SOLUTION OF FINITE ELEMENT EQUILIBRIUM EQUATIONS IN STATIC ANALYSIS

LECTURE 9

60 MINUTES

LECTURE 9 Solution of finite element equations in static analysis

Basic Gauss elimination

Static condensation

Substructuring

Multi-level substructuring

Frontal solution

$\underline{L}\ \underline{D}\ \underline{L}^T$ - factorization (column reduction scheme) as used in SAP and ADINA

Cholesky factorization

Out-of-core solution of large systems

Demonstration of basic techniques using simple examples

Physical interpretation of the basic operations used

TEXTBOOK: Sections: 8.1, 8.2.1, 8.2.2, 8.2.3, 8.2.4,

Examples: 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9, 8.10

SOLUTION OF EQUILIBRIUM EQUATIONS IN STATIC ANALYSIS

$\underline{K}\ \underline{U} = \underline{R}$

- Iterative methods, e.g. Gauss-Seidel
- Direct methods

  these are basically variations of Gauss elimination

  - static condensation
  - substructuring
  - frontal solution
  - $\underline{L}\ \underline{D}\ \underline{L}^T$ factorization
  - Cholesky decomposition
  - Crout
  - column reduction (skyline) solver

THE BASIC GAUSS ELIMINATION PROCEDURE

Consider the Gauss elimination solution of

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (8.2)$$

STEP 1: Subtract a multiple of equation 1 from equations 2 and 3 to obtain zero elements in the first column of K .

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & -\frac{16}{5} & \frac{29}{5} & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{8.3}$$

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & -\frac{20}{7} & \frac{65}{14} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ -\frac{5}{14} \end{bmatrix} \tag{8.4}$$

STEP 3:

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & 0 & \frac{5}{6} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ \frac{7}{6} \end{bmatrix} \quad (8.5)$$

Now solve for the unknowns U_4, U_3, U_2 and U_1:

$$U_4 = \frac{\frac{7}{6}}{\frac{5}{6}} = \frac{7}{5} \; ; \quad U_3 = \frac{\frac{8}{7} - (-\frac{20}{7})\,U_4}{\frac{15}{7}} = \frac{12}{5}$$

$$U_2 = \frac{1 - (-\frac{16}{5})\,U_3 - (1)\,U_4}{\frac{14}{5}} = \frac{13}{5} \quad (8.6)$$

$$U_1 = \frac{0 - (-4)\frac{19}{35} - (1)\frac{36}{15} - (0)\,\frac{7}{5}}{5} = \frac{8}{5}$$

STATIC CONDENSATION

Partition matrices into

$$\begin{bmatrix} \underline{K}_{aa} & \underline{K}_{ac} \\ \underline{K}_{ca} & \underline{K}_{cc} \end{bmatrix} \begin{bmatrix} \underline{U}_{a} \\ \underline{U}_{c} \end{bmatrix} = \begin{bmatrix} \underline{R}_{a} \\ \underline{R}_{c} \end{bmatrix} \qquad (8.28)$$

Hence

$$\underline{U}_c = \underline{K}_{cc}^{-1} \left(\underline{R}_c - \underline{K}_{ca} \, \underline{U}_a \right)$$

and

$$\underbrace{\left(\underline{K}_{aa} - \underline{K}_{ac} \, \underline{K}_{cc}^{-1} \, \underline{K}_{ca} \right)}_{\overline{\underline{K}}_{aa}} \underline{U}_a = \underline{R}_a - \underline{K}_{ac} \, \underline{K}_{cc}^{-1} \, \underline{R}_c$$

Example

$\underline{K}_{cc}$ $\qquad$ $\underline{K}_{ca}$

$$\left[\begin{array}{c|ccc} 5 & -4 & 1 & 0 \\ \hline -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{array}\right] \left[\begin{array}{c} U_1 \\ \hline U_2 \\ U_3 \\ U_4 \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline 1 \\ 0 \\ 0 \end{array}\right]$$

$\underline{K}_{ac}$ $\qquad$ $\underline{K}_{aa}$

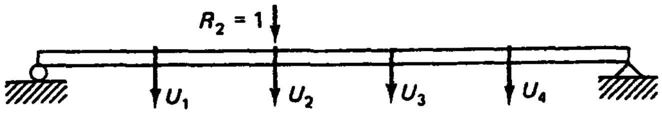
Hence (8.30) gives

$$\overline{\underline{K}}_{aa} = \begin{bmatrix} 6 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix} - \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} [1/5] \; [-4 \quad 1 \quad 0]$$

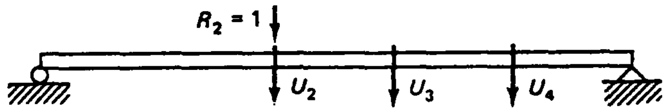
so that

$$\overline{\underline{K}}_{aa} = \begin{bmatrix} \frac{14}{5} & -\frac{16}{5} & 1 \\ -\frac{16}{5} & \frac{29}{5} & -4 \\ 1 & -4 & 5 \end{bmatrix}$$

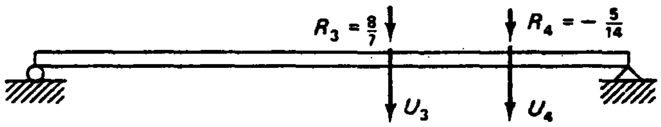
and we have obtained the 3x3 unreduced matrix in (8.3)



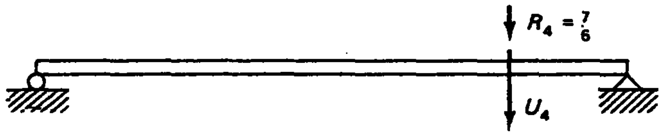
$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} \frac{14}{5} & -\frac{16}{5} & 1 \\ -\frac{16}{5} & \frac{29}{5} & -4 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \\ U_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} \frac{15}{7} & -\frac{20}{7} \\ -\frac{20}{7} & \frac{65}{14} \end{bmatrix} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} \frac{8}{7} \\ -\frac{5}{14} \end{bmatrix}$$

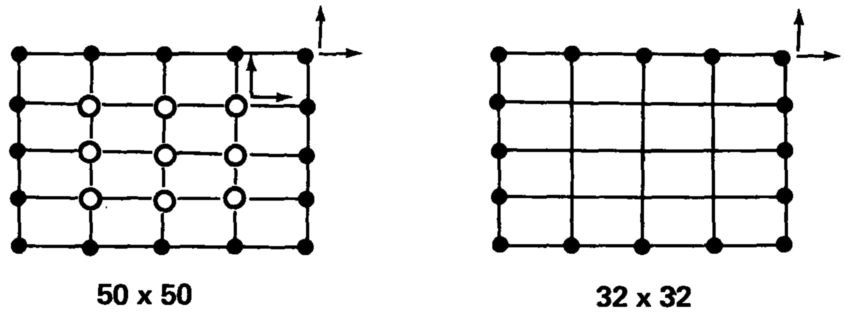


$$\begin{bmatrix} \frac{5}{6} \end{bmatrix} \begin{bmatrix} U_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{6} \end{bmatrix}$$

Fig. 8.1 Physical systems considered in the Gauss elimination solution of the simply supported beam.

SUBSTRUCTURING

- We use static condensation on the internal degrees of freedom of a substructure
- the result is a new stiffness matrix of the substructure involving boundary degrees of freedom only



Example

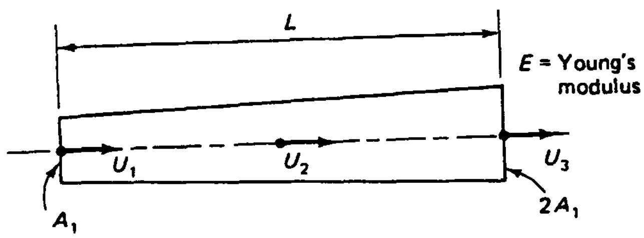


Fig. 8.3. Truss element with linearly varying area.

We have for the element,

$$\frac{EA_1}{6L}\begin{bmatrix} 17 & -20 & 3 \\ -20 & 48 & -28 \\ 3 & -28 & 25 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

First rearrange the equations

$$\frac{EA_1}{6L}\begin{bmatrix} 17 & 3 & -20 \\ 3 & 25 & -28 \\ -20 & -28 & 48 \end{bmatrix}\begin{bmatrix} U_1 \\ U_3 \\ U_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_3 \\ R_2 \end{bmatrix}$$

Static condensation of U_2 gives

$$\frac{EA_1}{6L}\left\{\begin{bmatrix} 17 & 3 \\ 3 & 25 \end{bmatrix} - \begin{bmatrix} -20 \\ -28 \end{bmatrix}\left[\frac{1}{48}\right][-20 \quad -28]\right\}\begin{bmatrix} U_1 \\ U_3 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 + \frac{20}{48} R_2 \\ R_3 + \frac{28}{48} R_2 \end{bmatrix}$$

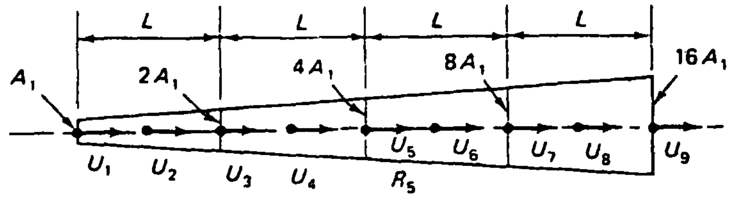
or

$$\frac{13}{9}\frac{EA_1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_3 \end{bmatrix} = \begin{bmatrix} R_1 + \frac{5}{12} R_2 \\ R_3 + \frac{7}{12} R_2 \end{bmatrix}$$

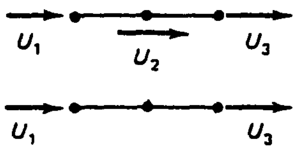
and

$$U_2 = \frac{1}{24}\left(\frac{3L}{EA_1} R_2 + 10\, U_1 + 14\, U_3\right)$$

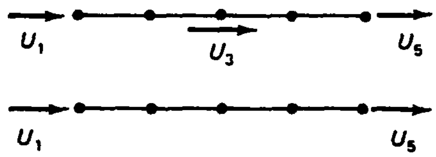
Multi-level Substructuring



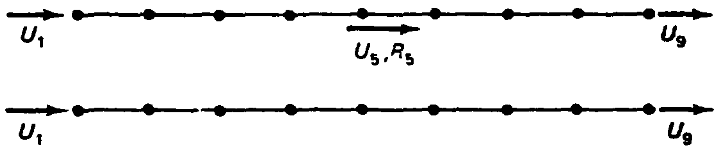
Bar with linearly varying area



(a) First-level substructure



(b) Second-level substructure



(c) Third-level substructure and actual structure.

Fig. 8.5. Analysis of bar using substructuring.

Frontal Solution

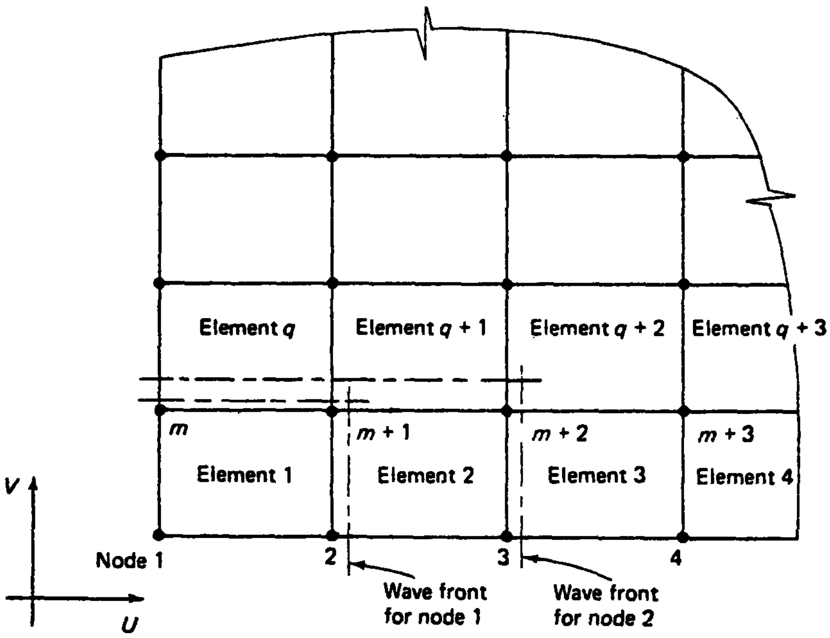


Fig. 8.6. Frontal solution of plane stress finite element idealization.

- **The frontal solution consists of successive static condensation of nodal degrees of freedom.**
- **Solution is performed in the order of the element numbering .**
- **Same number of operations are performed in the frontal solution as in the skyline solution, if the element numbering in the wave front solution corresponds to the nodal point numbering in the skyline solution.**

$\underline{L}\ \underline{D}\ \underline{L}^T$ FACTORIZATION

- is the basis of the skyline solution (column reduction scheme)

- Basic Step

$$\underline{L}_1^{-1}\ \underline{K} = \underline{K}_1$$

Example:

$$\begin{bmatrix} 1 & & & \\ \frac{4}{5} & 1 & & \\ -\frac{1}{5} & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & -\frac{16}{5} & \frac{29}{5} & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

We note

$$\underline{L}_1^{-1} = \begin{bmatrix} 1 & & & \\ \frac{4}{5} & 1 & & \\ -\frac{1}{5} & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad \underline{L}_1 = \begin{bmatrix} 1 & & & \\ -\frac{4}{5} & 1 & & \\ \frac{1}{5} & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Proceeding in the same way

$$\underline{L}_{n-1}^{-1}\ \underline{L}_{n-2}^{-1}\ \cdots\ \underline{L}_{2}^{-1}\ \underline{L}_{1}^{-1}\ \underline{K} = \underline{S}$$

$$\underline{S} = \left[\begin{array}{cccccccc} x & x & x & x & \cdots & x \\ & x & x & x & \cdots & x \\ & & x & \cdots & \cdots & x \\ & & & x & \cdots & x \\ & & & & x & \\ & & & & & \ddots\ x \end{array}\right] \left.\vphantom{\begin{array}{c} x \\ x \\ x \\ x \end{array}}\right\} \text{upper triangular matrix}$$

Hence

$$\underline{K} = (\underline{L}_1\ \underline{L}_2\ \cdots\ \underline{L}_{n-2}\ \underline{L}_{n-1})\ \underline{S}$$

or

$$\underline{K} = \underline{L}\ \underline{S}\ ;\ \underline{L} = \underline{L}_1\ \underline{L}_2\ \cdots\ \underline{L}_{n-2}\ \underline{L}_{n-1}$$

Also, because $\underline{K}$ is symmetric

$$\underline{K} = \underline{L}\ \underline{D}\ \underline{L}^T\ ;$$

where

$$\underline{D} = \text{ diagonal matrix }\ ;\ d_{ii} = s_{ii}$$

In the Cholesky factorization, we use

$$\underline{K} = \underline{\tilde{L}}\,\underline{\tilde{L}}^T$$

where

$$\underline{\tilde{L}} = \underline{L}\,\underline{D}^{\frac{1}{2}}$$

SOLUTION OF EQUATIONS

Using

$$\underline{K} = \underline{L}\,\underline{D}\,\underline{L}^T \qquad (8.16)$$

we have

$$\underline{L}\,\underline{V} = \underline{R} \qquad (8.17)$$

$$\underline{D}\,\underline{L}^T\,\underline{U} = \underline{V} \qquad (8.18)$$

where

$$\underline{V} = \underline{L}_{n-1}^{-1} \cdots \underline{L}_2^{-1}\,\underline{L}_1^{-1}\,\underline{R} \qquad (8.19)$$

and

$$\underline{L}^T\,\underline{U} = \underline{D}^{-1}\underline{V} \qquad (8.20)$$

COLUMN REDUCTION SCHEME

$$\begin{bmatrix} 5 & -4 & 1 & \\ & 6 & -4 & 1 \\ & & 6 & -4 \\ & & & 5 \end{bmatrix}$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 5 & -\frac{4}{5} & 1 & \\ & \frac{14}{5} & -4 & 1 \\ & & 6 & -4 \\ & & & 5 \end{array}\right] \quad \left[\begin{array}{cc|cc} 5 & -\frac{4}{5} & 1 & \\ & \frac{14}{5} & -4 & 1 \\ & & 6 & -4 \\ & & & 5 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 5 & -\frac{4}{5} & \frac{1}{5} & \\ & \frac{14}{5} & -\frac{8}{7} & 1 \\ & & \frac{15}{7} & -4 \\ & & & 5 \end{array}\right] \quad \left[\begin{array}{ccc|c} 5 & -\frac{4}{5} & \frac{1}{5} & \\ & \frac{14}{5} & -\frac{8}{7} & 1 \\ & & \frac{15}{7} & -4 \\ & & & 5 \end{array}\right]$$

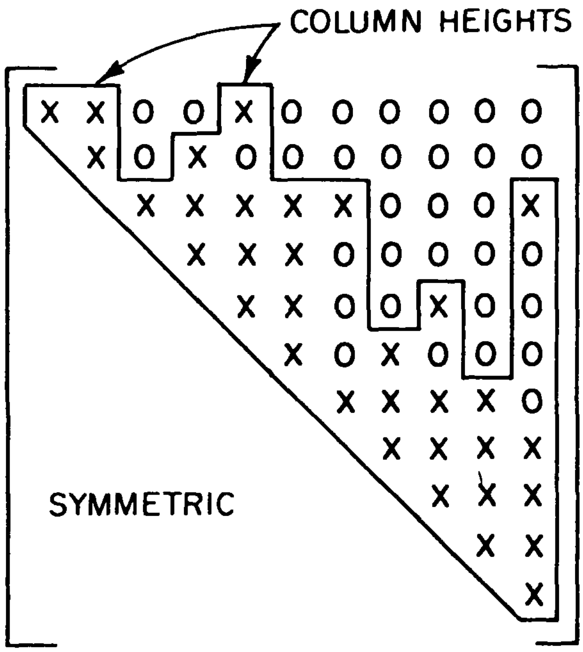
$$\downarrow$$

$$\begin{bmatrix} 5 & -\frac{4}{5} & \frac{1}{5} & \\ & \frac{14}{5} & -\frac{8}{7} & \frac{5}{14} \\ & & \frac{15}{7} & -\frac{4}{3} \\ & & & \frac{5}{6} \end{bmatrix}$$

$\underline{L}^T$

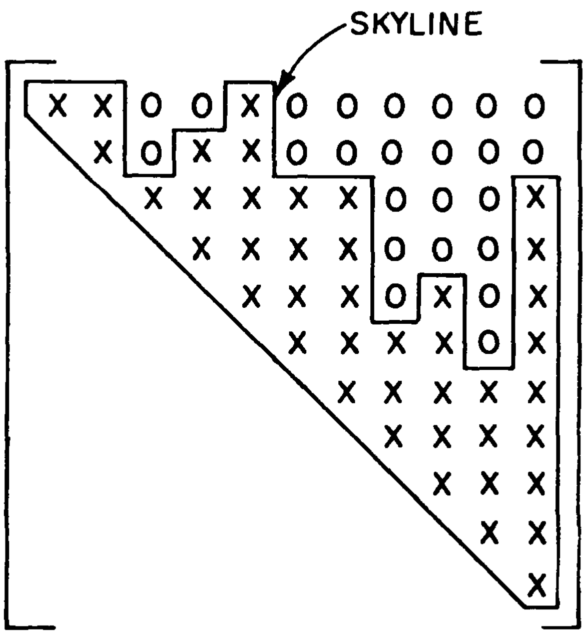
$\underline{D}$

X = NONZERO ELEMENT
O = ZERO ELEMENT



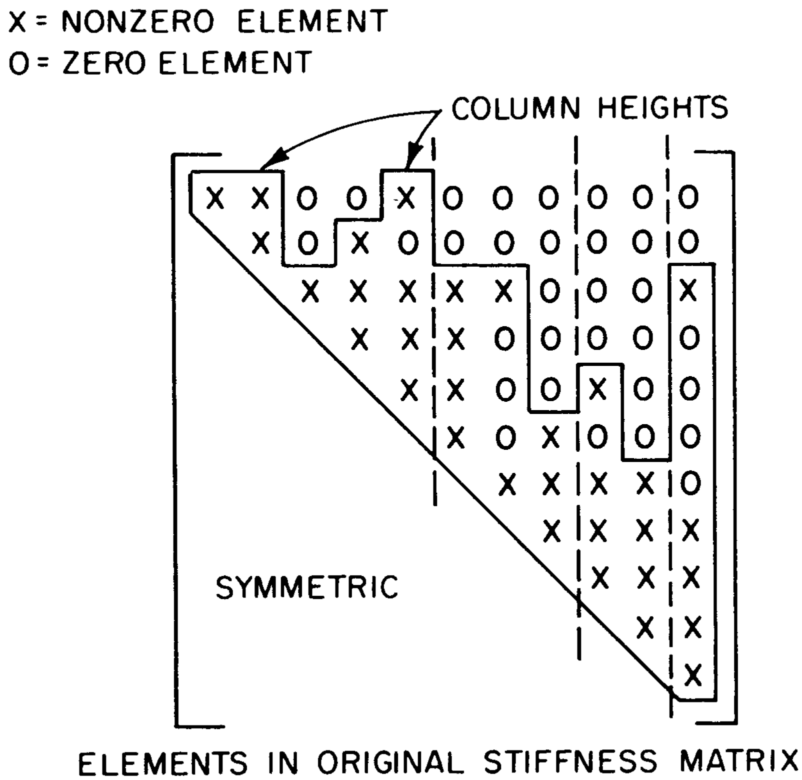
ELEMENTS IN ORIGINAL STIFFNESS MATRIX

Typical element pattern in a stiffness matrix



ELEMENTS IN DECOMPOSED STIFFNESS MATRIX

Typical element pattern in a stiffness matrix



Typical element pattern in a stiffness matrix using block storage.