></p> <p>(d) <math>\tilde{T}_s = -\gamma(k_y^2 + k_z^2) \tilde{\xi}</math></p>
	<p>(e) <math>\vec{n} = \vec{i}_r - \frac{1}{R} \frac{\partial \xi}{\partial \theta} \vec{i}_\theta - \frac{\partial \xi}{\partial z} \vec{i}_z</math></p> <p>(f) <math>(\vec{T}_s)_r = \gamma \left[ -\frac{1}{R} + \frac{\xi}{R^2} + \frac{1}{R^2} \frac{\partial^2 \xi}{\partial \theta^2} + \frac{\partial^2 \xi}{\partial z^2} \right]</math></p> <p>(g) <math>\xi = \text{Re } \tilde{\xi} \exp -j(m\theta + kz)</math></p> <p>(h) <math>\tilde{T}_s = \frac{\gamma}{R^2} \left[ (1 - m^2) - (kR)^2 \right] \tilde{\xi}</math></p>
	<p>(i) <math>\vec{n} = \vec{i}_r - \frac{1}{R} \frac{\partial \xi}{\partial \theta} \vec{i}_\theta - \frac{1}{R \sin \theta} \frac{\partial \xi}{\partial \phi} \vec{i}_\phi</math></p> <p>(j) <math>(\vec{T}_s)_r = \gamma \left[ -\frac{2}{R} + \frac{2\xi}{R^2} + \frac{1}{R^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \xi}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 \xi}{\partial \phi^2} \right]</math></p> <p>(k) <math>\xi = \text{Re } \tilde{\xi} P_n^m(\cos \theta) e^{-jm\phi}</math></p> <p>(l) <math>\tilde{T}_s = -\frac{\gamma}{R^2} (n-1)(n+2) \tilde{\xi}</math></p>

Substitution of Eq. 11 into Eq. 3 gives an alternative expression for the surface tension surface force density:

$$\vec{T}_s = -\gamma(\nabla \cdot \vec{n})\vec{n} \quad (12)$$

The use of this expression for relating  $\vec{T}_s$  to interfacial deformations, as summarized in Table 7.6.2, is now illustrated for the cylindrical coordinate configuration. The interface is then described by

$$F = r - R - \xi(\theta, z, t) = 0$$

If terms that are quadratic in the perturbation amplitude  $\xi$  are ignored, it follows from Eq. 7.5.3 that  $\vec{n}$  is given by Eq. (e) of Table 7.6.2. In turn,

$$\nabla \cdot \vec{n} = \frac{1}{r} - \frac{1}{r^2} \frac{\partial^2 \xi}{\partial \theta^2} - \frac{\partial^2 \xi}{\partial z^2} \quad (13)$$

Consistent with the small amplitude is the approximation  $r^{-1} \approx R^{-1} - \xi/R^2$ . Thus,  $\vec{T}_s$  is as given by Eq. (f) of Table 7.6.2. For  $\xi = 0$ , there is an equilibrium surface force density acting radially inward, tending to compress what is inside the surface much as if it were enclosed by a membrane under tension.

Also summarized in Table 7.6.2 are the complex amplitudes of  $\vec{T}_s$ . In the Cartesian and circular cylindrical geometries these are found by straightforward substitution. However, in the spherical case, Eq. (k) is obtained by using the fact that  $P_n^m$  is a solution to Eq. 2.16.31a.

## 7.7 Boundary and Jump Conditions

It can be taken as phenomenologically based fact that there is neither tangential nor normal velocity of a fluid adjacent to a fixed rigid impermeable wall. Thus, boundary conditions for such a wall are

$$\vec{n} \cdot \vec{v} = 0 \quad (1)$$

$$\vec{n} \times \vec{v} = 0 \quad (2)$$

where  $\vec{n}$  is the normal to the boundary.

The condition on the tangential component of  $\vec{v}$  results because of the friction between wall and fluid, i.e., because of the fluid viscosity. If the fluid is modeled as inviscid, it is consistent to ignore the tangential velocity boundary condition. An inviscid model pictures the fluid as slipping adjacent to a fixed boundary. The extent of the error is investigated in Sec. 7.18.

The jump conditions at an interface between fluids are deduced from the integral laws, much as in Sec. 2.10 for the electromagnetic fields. But, before this can be done, it is necessary to specify the order of the singularity in mass density, pressure and velocity that is included in the interfacial model. It is assumed here that there is no surface mass density, that the density takes at most a step discontinuity. So also does the pressure, and in fact mechanical stresses including viscosity (Sec. 7.15) are assumed to be at most a step singularity. Because the viscous stresses depend on the spatial rates of change of the velocity (the strain rates), a self-consistent model for the interface requires that the velocity be continuous. But, in the inviscid limit, only the normal velocity must be continuous. That this is all required if the fluids are to have a common surface of demarcation can be seen from the relation between fluid velocity at the interface and interfacial geometry, Eq. 7.5.4. At a given location on the interface,  $\nabla F$  has a normal direction. Hence, Eq. 7.5.4 involves only the velocity normal to the interface. Because the expression must hold whether  $\vec{v}$  is evaluated on one or the other side of the interface, it is clear that the normal component of  $\vec{v}$  must be continuous:

$$\vec{n} \cdot \llbracket \vec{v} \rrbracket = 0 \quad (3)$$

Conditions implied by the integral laws follow by using the same incremental volume of fixed identity used for some of the jump conditions in Sec. 2.10 and shown in Fig. 2.10.1. Because there is no surface mass density, mass conservation, Eq. 7.2.1, is automatically satisfied. Formally, this is seen from Eqs. 2.10.14 and 2.10.15 by replacing the free charge density with the mass density.

It is perhaps tempting to require that the mass flux  $\rho \vec{v}$  normal to the interface be continuous. But, the interface considered here is composed of given fluid particles and deforms with the fluid.

The integral momentum-conservation law, expressed as Eq. 7.3.1, makes it clear that for similar reasons there is no contribution of the inertia (represented by the left-hand side) to the interfacial

boundary condition. On the right, those force densities that are spatial impulses (surface force densities) make contributions in the limit  $\Delta \rightarrow 0$ . It is convenient to represent the mechanical and electrical surface force densities by writing them as the divergence of stress tensors,  $T_{ij}^m$  and  $T_{ij}^e$ . For an inviscid fluid,  $T_{ij}^m$  is  $-p\delta_{ij}$  given by Eq. 7.4.2 while  $T_{ij}^e$  is one of the tensors summarized in Table 3.10.1. The contribution of surface tension has already been written as a surface force density, Eq. 7.6.9 or Eq. 7.6.12. With the use of the tensor form of Gauss' theorem, Eq. 3.9.4, the integral momentum law therefore becomes

$$\oint_S (T_{ij}^m + T_{ij}^e)(i_n)_j da + \int_A (T_s)_i da = 0 \quad (4)$$

In the limit where A is incremental, the force (or stress) jump condition results:

$$\llbracket T_{ij}^m + T_{ij}^e \rrbracket n_j + (T_s)_i = 0 \quad (5)$$

This expression will be used with viscous fluids as well, but consider its special form for inviscid fluids and a clean interface so that  $T_{ij}^m$  is given by Eq. 7.4.2 and  $\vec{T}_s$  is given by Eq. 7.6.12:

$$\llbracket p \rrbracket n_i = \llbracket T_{ij}^e \rrbracket n_j - \gamma(\nabla \cdot \vec{n})n_i \quad (6)$$

This vector jump condition has three components. Note that the pressure and surface tension contributions are normal to the interface. This makes it clear that to be consistent with the inviscid and clean interface model, the first term on the right, the surface force density of electric or magnetic origin, must also have no shearing components. Electromagnetic properties of interfaces meeting this requirement are taken up in Sec. 8.2.

### 7.8 Bernoulli's Equation and Irrotational Flow of Homogeneous Inviscid Fluids

In this section, external force densities take the form of the gradient of a scalar. Examples include the gravitational force density on a fluid having uniform density  $\rho$ . With  $\vec{g}$  defined as the directed gravitational acceleration and  $\vec{r} \equiv x\vec{i}_x + y\vec{i}_y + z\vec{i}_z$ , this force density is

$$\vec{F}^g = \rho\vec{g} = \nabla(\rho\vec{g} \cdot \vec{r}) \quad (1)$$

Note that  $\rho$  must be uniform, or the last equality does not hold.

In general, electric and magnetic force densities do not take the form of the gradient of a scalar. However, in many important situations, they are approximated by such a form. In fact, as illustrated in Chap. 8, it is often desirable to design a system so that this is the case. Thus, looking forward to such examples, the force densities of electric and magnetic origin are written as

$$\vec{F}^e = -\nabla\mathcal{E} \quad (2)$$

With these contributions to  $F_{ex}$ , the force equation, Eq. 7.4.4, becomes

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + \vec{v} \cdot \nabla\vec{v}\right) + \nabla p = \nabla(\rho\vec{g} \cdot \vec{r} - \mathcal{E}) \quad (3)$$

A vector identity\* makes it possible to rewrite Eq. 3 in a form that makes evident the contribution of vorticity  $\vec{\omega} \equiv \nabla \times \vec{v}$ , to the dynamics:

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + \vec{\omega} \times \vec{v}\right) + \nabla(p + \frac{1}{2}\rho\vec{v} \cdot \vec{v} - \rho\vec{g} \cdot \vec{r} + \mathcal{E}) = 0 \quad (4)$$

Bernoulli's equation is a statement of invariance for a combination of dynamical quantities that represent the total energy. It is important to recognize that there are two essentially different circumstances under which similar equations apply.

First, consider points (a) and (b) in the flow, as sketched in Fig. 7.8.1, that can be joined by a streamline (not a particle line but rather a line always tangent to the instantaneous velocity vector  $\vec{v}$ ). Then, integration of Eq. 4 along the line C gives no contribution from the second term, which must be perpendicular to the velocity  $\vec{v}$ , and hence the direction of integration. Further, in view of Eq. 2.6.1, the remaining terms integrate to

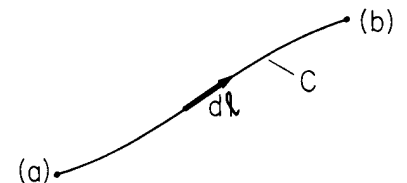


Fig. 7.8.1. Points (a) and (b) are joined by a streamline.

\*  $(\vec{v} \cdot \nabla)\vec{v} = (\nabla \times \vec{v}) \times \vec{v} + \frac{1}{2}\nabla(\vec{v} \cdot \vec{v})$ .

$$\rho \int_a^b \frac{\partial \vec{v}}{\partial t} \cdot d\vec{\ell} + [p + \frac{1}{2} \rho \vec{v} \cdot \vec{v} - \rho \vec{g} \cdot \vec{r} + \xi]_a^b = 0 \quad (5)$$

This form of Bernoulli's equation applies to any two points joined by a streamline, regardless of the flow. Reference 8 of Appendix C gives experimental demonstrations of Bernoulli's law.

Second, consider irrotational flows, defined as having no vorticity,  $\vec{\omega} = 0$ . Then, it is appropriate to define a velocity potential  $\Theta$

$$\vec{v} = -\nabla\Theta \quad (6)$$

and integration of Eq. 4 between fixed points a and b gives

$$[-\rho \frac{\partial \Theta}{\partial t} + p + \frac{1}{2} \rho \vec{v} \cdot \vec{v} - \rho \vec{g} \cdot \vec{r} + \xi]_a^b = 0 \quad (7)$$

This expression is restricted to irrotational flows, but applies to arbitrary fixed points a and b.

The importance of irrotational flows stems from the theorem on vorticity of Helmholtz and Kelvin. If at some instant fluid of fixed identity sustains an irrotational flow, then for this same material the irrotational condition prevails at a later instant. For example, if the flow was initiated from a static (and hence irrotational) condition, it must be irrotational.

Proof of this theorem follows by taking the curl of Eq. 4 and observing that the curl of a gradient is identically zero:

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega} \times \vec{v}) = 0 \quad (8)$$

If the vorticity,  $\vec{\omega}$ , is replaced by the magnetic flux density,  $\vec{B}$ , this expression is the same as that governing the magnetic field in a deforming perfect conductor, Eq. 6.2.3 in the limit  $\sigma \rightarrow \infty$ . Thus, the theorem on flux conservation for a perfectly conducting surface of fixed identity, Eq. 6.2.4, with  $\sigma \rightarrow \infty$ , becomes the theorem

$$\frac{d}{dt} \int_S \vec{\omega} \cdot \vec{n} da = 0 \quad (9)$$

The vorticity linking a material surface S as it deforms with the flow is conserved. If there is no initial vorticity in a given region, the same material will have no vorticity in whatever region it occupies at a later time.

Conservation of mass requires that the flow be solenoidal (Eq. 7.2.5); this combines with the condition for irrotational flow (Eq. 6) to show that the velocity potential is governed by Laplace's equation

$$\nabla^2 \Theta = 0 \quad (10)$$

If boundary conditions involve only  $\vec{v}$  (and hence  $\Theta$ ), this equation defines the flow distribution. With  $\Pi$  defined as a function of time alone set by flow conditions at a reference point, the associated pressure distribution follows from Eq. 7,

$$p = \rho \frac{\partial \Theta}{\partial t} - \frac{1}{2} \rho \vec{v} \cdot \vec{v} + \rho \vec{g} \cdot \vec{r} - \xi + \Pi \quad (11)$$

Although p is a nonlinear function of the velocity, it can be determined in such a problem "after the fact," once  $\vec{v}$  has been found by solving a linear problem. That is, Laplace's equation is linear, in that superimposed solutions are also solutions. But, note that the pressure must be evaluated using the total velocity. Because Eq. 11 is a nonlinear function of  $\vec{v}$ , the pressure does not satisfy the conditions for superposition.

The flux potential relations derived in Sec. 2.16 for electric and magnetic cases are equally applicable here. With the identification  $D_n/\epsilon \rightarrow v_n$  and  $\Phi \rightarrow \Theta$ , the transfer relations and associated bulk distributions of Sec. 2.16 summarize solutions to Eq. 10 in Cartesian, cylindrical and spherical coordinates.

A Capillary Static Equilibrium: The static equilibrium illustrated in Fig. 7.6.2 is described by combining Bernoulli's equation with the capillary surface force density discussed in Sec. 7.6. The object is to find the interfacial profile,  $\xi(r)$ , of the water-air interface. Points (b) and (c) are related by Eq. 7, evaluated with  $\partial/\partial t = 0$ ,  $\vec{v} = 0$ ,  $\vec{g} = -g\vec{i}_z$  and  $\xi = 0$ :

$$p_c = p_b + \rho g \xi \quad (12)$$

where  $\rho$  is the mass density of water. The mass density of the air is  $10^3$  times less than that of the water, so its contribution is ignored in connecting points (a) and (d) via Eq. 7 through the air:

$$p_a = p_d \quad (13)$$

These two bulk relations are augmented by boundary conditions that relate the pressures on opposite sides of the interface. At the bottom of the meniscus, the  $z$  component of Eq. 7.7.6 is evaluated. It is assumed that the glass plates are perfectly wetted by the water and that the meniscus curvature is dominated by variations of the interface in the azimuthal direction. With the shape of the meniscus over the gap between plates approximated as being essentially circular, the local radius of curvature is approximately  $\alpha r/2$  and Eq. 7.7.6 becomes

$$-(p_a - p_b) = -\gamma \left( \frac{2}{\alpha r} \right) \quad (14)$$

The balance of surface force densities at (c-d), where the interface is flat, shows that

$$p_d - p_c = 0 \quad (15)$$

The pressures can be eliminated by adding Eqs. 12-15 and the result solved for  $\xi$ :

$$\xi = \left( \frac{2\gamma}{\alpha \rho g} \right) \frac{1}{r} \quad (16)$$

This is essentially the interfacial radial profile shown in Fig. 7.6.2.

### 7.9 Pressure-Velocity Relations for Inviscid, Incompressible Fluid

Just as the electrical transfer relations introduced in Sec. 2.16 are a convenient building block for modeling complex systems, the mechanical relations derived in this section are useful in a variety of mechanical and electromechanical situations. They are restricted to perturbations described by the inviscid model of Sec. 7.8. The fluid is homogeneous and incompressible so that  $\rho$  is a constant. The transfer relations relate dynamical perturbations from a stationary equilibrium. In making use of the relations in a specific problem, it is important to first establish that the stationary (in special cases, static) conditions are satisfied.

Streaming Planar Layer: Consider first the planar layer of fluid shown in Table 7.9.1, having as a stationary state a uniform velocity in the  $z$  direction. Gravity acts in the  $-x$  direction, so  $\vec{g} = -g\hat{i}_x$ . The velocity takes the form

$$\vec{v} = U\hat{i}_z - \nabla\Theta' \quad (1)$$

The equilibrium part has the velocity potential  $-Uz$ , which satisfies Laplace's equation, Eq. 7.8.10. By superposition, the perturbation  $\Theta'$  must also satisfy this equation. Thus  $\Theta'$  is described by the same derivation given in Sec. 2.16, Eqs. 2.16.11-2.16.16. With the identification  $D_x/\epsilon \rightarrow v_x$  and  $\Phi \rightarrow \Theta$ , the transfer relations of Table 2.16.1, Eq. (b), become

$$\begin{bmatrix} \hat{\Theta}^\alpha \\ \hat{\Theta}^\beta \end{bmatrix} = \frac{1}{\gamma} \begin{bmatrix} -\coth \gamma \Delta & \frac{1}{\sinh \gamma \Delta} \\ \frac{-1}{\sinh \gamma \Delta} & \coth \gamma \Delta \end{bmatrix} \begin{bmatrix} \hat{v}_x^\alpha \\ \hat{v}_x^\beta \end{bmatrix} \quad (2)$$

Here it is understood that the complex amplitudes represent the perturbation. Because the next step brings in a time-rate of change, the time dependence has been specified in Eq. 2, as indicated by replacing  $\sim$  with  $\wedge$ . That is,

$$\Theta' = \text{Re } \hat{\Theta}(x) e^{j(\omega t - k_y y - k_z z)}; \quad \vec{v} = \text{Re } \hat{\vec{v}}(x) e^{j(\omega t - k_y y - k_z z)} + U\hat{i}_z \quad (3)$$

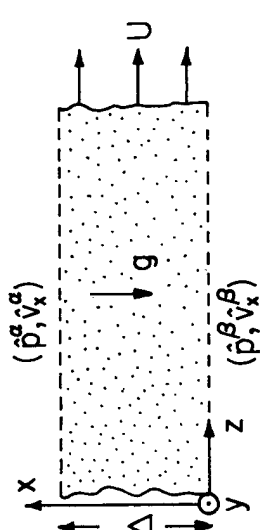
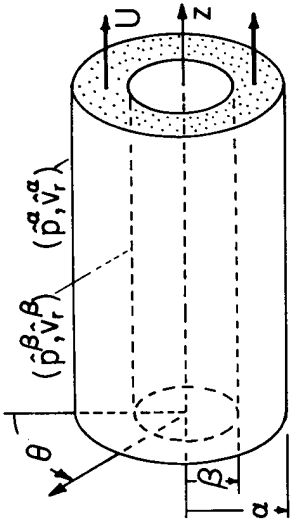
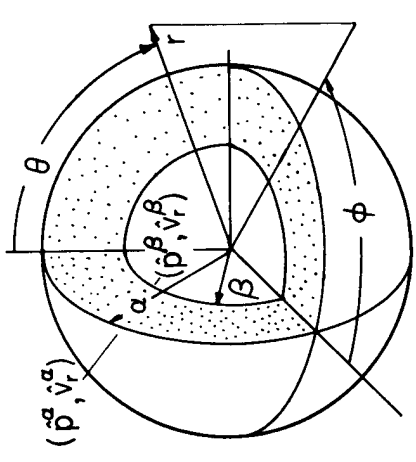
To linear terms in the perturbations, Bernoulli's equation (Eq. 7.8.11) gives the pressure

$$p = -\frac{1}{2} \rho U^2 - \mathcal{E} + \Pi - \rho g x + \rho \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial z} \right) \Theta' \quad (4)$$

In terms of complex amplitudes, this expression becomes

$$p = -\frac{1}{2} \rho U^2 - \mathcal{E} + \Pi - \rho g x + \text{Re } \hat{p}(x) e^{j(\omega t - k_y y - k_z z)} \quad (5)$$

Table 7.9.1. Pressure-velocity relations for perturbations of inviscid fluid.

Cartesian	Cylindrical	Spherical
		
$p = \Pi - \frac{1}{2} \rho U^2 - \rho g x + \text{Re } \hat{p}(x) e^{j(\omega t - k_y y - k_z z)} \quad (a)$	$p = \Pi - \frac{1}{2} \rho U^2 - \rho g + \text{Re } \hat{p}(r) e^{j(\omega t - m\theta - kz)} \quad (d)$	$p = \Pi - \rho g + \text{Re } \hat{p}(r) P_n^m(\cos \theta) e^{j(\omega t - m\phi)} \quad (g)$
$\hat{p}(x) = j(\omega - k_z U) \rho \hat{\phi}(r) \quad (b)$	$\hat{p}(r) = j(\omega - kU) \rho \hat{\phi}(r) \quad (e)$	$\hat{p}(r) = j\omega \rho \hat{\phi}(r) \quad (h)$
$\begin{bmatrix} \hat{p}^\alpha \\ \hat{p}^\beta \end{bmatrix} = \frac{j(\omega - k_z U) \rho}{\gamma} \begin{bmatrix} -\coth \gamma \Delta & \frac{1}{\sinh \gamma \Delta} \\ \frac{-1}{\sinh \gamma \Delta} & \coth \gamma \Delta \end{bmatrix} \begin{bmatrix} \hat{v}_x^\alpha \\ \hat{v}_x^\beta \end{bmatrix} \quad (c)$ $\gamma \equiv \sqrt{k_y^2 + k_z^2}$	$\begin{bmatrix} \hat{p}^\alpha \\ \hat{p}^\beta \end{bmatrix} = j(\omega - kU) \rho \begin{bmatrix} F_m(\beta, \alpha) G_m(\alpha, \beta) \\ G_m(\beta, \alpha) F_m(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \hat{v}_r^\alpha \\ \hat{v}_r^\beta \end{bmatrix} \quad (f)$ $\gamma \equiv \sqrt{k^2 - \frac{(\omega - kU)^2}{a^2}}$	$\begin{bmatrix} \hat{p}^\alpha \\ \hat{p}^\beta \end{bmatrix} = j\omega \rho \begin{bmatrix} F_n(\beta, \alpha) G_n(\alpha, \beta) \\ G_n(\beta, \alpha) F_n(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \hat{v}_r^\alpha \\ \hat{v}_r^\beta \end{bmatrix} \quad (i)$
<p>Compressible:</p> $\gamma \equiv \sqrt{k_y^2 + k_z^2 - \frac{(\omega - k_z U)^2}{a^2}}$	<p>(See Table 2.16.2 for <math>F_m</math> and <math>G_m</math>)</p> <p>Compressible; replace <math>k \rightarrow \gamma</math> in <math>F_m</math> and <math>G_m</math>:</p> $\gamma \equiv \sqrt{k^2 - \frac{(\omega - kU)^2}{a^2}}$	<p>(See Table 2.16.3 for <math>F_n</math> and <math>G_n</math>)</p>

where

$$\hat{p}(\mathbf{x}) = j(\omega - k_z U) \rho \hat{\theta}(\mathbf{x}) \quad (6)$$

Note that the first four terms in Eq. 5, the "equilibrium" pressure, are independent of time; but, because of the gravitational force, this pressure is a linearly decreasing function of altitude,  $x$ .

With the understanding that it is only the part of the pressure that is a function of time at a fixed location  $(x, y, z)$  that is being described (the last term in Eq. 5), Eq. 6 is used to write Eq. 2 as the pressure-velocity relations summarized in Table 7.9.1.

Streaming Cylindrical Annulus: In the cylindrical configuration of Table 7.9.1, the fluid again assumes a stationary state of streaming in the  $z$  direction with the uniform velocity  $U$ . However, it is assumed that the effects of gravity are negligible. The relations summarized in Table 7.9.1 follow by exploiting the flux-potential relations of Table 2.16.2. The reasoning is identical to that for the planar relations.

Static Spherical Shell: In the spherical configuration, it is assumed that the fluid equilibrium is static, so that the perturbation velocity is the total velocity. Also, the effects of gravity are ignored. Then, the relations summarized in Table 7.9.1 follow from those of Table 2.16.3 using the reasoning already described.

### 7.10 Weak Compressibility

To specify the relationship between mass density and the other dynamical variables, it is helpful to distinguish between those tied to the material and to a given position in space. Thus, a constitutive law relating mass density to extensive variables,  $\alpha_i$ , and pressure,  $p$ , takes the form

$$\rho = \rho(\alpha_1, \dots, \alpha_m, p) \quad (1)$$

One of the  $\alpha$ 's might be a concentration (perhaps of salt in water) or might be the entropy density. In general, these variables are themselves described by still other laws that bring in additional rate processes. For example, the molecular diffusion in the face of material convection governs the concentrations, while heat conduction and convection determines the distribution of entropy. Coupling to additional subsystems is avoided (and hence closure of the laws needed to describe the dynamics obtained) by taking the  $\alpha_i$ 's as being conserved by fluid of fixed identity. Just as Eq. 7.2.1 then implies Eq. 3, it follows that

$$\frac{\partial \alpha_i}{\partial t} + \nabla \cdot \alpha_i \vec{v} = 0 \quad (2)$$

The pressure is not carried in this fashion by the material. Its role is simplified by confining the discussion to excursions of pressure that can be described as linear perturbations from a reference pressure  $p_r$ . Thus, Eq. 1 is specialized to

$$\rho = \rho(\alpha_1, \dots, \alpha_m, p_r) + \frac{1}{a} (p - p_r) \quad (3)$$

where  $a$ , defined by

$$a^{-1} = \sqrt{\left(\frac{\partial \rho}{\partial p}\right) \alpha_i}'s = \text{constant} \quad (4)$$

is taken as being independent of  $p$ , and is identified in the next section as the velocity of an acoustic wave.

If coupling to the thermodynamic subsystem were self-consistently included in the model (Sec. 7.23), it would be found that for processes having rates typical of acoustic applications, it is the entropy density that is held fixed (possibly along with other  $\alpha_i$ 's) in Eq. 4.

### 7.11 Acoustic Waves and Transfer Relations

Compressibility gives rise to time delays associated with the propagation of acoustic waves. For many purposes, acoustic phenomena can be represented in terms of small perturbations from an equilibrium of uniform density  $\rho_0$  and pressure  $p_0$ . In most acoustic applications, the equilibrium is also static, but to be able to represent doppler-related phenomena, included in this section is the possibility that the fluid streams with a uniform  $z$ -directed velocity,  $U$ .

The equations of motion that relate perturbations  $v'$ ,  $\rho'$ , and  $p'$  in the velocity, density, and

pressure, respectively, are conservation of mass and momentum, Eqs. 7.2.3 and 7.4.4 with  $\vec{F}_{ex} = 0$ . Written to linear terms in the perturbation quantities, these are

$$\rho_0 \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial z} \right) \vec{v}' + \nabla p' = 0 \quad (1)$$

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial z} \right) \rho' + \rho_0 \nabla \cdot \vec{v}' = 0 \quad (2)$$

The equation of state, Eq. 7.10.3, provides the third relation. It follows that

$$p' = a^2 \rho' \quad (3)$$

Typical values of the acoustic velocity,  $a$ , as well as the mass density and the acoustic impedance (to be defined in Sec. 7.13) are given in Table 7.11.1.

Table 7.11.1. Sound velocity, mass density and acoustic impedance for common fluids.<sup>1</sup>

Fluid	Temperature T°C	Acoustic velocity a (m/sec)	Mass density $\rho$ (kg/m <sup>3</sup> )	Acoustic impedance $Z_0 \left( \frac{\text{n-sec}}{\text{m}} \right)$
<b>Gases</b>				
Air	0	331.45	1.293	429
He	0	970	0.1785	173
CO <sub>2</sub>	0	258	1.977	509
H <sub>2</sub>	0	1269	0.08988	114
O <sub>2</sub>	0	317	1.429	452
N <sub>2</sub>	0	337	1.250	421
<b>Liquids</b>				
Water	17	$1.43 \times 10^3$	$0.999 \times 10^3$	$1.43 \times 10^6$
Benzene	20	$1.32 \times 10^3$	$0.879 \times 10^3$	$1.16 \times 10^6$
Glycerine	20	$1.92 \times 10^3$	$1.26 \times 10^3$	$2.42 \times 10^6$
Mercury	20	$1.45 \times 10^3$	$1.35 \times 10^4$	$1.96 \times 10^7$

The operators in Eqs. 1 and 2 are linear, and have constant coefficients. Thus, the velocity can be eliminated as a variable between the divergence of Eq. 1 and the convective derivative of Eq. 2, to obtain

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial z} \right)^2 p' = a^2 \nabla^2 p' \quad (4)$$

The second convective derivative on the left is the second derivative with respect to time for an observer moving with the velocity  $U$  in the  $z$  direction. Hence, in that moving frame, Eq. 4 is the wave equation and shows that waves have the velocity,  $a$ , relative to the fluid.

Pressure-Velocity Relations for Planar Layer: In the prototype configuration of Fig. 7.11.1, a layer of compressible but inviscid fluid fills the planar region between the  $\alpha$  and  $\beta$  planes.

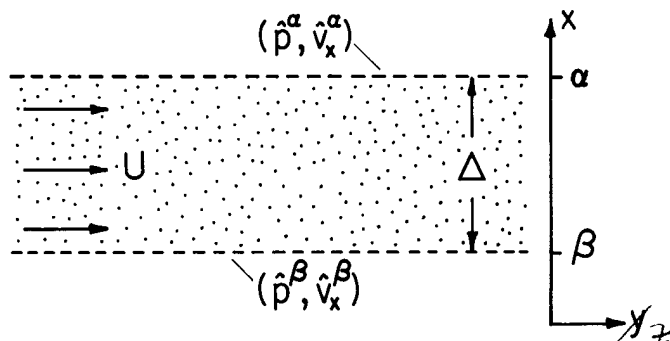


Fig. 7.11.1.

A layer of compressible fluid is bounded from above and below by surfaces having the perturbation deflections  $\xi^\alpha$  and  $\xi^\beta$ . The pressures just inside the fluids adjacent to these surfaces are  $p^\alpha$  and  $p^\beta$ , respectively.

1. L. L. Beranek, Acoustic Measurements, John Wiley & Sons, New York, 1949, pp. 40-46.



Solutions to Eqs. 1-4 take the form  $p' = \text{Re } \hat{p}(x) e^{j(\omega t - k_y y - k_z z)}$ . From Eq. 4, it follows that

$$\frac{d^2 \hat{p}}{dx^2} - \gamma^2 \hat{p} = 0 \quad (5)$$

where

$$\gamma \equiv \sqrt{k_y^2 + k_z^2 - \frac{(\omega - k_z U)^2}{a^2}} \quad (6)$$

The program is now the same as in Sec. 2.16. With perturbation pressures at  $x = \alpha$  and  $x = \beta$  denoted by  $(\hat{p}^\alpha, \hat{p}^\beta)$ , the solution to Eq. 5 is

$$\hat{p}(x) = \frac{1}{\sinh \gamma \Delta} [\hat{p}^\alpha \sinh \gamma(x - \beta) - \hat{p}^\beta \sinh \gamma(x - \alpha)] \quad (7)$$

The x component of Eq. 1 then gives  $\hat{v}_x$  as

$$\begin{aligned} \hat{v}_x &= \frac{j}{\rho_0 (\omega - k_z U)} \frac{d\hat{p}}{dx} \\ &= \frac{j\gamma}{\rho_0 (\omega - k_z U) \sinh \gamma \Delta} \{ \hat{p}^\alpha \cosh \gamma(x - \beta) - \hat{p}^\beta \cosh \gamma(x - \alpha) \} \end{aligned} \quad (8)$$

Evaluation of this expression at  $x = \alpha$  gives  $\hat{v}_x^\alpha(\hat{p}^\alpha, \hat{p}^\beta)$  and at  $x = \beta$  gives  $\hat{v}_x^\beta(\hat{p}^\alpha, \hat{p}^\beta)$ . This pair of equations is then inverted to give transfer relations (c) of Table 7.9.1, but with  $\gamma$  as defined by Eq. 6.

Pressure-Velocity Relations for Cylindrical Annulus: The same arguments as just outlined extend the cylindrical relations of Table 7.9.1 to include acoustic phenomena. With the substitution  $p' = \text{Re } \hat{p}(r) \exp j(\omega t - m\theta - kz)$ , Eq. 4 reduces to Bessel's equation, Eq. 2.16.19, with  $\phi \rightarrow \hat{p}$  and  $k^2 \rightarrow \gamma^2$  where

$$\gamma^2 \equiv k^2 - \frac{(\omega - k_z U)^2}{a^2} \quad (9)$$

Thus, solutions for  $\hat{p}(r)$  take the form of Eq. 2.16.25. From the radial component of Eq. 1,  $\hat{v}_r$  is then evaluated at the  $\alpha$  and  $\beta$  surfaces. The resulting transfer relations are the same as Eq. (f) of Table 7.9.1 if the functions  $F_m$  and  $G_m$  are evaluated replacing  $k \rightarrow \gamma$ . Because  $\gamma$  depends on the layer properties, these functions are now designated by three arguments. For example  $F_m(x, y, \gamma)$  is  $F_m$  as summarized in Table 2.16.2 with  $k \rightarrow \gamma$ .

## 7.12 Acoustic Waves, Guides and Transmission Lines

In the configuration shown in Fig. 7.12.1, fluid having a static equilibrium is confined between a rigid wall at  $x = 0$  and a deformable one at  $x = d + \xi$ . In addition to this transverse drive, a longitudinal excitation can be imposed at  $z = 0$  and an acoustic load attached at  $z = \ell$ . In this section it is assumed that all excitations have the same real frequency  $\omega$  and that sinusoidal steady-state conditions are established.

In specific terms, the acoustic response to the transverse drive demonstrates effects of compressibility on interactions across a layer of fluid. The compressible and inertial quasistatic limits discussed in general terms in Sec. 7.22, are exemplified by this response.

The eigenmodes of the response to the transverse drive represent fluid motions between rigid plates. The structure is then a planar acoustic waveguide. In a typical guide, a source having the frequency  $\omega$  excites the system at one longitudinal boundary ( $z = 0$ ) and a load exists at another ( $z = \ell$ ). Both source and load are often electromechanical. If the frequency is lower than cutoff frequency determined in the following, interactions between longitudinal boundaries

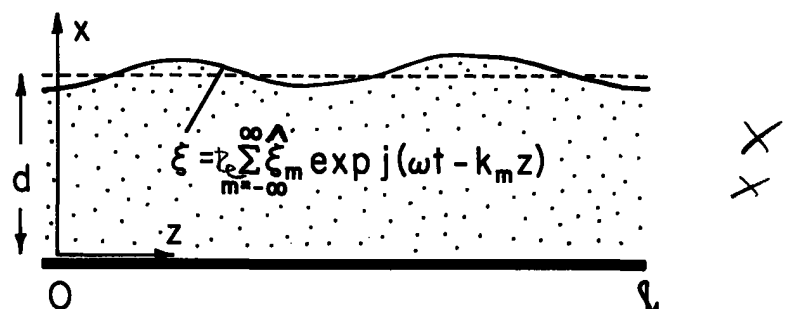


Fig. 7.12.1. Planar region is excited from transverse boundary at  $x = d + \xi$ . Longitudinal boundary conditions typically represent a load at  $z = \ell$  and a source at  $z = 0$ .

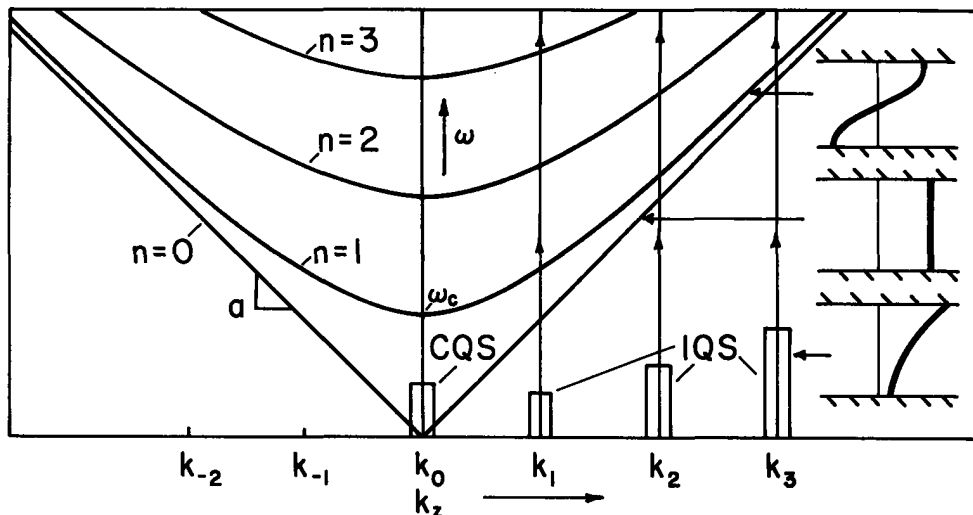


Fig. 7.12.2. Regions of  $\omega$ - $k_z$  plane characterize  $x$  dependence of response to transverse drive of each Fourier mode as driving frequency is raised.

can be represented in terms of the principal mode. This section carries the associated subject of acoustic transmission lines far enough to make clear the analogy with electromagnetic transmission lines.

Response to Transverse Drive: It follows from Eq. 7.5.5 that to linear terms the deformation of the upper boundary stipulates the velocity in the plane  $x = d$ . So, transverse boundary conditions are

$$\hat{v}_x^a = j\omega\hat{\xi}, \quad \hat{v}_x^b = 0 \quad (1)$$

Here,  $\hat{\xi}$  is any one of the Fourier amplitudes,  $\hat{\xi}_m$ , specified in Fig. 7.12.1. It follows from Eq. (c) of Table 7.9.1 (with  $\gamma$  defined by Eq. 7.11.6) that the pressure amplitudes at the upper and lower boundaries are

$$\begin{bmatrix} \hat{p}_a \\ \hat{p}_b \end{bmatrix} = \frac{\omega^2 \rho_0 \hat{\xi}}{\gamma \sinh \gamma d} \begin{bmatrix} \cosh \gamma d \\ 1 \end{bmatrix} \quad (2)$$

These in turn are substituted into Eq. 7.11.7 to show that the pressure distribution over the duct cross section is

$$p = \text{Re} \sum_{m=-\infty}^{+\infty} \omega^2 \rho_0 \frac{\cosh \gamma_m x}{\gamma_m \sinh \gamma_m d} \hat{\xi}_m e^{j(\omega t - k_m z)} \quad (3)$$

where

$$\gamma_m = \sqrt{k_m^2 - \frac{\omega^2}{a^2}}$$

For the moment, consider that the system extends to "infinity" in the  $z$  direction, or alternatively that it closes on itself, so that the additional response from the longitudinal boundary conditions is absent. With the expression for  $\gamma_m$  given with Eq. 3 in view, the  $x$  dependence of each Fourier component can be pictured with the help of Fig. 7.12.2. At very low frequency, and for Fourier components other than  $m = 0$ ,  $\gamma_m \rightarrow k$ . Thus, the  $x$  distribution is the decaying function familiar from the incompressible case. These low-frequency  $m \neq 0$  components are termed the inertial (or incompressible) quasistatic (IQS) response. Note that they are the result of the part of the excitation that automatically conserves volume. The  $m = 0$  part results from the "d-c" component of the surface displacement and so does not conserve volume. Nevertheless, at low frequencies the  $m = 0$  component has a quasistatic nature. For this component, Eq. 3 takes the limiting form

$$\hat{p}_0 \rightarrow -a^2 \rho_0 \left( \frac{\hat{\xi}_0}{d} \right) \quad (4)$$

At low frequencies, this compressible quasistatic (CQS) response has a pressure that is uniformly distributed over the layer cross section. It is just what would be expected as the pressure distribution

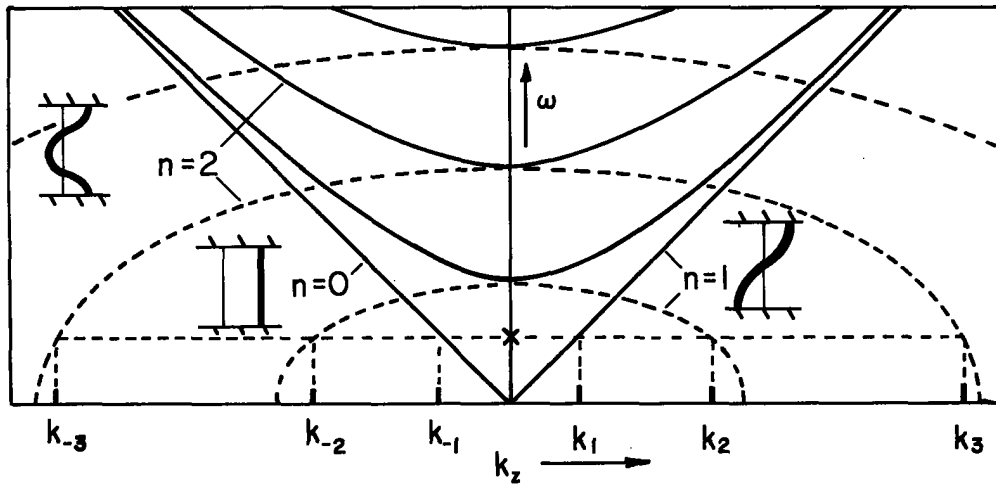


Fig. 7.12.3. Dispersion relation showing complex  $k_z$  for real  $\omega$ . At the frequency shown, all but the  $n=0$  modes are evanescent (cutoff).

in a fluid region slowly driven by vertical displacement of a horizontal piston.

As the frequency is raised, each  $m \neq 0$  component takes on a uniform distribution at the frequency  $|\omega| = a|k_m|$  (and hence  $\gamma_m = 0$ ). For higher frequencies,  $\gamma_m$  is purely imaginary and the distribution becomes oscillatory. The curves shown in Fig. 7.12.2 are for  $\gamma_m d = jn\pi$ , where the frequency follows from Eq. 3 as

$$\omega = a \sqrt{k_m^2 + \left(\frac{n\pi}{d}\right)^2} \quad (5)$$

and the transverse pressure distribution is  $n$  half-wavelengths. These curves also denote resonances in the driven response, as is evident from the fact that the denominator of Eq. 2 vanishes as the frequency meets the condition of Eq. 5, so that  $\gamma_m d = jn\pi$ .

**Spatial Eigenmodes:** Longitudinal conditions are satisfied by adding to the transverse driven response the eigenmodes consistent with both transverse boundaries being rigid (with  $\xi = 0$ ). From Eq. 2,

$$\gamma d = jn\pi \quad (6)$$

where now  $k_z$  is a complex eigenvalue determined by combining Eq. 6 with the definition of  $\gamma_1$

$$k_n = \pm \sqrt{\frac{\omega^2}{a^2} - \left(\frac{n\pi}{d}\right)^2} \quad (7)$$

Thus, the spatial transient response to the longitudinal boundary conditions is composed of two or more propagating modes (real longitudinal wavenumbers) and an infinite number of evanescent modes. These wavenumbers are shown graphically in Fig. 7.12.3, where complex values of  $k_z$  are drawn for real values of  $\omega$ . The  $n$ th mode is evanescent or cut off below the frequency.

$$\omega_c = a \left(\frac{n\pi}{d}\right) \quad (8)$$

These spatial evanescent plus propagating eigenmodes form an orthogonal set that can be used to satisfy longitudinal boundary conditions having an arbitrary dependence on  $x$ .

**Acoustic Transmission Lines:** The  $n = 0$  mode has no cutoff frequency and propagates without dispersion at the velocity  $a$ , regardless of frequency. Such a mode is termed the "principal" mode. It is distinguished by having a pressure and velocity independent of  $x$  and  $y$ , and hence no transverse components of velocity anywhere. The principal mode is independent of the tube cross section. It exists in tubes of arbitrary geometry and is comprised of the same fluid motion as for a plane wave in free space. These principal modes are the most common in acoustic systems, and are conveniently pictured in terms of transmission line theory analogous to that used for TEM waves on electromagnetic transmission lines.<sup>1</sup>

1. P. C. Magnusson, Transmission Lines and Wave Propagation, Allyn and Bacon, Boston, Mass., 1970, pp. 57-111.



A few further steps show how impedance concepts apply to the principal mode. With the understanding that  $k_1 = k$ ,

$$p = \text{Re} \hat{P} e^{j\omega t} = \text{Re} [\hat{p}^+ e^{-jkz} + \hat{p}^- e^{jkz}] e^{j\omega t} \quad (9)$$

From Eq. 7.11.1 it follows that

$$v_z = \text{Re} \hat{V} e^{j\omega t} = \text{Re} \frac{1}{Z_0} [\hat{p}^+ e^{-jkz} - \hat{p}^- e^{jkz}] e^{j\omega t} \quad (10)$$

where the characteristic acoustic impedance is defined as

$$Z_0 \equiv a\rho_0 \quad (11)$$

The (specific) acoustic impedance is defined as the ratio  $\hat{P}/\hat{V}$ , and is given by taking the ratio of complex amplitudes given by Eqs. 9 and 10, and then dividing through by  $\hat{p}^+$ :

$$z = \frac{\hat{P}}{\hat{V}} = Z_0 \left[ \frac{1 + \hat{\Gamma} e^{2jkz}}{1 - \hat{\Gamma} e^{2jkz}} \right] \quad (12)$$

The reflection coefficient  $\hat{\Gamma}$  has been defined as the ratio of reflected to forward wave amplitudes

$$\hat{\Gamma} = \frac{\hat{p}^-}{\hat{p}^+} \quad (13)$$

In terms of the impedance function, the analysis of a system proceeds by specifying the load impedance at  $z = l$ . For example, if there is a rigid wall at  $z = l$ ,  $v_z = 0$  and the impedance is infinite. Or, if the load is an absorber, then  $P/\hat{V}$  is a real number. Given the load impedance at  $z = l$ , Eq. 12 can be inverted to find the reflection coefficient  $\hat{\Gamma}$ . Then, the impedance at any other point on the line can be determined by using Eq. 12 evaluated using the appropriate values of  $z$  and the previously determined value of reflection coefficient. The Smith chart, familiar in the theory of electromagnetic transmission lines, is a graphical representation of the calculation outlined here.

From Eq. 12, it is clear that if the reflection coefficient is to vanish, so that there is only a forward wave, then the load impedance must be  $Z_0$ . This line is then "matched." If there is no reflected wave,  $Z_0$  has the physical significance of being  $\hat{P}/\hat{V}$  at any position  $z$ . Typical values of the characteristic (specific) acoustic impedance are given in Table 7.11.1. For a given velocity response,  $Z_0$  typifies the required pressure excursion. Values of  $Z_0$  in liquids are typically 3000 times greater than in gases.

### 7.13 Experimental Motivation for Viscous Stress Dependence on Strain Rate

Shear stress is exhibited by common fluids in motion, but not at rest. For most static fluids, the isotropic pressure of Sec. 7.4 is all that remains of the mechanical stress exerted on an element of fluid by its surroundings.

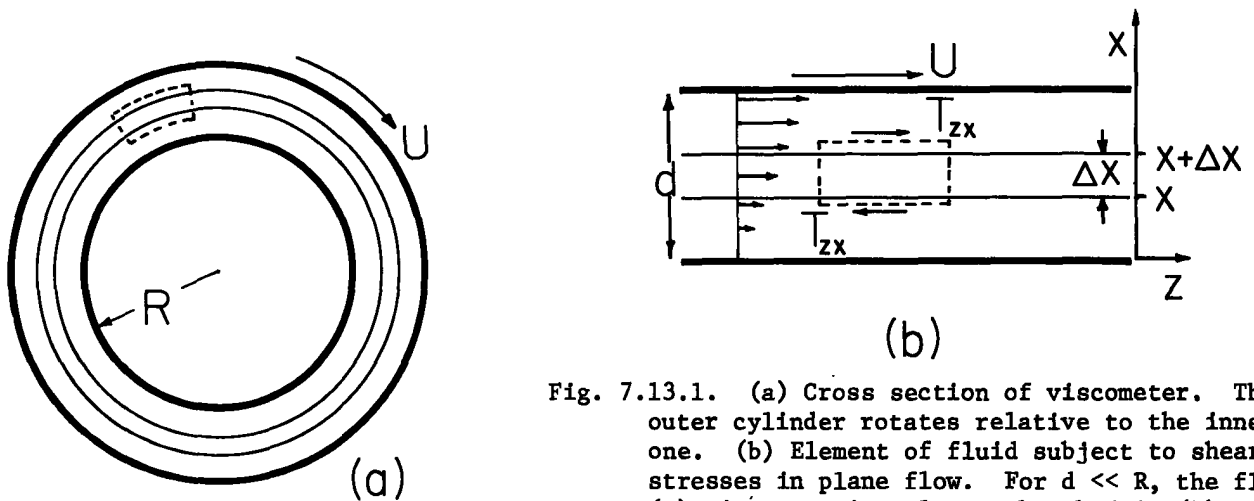


Fig. 7.13.1. (a) Cross section of viscometer. The outer cylinder rotates relative to the inner one. (b) Element of fluid subject to shear stresses in plane flow. For  $d \ll R$ , the flow, (a), is approximately as sketched in (b).

Typical of experiments that establish how a shear stress is transmitted across fluid layers suffering finite rate deformations is the Couette viscometer shown in Fig. 7.13.1a. The pair of concentric cylinders is arranged with the inside cylinder fixed and the outside one rotating at a constant peripheral velocity  $U$ . With the inner cylinder mounted on a torsion spring, static azimuthal deflections are a measure of the torque, and hence the shear stress, exerted by the surrounding fluid.

If the spacing  $d$  is small compared to the radius, a section of the annular region filled by fluid and bounded by the cylinders assumes the planar appearance of Fig. 7.13.1b. For common fluids, it is experimentally observed that the force per unit area,  $T_z$ , transmitted to the fixed inner plate by the moving outer one has the dependence on  $U$  and  $d$ ,

$$T_z = \eta \left( \frac{U}{d} \right) \quad (1)$$

with  $\eta$  a constant defined as the absolute viscosity or the first coefficient of viscosity. Typical values of  $\eta$  and the kinematic viscosity  $\nu \equiv \eta/\rho$  are given in Table 7.13.1.

Table 7.13.1. Typical viscosities of liquids and gases at 20°C and atmospheric pressure.\*

	Absolute viscosity $\eta$ (kg/sec m)	Mass density $\rho$ (kg/m <sup>3</sup> )	Kinematic viscosity $\nu$ (m <sup>2</sup> /sec)
Water	$1.002 \times 10^{-3}$	$1.00 \times 10^3$	$1.002 \times 10^{-6}$
Mercury	$1.55 \times 10^{-3}$	$13.6 \times 10^3$	$1.14 \times 10^{-7}$
Heptane	$0.409 \times 10^{-3}$	$0.684 \times 10^3$	$5.99 \times 10^{-7}$
Glycerin	1.49	$1.26 \times 10^3$	$1.18 \times 10^{-3}$
Carbon tetrachloride	$0.969 \times 10^{-3}$	$1.59 \times 10^3$	$6.09 \times 10^{-7}$
Corn oil	$5.5 \times 10^{-2}$	$0.914 \times 10^3$	$6.02 \times 10^{-5}$
Cerelox-117 alloy	$\sim 5 \times 10^{-4}$	$8.8 \times 10^3$	$\sim 6 \times 10^{-8}$
Olive oil	0.138	$0.918 \times 10^3$	$1.51 \times 10^{-4}$
Turpentine	$1.487 \times 10^{-3}$	$0.87 \times 10^3$	$1.71 \times 10^{-6}$
Air	$1.83 \times 10^{-5}$	1.20	$1.53 \times 10^{-5}$
Carbon dioxide	$1.48 \times 10^{-5}$	1.98	$7.47 \times 10^{-6}$
Hydrogen	$0.87 \times 10^{-5}$	0.09	$9.67 \times 10^{-5}$
Oxygen	$2.02 \times 10^{-5}$	1.43	$1.41 \times 10^{-5}$
<p>* <u>Conversion:</u> <math>\eta_{\text{mks}}</math> (kg/sec m) = 0.1 <math>\eta_{\text{cgs}}</math> (Poise); Poise <math>\equiv</math> gm/sec cm  <math>\nu_{\text{mks}}</math> (m<sup>2</sup>/sec) = <math>10^{-4}</math> <math>\nu_{\text{cgs}}</math> (Stoke); Stoke <math>\equiv</math> cm<sup>2</sup>/sec</p>			

Even with common fluids, at sufficiently large rotational velocities, Eq. 1 no longer holds. The planar motions are replaced by two- and three-dimensional ones, and eventually turbulence (motions that are never steady). The postulated viscometer flow is unstable at high velocities. The result is a complex flow, not the one postulated here.

The inverse dependence of  $T_z$  on  $d$  in Eq. 1 suggests that any pair of planes in the fluid are equivalent to the plates. Instead of  $d$ , the spacing is  $\Delta x$ , and instead of  $U$ , the relative velocity is the difference  $v_z(x + \Delta x) - v_z(x)$ . With  $T_{zx}$  the shear stress transmitted to the layer from the fluid above, Eq. 1 suggests that

$$T_{zx} = \eta \left[ \frac{v_z(x + \Delta x) - v_z(x)}{\Delta x} \right] \quad (2)$$

The incremental layer must itself be in force equilibrium. For the incremental volume shown in Fig. 7.13.1b this means that the shear stress exerted on the layer by the fluid below is equal in magnitude to that given by Eq. 2 and that normal stresses acting in the  $z$  direction on the right and left surfaces cancel. In the viscometer, this is assured by the rotational symmetry of the flow, which excludes variations in the  $z$  direction.

In the limit  $\Delta x \rightarrow 0$ , Eq. 2 becomes

$$T_{zx} = \eta \frac{\partial v_z}{\partial x} \quad (3)$$

This simple but important example supports the postulate that viscous stresses are linear functions of spatial velocity derivatives.

It also illustrates the steps involved in finding the stresses on an arbitrary volume of fluid. First, the particular spatial derivatives that can reasonably give rise to mechanical stresses are defined as the components of the strain-rate tensor. Then, appeal is made to conditions of isotropy and experiments like the Couette viscometer to relate the strain-rate tensor to the stress. To carry the derivation one step further, the divergence of the viscous stress tensor finally gives the required viscous force density. These three steps are carried out in the next three sections.

#### 7.14 Strain-Rate Tensor

Consider the difference in fluid velocity at two points separated by the incremental distance  $\Delta \vec{r}$ , as shown in Fig. 7.14.1. The  $i$ th component, expanded in a Taylor expansion about the position  $\vec{r}$ , is

$$v_i(\vec{r} + \Delta \vec{r}, t) - v_i(\vec{r}, t) = v_i(\vec{r}, t) + \frac{\partial v_i}{\partial x_j}(\vec{r}, t) \Delta x_j - v_i(\vec{r}, t) \quad (1)$$

As  $\Delta x_j \rightarrow 0$ , all that remains in this expression is the second term, which can be written identically as

$$v_i(\vec{r} + \Delta \vec{r}, t) - v_i(\vec{r}, t) = \frac{1}{2} \left[ \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right] \Delta x_j + \dot{e}_{ij} \Delta x_j \quad (2)$$

where  $\dot{e}_{ij}$  is the strain-rate tensor, defined as

$$\dot{e}_{ij} \equiv \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (3)$$

Just as translational fluid motions cannot give rise to a viscous stress, neither can combinations of the spatial velocity derivatives that represent a pure rotation. Note that the first term in Eq. 2 is composed of a sum on products of  $\Delta x_j$  and components of the curl  $\vec{\nabla} \times \vec{v}$ . Thus, it represents relative fluid motion in the neighborhood of  $\vec{r}$  that is circulating about the point. This combination of spatial derivatives is not expected to be proportional to the viscous stress. Thus the strain rate, Eq. 3, is identified as that combination of the spatial derivatives that should be proportional to the stress components.

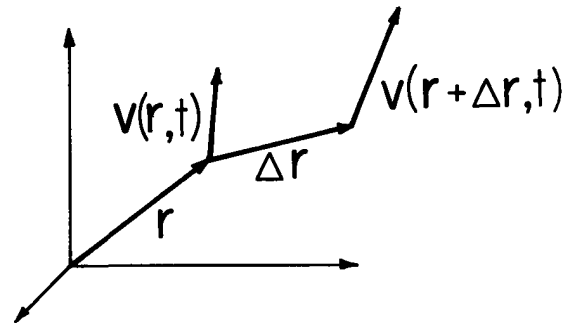


Fig. 7.14.1. Positions in the flow separated by an incremental distance  $\Delta \vec{r}$ .

The components of the strain rate take on physical significance if associated with the types of flow shown in Fig. 7.14.2. The diagonal components  $i = j$  represent dilatational motion, while the components  $i \neq j$  stand for relative motions such that fluid particles located on initially perpendicular lines are found an instant later on lines at an acute angle.

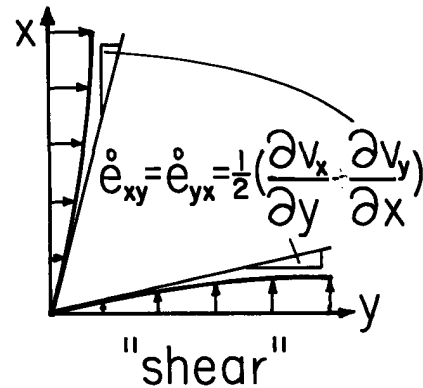
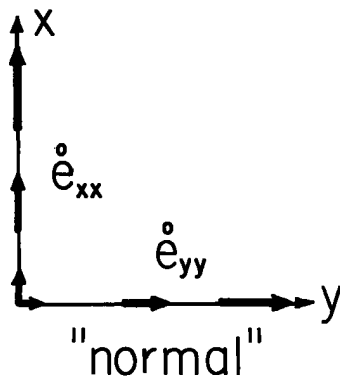


Fig. 7.14.2. Illustration of the geometric significance of "normal" and "shear" strain rates.

The viscous force density is a mechanism for introducing vorticity and hence local circulation to a flow. This point is developed in Sec. 7.18. That the viscous stress is here postulated to be independent of local rotation is a seeming contradiction. The stress tensor must be distinguished from its tensor divergence, the force density. Even though the vorticity is not linked to the local viscous stress by linear constitutive laws, its spatial rates of change are an essential part of the force density.

**Fluid Deformation Example:** The plane flow shown in Fig. 7.13.1b is  $\vec{v} = U(x/d)\vec{i}_z$ . That the flow has translational, rotational and strain-rate parts is illustrated by following the same procedure of adding and subtracting equal parts used in going from Eq. 1 to Eq. 2:

$$U \frac{x}{d} \vec{i}_z = \frac{U}{2} \vec{i}_z + \frac{U}{4} \left[ \left( \frac{2x}{d} - 1 \right) \vec{i}_z - \left( \frac{2z}{d} \right) \vec{i}_x \right] + \frac{U}{4} \left[ \left( \frac{2x}{d} - 1 \right) \vec{i}_z + \left( \frac{2z}{d} \right) \vec{i}_x \right] \quad (4)$$

The respective terms have the physical significance shown in Fig. 7.14.3.

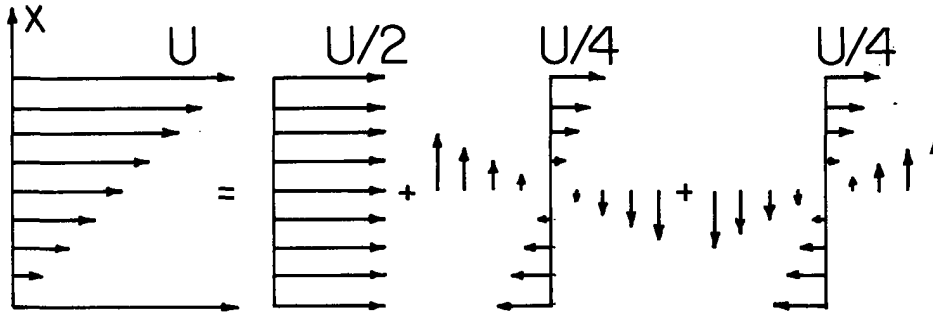


Fig. 7.14.3. Plane shear flow divided into translation, rotation and strain-rate flow.

**Strain Rate as a Tensor:** A discussion of the tensor character of the stress is given in Sec. 3.9. To similarly prove that  $e_{ij}$  transforms from one coordinate system to another in accordance with

$$e'_{ij} = a_{ik} a_{jl} e_{kl} \quad (5)$$

the vector nature of  $\vec{v}$  is exploited:

$$v'_i = a_{ij} v_j \quad (6)$$

It follows from this relation and the definition of the direction cosines  $a_{ij}$  (Eq. 3.9.7 and discussion following Eq. 3.9.11) that

$$\frac{\partial v'_i}{\partial x'_j} = a_{ik} \frac{\partial v_k}{\partial x'_j} = a_{ik} \frac{\partial v_k}{\partial x_l} \frac{\partial x_l}{\partial x'_j} = a_{ik} a_{jl} \frac{\partial v_k}{\partial x_l} \quad (7)$$

From this expression, the definition of  $e'_{ij}$ , Eq. 3 written in the primed frame of reference, becomes

$$e'_{ij} = \frac{1}{2} \left( \frac{\partial v'_i}{\partial x'_j} + \frac{\partial v'_j}{\partial x'_i} \right) = a_{ik} a_{jl} \frac{1}{2} \left( \frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \right) \quad (8)$$

and Eq. 5 follows. The tensor nature of  $e'_{ij}$  is exploited in the next section.

### 7.15 Stress-Strain-Rate Relations

It is a postulate that the fluids of interest can be described by a linear relationship between viscous stress and strain rate. With  $c_{ijkl}$  coefficients defined as properties of the fluid, the most general linear constitutive relation is

$$T_{ij} = c_{ijkl} e_{kl} \quad (1)$$

Even though these properties must be deduced in the laboratory, the number that must actually be measured can be greatly reduced by exploiting the isotropy of the fluid.

All arguments in this section pertain to relations at a given fixed location in the fluid. The coordinate systems (primed and unprimed) have a common origin at this point, as suggested by Fig. 3.9.3. The fluid is in general not necessarily homogeneous. The properties  $c_{ijkl}$  can be functions of position.

At any given point in an isotropic material, the properties do not depend on the coordinates. Hence, in a primed frame of reference, the constitutive law of Eq. 1 is

$$T'_{ij} = c'_{ijkl} e'_{k\ell} \quad (2)$$

and isotropy requires that the properties are the same:

$$c_{ijkl} = c'_{ijkl} \quad (3)$$

For example, if shear stress and shear strain rate ( $T_{ij}, e_{ij}$ ) are related by a viscosity coefficient in one coordinate system, the same components ( $T'_{ij}, e'_{ij}$ ) will be related by the same coefficient in the primed frame of reference.

**Principal Axes:** For any tensor there is a coordinate system in which it has only normal components. To see this first observe that the stress, having components  $T_{ij}$  in the unprimed frame of reference, gives rise to the traction  $T_i = T_{ij}n_j$  on a surface having the normal vector  $\vec{n}$  (Eq. 3.9.5). Suppose that a plane is defined such that the traction is in the normal direction, and has magnitude  $T$ . Then

$$T_{ij}n_j = Tn_i = Tn_j\delta_{ij} \quad (4)$$

With the components of  $\vec{n}$  regarded as the unknowns, by setting  $i = 1, 2$  and  $3$ , this expression is three equations:

$$\begin{bmatrix} T_{11} - T & T_{12} & T_{13} \\ T_{21} & T_{22} - T & T_{23} \\ T_{31} & T_{32} & T_{33} - T \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \quad (5)$$

These homogeneous relations have a solution if the determinant of the coefficients vanishes. This condition gives three eigenvalues,  $T = T_1, T_2, T_3$ , which are the normal components of stress in three directions.

To actually find one of these directions, the associated eigenvalue  $T$  is inserted into Eqs. 5, a value of  $n_1$  is assumed and any pair of the expressions then solved for  $n_2$  and  $n_3$ . The magnitudes of these components of  $\vec{n}$  are then adjusted so that  $|\vec{n}| = 1$ .

That the three directions found in this way are orthogonal follows from Eq. 4, which gives the traction associated with each of the eigenvalues. Suppose that the eigenvalues  $T_a$  and  $T_b$ , respectively, give the normal vectors  $\vec{n} = \vec{a}$  and  $\vec{n} = \vec{b}$ . Then, from Eq. 4

$$T_{ij}a_j = T_a a_i \quad (6)$$

$$T_{ij}b_j = T_b b_i \quad (7)$$

Multiplication of Eq. 6 by  $b_i$  and of Eq. 7 by  $a_i$  and subtraction gives

$$b_i T_{ij} a_j - a_i T_{ij} b_j = (T_a - T_b) a_i b_i \quad (8)$$

Each of the indices is summed, so they are dummy variables which can be relabeled. In the first term on the left,  $i$  and  $j$  can be interchanged. Then, so long as  $T_{ij}$  is symmetric, it is clear that the terms on the left cancel. Provided that the eigenvalues  $T_a$  and  $T_b$  are distinct, it follows that  $a_i b_i = \vec{a} \cdot \vec{b} = 0$ . These axes, shown here to be orthogonal, are called the principal axes.

**Strain-Rate Principal Axes the Same as for Stress:** The strain rate, like the stress, is a symmetric tensor. This is shown in Sec. 7.14. Suppose that the unprimed coordinates are the principal axes for the strain rate. Then, according to Eq. 1, the shear stress  $T_{yz}$  is

$$T_{yz} = c_{yzxx} e_{xx} + c_{zyyy} e_{yy} + c_{yzzz} e_{zz} \quad (9)$$



The axes in a primed coordinate system gotten from this one by rotating it  $180^\circ$  about the z axis must also be principal axes. Hence, hence Eq. 2 becomes

$$T'_{yz} = c'_{yzxx} \dot{e}'_{xx} + c'_{yzyy} \dot{e}'_{yy} + c'_{yzzz} \dot{e}'_{zz} \quad (10)$$

Formally, the transformation of the stress and strain rate tensors between these coordinates is  $T'_{ij} = a_{ik} a_{jl} T_{kl}$  (Eq. 3.9.11) and  $\dot{e}'_{ij} = a_{ik} a_{jl} \dot{e}_{kl}$  (Eq. 7.14.6), where

$$a_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (11)$$

so with the use of the isotropy condition, Eq. 3, Eq. 10 becomes

$$-T'_{yz} = c_{yzxx} \dot{e}_{xx} + c_{yzyy} \dot{e}_{yy} + c_{yzzz} \dot{e}_{zz} \quad (12)$$

Comparison of this expression with Eq. 9 shows that  $T'_{yz} = 0$ . Similar arguments show that the other shear stress components are zero.

It is concluded that in a coordinate system where the strain rate has only normal components, the stress must also be normal.

Principal Coordinate Relations: That the stress and strain rate have the same principal axes effectively reduces the number of independent coefficients to nine, because in such a coordinate system (now the primed system) Eq. 1 reduces to

$$\begin{bmatrix} T'_{xx} \\ T'_{yy} \\ T'_{zz} \end{bmatrix} = \begin{bmatrix} c_{xxxx} & c_{xxyy} & c_{xxzz} \\ c_{yyxx} & c_{yyyy} & c_{yyzz} \\ c_{zzxx} & c_{zzyy} & c_{zzzz} \end{bmatrix} \begin{bmatrix} \dot{e}'_{xx} \\ \dot{e}'_{yy} \\ \dot{e}'_{zz} \end{bmatrix} \quad (13)$$

But, the isotropy requires a further reduction in this number. For the x axis, it is clear that either  $\dot{e}_{yy}$  or  $\dot{e}_{zz}$  must have the same effect on  $T_{xx}$ . Hence, the first of Eqs. 13 reduces to the first of the following relations

$$\begin{bmatrix} T'_{xx} \\ T'_{yy} \\ T'_{zz} \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_2 \\ k_2 & k_1 & k_2 \\ k_2 & k_2 & k_1 \end{bmatrix} \begin{bmatrix} \dot{e}'_{xx} \\ \dot{e}'_{yy} \\ \dot{e}'_{zz} \end{bmatrix} \quad (14)$$

Because of the isotropy there is no distinction between the x axis and the other two. The same coefficient relates  $T'_{yy}$  to  $\dot{e}'_{yy}$  as relates  $T'_{xx}$  to  $\dot{e}'_{xx}$ , for example. To complete the last step in the deduction of the stress-strain rate relations, observe that Eq. 14 can also be written as

$$\begin{aligned} T'_{xx} &= k_2 \dot{e}'_{nn} + (k_1 - k_2) \dot{e}'_{xx} \\ T'_{yy} &= k_2 \dot{e}'_{nn} + (k_1 - k_2) \dot{e}'_{yy} \\ T'_{zz} &= k_2 \dot{e}'_{nn} + (k_1 - k_2) \dot{e}'_{zz} \end{aligned} \quad (15)$$

where  $\dot{e}'_{nn} = \nabla \cdot \vec{v}$  is the same number regardless of the coordinates used in the evaluation.

Isotropic Relations: The constitutive laws expressed in the form of Eq. 15 are now transformed to the arbitrary unprimed frame by using the transformation law  $T'_{ij} = a_{ki} a_{lj} T_{kl}$  (Eq. 3.9.11 and subsequent discussion):

$$\begin{aligned} T_{ij} &= a_{xi} a_{xj} [k_2 \dot{e}'_{nn} + (k_1 - k_2) \dot{e}'_{xx}] \\ &\quad + a_{yi} a_{yj} [k_2 \dot{e}'_{nn} + (k_1 - k_2) \dot{e}'_{yy}] \\ &\quad + a_{zi} a_{zj} [k_2 \dot{e}'_{nn} + (k_1 - k_2) \dot{e}'_{zz}] \end{aligned} \quad (16)$$

Because  $a_{ki}a_{kj} = \delta_{ij}$  (Eq. 3.9.14 and discussion following Eq. 3.9.11) and  $e_{ij}^0 = a_{ki}a_{lj}e_{kl}^0$ , it follows from Eq. 16 that

$$T_{ij} = k_2 e_{nn}^0 \delta_{ij} + (k_1 - k_2) e_{ij}^0 \quad (17)$$

To be consistent with the coefficient of viscosity defined with Eq. 7.13.3, it is observed that for that plane flow situation, all components of  $e_{ij}^0 = 0$ , except  $e_{zx}^0 = e_{xz}^0 = (\partial v_z / \partial x) / 2$ . Thus, Eq. 7.13.3 is  $T_{zx} = 2\eta e_{zx}^0$ , and Eq. 17 reduces to this expression if

$$k_1 - k_2 = 2\eta \quad (18)$$

By convention, a second coefficient of viscosity,  $\lambda$ , is defined such that

$$k_2 = \lambda - \frac{2}{3} \eta \quad (19)$$

Thus, the viscous stress-strain-rate relations for an isotropic fluid are

$$T_{ij} = (\lambda - \frac{2}{3} \eta) \delta_{ij} e_{kk}^0 + 2\eta e_{ij}^0 \quad (20)$$

In general, the viscosities  $\eta$  and  $\lambda$  are functions of position.

### 7.16 Viscous Force Density and the Navier-Stokes Equation

The total mechanical stress,  $S_{ij}$ , is the sum of the viscous stress given by Eq. 7.15.20 and the isotropic pressure stress remaining with strain rate absent (Eq. 7.4.2). In terms of the strain rate

$$S_{ij} = -p \delta_{ij} + 2\eta e_{ij}^0 + (\lambda - \frac{2}{3} \eta) \delta_{ij} e_{kk}^0 \quad (1)$$

while substitution for  $e_{ij}^0$  from Eq. 7.14.3 gives

$$S_{ij} = -p \delta_{ij} + \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + (\lambda - \frac{2}{3} \eta) \frac{\partial v_k}{\partial x_k} \delta_{ij} \quad (2)$$

The tensor divergence of this expression (Eq. 3.9.1) is the force density required for writing the force balance equation. In taking this divergence,  $\eta$  and  $\lambda$  are for the first time taken as constants. The  $i$ th component is

$$F_i^v = \frac{\partial S_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j} + (\lambda + \frac{1}{3} \eta) \frac{\partial}{\partial x_i} \left( \frac{\partial v_k}{\partial x_k} \right) \quad (3)$$

and translated into vector notation

$$\vec{F}^v = -\nabla p + \eta \nabla^2 \vec{v} + (\lambda + \frac{1}{3} \eta) \nabla (\nabla \cdot \vec{v}) \quad (4)$$

With the use of a vector identity ( $\nabla^2 \vec{v} = \nabla (\nabla \cdot \vec{v}) - \nabla \times \nabla \times \vec{v}$ ), the essential role of vorticity becomes apparent:

$$\vec{F}^v = -\nabla p - \eta \nabla \times (\nabla \times \vec{v}) + (\lambda + \frac{4}{3} \eta) \nabla (\nabla \cdot \vec{v}) \quad (5)$$

Note that in an incompressible fluid, the last term in both Eqs. 4 and 5 vanishes.

With  $\vec{F}_{ex}$  denoting the sum of all force densities other than the internal ones due to pressure and viscosity, the force equation, Eq. 7.4.4, becomes

$$\rho \frac{D\vec{v}}{Dt} + \nabla p = \vec{F}_{ex} + \eta \nabla^2 \vec{v} + (\lambda + \frac{1}{3} \eta) \nabla (\nabla \cdot \vec{v}) \quad (6)$$

This form of the momentum conservation law is termed the Navier-Stokes equation.

## 7.17 Kinetic Energy Storage, Power Flow and Viscous Dissipation

A statement of kinetic energy conservation is made by starting with the  $i$ th component of the force equation, written using a vector identity\*

$$\rho \left\{ \frac{\partial v_i}{\partial t} + [(\nabla \times \vec{v}) \times \vec{v}]_i + \frac{\partial}{\partial x_i} \left( \frac{1}{2} v_j v_j \right) \right\} = (F_{ex})_i + \frac{\partial}{\partial x_j} (S_{ij}) \quad (1)$$

Dot multiplication of this expression by  $\vec{v}$  eliminates the second term on the left, and mass conservation, Eq. 7.2.3, makes it possible to manipulate the remaining inertial terms so that they take the form required for a conservation statement (for example, the form of Eq. 3.13.13):

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho v_i v_i \right) + \frac{\partial}{\partial x_i} \left[ \left( \frac{1}{2} \rho v_j v_j \right) v_i \right] = (F_{ex})_i v_i + \frac{\partial}{\partial x_j} (v_i S_{ij}) - S_{ij} \frac{\partial v_i}{\partial x_j} \quad (2)$$

The viscous stress and pressure term on the right has also been written as a perfect divergence minus what is required to make it agree with the original expression. Integration of Eq. 2 over an arbitrary volume  $V$  then results in perfect divergence terms on the left and right that, by virtue of the tensor form of Gauss' theorem, Eq. 9.6.2, can be converted to surface integrals:

$$\int_V \frac{\partial}{\partial t} \frac{1}{2} \rho \vec{v} \cdot \vec{v} dV + \oint_S \left( \frac{1}{2} \rho \vec{v} \cdot \vec{v} \right) \vec{v} \cdot \vec{n} da = \int_V \vec{F}_{ex} \cdot \vec{v} dV + \oint_S v_i S_{ij} n_j da - \int_V S_{ij} \frac{\partial v_i}{\partial x_j} dV \quad (3)$$

The volume  $V$  can either be fixed in space, or be one of fixed identity. In the latter case, where the surface  $S$  moves with the material itself, what is on the left in Eq. 3 will be recognized as the rate of change with respect to time of the volume integral of the kinetic energy density  $\rho \vec{v} \cdot \vec{v} / 2$  (see the scalar form of the generalized Leibnitz rule, Eq. 2.6.5).

According to Eq. 3, the rate of increase of the total kinetic energy in  $V$  is equal to the rate at which the external force density does work through the volume, plus the rate at which stresses (that balance the viscous and pressure stresses) do work on the volume through the surface  $S$ , minus the last term. That this last term apparently represents a part of the input power that does not go into kinetic energy suggests that it is power leaving the kinetic energy subsystem in the form of heat (viscous dissipation) to be stored in the internal energy of the fluid. To support this interpretation, note that reindexing and then exploiting the symmetry of  $S_{ij}$  gives

$$S_{ij} \frac{\partial v_i}{\partial x_j} \equiv S_{ji} \frac{\partial v_j}{\partial x_i} = S_{ij} \frac{\partial v_j}{\partial x_i} \quad (4)$$

Thus,

$$S_{ij} \frac{\partial v_i}{\partial x_j} = S_{ij} \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = S_{ij} \hat{e}_{ij} \quad (5)$$

With use made of Eq. 7.16.1 to write  $S_{ij}$  in terms of the strain rate, it follows from some algebraic manipulation that

$$S_{ij} \hat{e}_{ij} = -p \vec{\nabla} \cdot \vec{v} + \phi_v \quad (6)$$

where the positive definite quantity

$$\phi_v \equiv \lambda (\hat{e}_{kk}^o)^2 + 4\eta (\hat{e}_{xy}^o{}^2 + \hat{e}_{yz}^o{}^2 + \hat{e}_{zx}^o{}^2) + \frac{2}{3} \eta [(\hat{e}_{xx}^o - \hat{e}_{yy}^o)^2 + (\hat{e}_{yy}^o - \hat{e}_{zz}^o)^2 + (\hat{e}_{zz}^o - \hat{e}_{xx}^o)^2] \quad (7)$$

is identified as the viscous dissipation density. In terms of this density, the integral statement of kinetic power flow (Eq. 3) becomes the statement that the rate of doing work on the fluid is equal to the rate of increase of kinetic energy (the first two terms on the right), plus the rate of increase of energy stored internally by compressing the fluid (the third term on the right), plus the viscous dissipation:

$$\int_V \vec{F}_{ex} \cdot \vec{v} dV + \oint_S v_i S_{ij} n_j da = \int_V \frac{\partial}{\partial t} \left( \frac{1}{2} \rho \vec{v} \cdot \vec{v} \right) dV + \oint_S \left( \frac{1}{2} \rho \vec{v} \cdot \vec{v} \right) \vec{v} \cdot \vec{n} da - \int_V p \vec{\nabla} \cdot \vec{v} dV + \int_V \phi_v dV \quad (8)$$

In general, by mechanisms such as heat conduction, some of the internal energy can be dissipated. But according to the "weak compressibility" model introduced in Sec. 7.10, dilatations result in energy

\*  $\vec{v} \cdot \nabla \vec{v} = (\nabla \times \vec{v}) \times \vec{v} + \frac{1}{2} \nabla (\vec{v} \cdot \vec{v})$

storage. This is clarified by first using mass conservation, Eq. 7.2.3, and then using Eqs. 7.10.2 and 7.10.3 to write the compressibility energy storage term as

$$-p\vec{\nabla}\cdot\vec{v} = \frac{p}{\rho} \frac{D\rho}{Dt} = \frac{p}{\rho} \frac{\partial\rho}{\partial p} \frac{Dp}{Dt} \quad (9)$$

Given the constitutive law of Eq. 7.10.3, an energy density  $W_c$  can be defined:

$$W_c = \int_{p_r}^p \frac{p}{\rho} \frac{\partial\rho}{\partial p} dp = (p-p_r) - [a^2 \rho(\alpha_1, \dots, \alpha_m, p_r) - p_r] \ln \left[ 1 + \frac{(p-p_r)}{a^2 \rho(\alpha_1, \dots, \alpha_m, p_r)} \right] \quad (10)$$

such that Eq. 9 is

$$-p\vec{\nabla}\cdot\vec{v} = \frac{DW_c}{Dt} \quad (11)$$

Hence, what is added up by the volume integration of Eq. 11, called for in Eq. 8, is the time-rate of change of an energy density  $W_c$  as measured by a fluid particle of fixed identity.

### 7.18 Viscous Diffusion

The theme of this section is the interplay between inertial and viscous forces. Approximations underlying relations derived in Secs. 7.19 - 7.21 are established here.

Throughout, the fluid is presumed incompressible, so that

$$\vec{\nabla}\cdot\vec{v} = 0 \quad (1)$$

Even more, the mass density is uniform, as is also the viscosity.

External forces are represented by scalar and vector potentials:

$$\vec{F}_{ex} = -\vec{\nabla}\xi + \vec{\nabla} \times \vec{G} \quad (2)$$

and the Navier-Stokes's equation, Eq. 7.6.6 (written using 7.6.5 rather than 7.6.4), becomes

$$\rho \left[ \frac{\partial\vec{v}}{\partial t} + (\vec{\nabla} \times \vec{v}) \times \vec{v} \right] + \vec{\nabla} \left( \frac{1}{2} \rho \vec{v} \cdot \vec{v} + p + \xi \right) = -\eta \vec{\nabla} \times (\vec{\nabla} \times \vec{v}) + \vec{\nabla} \times \vec{G} \quad (3)$$

Convective Diffusion of Vorticity: It is shown in Sec. 7.8 that in an inviscid fluid, the net vorticity linking a surface of fixed identity is conserved. The basis for proving that this is so, the force equation written in terms of the vorticity  $\vec{\omega} \equiv \vec{\nabla} \times \vec{v}$  (Eq. 7.8.3), is now examined to identify viscous stresses and other rotational forces (represented by  $\vec{G}$ ) as generators of vorticity. The curl of Eq. 3 is

$$\frac{\partial\vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{v}) = -\frac{\eta}{\rho} \vec{\nabla} \times (\vec{\nabla} \times \vec{\omega}) + \frac{1}{\rho} \vec{\nabla} \times (\vec{\nabla} \times \vec{G}) \quad (4)$$

Without the external force, comparison of this expression to that governing magnetic diffusion in a deforming conductor (Eq. 6.2.6) shows a complete analogy. The role of the vorticity,  $\vec{\omega}$ , is played by the magnetic flux density. Just as the magnetic flux linking a surface of fixed identity is dissipated by joule heating, viscous losses tend to dissipate the net vorticity. This is stated formally by integrating Eq. 4 over a surface of fixed identity and exploiting the generalized Leibnitz rule for surface integrals, Eq. 2.6.4:

$$\frac{d}{dt} \int_S \vec{\omega} \cdot \vec{n} da = -\frac{\eta}{\rho} \oint_C (\vec{\nabla} \times \vec{\omega}) \cdot \vec{d}\ell + \frac{1}{\rho} \oint_C \vec{\nabla} \times \vec{G} \cdot \vec{d}\ell \quad (5)$$

In the neighborhood of a fixed wall, for example, an inviscid fluid can slip. In a real fluid, the tangential velocity must vanish. The modification of velocity in the neighborhood of the boundary enters through the viscosity term on the right in Eq. 5 to generate vorticity.

In Chap. 6, the material deformation represented by  $\vec{v}$  is given, and so the magnetic analogue to Eq. 4 is linear. In the vorticity equation,  $\vec{\omega}$  really represents the unknown  $\vec{v}$ , and so Eq. 4 is not linear. But, two important approximations are now identified in which linear differential equations do describe flows. Because  $\vec{v}$  is solenoidal, it is first convenient to represent it in terms of a vector potential, familiar from Sec. 2.18,

$$\vec{v} = \vec{\nabla} \times \vec{A}_v; \quad \vec{\nabla} \cdot \vec{A}_v = 0 \quad (6)$$

Substitution into Eq. 3 then gives

$$\nabla \pi + \nabla \times \vec{C} + \rho \nabla \times (\nabla \times \vec{A}_v) \times (\nabla \times \vec{A}_v) = 0 \quad (7)$$

where

$$\pi = p + \frac{1}{2} \rho \vec{v} \cdot \vec{v} + \xi$$

$$\vec{C} = \rho \frac{\partial \vec{A}_v}{\partial t} - \eta \nabla^2 \vec{A}_v - \vec{G}$$

Perturbations from Static Equilibria: In the equilibrium state,  $\vec{A}_v = 0$ . For incremental flows, the third term in Eq. 7, which is proportional to the product of perturbation quantities, can be ignored. The curl of the remaining terms gives a fourth order expression for  $\vec{A}_v$ :

$$\nabla \times \nabla \times \left[ \rho \frac{\partial \vec{A}_v}{\partial t} - \eta \nabla^2 \vec{A}_v - \vec{G} \right] = 0 \quad (8)$$

Given  $\vec{A}_v$ , and hence  $\vec{C}$ ,  $\pi$  is determined by integrating the first two terms of Eq. 7 between some reference point  $\vec{r}_0$  and the position  $\vec{r}$  of interest,

$$\int_{\vec{r}_0}^{\vec{r}} \nabla \pi \cdot d\vec{\ell} = \pi(\vec{r}) - \pi(\vec{r}_0) = - \int_{\vec{r}_0}^{\vec{r}} \nabla \times \vec{C} \cdot d\vec{\ell} \quad (9)$$

Thus, the relation between pressure and the vector potential is

$$p = p(\vec{r}_0) + \xi(\vec{r}_0) - \xi(\vec{r}) - \int_{\vec{r}_0}^{\vec{r}} \nabla \times \vec{C} \cdot d\vec{\ell} \quad (10)$$

where the dynamic pressure term,  $\rho \vec{v} \cdot \vec{v} / 2$ , is dropped from  $\pi$  because it is the square of a perturbation.

Equations 8 and 10 are used in Sec. 7.19 to derive general relations that are used extensively in the following chapters. Further physical insights are the objective of Sec. 7.20.

Low Reynolds Number Flows: The terms that make Eq. 7 nonlinear arise because of the inertial force density. For flows that are slow enough that viscous diffusion is complete, this force density has a negligible effect. The third term in Eq. 7 is then ignorable for a reason other than its nonlinearity. Indeed, the terms in  $\pi$  and  $\vec{C}$  involving the mass density are also negligible.

To clarify what is meant by this "creep-flow" approximation, external forces are not considered. The Navier-Stokes's equation, Eq. 7.16.6, is written in terms of normalized variables:

$$(x, y, z) = (\underline{x}, \underline{y}, \underline{z}) \ell, \quad t = \underline{t} \tau, \quad \vec{v} = \underline{v} u, \quad p = \underline{p} \frac{\eta u}{\ell} \quad (11)$$

$$\frac{\tau}{\underline{\tau}} \frac{\partial \underline{v}}{\partial \underline{t}} + R_y \underline{v} \cdot \nabla \underline{v} = -\nabla p + \nabla^2 \underline{v} \quad (12)$$

where

$$\tau_v \equiv \frac{\rho \ell^2}{\eta} = \text{viscous diffusion time}$$

$$R_y \equiv \frac{\rho u \ell}{\eta} = \text{Reynolds number}$$

Shear stresses set a fluid into motion in spite of its inertia at a rate typified by the viscous diffusion time. If processes of interest occur on a time scale  $\tau$  that is long compared to this time, then the effect of the first inertial term in Eq. 12 is ignorable. The Reynolds number, which is the ratio of  $\tau_v$  to a residence time  $\ell/u$ , represents the importance of inertia relative to viscosity for processes that are typified by a velocity rather than a time. Examples are flows in the steady state. Alternatively,  $R_y$  typifies the ratio of an inertial force density to a viscous force density.

In the "low Reynolds number approximation," the terms on the left in Eq. 12 are neglected. This expression is equivalent to the curl of Eq. 7 without its inertial terms:

$$\nabla \times \nabla \times (\eta \nabla^2 \vec{A}_v + \vec{G}) = 0 \quad (13)$$

The pressure then follows from Eq. 10 with the inertial terms omitted:

$$p = p(\vec{r}_0) + \xi(\vec{r}_0) - \xi(\vec{r}) + \int_{\vec{r}_0}^{\vec{r}} \nabla \times (\eta \nabla^2 \vec{A}_v + \vec{G}) \cdot d\vec{\ell} \quad (14)$$

Without compromise concerning the amplitude of the flow, these linear expressions are used to predict flows that are extremely viscous, that involve extremely small dimensions or that occur over long periods of time. They are applied in Secs. 7.20 and 7.21.

### 7.19 Perturbation Viscous Diffusion Transfer Relations

Consider small-amplitude motions in the x-y plane of a viscous fluid with no external rotational forces ( $\vec{G} = 0$ ). Then, in Cartesian coordinates, the vector potential reduces to just the z component, with amplitude  $A_v$ , and Eq. 7.18.8 reduces to the single scalar equation

$$\nabla^2 \left( \rho \frac{\partial A_v}{\partial t} - \eta \nabla^2 A_v \right) = 0 \quad (1)$$

Here, a vector identity\* and the solenoidal character of  $A_v$  have been used (Eq. 7.18.6). This is the first of the four symmetric configurations summarized by Table 2.18.1 that are represented by a single component of the vector potential. The others are handled as illustrated by the Cartesian case considered now.

With the objective of obtaining relations that can be adapted to a variety of physical situations, consider the motions within a planar region having thickness  $\Delta$ , as shown in Fig. 7.19.1.

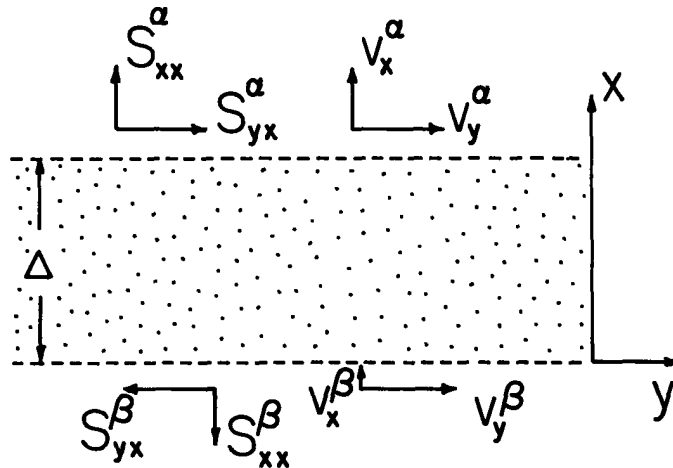


Fig. 7.19.1

Planar region filled by viscous fluid with stress components ( $S_{xx}, S_{yx}$ ) and velocity components ( $v_x, v_y$ ) in the  $\alpha$  and  $\beta$  planes related by Eq. 13.

For perturbations having the form  $\text{Re } \hat{A}_v(x) \exp j(\omega t - ky)$ , Eq. 1 requires that the complex amplitude,  $A_v(x)$ , satisfy a fourth-order differential equation that has two solutions familiar from the incompressible inviscid fluid model of Sec. 7.9,

$$\left( \frac{d^2}{dx^2} - k^2 \right) \left( \frac{d^2}{dx^2} - \gamma^2 \right) \hat{A}_v = 0 \quad (2)$$

where

$$\gamma^2 = k^2 + j \frac{\omega \rho}{\eta}$$

The other two are solutions to the diffusion equation, familiar from magnetic diffusion as discussed in Sec. 6.5. Thus,

$$\hat{A}_v = \hat{A}_1 \sinh kx + \hat{A}_2 \sinh k(x - \Delta) + \hat{A}_3 \sinh \gamma x + \hat{A}_4 \sinh \gamma(x - \Delta) \quad (3)$$

The two lengths that typify the interactions between  $\alpha$  and  $\beta$  surfaces are evident in this equation. For the first two solutions, which represent pressure attenuation across the layer, the length is  $2\pi/k$ . Identification of these components in Eq. 3 with the pressure follows from taking the gradient and then the divergence of Eq. 7.18.10 to show that  $p$  satisfies Laplace's equation. The last two terms bring in the second length scale,  $2\pi/|\gamma|$ , which is at most  $2\pi/k$  and at least the viscous skin depth defined (analogous to the magnetic skin depth, Eq. 6.2.10) as

$$\delta = \sqrt{\frac{2\eta}{\omega \rho}} \quad (4)$$

\*  $\nabla \times \nabla \times \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$ ;  $\nabla \cdot \vec{F} \equiv 0$

This length, which represents the transmission of shear stress across the layer through the action of viscosity inhibited by the fluid inertia, is shown as a function of frequency for some typical fluids in Fig. 7.19.2. The viscosity and mass density are taken from Table 7.13.1. Even with relatively modest frequencies, the viscous skin depth can be quite short.

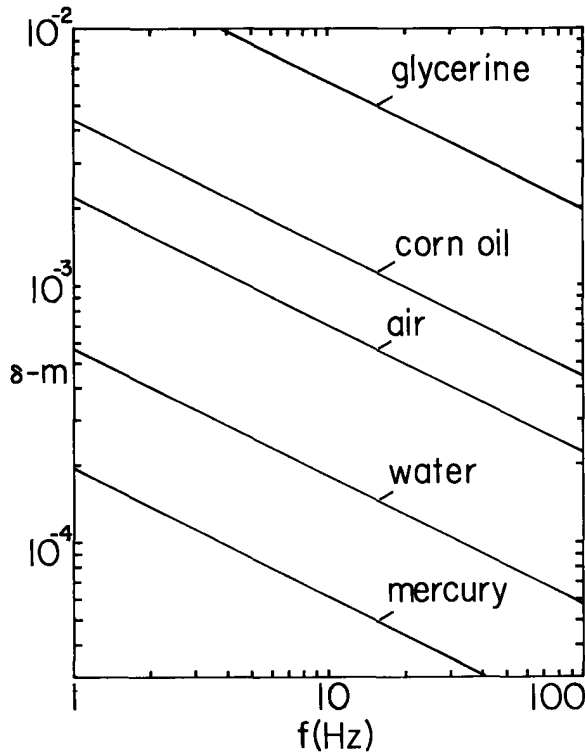


Fig. 7.19.2

Viscous skin depth as function of frequency.

In the remainder of this section, the relationships between the velocities in the  $\alpha$  and  $\beta$  planes and the stress components in these planes are determined. First, this is done without further approximations. Then, the interaction between boundary layers is illustrated by taking the limit  $\delta \ll \Delta$ , so that the transmission of stresses across the layer is through the pressure modes alone. Finally, useful relations are derived between stress and velocity with not only  $\delta \ll \Delta$ , but  $k\Delta \ll 1$ , so that the surfaces are uncoupled.

Layer of Arbitrary Thickness: The velocity components are written in terms of the coefficients  $A_i$  by taking the curl of  $\vec{A}$ , Eq. 3 (Eq. (b), Table 2.18.1). Evaluated at the respective planes  $x = \Delta$  and  $x = 0$ , these are

$$\begin{bmatrix} \hat{v}_x^\alpha \\ \hat{v}_x^\beta \\ \hat{v}_y^\alpha \\ \hat{v}_y^\beta \end{bmatrix} = \begin{bmatrix} -jk \sinh k\Delta & 0 & -jk \sinh \gamma\Delta & 0 \\ 0 & jk \sinh k\Delta & 0 & jk \sinh \gamma\Delta \\ -k \cosh k\Delta & -k & \cosh \gamma\Delta & -\gamma \\ -k & -k \cosh k\Delta & -\gamma & -\gamma \cosh \gamma\Delta \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \\ \hat{A}_4 \end{bmatrix} \quad (5)$$

Inversion of these equations is the first chore in determining the transfer relations. Cramer's rule gives

$$[\hat{A}] = [M][\hat{v}] \quad (6)$$

where  $[\hat{A}]$  and  $[\hat{v}]$  are the column matrices and  $[M]$  is the inverse of the  $4 \times 4$  matrix appearing in Eq. 5. Even though it is the velocity and stress amplitudes that are usually used when the transfer functions represent a piece of a more complex system, the entries in  $M$  are worth saving so that the distribution with  $x$  can be reconstructed from the velocity amplitudes:

$$\begin{aligned}
M_{11} &= -M_{22} = jk\gamma \sinh \gamma\Delta [\gamma \sinh k\Delta \sinh \gamma\Delta + k(1 - \cosh k\Delta \cosh \gamma\Delta)]/D \\
M_{12} &= -M_{21} = jk^2\gamma \sinh \gamma\Delta (\cosh k\Delta - \cosh \gamma\Delta)/D \\
M_{13} &= M_{24} = k^2 \sinh \gamma\Delta (\gamma \sinh k\Delta \cosh \gamma\Delta - k \cosh k\Delta \sinh \gamma\Delta)/D \\
M_{14} &= M_{23} = k^2 \sinh \gamma\Delta (k \sinh \gamma\Delta - \gamma \sinh k\Delta)/D \\
M_{31} &= -M_{42} = jk^2 \sinh k\Delta [\gamma(1 - \cosh \gamma\Delta \cosh k\Delta) + k \sinh \gamma\Delta \sinh k\Delta]/D \\
M_{32} &= -M_{41} = jk^2\gamma \sinh k\Delta (\cosh \gamma\Delta - \cosh k\Delta)/D \\
M_{33} &= M_{44} = k^2 \sinh k\Delta (k \cosh k\Delta \sinh \gamma\Delta - \gamma \sinh k\Delta \cosh \gamma\Delta)/D \\
M_{43} &= M_{34} = k^2 \sinh k\Delta (k \sinh \gamma\Delta - \gamma \sinh k\Delta)/D
\end{aligned} \tag{7}$$

where

$$D \equiv k^4 \sinh k\Delta \sinh \gamma\Delta \left[ \frac{2\gamma}{k} (1 - \cosh \gamma\Delta \cosh k\Delta) + \sinh k\Delta \sinh \gamma\Delta \left( \frac{\gamma^2}{k^2} + 1 \right) \right]$$

The stress components are written in terms of the velocity components and the pressure using Eq. 2 (with the last term omitted because  $\nabla \cdot \vec{v} = 0$ ):

$$\hat{S}_{xx} = -\hat{p} + 2\eta \left( \frac{d\hat{v}_x}{dx} \right) \tag{8}$$

$$\hat{S}_{yx} = \eta \left( \frac{d\hat{v}_y}{dx} - jk\hat{v}_x \right) \tag{9}$$

With the objective of evaluating these in terms of the A's, ( $\hat{v}_x, \hat{v}_y$ ), found earlier from Eq. 3, are substituted into these expressions. But, the pressure must also be expressed in terms of the A's by using Eq. 3 to evaluate Eq. 7.18.10. With p defined as  $\Pi_0$  at (x,y) = (0,0), the line integration results in

$$p = \Pi_0 + \text{Re } \omega\rho(\hat{A}_1 + \hat{A}_2 \cosh k\Delta)e^{j\omega t} + \text{Re } \hat{p}(x)e^{j(\omega t - ky)} \tag{10}$$

where the complex amplitude representing the part of p that depends on (x,y,t) is

$$\hat{p} = -\omega\rho[\hat{A}_1 \cosh kx + \hat{A}_2 \cosh k(x - \Delta)] \tag{11}$$

Note that the definition of  $\vec{C}$ , Eq. 7.18.7, insures that the Laplacian solutions contribute to p, to the exclusion of the diffusion solutions.

With the stress components expressed as functions of x in terms of the A's, they are evaluated at the respective planes, to obtain

$$\begin{bmatrix} \hat{S}_{xx}^\alpha \\ \hat{S}_{xx}^\beta \\ \hat{S}_{yx}^\alpha \\ \hat{S}_{yx}^\beta \end{bmatrix} = \eta \begin{bmatrix} -j(k^2 + \gamma^2)\cosh k\Delta & -j(k^2 + \gamma^2) & -2j\gamma k \cosh \gamma\Delta & -2j\gamma k \\ -j(k^2 + \gamma^2) & -j(k^2 + \gamma^2)\cosh k\Delta & -2j\gamma k & -2j\gamma k \cosh \gamma\Delta \\ -2k^2 \sinh k\Delta & 0 & -(\gamma^2 + k^2)\sinh \gamma\Delta & 0 \\ 0 & 2k^2 \sinh k\Delta & 0 & (\gamma^2 + k^2)\sinh \gamma\Delta \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \\ \hat{A}_4 \end{bmatrix} \tag{12}$$

In compact notation, this expression is equivalent to  $[\hat{S}] = [N] [\hat{A}]$ . Finally, the transfer relations are obtained by substituting Eq. 6 for the column matrix  $[\hat{A}]$  in Eq. 12 and performing the multiplication  $[N] [M] \equiv \eta [P]$ :

$$\begin{bmatrix} \hat{S}_{xx}^\alpha \\ \hat{S}_{xx}^\beta \\ \hat{S}_{yx}^\alpha \\ \hat{S}_{yx}^\beta \end{bmatrix} = \eta [P_{ij}] \begin{bmatrix} \hat{v}_x^\alpha \\ \hat{v}_x^\beta \\ \hat{v}_y^\alpha \\ \hat{v}_y^\beta \end{bmatrix} \tag{13}$$



where

$$\begin{aligned}
 F &= \frac{2\gamma}{k} (1 - \cosh \gamma\Delta \cosh k\Delta) + \sinh \gamma\Delta \sinh k\Delta \left[ \left(\frac{\gamma}{k}\right)^2 + 1 \right] \\
 P_{11} = -P_{22} &= k \left[ 1 - \left(\frac{\gamma}{k}\right)^2 \right] \left[ \frac{\gamma}{k} \cosh \gamma\Delta \sinh k\Delta - \left(\frac{\gamma}{k}\right)^2 \cosh k\Delta \sinh \gamma\Delta \right] / F \\
 P_{12} = -P_{21} &= \gamma \left[ 1 - \left(\frac{\gamma}{k}\right)^2 \right] \left[ \frac{\gamma}{k} \sinh \gamma\Delta - \sinh k\Delta \right] / F \\
 P_{13} = -P_{31} = P_{24} = -P_{42} &= jk \left\{ \frac{\gamma}{k} \left[ 3 + \left(\frac{\gamma}{k}\right)^2 \right] \left[ 1 - \cosh k\Delta \cosh \gamma\Delta \right] + \left[ 1 + 3\left(\frac{\gamma}{k}\right)^2 \right] \sinh k\Delta \sinh \gamma\Delta \right\} / F \\
 P_{14} = P_{23} = P_{32} = P_{41} &= jk \left(\frac{\gamma}{k}\right) \left[ 1 - \left(\frac{\gamma}{k}\right)^2 \right] (\cosh \gamma\Delta - \cosh k\Delta) / F \\
 P_{33} = -P_{44} &= k \left[ \left(\frac{\gamma}{k}\right)^2 - 1 \right] \left( \frac{\gamma}{k} \sinh k\Delta \cosh \gamma\Delta - \sinh \gamma\Delta \cosh k\Delta \right) / F \\
 P_{34} = -P_{43} &= k \left[ \left(\frac{\gamma}{k}\right)^2 - 1 \right] (\sinh \gamma\Delta - \frac{\gamma}{k} \sinh k\Delta) / F
 \end{aligned}$$

These transfer relations are used to describe a variety of problems, not only of a fluid-mechanical nature, but involving electromechanical coupling that can be relegated to deformable interfaces. Examples are given in Chaps. 8 and 9.

**Short Skin-Depth Limit:** By way of illustrating the two lengths typifying the dynamics of the viscous layer, suppose that the viscous skin depth is small so that  $\delta \ll \Delta$  and hence  $|\gamma\Delta| \gg 1$ , but that  $k\Delta$  is arbitrary. Then, viscous diffusion is confined to boundary layers adjacent to the  $\alpha$  and  $\beta$  planes. Instead of Eq. 3, solutions exploiting the approximation would conveniently take the form

$$\hat{A}_v = \hat{A}_5 \sinh kx + \hat{A}_6 \sinh k(x - \Delta) + \hat{A}_7 e^{-\gamma x} + \hat{A}_8 e^{\gamma(x-\Delta)} \quad (14)$$

where it is understood that  $\gamma$  is defined such that  $\text{Re} \gamma > 0$ . The diffusion terms are respectively negligible when evaluated in the  $\alpha$  and  $\beta$  planes. This could be exploited in simplifying a derivation of the transfer relations for this limiting case, one that parallels that begun with Eq. 3. Because the result is easily found by taking the appropriate limit of Eq. 13, it suffices to draw attention to the apparent role of the pressure in coupling one viscous boundary layer to the other. Even though the viscous skin depth is short compared to the layer thickness, the coupling between planes afforded by the pressure results in diffusion motions at one plane caused by excitations at the other. For example, the shear stress  $\hat{S}_{yx}^\alpha$  in the  $\alpha$  plane caused by a shearing velocity  $\hat{v}_y^\beta$  in the  $\beta$  plane is proportional to  $\eta P_{34}$ . From Eq. 13, even in the limit  $|\gamma\Delta| \gg 1$ , but  $k\Delta \sim 1$ ,

$$P_{34} = \eta k / \sinh k\Delta \quad (15)$$

It is only in the limit  $k\Delta \gg 1$ , so that the pressure perturbations cannot penetrate the layer, that the shearing interactions across the layer cease.

**Infinite Half-Space of Fluid:** With both  $|\gamma\Delta| \gg 1$  and  $k\Delta \gg 1$ , motions in one plane are uncoupled from those in the other. With the understanding that  $\text{Re} \gamma > 0$ , and that upper signs refer to a lower half space bounded from above by the  $\alpha$  plane while lower signs are for an upper half space bounded from below by the  $\beta$  plane, appropriate solutions to Eq. 2 are

$$A_v = \hat{A}_1 e^{\frac{+kx}{2}} + \hat{A}_2 e^{\frac{+\gamma x}{2}} \quad (16)$$

Transfer relations are determined following the same steps just outlined. First, the velocity amplitudes are written in terms of  $(\hat{A}_1, \hat{A}_2)$  and then these relations are inverted to obtain

$$\begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} = \frac{1}{k-\gamma} \begin{bmatrix} -j \frac{\gamma}{k} & \mp 1 \\ j & \pm 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \hat{v}_x^\beta \\ \alpha \\ \hat{v}_y^\beta \end{bmatrix} \quad (17)$$

In terms of the potential amplitudes, the respective stress components are

$$\begin{bmatrix} \alpha \\ \hat{S}_{xx}^\beta \\ \alpha \\ \hat{S}_{yx}^\beta \end{bmatrix} = \begin{bmatrix} \mp j\eta(k^2 + \gamma^2) & \mp 2j\eta\gamma k \\ -2\eta k^2 & -\eta(\gamma^2 + k^2) \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} \quad (18)$$

Finally, the transfer relations follow by combining Eqs. 17 and 18:

$$\begin{bmatrix} \alpha \\ \hat{S}_{xx}^\beta \\ \alpha \\ \hat{S}_{yx}^\beta \end{bmatrix} = \begin{bmatrix} \pm\eta \frac{\gamma}{k}(\gamma + k) & -j\eta(\gamma - k) \\ j\eta(\gamma - k) & \pm\eta(\gamma + k) \end{bmatrix} \begin{bmatrix} \alpha \\ \hat{v}_x^\beta \\ \alpha \\ \hat{v}_y^\beta \end{bmatrix} \quad (19)$$

Remember that  $k$  has been assumed positive. If a wave propagating in the  $-y$  direction is to be represented, the derivation can be repeated with  $k \rightarrow -k$ . For this negative traveling wave, Eq. 19 is altered by a sign reversal of the two off-diagonal terms.

## 7.20 Low Reynolds Number Transfer Relations

In terms defined with Eq. 7.18.12, the inertial force density is negligible compared to that due to viscosity if the viscous diffusion time is short compared to times of interest, or equivalently, if the Reynolds number is small:

$$\tau_v = \rho l^2 / \eta \ll 1; \quad R_y = \rho u l / \eta \ll 1 \quad (1)$$

In this extreme, the dynamic response is a sequence of stationary states. The governing volume equation Eq. 7.18.13, is written as the biharmonic equation using a vector identity,\*

$$\nabla^2 (\eta \nabla^2 \vec{A}_v + \vec{G}) = 0 \quad (2)$$

It involves no time rates of change. The flow is therefore an arbitrary function of time determined by boundary conditions and the external rotational force density. The flow at any instant can adjust itself throughout the volume without the time delays associated with viscous diffusion.† A consequence is flow reversibility. For a graphic demonstration, see Reference 6, Appendix C. Moreover, so long as the conditions of Eq. 1 prevail, the amplitude of the response is also arbitrary. There is no implied linearization. Finally, because Eq. 2 is linear, a superposition of solutions is also a solution.

The vector potential reduces to a single scalar component for the configurations of Table 2.18.1. In the following subsections, two of these are considered. First, the dynamics of a planar layer is revisited and then the transfer relations for axisymmetric flows in spherical geometry are derived.

Planar Layer: With  $\vec{G} = 0$  and  $\vec{A}_v = \text{Re} \tilde{A}_v(x, t) e^{-jky_1}$ , Eq. 2 requires that the  $x$  dependence satisfy

$$\left( \frac{d^2}{dx^2} - k^2 \right)^2 \tilde{A}_v = 0 \quad (3)$$

Formally, this is the limit  $\omega\rho/\eta \ll k^2$  and hence  $\gamma \rightarrow k$  of Eq. 7.19.2 (but of course the underlying approximations do not limit the solution to small amplitudes). Because the viscous and Laplacian roots of Eq. 3 have now degenerated into the same roots, two solutions are linear combinations of  $\exp(\pm kx)$  and the other two are combinations of  $x \exp(\pm kx)$ :

$$\tilde{A}_v = \tilde{A}_1 \sinh kx + \tilde{A}_2 \left( \frac{x}{\Delta} \right) \sinh kx + \tilde{A}_3 \sinh k(x-\Delta) + \tilde{A}_4 \left( \frac{x}{\Delta} \right) \sinh k(x-\Delta) \quad (4)$$

By contrast with the amplitudes of Sec. 7.19, the  $\tilde{A}_i$ 's are arbitrary functions of time.

The outline for finding the transfer relations for the planar layer shown in Fig. 7.19.1 is now the same as illustrated in Sec. 7.19. With the caveat that the result does not have the same limitations as the viscous diffusion relations, it is possible to obtain the transfer relations as a limit of

\*  $\nabla_x \nabla_x \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$ ;  $\nabla \cdot \vec{F} \equiv 0$

the results of Sec. 7.19 in which  $\omega\rho/\eta \rightarrow 0$ . As a practical matter, it is perhaps easier to repeat the derivation.

For reference, the potential amplitudes of Eq. 4 are related to the velocity amplitudes by

$$\begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{A}_3 \\ \tilde{A}_4 \end{bmatrix} = [Q_{ij}] \begin{bmatrix} \tilde{v}_x^\alpha \\ \tilde{v}_x^\beta \\ \tilde{v}_y^\alpha \\ \tilde{v}_y^\beta \end{bmatrix} \quad (5)$$

where

$$\begin{aligned} H &= [\sinh^2(k\Delta) - (k\Delta)^2]/4k \\ Q_{11} &= j[\sinh(k\Delta) + k\Delta \cosh(k\Delta)]/4k^2 H \\ Q_{21} &= Q_{42} = -j\Delta[\cosh(k\Delta)\sinh(k\Delta) + k\Delta]/4k \sinh(k\Delta) H \\ Q_{41} &= -Q_{12} = Q_{22} = j\Delta[\sinh(k\Delta) + k\Delta \cosh(k\Delta)]/4k \sinh(k\Delta) H \\ Q_{32} &= j[(k\Delta)^2 - \sinh^2(k\Delta)]/4k^2 \sinh(k\Delta) H \\ Q_{13} &= -Q_{23} = Q_{44} = \Delta \sinh k\Delta/4kH \\ Q_{14} &= -Q_{24} = Q_{43} = \Delta^2/4H \\ Q_{31} &= Q_{33} = Q_{34} = 0 \end{aligned}$$

The stress-velocity transfer relations are then

$$\begin{bmatrix} \tilde{S}_{xx}^\alpha \\ \tilde{S}_{xx}^\beta \\ \tilde{S}_{yx}^\alpha \\ \tilde{S}_{yx}^\beta \end{bmatrix} = \eta [P_{ij}] \begin{bmatrix} \tilde{v}_x^\alpha \\ \tilde{v}_x^\beta \\ \tilde{v}_y^\alpha \\ \tilde{v}_y^\beta \end{bmatrix} \quad (6)$$

where

$$\begin{aligned} P_{11} &= -P_{22} = [\frac{1}{4} \sinh(2k\Delta) + \frac{k\Delta}{2}]/H \\ P_{33} &= -P_{44} = [\frac{1}{4} \sinh(2k\Delta) - \frac{k\Delta}{2}]/H \\ P_{21} &= -P_{12} = [\frac{k\Delta}{2} \cosh(k\Delta) + \frac{1}{2} \sinh(k\Delta)]/H \\ P_{31} &= -P_{13} = -P_{24} = P_{42} = j(k\Delta)^2/2H \\ P_{14} &= P_{23} = P_{32} = P_{41} = -j(\frac{k\Delta}{2}) \sinh k\Delta/H \\ P_{43} &= -P_{34} = \frac{1}{2}[\sinh(k\Delta) - k\Delta \cosh(k\Delta)]/H \end{aligned}$$

Application of these relations is illustrated in Chap. 9

Axisymmetric Spherical Flows: To describe motions around small particles, bubbles and the like, creep flows are now considered in spherical coordinates. The relations developed are limiting forms of those for a spherical shell. First, stress-velocity relations are obtained relating variables on

a spherical surface of radius  $\alpha$  enclosing the region of interest. (The shell's inner radius  $\beta \rightarrow 0$ .) Then, they are found for an infinite region exterior to a surface of radius  $\beta$  ( $\alpha \rightarrow \infty$ ).

In spherical coordinates, flows with no azimuthal dependence are described by the vector potential of Table 2.18.1:

$$\vec{A}_v = \frac{\Lambda(r, \theta)}{r \sin \theta} \hat{t}_\phi \quad (7)$$

In substituting this form into Eq. 2 (with  $\vec{G} = 0$ ), observe that

$$\nabla^2 \left( \frac{\Lambda \hat{t}_\phi}{r \sin \theta} \right) = \frac{1}{r \sin \theta} \left[ \frac{\partial^2 \Lambda}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Lambda}{\partial \theta} \right) \right] \hat{t}_\phi \quad (8)$$

To evaluate Eq. 2, the vector Laplacian is now taken of this expression. Because it takes the same form as Eq. 7, with the quantity in [ ] playing the role of  $\Lambda$ , it follows that Eq. 2 reduces to

$$\frac{1}{r \sin \theta} \left[ \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right] \Lambda = 0 \quad (9)$$

That variable separable solutions to Eq. 9 take the form

$$\Lambda = \sin \theta P_n^1(\cos \theta) \tilde{\Lambda}(r, t) \quad (10)$$

can be seen by observing from Eqs. 2.16.31a and 2.16.34 that the Legendre polynomial  $P_n^1$  satisfies

$$\frac{d}{d\theta} \left[ \frac{1}{\sin \theta} \frac{d}{d\theta} (P_n^1 \sin \theta) \right] = \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dP_n^1}{d\theta} \right) - \frac{P_n^1}{\sin^2 \theta} = -n(n+1)P_n^1 \quad (11)$$

Hence, substitution of Eq. 10 into Eq. 9 results in a fourth-order differential equation determining the radial dependence:

$$\left[ \frac{d^2}{dr^2} - \frac{n(n+1)}{r^2} \right] \tilde{\Lambda} = 0 \quad (12)$$

Further substitution shows that two solutions to Eq. 12 are of the form  $r^q$ , where  $q = 1 + n$  and  $-n$ . Two more solutions follow as  $r^2 r^q$ , so that the radial dependence is expressed in terms of four time-dependent amplitudes,  $\tilde{\Lambda}_j$ ,

$$\tilde{\Lambda} = \tilde{\Lambda}_1 \left( \frac{r}{R} \right)^{n+1} + \tilde{\Lambda}_2 \left( \frac{r}{R} \right)^{-n} + \tilde{\Lambda}_3 \left( \frac{r}{R} \right)^{2-n} + \tilde{\Lambda}_4 \left( \frac{r}{R} \right)^{n+3} \quad (13)$$

The radius  $R$  will be identified with either  $\alpha$  or  $\beta$ .

The velocity components are evaluated from Eq. 13 by using Eq. (k) of Table 2.18.1:

$$v_r = \tilde{v}_r \left\{ \frac{1}{\sin \theta} \frac{d}{d\theta} [\sin \theta P_n^1(\cos \theta)] \right\}; \quad \tilde{v}_r \equiv \frac{\tilde{\Lambda}}{r^2} \quad (14)$$

$$v_\theta = \tilde{v}_\theta P_n^1(\cos \theta); \quad \tilde{v}_\theta \equiv -\frac{1}{r} \frac{d\tilde{\Lambda}}{dr} \quad (15)$$

The  $\theta$  dependences of the two components differ. For convenience, these are summarized in Table 7.20.1. The amplitudes  $\tilde{v}_\theta$  and  $\tilde{v}_r$  are multiplied by the respective functions from Table 7.20.1 to recover the  $\theta$  dependence.

Flow within a volume enclosed by a spherical surface having radius  $\alpha$  includes the origin. Because the velocity implied by the second and third terms in Eq. 13 is singular at the origin, these terms are excluded. Evaluation of Eqs. 14 and 15 at  $r = \alpha$  then gives a pair of expressions in  $(\tilde{\Lambda}_1, \tilde{\Lambda}_4)$  which can be inverted to obtain

$$\begin{bmatrix} \tilde{\Lambda}_1 \\ \tilde{\Lambda}_4 \end{bmatrix} = \begin{bmatrix} \frac{(n+3)\alpha^2}{2} & \frac{\alpha^2}{2} \\ -\frac{(n+1)\alpha^2}{2} & -\frac{\alpha^2}{2} \end{bmatrix} \begin{bmatrix} \tilde{v}_r^\alpha \\ \tilde{v}_\theta^\alpha \end{bmatrix} \quad (16)$$

Table 7.20.1. Angular dependence of velocity and stress functions.

n	$v_\theta$ and $S_{\theta r}$ $P_n^1(\cos \theta)$	$v_r$ and $S_{rr}$ $\frac{1}{\sin \theta} \frac{d}{d\theta} [\sin \theta P_n^1(\cos \theta)]$
1	$\sin \theta$	$2\cos \theta$
2	$3\sin \theta \cos \theta$	$3(3\cos^2 \theta - 1)$
3	$\frac{3}{2} \sin \theta (5\cos^2 \theta - 1)$	$15\cos \theta (1 - 2\sin^2 \theta) - 3\cos \theta$
4	$\frac{5}{2} \sin \theta (7\cos^3 \theta - 3\cos \theta)$	$5\cos \theta (7\cos^3 \theta - 3\cos \theta) + \frac{5}{2} \sin^2 \theta (3 - 21\cos^2 \theta)$

For the flow in the region exterior to the surface having radius  $r = \beta$ , contributions to Eq. 13 that are singular as  $r \rightarrow \infty$  are excluded. The  $n=1$  mode is special, in that it represents flow that is uniform in the  $z$  direction far from  $r = \beta$ . Thus, the second and third terms in Eq. 13 contribute for all values of  $n$ , but the first term also contributes when  $n=1$ . For a uniform parallel flow,  $\vec{v} = U\hat{i}_z$  at infinity, and it follows that for  $n=1$ ,  $\tilde{\Lambda}_1 = U\beta^2/2$ . Two equations for  $(\tilde{\Lambda}_2, \tilde{\Lambda}_3)$  are then written by evaluating Eqs. 14 and 15 at  $r = \beta$ . These are inverted to obtain

$$\begin{bmatrix} \tilde{\Lambda}_2 \\ \tilde{\Lambda}_3 \end{bmatrix} = \begin{bmatrix} \frac{(2-n)\beta^2}{2} & \frac{\beta^2}{2} \\ \frac{n\beta^2}{2} & -\frac{\beta^2}{2} \end{bmatrix} \begin{bmatrix} \tilde{v}_r^\beta - \frac{1}{2} U\delta_{1n} \\ \tilde{v}_\theta^\beta + U\delta_{1n} \end{bmatrix} \quad (17)$$

In spherical coordinates, the stress components are

$$S_{rr} = -p + 2\eta \frac{\partial v_r}{\partial r} \quad (18)$$

$$S_{\theta r} = \eta \left[ r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (19)$$

To evaluate the pressure in terms of the  $\tilde{\Lambda}_i$ 's, Eq. 7.18.14 is evaluated using Eq. 8. The line integration can be carried out along the  $\phi, \theta$  and finally  $r$  directions. Because the integration is only a function of the end points, it is clear that the  $\theta$ -dependent part comes from the last integration. Thus,

$$\tilde{p} = \eta \int \frac{1}{r^2} \left[ \frac{d^2 \tilde{\Lambda}}{dr^2} - \frac{n(n+1)}{r^2} \tilde{\Lambda} \right] dr \quad (20)$$

with the  $\theta$  dependence the same as for  $v_r$ . Equations 18 and 19 are now evaluated, first at  $r = R = \alpha$  (the region  $r < R$ , where  $\tilde{\Lambda}_2$  and  $\tilde{\Lambda}_3 = 0$ ):

$$\begin{bmatrix} \tilde{S}_{rr}^\alpha \\ \tilde{S}_{\theta r}^\alpha \end{bmatrix} = \frac{\eta}{\alpha^3} \begin{bmatrix} 2(n-1) & \frac{2}{n}(n^2-n-3) \\ 2(n+1)(1-n) & -2n(n+2) \end{bmatrix} \begin{bmatrix} \tilde{\Lambda}_1 \\ \tilde{\Lambda}_4 \end{bmatrix} \quad (21)$$

and then at  $r = R = \beta$  (the region  $r > R$ , where  $\tilde{\Lambda}_1 = U\beta^2/2$ ,  $\tilde{\Lambda}_4 = 0$ ):

$$\begin{bmatrix} \tilde{S}_{rr}^\beta \\ \tilde{S}_{\theta r}^\beta \end{bmatrix} = \frac{\eta}{\beta^3} \begin{bmatrix} -2(n+2) & \frac{-2(n^2+3n-1)}{n+1} \\ -2n(n+2) & 2(1-n^2) \end{bmatrix} \begin{bmatrix} \tilde{\Lambda}_2 \\ \tilde{\Lambda}_3 \end{bmatrix} \quad (22)$$

Note that the term in  $\tilde{\Lambda}_1$  does not contribute to Eq. 22 because its coefficient is zero for  $n = 1$ .

To recover the  $\theta$  dependence, the amplitudes  $\tilde{S}_{rr}$  and  $\tilde{S}_{\theta r}$  are respectively multiplied by the functions summarized in Table 7.20.1. That the  $\theta$  dependence of  $S_{\theta r}$  is indeed simply  $P'_n(\cos \theta)$  is shown by making use of Eq. 11.

The stress-velocity transfer relations now follow by substituting Eqs. 16 and 17 respectively into Eqs. 21 and 22:

$$\begin{bmatrix} \tilde{S}_{rr}^\alpha \\ \tilde{S}_{\theta r}^\alpha \end{bmatrix} = \frac{\eta}{\alpha} \begin{bmatrix} \frac{1}{n}(2n^2+n+3) & \frac{3}{n} \\ 3(n+1) & (2n+1) \end{bmatrix} \begin{bmatrix} \tilde{v}_r^\alpha \\ \tilde{v}_\theta^\alpha \end{bmatrix} \quad (23)$$

$$\begin{bmatrix} \tilde{S}_{rr}^\beta \\ \tilde{S}_{\theta r}^\beta \end{bmatrix} = \frac{-\eta}{\beta} \begin{bmatrix} \frac{(2n^2+3n+4)}{n+1} & \frac{3}{n+1} \\ 3n & (2n+1) \end{bmatrix} \begin{bmatrix} \tilde{v}_r^\beta - \frac{1}{2} U \delta_{1n} \\ \tilde{v}_\theta^\beta + U \delta_{1n} \end{bmatrix} \quad (24)$$

The stress and velocity components in these relations are multiplied by the functions of  $\theta$  given in Table 7.20.1 to recover the  $\theta$  dependence. Application of these relations is illustrated in Sec. 7.21.

### 7.21 Stokes's Drag on a Rigid Sphere

Certainly the most celebrated low Reynolds number flow is that around a rigid sphere placed in what would otherwise be a uniform flow. Of particular interest is the total drag force on the sphere, found by integrating the  $z$  component of the traction,  $S_{rr} \cos \theta - S_{\theta r} \sin \theta$ , over its surface,

$$f_z = \int_0^\pi [S_{rr} \cos \theta - S_{\theta r} \sin \theta] 2\pi R^2 \sin \theta d\theta \quad (1)$$

The exterior  $n=1$  flow of Sec. 7.20 is now identified with that around the sphere. The uniform  $z$ -directed velocity far from the sphere is  $U$ . Because the sphere surface at  $r = R$  is rigid, both velocity components vanish there. In Eq. 7.20.24,

$$\tilde{v}_r^\beta = 0, \quad \tilde{v}_\theta^\beta = 0 \quad (2)$$

and the stress components are

$$\begin{bmatrix} \tilde{S}_{rr}^\beta \\ \tilde{S}_{\theta r}^\beta \end{bmatrix} = \frac{-\eta}{2R} \begin{bmatrix} -\frac{3U}{2} \\ 3U \end{bmatrix} \quad (3)$$

Using these amplitudes, as well as the  $\theta$  dependence given in Table 7.20.1, Eq. 1 becomes

$$f_z = 6\pi\eta RU \quad (4)$$

For a particle falling through a static fluid,  $U$  is the particle velocity. This "Stokes's drag" force is a good approximation, provided the Reynolds number based on the particle diameter is small compared to unity.

### 7.22 Lumped Parameter Thermodynamics of Highly Compressible Fluids

That additional laws are required to model highly compressible fluids is evident from the appearance of additional dependent variables in the constitutive law for the mass density. In this section, certain constitutive laws and thermodynamic relations are introduced. In Sec. 7.23 these are used to formulate integral and differential statements of energy conservation for the internal energy subsystem.

These laws are used extensively in Secs. 9.15-9.19.

Mechanical Equations of State: For a weakly compressible fluid, as defined in Sec. 7.10, the mass density is a function of pressure and parameters reflecting either the fluid composition or state. That air is buoyant when heated at constant pressure makes it evident that the mass density also depends on temperature. A commonly used mechanical constitutive law, representing a single-component perfect gas, is

$$p = \rho RT \tag{1}$$

The temperature, T, is measured in degrees Kelvin ( $T_{\text{Kelvin}} = T_{\text{centigrade}} + 273.15$ ). The gas constant, R, is  $R = R_g/M$ , where  $R_g = 8.31 \times 10^3$  is the universal gas constant and M is the molecular weight of the fluid. Using  $N_2$  as an example, the molecular weight is 28,  $R = 297$  and it follows from Eq. 1 that at atmospheric pressure ( $p=1.013 \times 10^5 \text{ n/m}^2$ ) and  $20^\circ\text{C}$  the mass density in mks units is  $\rho=p/RT=1.16 \text{ kg/m}^3$ . *Rg unit*

Energy Equation of State for a Perfect Gas: The specific internal energy,  $W_T$ , is defined as the energy per unit mass stored in the thermal motions of the molecules. In a perfect gas, it depends only on the temperature. Incremental changes in internal energy and temperature are related by *Joules*

$$\delta W_T = c_v \delta T \tag{2}$$

and a simple constitutive law takes the specific heat at constant volume,  $c_v$ , as being constant over the temperature range of interest.

Conservation of Internal Energy in CQS Systems: There is a formal correspondence between conservation of energy statements exploited in describing lumped-parameter electromechanical coupling in Sec. 3.5 and used now for thermal-mechanical coupling in a fluid. As a reminder, suppose an EQS electromechanical subsystem having single electrical and mechanical degrees of freedom is represented electrically by a charge q at the potential v and mechanically by the displacement  $\xi$  of material subject to the force of electrical origin f. Energy conservation for a subsystem defined as being free of dissipation is expressed by

$$v \delta q = \delta w + f \delta \xi \tag{3}$$

where w is the electrical energy stored.

Now, consider the thermal lumped-parameter system exemplified by Fig. 7.22.1. The first law of thermodynamics, conservation of energy for this subsystem, states that an increment of heat,  $\delta q_T$  (measured in joules) goes either into increasing the energy stored, or into doing mechanical work on an external system

$$\delta q_T = \delta w_T + p \delta v \tag{4}$$

Here,  $w_T$  plays the role of w and is energy stored in kinetic (thermal) motions at the molecular level. The mechanical work done is expressed in terms of the change in the total volume, v, and the pressure, p. That this term plays the role of the last term in Eq. 3 is seen by considering the work done by the displacement of p pistons in Fig. 7.22.1. With  $A_i$  the area of the ith piston, the net change in volume is

$$\delta v = \sum_{i=1}^p A_i \delta \xi_i \tag{5} \quad \text{CQS = compressible gas}$$

Because the gas is quasistatic (in the CQS sense of Sec. 7.25) the pressure exerted on each of the pistons is the same. Thus,

$$\sum_{i=1}^p p_i A_i \delta \xi_i = p \sum_{i=1}^p A_i \delta \xi_i = p \delta v \tag{6}$$

so that  $p \delta v$  is indeed the mechanical work resulting from the net motions of the pistons.

Comparison of Eqs. 3 and 4 makes it natural to represent the incremental heat addition in terms of two variables. One of these, the potential, v, in the electrical analogue represented the intensity through which the heat addition is made and is the temperature, T. The other variable, defined as the entropy s, is analogous to the charge. It expresses the quantity or extent of the heat addition in units of joules/°K. With the understanding that the incremental heat addition is indeed to a "conservative system" (that the thermal input can be recovered), the statement of energy conservation, Eq. 4, becomes

$$T \delta s = \delta w_T + p \delta v \tag{7}$$

In working with a continuum, it is convenient to use extensive variables that are normalized to the mass density. This is accomplished in the lumped-parameter system now being considered by dividing Eq. 7 by the (constant) total mass of the system. Thus, Eq. 7 becomes

$$T\delta S = \delta W_T + p\delta V \quad (8)$$

where the entropy per unit mass or specific entropy is  $S$ , and the specific volume  $V = 1/\rho$  will be recognized as the reciprocal mass density.

Just as it is natural to think of  $(q, \xi)$  as independent variables in Eq. 1,  $(S, V)$  are independent variables in Eq. 8. Thus, the specific thermal energy is a state function  $W_T(S, V)$ . The coenergy,  $W'(v, \xi)$ , is introduced in the electromechanical system if it is more convenient to use the potential rather than the charge as an independent variable. With a similar motivation, energy-function alternatives to  $W_T$  are often introduced.

Where  $p$  is a natural independent variable, the identity  $p\delta V = \delta(pV) - V\delta p$  converts Eq. 8 to

$$T\delta S = \delta H_T - V\delta p \quad (9)$$

where the specific enthalpy,  $H_T \equiv W_T + pV$ , is the convenient energy function. The specific enthalpy, like  $W_T$ , is a state function. But even more, for a perfect gas it is a function only of  $T$ . This is clear from the definition of  $H_T$ , the fact that for a perfect gas,  $W_T = W_T(T)$  and because (from Eq. 1)  $pV = (p/\rho) = RT$ .

An energy equation of state equivalent to Eq. 2 can be stated in terms of the specific enthalpy

$$\delta H_T = c_p \delta T \quad (10)$$

and since the specific enthalpy is a defined function, it is not surprising that specific heat at constant pressure,  $c_p$ , is related to  $c_v$  and  $R$ . To determine this relationship, write Eq. 9 using Eq. 10:

$$T\delta S = c_p \delta T - V\delta p \quad (11)$$

Subtract Eq. 8 evaluated using Eq. 2 from this relation and it follows that

$$(c_p - c_v)\delta T = \delta(pV) = R\delta T \quad (12)$$

where the second equality comes from Eq. 1. Thus,

$$c_p - c_v = R \quad (13)$$

### 7.23 Internal Energy Conservation in a Highly Compressible Fluid

In a moving fluid, the thermodynamic variables are generally functions of position and time. Strictly, neither the equations of state nor the thermodynamic statement of energy conservation from Sec. 7.22 applies to media in motion. The approach now taken in regard to the state equations is similar to that used in the latter part of Sec. 3.3 to broaden the application of Ohm's law to conductors in motion.

First, the laws must hold with the thermodynamic variables evaluated in the primed or moving frame of reference, at least for a fluid element undergoing uniform and constant translation. Equations of state are expressed in the laboratory frame of reference by transforming variables from the primed to the unprimed frame. The thermodynamic variables of temperature, specific entropy, etc., are scalars. They are the same in both reference frames, and hence the mechanical and energy equations of state, Eqs. 7.22.1 and 7.22.2, are used even if the fluid they describe is in motion.

The seeming ubiquity of these state equations should not obscure the underlying assumption that accelerations and relative deformations of the material have negligible effect on the mechanical and energy equations of state. The notion that the fluid can be described in terms of state functions rests on there being a local equilibrium condition for the internal energy subsystem. Because processes occur at a finite rate and in an accelerating frame of reference, extension of the first law to continuum systems rests on the assumption that each element of the medium reaches this equilibrium state at each

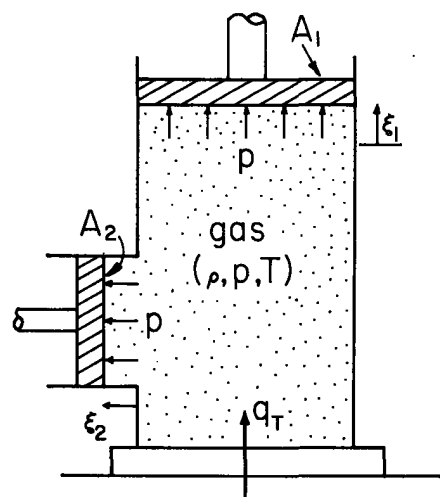


Fig. 7.22.1. Schematic view of lumped-parameter thermodynamic subsystem.



stage of the process.

A further assumption in what follows is that it is meaningful to separate the thermal and electric or magnetic subsystems. If the constitutive laws, Eqs. 7.22.1 and 7.22.2 for example, are modified by the electromagnetic fields, then this is not possible.

In this section, three subsystems are distinguished, each including dissipations. Two are the electric or magnetic and the mechanical subsystems. Each of these couples to the third, the internal energy subsystem. Given fluid of fixed identity filling a volume  $V$  enclosed by a surface  $S$ , the objective now is to write a continuum statement of internal energy conservation that makes the same physical statement as Eq. 7.22.8.

Power Conversion from Electromagnetic to Internal Form: To begin with, consider the inputs of heat to the volume. Whether the system is EQS or MQS, the electrical input of heat per unit volume is  $\vec{J}' \cdot \vec{E}'$ . To see this, observe from the conservation of energy statement for the electric (Eq. 2.13.16) or magnetic (Eq. 2.14.16) subsystem that the power density leaving that system is  $\vec{J}_f \cdot \vec{E}$ . This density either goes into the mechanical subsystem (into moving the fluid) or into the internal subsystem (into heating the fluid). Given the force densities, it is now possible to isolate the dissipation density. For an EQS system where polarization effects are negligible, the electrical dissipation must therefore be

$$\vec{J}_f \cdot \vec{E} - \rho_f \vec{E} \cdot \vec{v} = (\vec{J}'_f + \rho_f \vec{v}) \cdot \vec{E}' - \rho_f \vec{E}' \cdot \vec{v} = \vec{J}'_f \cdot \vec{E}' \quad (1)$$

Here, the EQS transformation laws (Eqs. 2.5.9a, 2.5.11a and 2.5.12a) have been used. For an MQS system without magnetization, the electrical dissipation density is

$$\vec{J}_f \cdot \vec{E} - \vec{J}_f \times \mu_0 \vec{H} \cdot \vec{v} = \vec{J}'_f \cdot (\vec{E}' - \vec{v} \times \mu_0 \vec{H}) - \vec{J}'_f \times \mu_0 \vec{H} \cdot \vec{v} = \vec{J}'_f \cdot \vec{E}' \quad (2)$$

where the MQS transformation laws (Eqs. 2.5.9b, 2.5.11b and 2.5.12b) and an identity\* have been used. Hence, the electrical dissipation density makes the same appearance for EQS and MQS systems.

Power Flow Between Mechanical and Internal Subsystems: Just as the statement of energy conservation is the basis for identifying the electrical dissipation density (Eqs. 1 and 2), the kinetic energy conservation statement, Eq. 7.17.8, makes it possible to identify the last two terms in that expression as power flowing from the mechanical system into the internal energy subsystem. Because the first of these two terms has been interpreted as heat generated by mechanical dissipation, it is now written on the left of the internal energy equation. However, the second of these terms represents mechanical power input in the form of a compression of the gas, and is therefore moved to the other side of the expression (with its sign of course reversed).

Integral Internal Energy Law: The continuum version of Eq. 7.22.8 is now written as

$$\int_V \vec{E}' \cdot \vec{J}'_f dV + \int_V \phi_v dV - \oint_S \vec{\Gamma}'_T \cdot \vec{n} da = \frac{d}{dt} \int_V \rho W_T dV + \int_V p \vec{v} \cdot \vec{v} dV \quad (3)$$

In addition to the first two heat input terms on the left, there has been added one representing the conduction of heat across the surface  $S$  and into the volume  $V$ . Typically, the thermal heat flux,  $\vec{\Gamma}'_T$ , is represented by a thermal conduction constitutive law, to be introduced in Sec. 10.2. On the right is the time rate of change of energy stored within the volume (which is one of fixed identity) plus the work done on the mechanical system through the expansion of the fluid.

Differential Internal Energy Law: To convert Eq. 3 to a differential statement, Gauss' theorem, Eq. 2.6.2, is used to write the surface integral as a volume integral. In addition, the generalized Leibnitz rule, Eq. 2.6.5, is used to take the time derivative inside the integral on the right. Then, conservation of mass, Eq. 7.2.3, is used to simplify that integrand. Because the volume  $V$  is arbitrary, it follows that

$$\vec{E}' \cdot \vec{J}'_f + \phi_v - \nabla \cdot \vec{\Gamma}'_T = \rho \frac{DW_T}{Dt} + p \vec{v} \cdot \vec{v} \quad (4)$$

Combined Internal and Mechanical Energy Laws: Especially in dealing with steady flows, it is often convenient to add the mechanical energy equation, Eq. 7.17.8, to the internal energy equation, Eq. 3:

$$\int_V \vec{E}' \cdot \vec{J}'_f dV + \oint_S v_i S_{ij} n_j da - \oint_S \vec{\Gamma}'_T \cdot \vec{n} da = \frac{d}{dt} \int_V \rho (W_T + \frac{1}{2} \vec{v} \cdot \vec{v}) dV \quad (5)$$

Here, Eqs. 1 or 2 have been used in reverse, with  $\vec{J}'$  taken as being of electric or magnetic origin. The surface integral is converted to a volume integral and the Leibnitz rule used on the right. Then, the

\*  $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C}$

integrands are equated to give

$$\vec{E} \cdot \vec{J}_f + \frac{\partial}{\partial x_j} (v_i S_{ij}) - \nabla \cdot \vec{T}'_T = \frac{\partial}{\partial t} \rho (W_T + \frac{1}{2} \vec{v} \cdot \vec{v}) + \nabla \cdot \rho \vec{v} (W_T + \frac{1}{2} \vec{v} \cdot \vec{v}) \quad (6)$$

Now, if the flow is steady so that  $\partial(\ )/\partial t = 0$ , substitution of  $S_{ij} = -p\delta_{ij} + T_{ij}^v$  gives

$$\vec{E} \cdot \vec{J}_f + \frac{\partial}{\partial x_j} (v_i T_{ij}^v) - \nabla \cdot \vec{T}'_T = \nabla \cdot [\rho \vec{v} (H_T + \frac{1}{2} \vec{v} \cdot \vec{v})] \quad (7)$$

where the pressure part of  $S_{ij}$  has been moved to the right and absorbed in the specific enthalpy,  $H_T \equiv W_T + p/\rho$  (Eq. 7.22.9).

Entropy Flow: That the energy equation, Eq. 4, is the continuum version of Eq. 7.22.8 is made evident if it is recognized from mass conservation, Eq. 7.2.3, that

$$\nabla \cdot \vec{v} = -\frac{1}{\rho} \frac{D\rho}{Dt} = \rho \frac{D}{Dt} \left( \frac{1}{\rho} \right) \quad (8)$$

Remember that the specific volume  $V \equiv 1/\rho$ . Thus, the right-hand side of Eq. 7.22.8 multiplied by  $\rho$  is the same as Eq. 4, provided that the variations  $\delta W_T$  and  $\delta V$  are replaced by convective derivatives of these functions. This suggests that the left side of Eq. 5 can be identified with  $T\delta S$ , so that Eq. 4 becomes

$$\rho T \frac{DS}{Dt} = \rho \frac{DW_T}{Dt} + \rho p \frac{D}{Dt} \left( \frac{1}{\rho} \right) \quad (9)$$

For an ideal gas, it follows from the mechanical and energy equations of state, Eqs. 7.22.1 and 7.22.2, that

$$\frac{DS}{Dt} = \frac{c_v}{T} \frac{DT}{Dt} + \rho R \frac{D}{Dt} \left( \frac{1}{\rho} \right) = c_v \left( \frac{\rho}{p} \right) \frac{D}{Dt} \left( \frac{p}{\rho} \right) + \rho R \frac{D}{Dt} \left( \frac{1}{\rho} \right) \quad (10)$$

Because  $R = c_p - c_v$  (Eq. 7.22.13), with  $\gamma \equiv c_p/c_v$ , this expression becomes

$$\frac{DS}{Dt} = c_v \left( \frac{1}{p} \frac{Dp}{Dt} - \frac{\gamma}{\rho} \frac{D\rho}{Dt} \right) = \frac{D}{Dt} [c_v \ln(p\rho^{-\gamma})] \quad (11)$$

It follows that along a particle line passing through a point where the properties are  $S = S_0$ ,  $p = p_0$  and  $\rho = \rho_0$ , the specific entropy of a perfect gas is

$$S = S_0 + c_v \ln \left[ \frac{p}{p_0} \left( \frac{\rho_0}{\rho} \right)^\gamma \right] \quad (12)$$

If in particular there are no heat additions to the element of fluid, so that the left side of Eq. 4 is zero, then the element of fluid sustains isentropic dynamics:  $S = S_0$  and the pressure and density are related by the isentropic equation of state,

$$\frac{p}{p_0} \left( \frac{\rho_0}{\rho} \right)^\gamma = 1 \Rightarrow \frac{p}{p_0} = \left( \frac{\rho}{\rho_0} \right)^\gamma \quad (13)$$

For isentropic flow, Eq. 13 represents an invariant along the trajectory of a given fluid element. If the volume of gas of interest originates where the properties are uniform, then Eq. 13 is equivalent to a constitutive law relating pressure and density throughout that volume. Thus, isentropic dynamics fall within the framework of the weakly compressible dynamics considered in Secs. 7.10 - 7.12. With the understanding that it is the specific entropy that is being held constant, the acoustic velocity follows from Eqs. 7.10.4 and Eq. 13 as

$$a = \sqrt{\left( \frac{\partial p}{\partial \rho} \right)_S} = \sqrt{\gamma \frac{\rho_0}{\rho} \rho_0^{\gamma-1}} = \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\gamma RT} \quad (14)$$

For a perfect gas, the acoustic velocity depends only on the temperature and ratio of specific heats. Note that if the dynamics were isothermal (constant temperature) rather than isentropic, the acoustic velocity would be  $a = \sqrt{RT}$ . Because  $\gamma$  ranges between unity and two, such a velocity would always be less than that for an isentropic process.

## 7.24 Overview

The fluid continuum developed in this chapter is capable of storing energy in two forms, the kinetic energy associated with the fluids having inertia and internal energy associated with its compressibility. Dissipation has been represented by the Newtonian model, in which stress is linearly related to strain rate. In summary, the differential laws are the equation of state, Eq. 7.10.3,

$$\rho = \rho(\alpha_1, \dots, \alpha_m, p_s) + \frac{p - p_s}{a^2} \quad (1)$$

conservation statements for the properties  $\alpha_i$ , Eq. 7.10.4,

$$\frac{\partial \alpha_i}{\partial t} + \nabla \cdot (\alpha_i \vec{v}) = 0 \quad (2)$$

conservation of mass, Eq. 7.2.3, which can be combined with Eqs. 1 and 2 to give

$$\frac{1}{a^2} \left( \frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p \right) = -(\rho - \alpha_i \frac{\partial \rho}{\partial \alpha_i}) \nabla \cdot \vec{v} \quad (3)$$

and conservation of momentum, Eq. 7.16.6,

$$\rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) + \nabla p = \vec{F}_{ex} + \eta \nabla^2 \vec{v} + \left( \lambda + \frac{1}{3} \eta \right) \nabla (\nabla \cdot \vec{v}) \quad (4)$$

The relations and approximations which have been developed are now placed in perspective by identifying the characteristic times underlying these laws and recognizing the hierarchy of these times implicit to the various models. The discussion is to the laws of fluid mechanics what that of Sec. 2.3 is to the laws of electrodynamics.

The laws are normalized by introducing dimensionless variables,

$$(x, y, z) = \ell (\underline{x}, \underline{y}, \underline{z}), \quad t = \tau \underline{t}, \quad \vec{v} = \frac{\ell}{\tau} \vec{v}(\ell/\tau), \quad \rho = R \underline{\rho} \quad (5)$$

With the objective a time-rate parameter expansion for the dependent variables, the pressure is given two different normalizations designed to make the zero order approximation all that is required in a wide range of physical situations. Thus,  $p$  is normalized to

reflect the dependence of density on pressure as represented by Eq. 1

$$p = a^2 R p \quad (6a)$$

and Eqs. 1-4 become

$$\rho = \rho(\alpha_1, \dots, \alpha_m, p_s) + p - p_s \quad (7a)$$

$$\frac{\partial \alpha_i}{\partial t} + \nabla \cdot (\alpha_i \vec{v}) = 0 \quad (8)$$

$$\left( \frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p \right) = -(\rho - \alpha_i \frac{\partial \rho}{\partial \alpha_i}) \nabla \cdot \vec{v} \quad (9a)$$

$$\begin{aligned} \beta \rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) + \nabla p \\ = \vec{F}_{ex} + \frac{\tau}{\ell} \left[ \nabla^2 \vec{v} + \left( \frac{\lambda}{\eta} + \frac{1}{3} \right) \nabla (\nabla \cdot \vec{v}) \right] \end{aligned} \quad (10a)$$

where

$$\begin{aligned} \vec{F}_{ex} &= \frac{\ell}{Ra^2} \underline{F}_{ex} \\ \tau_c &= \frac{\eta}{Ra^2} \end{aligned} \quad (11a)$$

reflect the dynamic pressure (inertia) appearing in Bernoulli's equation

$$p = \frac{R \ell^2}{\tau^2} \underline{p} \quad (6b)$$

$$\rho = \rho(\alpha_1, \dots, \alpha_m, p_s) + (p - p_s) \quad (7b)$$

$$\beta \left( \frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p \right) = -(\rho - \alpha_i \frac{\partial \rho}{\partial \alpha_i}) \nabla \cdot \vec{v} \quad (9b)$$

$$\begin{aligned} \rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) + \nabla p \\ = \vec{F}_{ex} + \frac{\tau}{\ell} \left[ \nabla^2 \vec{v} + \left( \frac{\lambda}{\eta} + \frac{1}{3} \right) \nabla (\nabla \cdot \vec{v}) \right] \end{aligned} \quad (10b)$$

$$\begin{aligned} \vec{F}_{ex} &= \frac{\tau^2}{R \ell} \underline{F}_{ex} \\ \tau_v &= \frac{R \ell^2}{\eta} \end{aligned} \quad (11b)$$

and

$$\beta = \left(\frac{\tau_a}{\tau}\right)^2; \tau_a \equiv \ell/a \quad (12)$$

The time-rate parameter  $\beta$  is the ratio of an acoustic wave transit time,  $\tau_a$ , to characteristic times of interest. The viscous dissipation brings in a second characteristic time, either  $\tau_c$  or  $\tau_v$ . The viscous diffusion time,  $\tau_v$ , is familiar from Sec. 7.18 (Eq. 7.18.12), where its analogy to the magnetic diffusion time is discussed. The viscous relaxation time,  $\tau_c$ , is analogous to the charge relaxation time. For example, both  $\tau_c$  and  $\tau_e$  are independent of the characteristic length. Moreover, as can be seen by substitution from Eqs. 11 and 12, the geometric mean of  $\tau_c$  and  $\tau_v$  is the acoustic transit time

$$\sqrt{\tau_v \tau_c} = \tau_a \quad (13)$$

The analogy to the electrodynamic relation between  $\tau_m$ ,  $\tau_e$  and  $\tau_{em}$ , Eq. 2.3.11, points to there being two quasistatic limits, each resulting because  $\beta \ll 1$ .

These can be identified by expanding the normalized dependent variables in power series in  $\beta$ . For example,

$$p = p_0 + \beta p_1 + \beta^2 p_2 + \dots \quad (14)$$

To zero order in  $\beta$ , Eqs. 7-10 become the quasistatic laws. In un-normalized form these are

<p style="text-align: center;">Compressible quasistatic (CQS)</p> $\rho = \rho(\alpha_1, \dots, \alpha_m, p_s) + \frac{p-p_s}{a^2} \quad (15a)$	<p style="text-align: center;">Incompressible (inertial) quasistatic (IQS)</p> $\rho = \rho(\alpha_1, \dots, \alpha_m, p_s) \quad (15b)$
---	--

$\frac{\partial \alpha_i}{\partial t} + \nabla \cdot (\alpha_i \vec{v}) = 0$	$\quad (16)$
--	--------------

$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad (17a)$	$\nabla \cdot \vec{v} = 0 \quad (17b)$
--	--

$\nabla p = \vec{F}_{ex} + \eta \nabla^2 \vec{v} + (\lambda + \frac{1}{3}\eta) \nabla (\nabla \cdot \vec{v}) \quad (18a)$	$\rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) + \nabla p = \vec{F}_{ex} + \eta \nabla^2 \vec{v} \quad (18b)$
---	--

where the ordering of characteristic times is respectively as indicated in Fig. 7.22.1.



Fig. 7.22.1. Ordering of  $\tau_a$ ,  $\tau_v$  and  $\tau_c$  and domain of mechanical quasistatics.

Which of the normalized laws, Eqs. 7-10, is used is arbitrary. However, if for example the left normalization were used for a configuration in which the quasistatic motions were incompressible, the zero-order approximation would be zero, and the appropriate solution would be first-order in  $\beta$ . Examples in which boundary conditions clearly require the CQS limit are those where the total volume of the fluid must change, as in the slow compression of a gas in a rigid-walled vessel by a piston. The IQS and CQS limits are identified for a specific problem, without viscous dissipation, in Sec. 7.12.

Usually, it is the IQS limit that is considered when  $\beta \ll 1$ . Note that with the exception of Secs. 7.10-7.12, Eqs. 15b - 18b have received most of the attention in this chapter. The inviscid incompressible model pertains to  $\tau_a \ll \tau \ll \tau_v$ . The low Reynolds number limit is one in which not only is  $\tau_a \ll \tau$ , but  $\tau_v \ll \tau$  as well.

Nature makes unlikely the CQS ordering of characteristic times. For  $\tau_v/\tau_c < 1$ , it is necessary that the characteristic length  $\ell < \eta/Ra$ . In air under standard conditions this length is a fraction of a micrometer. Because this is about the molecular mean free path, the continuum fluid model is of doubtful validity on a length scale small enough to make viscous relaxation important.