PROFESSOR: Ladies and gentlemen, welcome to this lecture on nonlinear finite element analysis of solids and structures. In this lecture, I'd like to discuss with you some solution methods that we employ to solve the finite element equations that we encounter in nonlinear dynamic analysis.

The solution methods that are, in general used, can be thought of to fall into these three categories; direct integration methods, mode superposition, and substructuring. In this lecture, I would like to talk about the explicit and implicit techniques, direct integration methods, that we use to solve the finite element equations. In the next lecture, we will talk about the mode superposition and substructuring techniques, and we will also consider example solutions.

When we want to solve the finite element equations in nonlinear dynamic analysis, we recognize that basically we have to operate on this set of equations. Here we have the inertia forces, the damping forces, the elastic forces, and the externally applied loads.

The inertia forces and damping forces, well, you know them quite well. They're written in terms of mass matrices, damping matrices, accelerations, velocities.

The elastic forces, here "elastic" in quotes, are actually the nodal point forces that are equivalent to the internal element stresses. And we talked, in the earlier lectures, about how we evaluate them using, in the updated Lagrangian formulation, Cauchy stresses. In the total Lagrangian formulation, the second Piola-Kirchhoff stresses et cetera. Please look up all the information that we talked about in the earlier lectures, if you want to study how we calculate these forces in large displacement, large deformation, large strain analysis.

This set of equations should hold at any time t during the response solution. And in direct integration analysis, we consider this set of equations at the discrete time points delta t apart, as shown down here. Of course, even in static analysis, we always worked with the data t time increment. But in static analysis, we did not include the inertia forces, and the damping forces in the analysis.

The issues that we like to discuss are, what are the basic procedures for obtaining the solutions at the discrete times? And then once we have discussed this question, we have to address the question of which procedure should be used for a given problem. If we use, so to say a bad procedure, "bad" in quotes for a given problem, the costs can be very high to solve the problem. Therefore, it can be important, can be very important to select the appropriate technique for the given problem.

Let's talk first about the explicit time integration techniques. And the one that I like to focus attention on is the central difference method. The central difference method is very widely used to solve nonlinear problems. It's an explicit technique, as we will just now see.

The basic equations that we are working with are listed here. This is the equilibrium equation at time t. Notice here is a tf, the force that corresponds to the internal element stresses at time t. Here are the damping forces, and here the inertia forces. We expand the velocity at time t as shown on the right hand side here. And we expand the acceleration at time t as shown on the right hand side here. These are the central difference formulas.

This method is mainly used for wave propagation problems. It's an explicit technique, and this is an important point, it's an explicit technique because we are looking at the equilibrium equation at time t to calculate the response for time t plus delta t. Any method that does just that, looks at the equilibrium equation at time t to obtain the solution for the response at time t plus delta t is termed an explicit method, an explicit integration method. So the central difference method is such an explicit method.

The important point to recognize right here is that we don't set up a k matrix, a

stiffness matrix. And that is one important fact that pertains to an explicit technique. Using these three equations, we can directly develop one equation. And that equation looks as shown up here.

Notice we have three equations and three unknowns which can be solved. The three unknowns being the displacement at time t plus delta t, the velocity at time t plus delta t, and the acceleration at time t plus delta t. Three equations and three unknowns. We can solve these equations and the governing equation that we obtain in that solution is listed right here.

Notice m over delta t squared. Notice c over 2 delta t here. And then the unknown displacement corresponding to time t plus delta t, and on the right hand side, we have what we call an effective load vector corresponding to time t. There is a hat here, and that hat means that there are a large number of terms to be taken into account. Namely, those corresponding to the inertia and the damping in the system.

Here we have tr hat is equal to tr, the actual loads applied corresponding to a time t. These are the externally applied loads to the nodal points. tf is the force vector corresponding to the internal element stresses, as I said earlier. Here we have an inertia contribution. Here another inertia contribution and a damping contribution.

But it is important to recognize that the right hand side is known. In other words, all the terms here are known because these terms only involve displacements corresponding to time t minus delta t, and displacements corresponding to time t. And therefore we know this tr hat, we can plug it in here on the right hand side and calculate the unknown displacements corresponding to time t plus delta t.

The method is most useful when m and c, the mass and damping matrices are diagonal. Because in that case, these equations here decouple as shown right here. Notice we can calculate the individual displacements components, or the displacement that's individual degrees of freedom, one after the other as shown here. Once we have evaluated tr hat, tr hat let's remember, contains these terms here. In other words, although m and c are diagonal, there is some coupling from the equation j into equation j minus 1 and into equation j plus 1 right here on the

right hand side, because the element contributions overlap the degrees of freedom. And therefore, this here is where the element coupling at the nodal points enters.

But once we have calculated this vector here, I should point to this vector here. Once we have calculated this vector, we know the individual components, then we enter here with that component corresponding to a degree of freedom i, and directly we can calculate the nodal point displacements corresponding to a degree of freedom i. That's an important point to recognize, that there is no need really to set up a k matrix, to trianglize a k matrix. We don't do that here, and that means that per time step, this solution method can be very effective.

Note that in the solution if the damping is 0, in other words, we don't have damping in the system, cii is equal to 0, we need that mii is greater than 0. Otherwise we can't solve the equations. Notice that tf is calculated by summing all the element contributions. Summing over m means summing over all the elements. And as I mentioned earlier, this is where coupling comes into the solution because one nodal point, one degree of freedom at that nodal point carries contribution from all the elements that surround that nodal point.

To start the solution, we need a special equation because, remember in the solution for time t plus delta t, we need the information, the displacements corresponding to time t, and corresponding to time t minus delta t.

So, if t is equal to 0, and we want to march ahead to delta t, in the first step-- we are in the first step-- we use this equation to calculate, to evaluate minus delta tu. And that means we can then directly start our explicit solution process.

The central difference method is only conditionally stable. The condition that has to be satisfied on the time step is given in this equation. Delta t critical is a critical time step, and delta t, the actual time step that is used in the solution, has to be smaller than that value. Smaller or equal to that value. Notice delta t critical is given as tn over pi, where tn is the smallest period in the finite element assemblage.

In nonlinear analysis, we have to recognize that tn changes during the time history.

Response, tn would become smaller if the system stiffens, due for example to large displacement effects. And tn would become larger if the system softens due to elasto-plasticity effects.

One interesting and most important point is that tn can be estimated. And it can be estimated, or we can find a bound on tn which we can then use to calculate the critical time step. And that is achieved as follows.

We notice that the omega n squared, omega n being 2 pi over tn radians per second omega n squared is smaller or equal to the maximum of the omega m, nms squared over all elements m. Now what does this mean? It means here that we're looking at omega nm squared of each of the elements, and we select the largest such value to put in here. And that then gives us an upper bound on omega n squared. The actual omega n that we need, because omega n gives us tn.

So we want to calculate the the largest frequency of all individual elements. And with that largest frequency, we enter in this equation here, to get the bound on tn, noting that in nonlinear analysis omega nm changes, just the way I said earlier tn will also change. But all the omega nm changes and tn correspondingly changes, this equation is always satisfied.

The time integration step that we can actually use then, is given by as this equation. We can use delta t is equal to 2 over omega nm maximum. The maximum radiant per second frequency that we have for any one of the elements in the complete mesh. Because that expression is smaller or equal than the critical time step. We could call this value here 2 over omega nm , the critical time step of element m. And therefore, what we're doing here really is, that we're taking the smallest of the critical time steps over all elements as that time step in the solution. And that time step we know is smaller or equal to delta t critical.

Let's look at the proof of this relationship, because it's an interesting proof and it gives us a bit more insight into why this relationship really holds. So I'd like to prove to you very briefly that this relationship here is valid. And we do so by using the Rayleigh quotient. Please look at the textbook where the Rayleigh quotient is

discussed, and I have to now assume that you are familiar with the Rayleigh quotient.

If you write down the Rayleigh quotient for omega n squared, it is this relationship here. Omega n being the largest frequency of the complete finite element mesh. Omega n is equal to 2 pi divided by tn. And tn was the smallest period in the mesh that we are interested in knowing. Let us define this relationship here, just the definition, and this relationship here. Once again, just the definition.

Then, by simply substituting from here and there into this relationship here, we surely can write this equation as shown here. We're summing over all of the us, and we're summing over all of the is. m going over all the elements. If you have a 1,000 elements in the mesh, then n would go from 1 to 1,000.

If we now look at the Rayleigh quotient for a single element, we can say that rho m is given by this relationship. Which is nothing else, using our definition of um and im, is nothing else than this relationship here. However, we can immediately recognize that rho m, this value here, must be smaller or equal to omega nm squared, where omega nm is the largest frequency radian per second of the element m. This is here a relationship that must hold because the Rayleigh quotient statement, or the Rayleigh quotient theorem, tells us that this must hold. And once again, please look up in the textbook a general proof, there's a general proof given that in fact this is a valid statement.

Where now we can use this statement, go back to this equation and immediately identify that this must hold. And notice once again I'm looking here at the element m. Of course, this relationship must hold for all elements, m going from 1 to the total number of elements you actually have in the mesh. This is the relationship we will be using, and we substitute that relationship in our earlier expression, for the omega n squared and directly arrive at this relationship here.

Now we look at this part here, and recognize that we can take this out of the summation sign by using the largest or the effect, we can take the effect out of the summation sign, I should've said, by taking the largest value of omega nm squared.

And that's being done here and now we can cancel these two terms resulting into this relationship which completes our proof.

It's an interesting proof, and once again if you're not familiar with the Rayleigh quotient, please look up the information in the textbook, after which I think you can quite easily follow this proof.

The important point now is that the largest frequencies of simple elements can be estimated analytically. Sometimes we can calculate them exactly, and here we have a case where we can calculate the largest frequency of the element exactly.

Here we have a two nodal truss element, m here, m here, equal to rho AL over 2. L being the length of the truss, A being the cross sectional area, rho being the mass density. If we write down the eigenvalue problem for this single element, we obtain this relationship here. Stiffness matrix times the eigenvector phi omega squared, which will be an eigenvalue, the mass matrix phi being the sought eigenvector.

And if we solve this eigenvalue problem, we obtain directly as the eigenvalues these two values here. There are only two of course, because this is a two by two matrix. And so is that. So we have only two eigenvalues. The first eigenvalue is 0, the rigid body mode in the system. And the second eigenvalue is given here, which is equal to omega n squared. And there is the relationship of that second eigenvalue obtained by some very simple calculations, and this is a rewriting of that equation, of that relationship here. c is a wave speed in the system. In fact, c squared is, as you can see here, e over rho.

It's this relationship that we can now use, very conveniently, to get a time step in a finite element mesh that will satisfy the critical time step criterion. In other words, if we have a finite element mesh that is consisting of an assemblage of such truss elements, then to get the time step for the explicit time integration, which will be a good time step, which will be smaller or equal to the critical time step, what we do is we simply calculate this relationship here for each of the elements. And this relationship will give us then, the proper time step by evaluating the smallest time step that we can use for any one of the elements.

In other words, we simply take 2 over omega n, which is written here, and we recognize that L over c, where L is the length of the element, is the time step that we can use in the analysis. Notice L over c, we want to use the smallest value of L over c coming from all of the elements. L would be an element length. c would be the wave speed through the element, and we want to use the smallest value of L over c, looking at all the elements, as the time step for our time integration.

Notice here that L over c is the time required for wave front to travel through the element. That is a physical interpretation of this L over c value.

Let's look at some modelling aspects regarding this technique. Let the applied wavelengths be Lw as shown here. Here schematically we show a wave. And let's see that how we would model the finite element system appropriately when we apply to that system this kind of wave.

Well, we first of all find the value tw, which is really the time required for the total wave to propagate across a particular point in the mesh, with the wave speed c of course. We then choose delta t via this relationship here, where n is the number of time steps used to represent the wave. With delta t now known, we directly get Le, which is the element lengths. Well, I should not quite say it's the element lengths, it's related to the element lengths.

For very simple problems, and particular for our problems in a model using truss elements, two nodal truss elements, actually Le would be the element lengths. But if we have more complex elements to model the finite elements solution, then this here, this Le value, would be related to the element lengths.

Note that in this formula here, n has to be chosen by the analyst to be a reasonable value so as to be able to represent the total wave appropriately. Here we have some observations. Note that in 1D analysis, c, the wave speed, is given as the square root out of e over rho. I mentioned that already earlier.

Notice that in nonlinear analysis, delta t must satisfy the stability limit throughout the solution. This is a very important point and in the next lecture we will look at some

examples that demonstrates what happens if you don't satisfy the stability limit throughout the solution. It may also be effective to change the time step during the analysis. Of course, this you would do in a nonlinear analysis, typically.

For the application. or for the use of the explicit time integration method, when you use the explicit time integration method, one generally uses lower-order elements. They are usually preferable. And in this particular case, you want to have Le here and Le there. Uniform meshing, uniform meshing.

For higher-order elements, you could also use the explicit time integration. And there again, you want to really have that this L is equal to that L star. If you don't have that condition, then this L here will govern the actual time step. The smallest length through the element will govern the time step. And if you use this value here for Le, with 8 there, you have a conservative value for Le. This 8 here enters into the solution because you have this midnode, and the midnode carries more stiffness then the corner nodes.

Let us look further at some observations regarding the method. In the analysis of this problem, where we have a beam subjected to an end step load as shown here, we can get the exact solution by simply choosing a finite element mesh of truss elements that satisfy this relationship here.

In other words, we represent this beam here, using two nodal truss elements. And these two nodal truss elements have a length equal to c times delta t. Of course, we're looking here at a linear elastic solution, meaning that c is constant. And we will get, this is the important point, we will get the exact solution for any number of truss elements that are used. In fact, we will look at this problem later on in the next lecture in more detail, and we will use this fact to obtain the exact solution, and we'll study also what happens if we change the time step. So I'd like to refer you to the next lecture in which I will be talking more about this problem.

Using the explicit time integration method, it is very important to use uniform meshing. And the reason for it is given right here. Namely, we want to have then a time step that is not unduly small in any region of the total mesh. If you don't use

uniform meshing, then the time step is governed by a particular part in the mesh, the time step size I should say, is governed by a particular part in the mesh. Namely, those elements which have the smallest Le, the way I talked about it earlier.

And that same time step would be used for the whole mesh, which means then, that time step, that small time step would actually be too small for the other parts of the mesh. Of course, one could say, well, why not use different time steps in the mesh? Namely, effective time steps for the elements that are used in the various regions of the mesh. That can be done, but then you have to use special coupling techniques to couple the time step integrations where you use in one mesh a small time step, in one part of the mesh a small time step, in the other products the mesh a larger time step. It can be done, but it needs special considerations regarding the coupling of the time integrations in the different parts of the mesh.

We may also want to note that a system with a very large bandwidth can be solved frequently, quite efficiently, with the central difference method, although the problem may not be a wave propagation problem. The reason for this fact is that in the explicit time integration, you do not set up a k matrix, and therefore, the large bandwidth that you're seeing in the k matrix is not really a hindrance. It is not being used because the k matrix is not being set up, and you have no factorization or LDL transpose factorization of the coefficient matrix. And therefore, this high cost that lies in the factorization of the coefficient matrix is not necessary, is not used, is not expensed in the explicit time integration.

And for that reason, it can be effective to use explicit time integration when there would be such a large bandwidth. Explicit time integration lends itself also very well to parallel processing. Notice that the basic equations that we are solving for in explicit time integration, in each time step are given as here. T plus delta tu is equal to something that I denote here by the three dots, times tr hat. And this tr hat vector we defined earlier.

So all we need to do is calculate these entries in the vector here to get our unknown

displacements. Well, that can be done quite effectively in parallel. We might want to calculate these entries here at the same time as we calculate say, these entries. Where this is here a particular element group, and that are the entries corresponding to an adjacent element group. And these are the entries corresponding to another element group. So we can, in other words, in parallel, process all the information that goes in here, that goes in here, and that goes in here, and so on.

Notice there's some coupling in that information, which goes across this red horizontal line. That coupling comes from the fm contributions, namely the element contributions that will be elements that lie across these equations, so to say. And that's where a coupling lies, and that has to be properly taken into account in the parallel processing. But in principle, explicit time integration because the equation that we are dealing with, as shown here, lends itself very well to parallel processing on computers.

Let us now look at the other technique that I wanted to discuss with you briefly in this lecture. And this is an implicit time integration method. It is a trapezoidal rule, which is very widely used in implicit time integration. But let's first look at the basic equation that we now are operating on. In implicit time integration, in any implicit time integration scheme, we are solving this equation here. At time t plus delta t to obtain the solution at time t plus delta t.

Notice this is the equilibrium equation at time t plus delta t. This is quite different from what we're doing in explicit time integration. In explicit time integration we are looking at the equilibrium equation at time t to obtain the solution at time t plus delta t. Here we are looking at the equilibrium equation, we are using the equilibrium equation at time t plus delta t to obtain the solution at time t plus delta t.

And that means that we have to deal with a stiffness matrix. And that stiffness matrix here is given as tk, corresponding to a modified Newton iteration. Notice that this equation will be solved to calculate an increment and displacement. And that increment in displacement is added to the displacement vector that we had from the

previous iteration to obtain an updated displacement vector.

This is one equation in the unknown displacements, velocities, and accelerations at time t plus delta t. And we need two more equations to solve for these three unknowns. In the trapezoidal rule, which as I mentioned just now, is very widely used for implicit time integration, these are the basic equations used for displacements and for velocities. If we rewrite these equations, we directly obtain the velocities and accelerations in this form.

Notice there is here the increment in displacement from time t to time t plus delta t. Similarly here. In nonlinear analysis, we are iterating for this displacement here, and therefore we rewrite these equations a bit. And that's being shown, that rewriting is shown on this viewgraph. Notice that we now have the superscript k here corresponding to the iteration k, and that we have split the displacement at time t plus delta t into a value that we know, corresponding to iteration k minus 1, and an increment that is unknown, delta uk. We do the same for the acceleration here.

Of course, these terms are as shown on the previous viewgraph. And now we have our velocity and acceleration corresponding to iteration k, in terms of known values and one unknown value, the incremental displacement.

We can now use these two equations and substitute into our equilibrium equation the equation that you saw on the first viewgraph, when we started to talk about the implicit time integration scheme. And the result of that substitution is given on this viewgraph. We have a coefficient matrix here, which we call tk hat times the incremental displacement vector, which is unknown. And on the right hand side of that equation, we have the externally applied loads, the nodal point forces corresponding to the internal element stresses at time t plus delta t, and at the end of iteration k minus one.

To evaluate these, we need the displacements corresponding to iteration k minus 1. And here we have inertia contributions and damping contributions. But notice all this information here is known once you know the displacements corresponding to t plus delta tu k minus 1.

We iterate on this equation until we satisfy certain equilibrium criteria, convergence criteria, and much the same way as we do it in static nonlinear analysis, and the way we discussed it in an earlier lecture. However, let us make a few observations.

First of all, we notice that as delta t gets smaller, the entries in this tk hat matrix, this effective stiffness matrix, increase. Why do they increase? Because we have delta t squared term, 1 over delta t squared term in front of the mass matrix. And the 1 over 2 delta t term in front of the damping matrix. As delta t becomes smaller, 1 over delta t squared becomes larger and that means that the entries in this matrix here become larger as delta t gets smaller.

The convergence characteristics of the equilibrium iterations are better than in static analysis. The reason being, that the entries in tk hat are generally larger, much larger than what we see in static analysis because of this 1 over delta t squared term, and 1 over 2 delta t term, that I just mentioned.

We also notice that the trapezoidal rule is unconditionally stable in linear analysis. The analysis that shows that the method is unconditionally stable is really performed on a linear system, and therefore the unconditional stability strictly holds only linear analysis.

For nonlinear analysis, we want to select delta t for accuracy, and we want to select delta t for convergence of the iteration. If we find, for example, that we have convergence difficulties in the iteration, then we may want to switch to a more powerful scheme. That's one approach, of course, just the same way as we are doing it in static analysis.

Or we may also recall, or we may identify actually, that delta t is too large, and we should decrease delta t. In particular, as a rule of thumb, I'd just like to share some experience with you. We find that if you use the BFGS method, which we discussed earlier in an earlier lecture, if you use the BFGS method for the equilibrium iteration, and if we have difficulties converging there, then very frequently the time step is too large, and should be decreased. Not just to obtain convergence in the BFGS

iterations, but to obtain an appropriate accuracy.

So we really look at the difficulties obtaining convergence in the iterations as also a signal to us to tell us that our time step delta t is probably too large. The convergence criteria that we are using are basically the same that we have seen already in the static analysis in an earlier lecture. In static analysis we only included this term here. In dynamic analysis we include now the mass and damping terms in our energy criterion.

We discussed this criterion earlier, of course not including the mass and damping effects. Now we do. Another convergence criterion is one on forces, and earlier we had only this term here. Now we include also the inertia and the damping terms.

Once again, if we are only having translational degrees of freedom, then here we have only nodal point forces, nodal point forces here. And our norm would be having the dimensions of a nodal point force.

If we have rotational degrees of freedom, then we still include here only the nodal point forces. We extract, in other words, the components corresponding to the translational degrees of freedom of this vector and that vector, and include those components only up here. We use RNORM that also has the units of the nodal point force, and we supplement this convergence criteria by another quotient, where we have on top the components corresponding to the rotational degrees of freedom only, and in the bottom, what we called RMNORM already earlier.

The point is that we want to have up here only those components that carry the same units as we have down here. Once for translations, once for rotations. And of course recall we're talking here about the Euclidean norm. which is defined as shown here. We went through this already earlier, when we talked about the convergence criteria in static analysis. We have also the convergence criterion on displacements. Notice this is what we saw already earlier in static analysis.

Once again, we use displacement components here to compare with the displacement. We introduced all of the displacement components here to compare

with this displacement and if we have rotational degrees of freedom, they would not be included here. Their components would not be included here. When you use DNORM here, they would, however, in a second check be included here and only included here, and then we would use here DMNORM. We went through that earlier already.

Some modelling hints regarding the use of the implicit time integration scheme. The integration scheme is generally used to analyze structural vibration problems, and we would proceed as follows. We identify the frequencies contained in the loading by a Fourier decomposition. We chose a finite element mesh that can accurately represent the static response and all important frequencies. And then we perform a direct integration using this time step here where tco is the smallest period in seconds that we want to integrate.

Now there's a lot of information here, and I'd like to refer you to an example, analysis of a tower that I will be presenting to you in the next lecture, where we will talk about these items quite a bit more. But this is basically the approach that we use, to model the problem and then also to perform the direct integration of the governing finite element equations.

As I said earlier already, the method is used primarily for structural vibration problems. It is effective, quite effective, using also higher-order elements. And if we do use higher-order elements, we do find frequently that method, or the total solution should be performed most effectively using a consistent mass matrix. I'm thinking here in particular, if you use very high-order elements such as cubic shall elements, isoparametric cubic shell elements, an element that we will talk about in a later lecture, then with such element you probably want to use a consistent mass matrix for the overall solution of the problem.

Notice that a structural vibration problem can be thought of as a static problem including inertia forces. And we already have noted that in structural analysis of static problems, the higher-order elements do quite well, and sometimes much better than the lower elements. And for this reason, they also do quite well in

dynamic analysis of structural vibration problems.

A typical problem that one might like to solve using this implicit time integration scheme, is schematically shown here. We have a tower that is subjected to a blast loading as shown here schematically, and we assume that only the structural vibrations of this tower are required to be solved.

In this case, we may very well obtain the response of the tower using something like about 100 steps to integrate the response. In fact, we will look at a problem of this nature in the next lecture, where we do actually analyze a tower using the implicit time integration scheme. And where we talk a little bit about how the tower would be modelled, and what one would look for to make sure that you have adequately calculated the response.

Finally, I'd like to also point out that it can be effective to use the combined technique of explicit and implicit integration. You might, for example, have a problem that shows an initial wave propagation and then a structural vibration. In this case, you might first want to use the explicit time integration to calculate the initial wave propagation response of the structure, and then restart to use an implicit time integration technique to calculate the vibrational response of the structure.

So in that case, you would basically combine explicit and implicit time integration, but the combination would be first explicit, and then implicit time integration. Another kind of combination is quite effective when you have to analyze problems where you have very stiff regions and very flexible regions. For the flexible regions of the problem, you might want to use explicit time integration, and for the stiff region you might want to use implicit time integration.

A typical analysis, for example, would be a fluid structure problem, where the fluid is very soft and the structure is very stiff. Using the implicit time integration for the structure and the explicit time integration for the fluid can be effective. Of course, then you would also have to couple these time integration schemes properly at the boundary between the fluid and the structure.

This brings me then to the end of what I wanted to say in this lecture. In the next lecture we will continue our discussion of solution methods for the solution of the nonlinear dynamic equilibrium equations. And we will also show examples, examples that demonstrate how the techniques that I discussed in this lecture are used to solve problems.

Thank you very much for your attention.