GILBERT STRONG: Well, you see spring has finally come to Boston. My sweater is gone and it's April the 16th, I think. It's getting late spring. So today, this video is the nice case, constant coefficient, linear equations, and the right hand side is an exponential. Those are the best.

And we've seen that before. In fact, let me extend, we saw it for first order equations, here it is for second order equations, and it could be an nth order equation. We could have the nth derivative and all lower derivatives. The first derivative, the function itself, 0-th derivative with coefficients, constant coefficients, equalling e to the st. That's what makes it easy.

And what do we do when the right hand side is e to the st? We look for a solution, a multiple of e to the st. Capital Y times e to the st, that's going to work. We just plug into the equation to find that transfer function capital Y. Can I just do that?

I'll do it for the nth degree. Why not? So will I plug this in for y, every derivative brings an s. Capital Y is still there, the exponential is going to be still there, and then there are all the s's that come down from the derivatives, and s's from the nth derivative. One s from the first derivative, no s from the constant term.

Do you see that equation is exactly like what we had before with as squared plus Bs plus C. We have quadratic equation, the most important case. Now, I'm including that with any degree equation, nth degree equation. And what's the solution for Y?

Because e to the st cancels e to the st. That whole thing equals 1. I divide by this and I get Y equal 1 over that key polynomial. It's a nth degree polynomial. And the 1 over it is called the transfer function. And that transfer function transfers the input-- e to the st-- to the output-- Ye to the st. It gives the exponential response.

Very nice formula. Couldn't be better. And you remember for second degree equations, our most important case is as squared Bs plus C. That's the solution almost every time. But one thing can go wrong.

One thing can go wrong. Suppose for the particular s, the particular exponent, in the forcing function, suppose that s in the forcing function is also one of the s's in the no solutions. You remember the no solutions, there are two s's-- s1 and s2-- that make this 0, that make that 0. Those are the s1 and s2 that go into the no solutions.

Now, if the forcing s is one of those no solution s's, we have a problem. Because this is 1 over 0 and we haven't got an answer yet. 1 over 0 has no meaning. So I have to, this is called resonance. Resonance is when the forcing exponent is one of the no exponents that make this 0.

And there are two of those for second order equations and there will be n different s's-- s1, s2, up to sn-- for nth degree equations. Those special s's, I could also call them poles of the transfer function. The transfer function has this in the denominator and when this is 0, that identifies a pole. So the s1, s2, to sn are the poles and we hope that this s is not one of those, but it could be.

And if it is, we need a new formula. So that's the only remaining case. This is a completely nice picture. We just need this last case with some resonance when s equals say s1. I'll just pick s1 and when A, I know that As 1 squared plus Bs 1 plus C is 0. So Y would be 1 over 0. And we can't live with that.

I've written here for the second degree equation same possibility for the nth degree, An s1 to the nth plus A0 equals 0. That would be a problem of resonance. In the nth degree equation, this gives us resonance, you see, because remember, the no solutions were e to the s1t was a no solution.

If I plug-in e to the s1t, the left side will give 0. So I can't get for equal to a forcing term on the right side. I need a new solution. I need a new y of t.

Can I tell you what it is? It's a typical case of L'Hopital's rule from calculus when we approach this bad situation, and we are getting a 1 over 0. Well, you'll see a 0 over 0 and that's L'Hopital's aim for. So where do we start?

A Y particular solution is this e to the st over this As squared plus Bs plus C. Right. That's our particular solution. If it works. Resonance is the case when this doesn't work because that's 0.

Now, that's a particular solution. I'll subtract off a no solution. I could do that. I still have a solution. So I subtract off e to the s1t. So S1 is, e to the s1t is a no solution. This is what I would call a very particular solution.

It's very particular because a t equals 0, it's 0. Do you see that, you see what's happening here. The question, resonance happens when s approaches s1. Resonance is s equal to, resonance itself is at the thing.

Now, we let s sneak up on s1 and we ask what happens to that formula. You see, we're sneaking up on resonance. At resonance, when s equals s1, that will be 0 and that will be 0. That's our problem. So approach it and you end up with the derivative of this divided by the derivative of this.

Do you remember L'Hopital? It was a crazy rule in calculus, but here it's actually needed. So as s goes to s1 this goes to 0 over 0, so I have to take derivative over derivative. So let me write the answer. Y resonant.

Can I call this the resonant solution when s equals s1. And what does it equal? Well, I take the s derivative of this and divide it by the s derivative of that. Derivative over derivative. The s derivative of that is te to the st. And the s derivative of this is 2As plus B.

And now, I have derivative over derivative, I can let s go to s1. So s goes to s1, this goes to s1, and I get an answer. The right answer. This is the correct solution and you notice everybody spots this t factor. That t factor is a signal to everyone that we're in a special case when two things happen to be equal.

Here the two things are the s and the s1. So that will work. So do you see the general picture? It's always this te to the st, t above. And down below we have the derivative of this polynomial at s equal s1. You know it's theoretically possible that we could have double resonance. We would have double resonance if that thing is 0.

If s1 was a double root, if s1 was a double root, then, well, that's just absurd, but it could happen. Then not only is that denominator 0, but after one use of L'Hopital, so we have to drag L'Hopital back from the hospital and say do it again. So we would have a second derivative. I won't write down that solution because it's pretty rare.

So what has this video done? Simply put on record the simplest case possible with a forcing function, e to the st. And above all, we've identified this transfer function. And let me just anticipate that if we need another way to solve these equations, instead of in the t domain, we could go to the Laplace transform in the s domain. We could solve it in the s domain.

And this is exactly what we'll meet when we take the Laplace transform. That will be the Laplace transform, which we have to deal with. So that transfer function is a fundamental, this polynomial tells us, its roots tell us the frequencies, s1, s2, and the no solutions. And then 1 over that tells us the right multiplier in the force solution.

So constant coefficients, exponential forcing, the best case possible. Thank you.