Solutions

BLOCK 1:

AN INTRODUCTION TO FUNCTIONS OF A COMPLEX VARIABLE

Pretest

1. $\frac{33}{65} + \frac{4}{65}\,i$.

2. We must be inside the circle centered at (1,1) with radius 2, but outside the circle centered at (0,2) with radius $\frac{3}{2}$.

3. (a) The upper half of the unit circle centered at the origin.

   (b) The entire unit circle centered at the origin. [The point $(1,\theta)$ is mapped into $(1,2\theta)$.]

4. $u = x^3 - 3xy^2,\ v = 3x^2y - y^3$

   $u_x = 3x^2 - 3y^2 = v_y$

   $u_y = -6xy = -v_x$

5. $v(x,y) = 3y - 2x + C$.

6. (a) $|z| < 1$.

   (b) $f(\frac{i}{12}) = 0.979 - 0.164i$.

7. The path (in the z-plane) which joins $z = -1$ to $z = 2$ must not pass through the origin ($z = 0$). As long as this condition is obeyed, the value of the integral is $-\frac{3}{2}$.

8. $2\pi i$.

Unit 1: A Prelude to the Complex Number System

1.1.1(L)

Here we use the indirect proof. We assume that there are integers whose quotient is $\sqrt{5}$. (This is the negation of what we would like to prove.) We then show that this assumption yields a contradiction, and this, in turn, means that our assumption is false. That is, we prove that the negation of the desired result is false, and this is logically equivalent to proving that the desired result is true.

The details are:

Assuming $\sqrt{5}$ is the quotient of two integers, we have that there are two integers m and n such that

$$\frac{m}{n} = \sqrt{5} \tag{1}$$

and since any fraction can be represented in lowest terms, we may, without loss of generality, assume that m and n were chosen so that $\frac{m}{n}$ was in lowest terms (which means that m and n share no non-trivial factors in common [1 and -1 are considered trivial factors of any integer]).

To clear our equation of radicals, we square both sides of (1) to obtain

$$\frac{m^2}{n^2} = 5$$

or

$$m^2 = 5n^2. \tag{2}$$

There are now two standard ways of showing that (2) contains the desired contradiction. The easier of the two utilizes the unique factorization theorem which states that a number can be decomposed uniquely (up to the order in which the factors are written) into a product of powers of primes. Using this as our basic tool, we see

1.1.1(L) continued

that the left side of (2) is a number which has an even number of factors of 5 (and observe that 0 is even [since it leaves no remainder when divided by 2] so that a number which is not divisible by 5 still has an even number of factors of 5). That is, when we square a number, we double the number of times that each factor appears and the double of any number is an even number. On the other hand, $5n^2$ must contain an odd number of factors of 5 since n^2 contains an even number and $5n^2$ has one more factor of 5 than does n^2. At any rate, since the two numbers m^2 and $5n^2$ have a different number of factors of the prime number 5, the unique factorization theorem tells us that m^2 cannot possibly equal $5n^2$ and this is the contradiction of (2) that establishes the result.

The second method is the one used by the ancient Greeks and is somewhat equivalent to the unique factorization theorem. It is based on the fact that if the product of two intergers is divisible by a prime number then at least one of the two numbers must be divisible by that prime number. The way we use this result is as follows. We see that since $5n^2$ is clearly divisible by 5 and since, by equation (2), $m^2 = 5n^2$, it follows that m^2 is divisible by 5. Since 5 is a prime number and it divides the product of two integers (after all, m^2 is the product of m and n), it must divide one of the numbers itself. Thus, it follows that m is also divisible by 5. In other words, there exists a number k_1 such that

$$m = 5k_1 \tag{3}$$

and this in turn means that $m^2 = 25\,k_1{}^2$. Putting this information into equation (2) and simplifying the result, we see that

$$5\,k_1{}^2 = n^2,$$

and using an argument similar to our previous one, it follows that n is also divisible by 5. This yields the desired contradiction since we chose m and n so that $\frac{m}{n}$ was in lowest terms and this in particular means that not both m and n can be divisible by 5.

1.1.2(L)

a. Again, we could use the technique used in proving that $\sqrt{5}$ was irrational, but there is an easier indirect proof in this case. Namely, we now use the fact that the sum of two rational numbers (or for that matter the difference, product, or quotient [except for dividing by 0] of two rational numbers) is a rational number. We then argue that suppose $3 + \sqrt{5}$ were rational. Let r denote this (rational) sum.

We would then have that

$$r = 3 + \sqrt{5}$$

or

$$\sqrt{5} = r - 3. \tag{1}$$

Since r and 3* are rational numbers, $r - 3$ is the difference of two rational numbers and hence is itself a rational number. Since $\sqrt{5}$ is equal to $r - 3$, it follows that $\sqrt{5}$ is rational and this is a contradiction based on our result in the previous exercise. This contradiction stemmed from the assumption that $3 + \sqrt{5}$ was rational. Hence, it must be that $3 + \sqrt{5}$ is not rational, that is, irrational.

b. The problem here is that the irrational numbers do not behave as nicely as the rational numbers. For example, $3 + \sqrt{5}$ and $\sqrt{5}$ are both irrational, but there difference is $(3 + \sqrt{5}) - \sqrt{5} = 3$ which is rational; or $\sqrt{5}$ is irrational but $\sqrt{5} \times \sqrt{5} = 5$ which is rational. This "unfortunate" facts of life tell us that we can expect to have trouble with the indirect proof here that worked so well in part (a).

The technique we use instead is to find an integral polynomial** equation which has $\sqrt{5} + \sqrt{2}$ as a root.

*All integers are themselves rational numbers since, in particular, every integer is the quotient of two integers, namely itself and 1. For example, $3 = 3 \div 1$.

**An integral polynomial is a polynomial whose coefficients are integers.

1.1.2(L) continued

To this end, we start with

$$x = \sqrt{5} + \sqrt{2} \tag{1}$$

(which is a polynomial equation but not an integral polynomial equation since $\sqrt{5} + \sqrt{2}$ is not an integer).

We rewrite (1) as

$$x - \sqrt{5} = \sqrt{2}$$

and square both sides to obtain

$$x^2 - 2\sqrt{5}x + 5 = 2$$

or

$$x^2 - 2\sqrt{5}x = -3. \tag{2}$$

We now rewrite (2) so that the term containing the radical is by itself. That is,

$$2\sqrt{5}x = x^2 + 3$$

and we again square both sides to obtain

$$20x^2 = x^4 + 6x^2 + 9$$

or

$$x^4 - 14x^2 + 9 = 0. \tag{3}$$

[As a check that $\sqrt{5} + \sqrt{2}$ is a root of $x^4 - 14x^2 + 9 = 0$, we have by the quadratic formula that

$$x^2 = \frac{14 \pm \sqrt{14^2 - 4(9)}}{2}$$

$$= 7 \pm 2\sqrt{10}. \tag{4}$$

1.1.2(L) continued

On the other hand,

$$(\sqrt{5} + \sqrt{2})^2 = 5 + 2\sqrt{10} + 2 = 7 + 2\sqrt{10},$$

which checks with (4). The other values of x which satisfy (4) are $-(\sqrt{5} + \sqrt{2})$ and $\pm(\sqrt{5} - \sqrt{2})$.]

The key point is that since (3) is an integral polynomial equation, any rational root $\frac{p}{q}$ must have the property that q is a divisor of 1 (the leading coefficient) and p is a divisor of 9 (the constant term). Since p and q are integers, q must be 1 or -1 while p must be either ±1, ±3, or ±9; so that the only possibilities for rational roots are

$$\frac{p}{q} = \pm 1, \pm 3, \pm 9.$$

A trivial check of equation (3) shows that none of these numbers is a root.*

Hence, every root of (3) is irrational. In particular, $\sqrt{5} + \sqrt{2}$ is a root of (3).

Therefore, $\sqrt{5} + \sqrt{2}$ is irrational.

1.1.3

a. If there exist integers m and n such that $m \div n = \sqrt[3]{7}$, it follows that

$$m^3 = 7n^3. \qquad (1)$$

m^3 has either 0, 3, 6, 9,... etc. factors of 7 in it (i.e., cubing a number triples the number of times each factor occurs) while $7n^3$, since 7 is a factor of $7n^3$, has either 1, 4, 7, 10,... etc. factors of 7 in it.

*Notice, of course, that this check is not necessary in order to conclude that $\sqrt{5} + \sqrt{2}$ is irrational. That is, whatever $\sqrt{5} + \sqrt{2}$ is, we know that it is not any of the numbers ±1, ±3, or ±9. Hence, since it is a root of (3) but not one of the possible rational ones, it must be irrational.

1.1.3 continued

Since 7 is a prime number, the unique factorization theorem tells us that m^3 and $7n^3$ cannot be equal since they have a different number 7's as factors.

b. Letting $x = \sqrt[3]{7} + \sqrt{5}$, we have that

$$x - \sqrt{5} = \sqrt[3]{7}.$$

Therefore,

$$(x - \sqrt{5})^3 = (\sqrt[3]{7})^3$$

or

$$x^3 - 3x^2\sqrt{5} + 3x(\sqrt{5})^2 - (\sqrt{5})^3 = 7.$$

Therefore,

$$x^3 - 3\sqrt{5}x^2 + 15x - 5\sqrt{5} = 7$$

or

$$x^3 - \sqrt{5}\,(3x^2 + 5) + 15x = 7.$$

Hence,

$$x^3 + 15x - 7 = \sqrt{5}\,(3x^2 + 5),$$

and squaring both sides, we obtain

$$(x^3 + 15x - 7)^2 = 5(3x^2 - 5)^2$$

or

$$x^6 + 225x^2 + 49 + 30x^4 - 14x^3 - 210x = 45x^4 + 150x^2 + 125,$$

or

1.1.3 continued

$$x^6 - 15x^4 - 14x^3 + 75x^2 - 210x - 76 = 0. \qquad (1)$$

c. The only possible rational roots of (1) are integral divisors of 76 and clearly $\sqrt[3]{7} + \sqrt{5}$ is not an integral (i.e., a whole number) let alone an integral divisor of 76, it follows therefore that since $\sqrt[3]{7} + \sqrt{5}$ is a root of (1), it must be irrational.

(Technically speaking, one should check all integral divisors of 76 to see whether (1) can have any rational roots. The process is tedious, but it turns out that (1) has no rational roots.)

1.1.4

a. Since

$$n = 0.324324\overline{324}\ldots\ldots, \qquad (1)$$

then

$$1000n = 324.324\overline{324}\ldots\ldots \quad . \qquad (2)$$

Subtracting (1) from (2) "cancels" the decimal portion and we obtain

$$999n = 324 \text{ (i.e., } 324.0000\ldots).$$

Hence,

$$n = \frac{324}{999}$$

$$= \frac{36}{111}$$

$$= \frac{12}{37}.$$

1.1.4 continued

Check

$$\begin{array}{r} 0.324 \\ 37\overline{)12.0000} \\ \underline{111} \\ 90 \\ \underline{74} \\ 160 \\ \underline{148} \\ 12 \end{array}$$

← and this means that the cycle 324 will repeat since we began by dividing 37 into 12.

b. Since each cycle contains one more 7 than the preceding cycle, $0.373773777\hat{7}\ldots$ cannot denote a rational number (since each rational number, in decimal form, must either eventually terminate or else repeat the <u>same</u> cycle of digits endlessly.

Hence, if we wish* to have $0.373773777\hat{7}\ldots$ be a number, we must "invent" the irrational numbers.

c. Consider $0.515115111\hat{1}$. This, for the same reason as in (b), is an irrational number. Yet

$$\begin{array}{l} 0.373773777\hat{7}\ldots \\ \underline{+0.515115111\hat{1}\ldots} \\ 0.88888888\overline{8}\ldots = \frac{8}{9}. \end{array}$$

Since $0.3737\hat{7}$ and $0.5151\hat{1}$ are irrational while $\frac{8}{9}$ is rational, we have another example of how the sum of two irrational numbers can be rational.

1.1.5

With

$n = 0.3737737773777\hat{7}$

*As usual, the choice is ours to make. That is, we could have elected to say that $0.373773777\hat{7}$ is not a (rational) number, but once we elect to make it a number, it means that we must augment the number system.

1.1.5 continued

we see that

$$U = 0.3737737774$$

exceeds n and is rational since it is a terminating decimal.

Similarly,

$$L = 0.3737737773$$

is a rational number which is less than n. Now,

$$U - L = 0.0000000001 = 10^{-10}.$$

Hence, the irrational number n is "between" the two rational numbers L and U whose difference is 10^{-10}. That is,

$U = 0.37377377740000\ldots$ ← These two

$n = 0.3737737773777\hat{7}\ldots$ numbers differ

$L = 0.37377377730000\ldots$ ← by 10^{-10}.

Unit 2: Complex Numbers from an Algebraic Point of View

1.2.1(L)

Our main aim in this exercise is to emphasize the role of preserving a given structure when we extend a number system.

a. We know that $i^2 = -1$, and if we want the usual rules of arithmetic (including the rules of exponents) to apply to the study of complex numbers then we must have that

$$i^3 = i^{2+1} = i^2 i^1. \tag{1}$$

In short, equation (1) is based on the fact that we want to preserve the structure that we multiply numbers with like bases by adding the exponents, even in the event that the base is complex.

If we now assume that $x^1 = x$ even when x is non-real, we see that (1) can be replaced by

$$i^3 = (-1)i \tag{2}$$

and if we now assume that $(-1)x = -x$ for all numbers x, real or otherwise, then (2) becomes

$$i^3 = -i. \tag{3}$$

It must be observed that there is nothing mystic about how the result of (3) comes into being. In terms of the basic theme of this entire course, we are free to insist upon any consistent set of assumptions and then to see where these assumptions validly lead us. In the present case, all we have done is shown that (3) follows inescapably from the assumptions we have elected to make.

1.2.1(L) continued

Continuing, we next find that

$$
\begin{aligned}
i^4 &= i^{3+1} \\
&= i^3 i^1 \\
&= i^3 i,
\end{aligned}
$$

and since by (3), $i^3 = -i$, it follows that

$$
\begin{aligned}
i^4 &= (-i)i \\
&= [(-1)i]i \\
&= (-1)[(i)(i)] \\
&= (-1)i^2
\end{aligned}
$$

and since $i^2 = -1$, it follows that

$$
\begin{aligned}
i^4 &= (-1)(-1) \\
&= 1. \qquad (4)
\end{aligned}
$$

While we do wish to keep belaboring the point, notice that in going from, say, $[(-1)i]i$ to $(-1)[(i)(i)]$, we are assuming that the complex numbers are structured so that they obey the associative rule for multiplication. We are not required to make such an assumption (even though it is a very sensible one to make), but without this assumption (or at least an equivalent one), we cannot conclude that $i^4 = 1$.

Equation (4) now affords us a very quick way of computing i^n for any natural number n. Namely, since $i^4 = 1$ and $1^m = 1$ for all m, it follows that

$$i^{4m} = (i^4)^{m*} = (1)^m = 1. \qquad (5)$$

We then utilize equation (5) by observing that every integer may be written uniquely in the form

$$4m + r \text{ where } r = 0, 1, 2, \text{ or } 3. \qquad (6)$$

* Notice again now we are assuming that the rule $(a^b)^c = a^{bc}$ which we accepted in our treatment of real numbers is also being accepted in our treatment of the complex numbers.

1.2.1(L) continued

[Notice that (6) is just a "fancy" way of saying that every integer leaves a remainder of either 0,1,2, or 3 when divided by 4.]

In any even, given the integer n, we write it as $n = 4m + r$, and we then use the rules of exponents to deduce that

$$\begin{aligned} i^n &= i^{4m + r} \\ &= i^{4m} i^r \\ &= (i^4)^m i^r \\ &= (i)^m i^r \\ &= (1) i^r, \end{aligned}$$

and assuming that the axion $1x = x$ is accepted even when x is non-real, we obtain

$$i^n = i^r, \tag{7}$$

where r is the remainder obtained when n is divided by 4.

In particular, in this exercise, we have

$235 = 4(58) + 3.$

Hence,

$$i^{235} = i^{4(58) + 3} = i^{4(58)} i^3 = (i^4)^{58} i^3 = (1)^{58} i^3 = 1(i^3)$$

so that by (3),

$$i^{235} = i^3 = -i. \tag{8}$$

The significant part of equation (8) is that it begins to appear that we never have to worry about powers of i other than i^0 and i^1 since all other powers of i are equivalent, within sign, to one of these two.

In summary

1.2.1(L) continued

$i^o = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i,$

and the cycle continues in this way (sort of like what happens when we take derivatives of sin x or cos x; the derivatives repeat in cycles of four).

b. The first thing we observe here is that $3 + 5i$ and $7 + 9i$ are each complex numbers. Why? Well because the complex numbers were invented so that we could find roots of $x^2 + 1 = 0$ and i is one such root, clearly i must be a complex number. Secondly, since every extended number system includes the previous number system, it follows that 3,5,7, and 9 are complex numbers simply by virtue of their being real numbers. Thirdly we would like the complex numbers to be closed with respect to addition and multiplication (that is, we want the sum as well as the product of two complex numbers to be a complex number). Therefore, for example, since 5 and i are complex numbers so also is 5i and since 3 is also a complex number so also is $3 + 5i$.

Now since we want the rules of real number arithmetic to remain valid in the complex number system, we no longer have any choice as to how $(3 + 5i) + (7 + 9i)$ must be defined. Namely,

$$(3 + 5i) + (7 + 9i) = 3 + 7 + 5i + 9i = 10 + 14i.$$

Clearly we may generalize this result to cover the case

$$(a + bi) + (c + di)$$

where a, b, c, and d are <u>real</u> numbers, to conclude that

$$(a, + bi) + (c + di) = (a + c) + (b + d)\ i. \qquad (1)$$

1.2.1(L) continued

Since a + c and b + d are also real numbers (a, b, c, and d are real), we see that sum of numbers of the form a + bi (where a and b are real numbers), is again a number of the same form.

Notice of course that our conclusions are strongly dependent on our choice that the arithmetic of the complex numbers obey the same structural rules as those obeyed by the arithmetic of the real numbers.

We shall pursue this idea further in part (c).

c. The fact that we want the product of two complex numbers to be a complex number does not tell us the recipe for multiplying complex numbers. Thus we must still make a decision as to how

$$(3 + 5i)(7 + 9i) \tag{1}$$

should be defined.

Again we invoke the structure of real numbers and we agree that since the rule

$$(a + b)(c + d) = ac + bd + bc + ad$$

is valid for real numbers, it will also be valid for the complex numbers. Once we make this agreement the procedure for forming the product in (1) becomes clear. Namely,

$$(3 + 5i)(7 + 9i) = (3)(7) + (5i)(9i) + (5i)7 + 3(9i),$$

and since we assume that the "usual" rules of arithmetic are still to be obeyed, it follows that

$$\begin{aligned}(3 + 5i)(7 + 9i) &= 21 + 45i^2 + 35i + 27i \\ &= 21 - 45 + i(35 + 27) \\ &= -24 + 62i. \end{aligned} \tag{2}$$

1.2.1(L) continued

The impact of (2) should now be fairly obvious. Notice, first of all, that there was nothing at all special about our choice of the numbers 3,5,7 and 9 in this exercise.

More generally, it is easy to show that for any real numbers a, b,c, and d,

$$\begin{aligned}(a + bi)(c + di) &= ac + bdi^2 + bci + adi \\ &= ac - bd + (bc + ad)i. \qquad (3)\end{aligned}$$

Since the real numbers are closed with respect to addition, subtraction, and multiplication, equation (3) tells us that since a,b,c, and d are real numbers, so also are ac - bd and bc + ad, so that equation (3) has the form

(real + [real]i)(real + [real]i) = real + [real]i

we see that numbers of the form a + bi where a and b are real are closed with respect to multiplication.

This result, coupled with the result of part (b) start to give us hope that the special numbers of the form a + bi with a and b real are at least an attractive substructure of the complex numbers since they are closed with respect to addition, subtraction, and multiplication. In fact, if we can now show that this class of numbers is an extension of the real numbers which contains i, and is closed with respect to division, it might be a shrewd move to define the complex numbers to be all those numbers of the form a + bi with a and b real real numbers, and the usual rules that also govern the arithmetic structure of the complex numbers.

d. Since 0i = 0 (if you don't mind too much, take these arithmetical results for granted, but for the interested student who wants to see a more rigorous demonstration in terms of structure and validity, we have supplied more details in the optional exercises), it follows that for every real number a, we may write it in the form a + 0i.

1.2.1(L) continued

In other words, if we define the complex numbers to be those numbers of the form $a + bi$ where a and b are real, then the real numbers are those for which $b = 0$.

In a similar way, notice that we can write i in the form

$i = 0 + 1i = 0 + i$,

For this reason we define $a + bi$ to be a "purely" imaginary number if $a = 0$. In this way every complex number (where we are now defining the complex numbers to be those of the form $a + bi$ with a and b real) is the sum of a real number and a purely imaginary number. Both the real numbers and the purely imaginary numbers are substructures of the complex numbers (and from a geometric point of view, which we shall discuss in more detail later in this unit, it turns out that the real numbers correspond to the arithmetic of the x-axis and the purely imaginary numbers to the arithmetic of the y-axis.

From the point of view of terminology, given the complex number $a + bi$, a is called the <u>real part</u> of the number and b is called the "purely" <u>imaginary part</u>. (It is worth noting that <u>by definition both</u> the real and imaginary parts of a complex number are <u>real</u> numbers. In particular it is b and not bi which is defined to be the imaginary part of the number.) Then a complex number is said to be real if its imaginary part is 0; and it is said to be purely imaginary if its real part is 0. Notice that the only way a complex number can be both real and purely imaginary is if both its real and imaginary parts are 0. This means that the number must be $0 + 0i$, and this is clearly 0.

e, Given the complex number $a + bi$, $a - bi$ is defined to be the <u>complex conjugate</u> of $a + bi$. In other words, when we change the sign of the imaginary part of a complex number we convert the complex number into its complex conjugate. Arithmetically, the complex conjugate has the following very helpful property. Namely,

1.2.1(L) continued

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - (bi)^2\\ &= a^2 - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2.\end{aligned}$$

Hence, if a and b are real numbers the product of a + bi and its complex conjugate a - bi is the non-negative real number, $a^2 + b^2$.

This result is very useful in forming the quotient of two numbers of the form a + bi where a and b are real. This brings us to part (f); namely:

f. $$\begin{aligned}\frac{3 + 5i}{7 + 9i} &= \frac{(3 + 5i)(7 - 9i)}{(7 + 9i)(7 - 9i)}\\ &= \frac{21 - 45i^2 + i(35 - 27)}{7^2 - (9i)^2}\\ &= \frac{21 + 45 + 8i}{49 + 81}\\ &= \frac{66 + 8i}{130}\\ &= \frac{66}{130} + \frac{8}{130}\,i.\end{aligned}$$

or,

$$\frac{3 + 5i}{7 + 9i} = \frac{33}{65} + \frac{4}{65}\,i \tag{1}$$

and the right side of (1) has the form a + bi where a and b are real.

More generally

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)}\\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}\\ &= \frac{ac + bd}{c^2 + d^2} + \frac{(bc - ad)i}{c^2 + d^2} \qquad (2)\end{aligned}$$

1.2.1(L) continued

and (2) has the form: real + (real)i unless $c = d = 0$ (in which case $c^2 + d^2 = 0$). Since $c = d = 0$ implies $c + di = 0$, equation (2) tells us that the quotient of a + bi and c + di is a complex number except if c + di is 0.

The main purpose of this exercise was to give you a better feeling for how the complex numbers were developed in terms of our theme, "The Game of Mathematics".

What we have shown in the various parts of this exercise is:

1. If we wish to impose the arithmetical structure that governed the real numbers, then all numbers of the form a + bi where a and b are real are closed with respect to addition, subtraction, multiplication, (raising to integral powers), and division (except as before, division by 0 is excluded).

2. These numbers include the roots of $x^2 + 1 = 0$ since these roots have been denoted by i and -i; and we have that i = 0 + 1i, while -i = 0 (-1)i. They also include the real numbers since any real number, x, may be written as x + 0i and both x and 0 are real.

3. Since the set of numbers of the form a + bi include the real numbers and extend the operations of arithmetic for the real numbers, they may be structurally called an extension of the real numbers.

4. The name given to this extension is The Complex Number System.

Notice that we are not saying that our definition of the complex number system is unique. One could have invented other extensions. What we are saying, however, is that our definition certainly "fills the bill" and other extension of the real number system which obeys the rules of ordinary arithmetic must contain what we have called the complex numbers. That is every number of the form a + bi with a and b real, must be part of any extension of the number system

1.2.1(L) continued

if it is to contain the solutions of $x^2 + 1 = 0$ (i.e., if the system is to obey the usual roles of arithmetic, then as soon as it contains a, b, and i it must, by closure, contain a + bi).

1.2.2

a. Since $(2 - 7i)(2 + 7i) = 4 + 49 = 53$, we have

$$\frac{5 - 3i}{2 - 7i} = \frac{(5 - 3i)(2 + 7i)}{(2 - 7i)(2 + 7i)}$$

$$= \frac{(10 - 21i^2) + i(-6 + 35)}{53}$$

$$= \frac{31 + 29i}{53}$$

$$= \frac{31}{53} + \frac{29}{53}i. \qquad (1)$$

b. If equation (1) holds then if division is the inverse of multiplication it means that

$$(2 - 7i)\left(\frac{31}{53} + \frac{29}{53}i\right) = 5 - 3i. \qquad (2)$$

[i.e., if $\frac{a}{b} = c$, then $bc = a$]

As a check, then

$$(2 - 7i)\left(\frac{31}{53} + \frac{29}{53}i\right) = 2\left(\frac{31}{53}\right) - 7\left(\frac{29}{53}\right)i^2 - 7i\left(\frac{31}{53}\right) + 2\left(\frac{29}{53}i\right)$$

$$= \frac{62}{53} + \frac{203}{53} - i\left(\frac{217}{53} - \frac{58}{53}\right)$$

so that,

$$(2 - 7i)\left(\frac{31}{53} + \frac{29}{53}i\right) = \frac{265}{53} - i\left(\frac{159}{53}\right)$$

$$= 5 - 3i$$

which checks with equation (2).

1.2.3(L)

Since the structure of the complex number system obeys the same rules as the structure for the real number system, the formula

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \tag{1}$$

must remain valid even when a and b are any complex numbers.

Applying equation (1) to the present exercise we obtain

$$\begin{aligned}(+ \sqrt{3}\, i)^3 &= 1^3 + 3(1)^2(\sqrt{3}\, i) + 3(1)(\sqrt{3}\, i)^2 + (\sqrt{3}\, i)^3 \\ &= 1 + 3\sqrt{3}\, i + 3(3i^2) + (\sqrt{3})^3 i^3 \end{aligned} \tag{2}$$

and since $i^2 = -1$, $i^3 = -i$, and $(\sqrt{3})^3 = 3\sqrt{3}$

it follows from equation (2) that

$$\begin{aligned}(1 + \sqrt{3}\, i)^3 &= 1 + 3\sqrt{3}\, i - 9 - 3\sqrt{3}\, i \\ &= -8. \end{aligned} \tag{3}$$

Thus far we cannot honestly say that this has been worthy of being called a "learning exercise" since the computations used in arriving at Equation (3) were very simple (at least in light of some of our other computations).

What is significant about equation (3) is the result it presents. Namely, equation (3) shows us that $1 + \sqrt{3}\, i$ is a cube root of -8. Now, in terms of real numbers the only cube root of -8 is -2. On the other hand -2 is a root of the integral polynomial equation

$$x^3 + 8 = 0. \tag{4}$$

The point is that from "elementary" algebra we know that

$$(x^3 + 8) = (x + 2)(x^2 - 2x + 4) \tag{5}$$

1.2.3(L) continued

Putting the result of (5) into (4), we see that

$$x^3 + 8 = 0$$

implies

$$(x + 2)(x^2 - 2x + 4) = 0.$$

Therefore

$$(x + 2) = 0 \tag{6}$$

or

$$x^2 - 2x + 4 = 0. \tag{7}$$

If (6) holds then $x = -2$, while if (7) holds the quadratic formula implies that

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4 - 16}}{2} \\ &= \frac{2 \pm \sqrt{-12}}{2} \\ &= \frac{2 \pm \sqrt{12(-1)}}{2} \\ &= \frac{2 \pm \sqrt{12}\sqrt{-1}}{2} \\ &= \frac{2 \pm 2\sqrt{3}\, i}{2} \\ &= 1 \pm \sqrt{3}\, i. \end{aligned} \tag{8}$$

[In fact we made up this exercise by deriving equation (8) from equation (4).]

1.2.3(L) continued

What equation (8) shows us is that the integral cubic equation $x^3 + 8 = 0$, which has only one real root, has three complex roots; namely 2, $1 + \sqrt{3}i$, and $1 - \sqrt{3}\,i$ (note that while $\sqrt{3}$ is irrational it is real since $(\sqrt{3})^2 = 3 > 0$).

This result will be generalized in the next unit, but for now it is hoped that you begin to appreciate how the complex numbers allow us to completely factor polynomials which are only partially reducible when we are restricted to the use of real numbers as coefficients.

1.2.4

$$x^6 - 9x^3 + 8 = (x^3 - 1)(x^3 - 8) \tag{1}$$

and

$$x^3 - 1 = (x - 1)(x^2 + x + 1) \tag{2}$$

while

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4). \tag{3}$$

Applying the results of (1), (2) and (3) to the equation

$$x^6 - 9x^3 + 8 = 0 \tag{4}$$

we obtain

$$(x - 1)(x^2 + x + 1)(x - 2)(x^2 + 2x + 4) = 0$$

whereupon, either

$$x - 1 = 0 \tag{5}$$

1.2.4 continued

or

$$x^2 + x + 1 = 0 \tag{6}$$

or

$$x - 2 = 0 \tag{7}$$

or

$$x^2 + 2x + 4 = 0. \tag{8}$$

From (5) and (7) we conclude that $x = 1$ and $x = 2$ are roots of equation (4), while from equations (6) and (8) we conclude that

$$x = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

and

$$x = \frac{-2 \pm \sqrt{4 - 16}}{2} = -1 \pm \sqrt{3}\, i$$

are also roots of equation (4).

Hence, the six complex roots of equation (4) are

$$1,\ 2,\ -\frac{1}{2} + \frac{\sqrt{3}}{2} i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2} i,\ -1 + \sqrt{3}\, i, \text{ and } -1 - \sqrt{3}\, i.$$

1.2.5(L)

In the early study of the quadratic equation, an interesting result is stated but it is not pursued in great depth. Namely if we look at

$$(x - r_1)(x - r_2) = 0 \tag{1}$$

it is trivial to see that the roots are r_1 and r_2.

On the other hand,

1.2.5(L) continued

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1r_2 \tag{2}$$

Combining (1) and (2) we see that the equation

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0$$

has r_1 and r_2 as its roots.

Thus, we may conclude that in a quadratic eqiation the coefficient of the linear term (i.e., the coefficient of x) is the negative of the sum of the two roots while the constant term is the product of the two roots.

This result generalizes to all (monic) polynomial equations. For example, suppose we start with the equation

$$(x - r_1)(x - r_2)(x - r_3) = 0. \tag{3}$$

Then, since $(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)\ x + r_1r_2$ it follows that

$$\begin{aligned}(x - r_1)(x - r_2)(x - r_3) &= [x^2 - (r_1 + r_2)\ x + r_1r_2](x - r_3)\\ &= x^3 - (r_1 + r_2 + r_3)x^2\\ &\qquad + (r_1r_2 + r_1r_3 + r_2r_3)x\\ &\qquad - r_1r_2r_3.\end{aligned} \tag{4}$$

Equations (4) shows us that in a monic cubic polynomial the coefficients of x^2, x and the constant term are determined by some rather interesting combinations of the roots. Namely, the coefficient of x^2 is the negative of the sume of the roots (a rather neat extension of what happened in the quadratic case), the coefficient of x is the sum of products of the roots taken two at a time and the constant term is the negative

1.2.5(L) continued

of the sum of the product of the roots taken three at a time.

Quite in general if $r_1, \ldots,$ and r_n are the roots of the nth degree monic polynomial equation

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_o = 0$$

then

$$a_{n-1} = -(r_1 + \ldots + r_n)$$

$$a_{n-2} = (r_1 r_2 + \ldots)$$

where the sum is taken over all possible combinations of the roots taken two at a time and this equals

$$\sum_{j=1}^{n} \sum_{i=1}^{n} r_i r_j \quad \text{where i and j vary from 1 to n but } i \neq j$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} r_i r_j - \sum_{i=}^{n} r_1^2$$ ← These are the terms for which $i = j$ so these are subtracted from our first sum which includes the terms for which $i = j$.

(where these last equalities are just to get used to various notations)

$$a_{n-3} = -(r_1 r_2 r_3 + \ldots) = \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} r_i r_j r_k, \; i \neq j, \; i \neq k, \; j \neq k$$

where the sum is taken over all possible combinations of the roots taken three at a time, and finally

$$a_o = \pm r_1 r_2 \ldots r_n,$$

the sign depending upon whether n is even or odd. (The sign is positive when n is even and negative when n is odd, as can be checked from our study of the quadratic and the cubic cases.)

1.2.5(L) continued

In the theory of polynomial equations the two coefficients a_{n-1} and a_o of a monic polynomial of degree n are given special names. a_{n-1} is called the <u>trace</u> of the polynomial and a_o is called the <u>norm</u> of the polynomial. From what we have just seen, the trace of a monic polynomial is the negative of the sum of roots of the equation when the polynomial is equated to 0 and the trace is the product of the roots if the polynomial has even degree and negative the product of the roots if the polynomial has odd degree.

The study of the norm and the trace of a polynomial plays an important role in many applied (as well as theoretical) branches of mathematics. Thus it is worth devoting an exercise just to introduce these terms. At the same time, however, to relate this result to our study of complex numbers, let us observe that the trace of a monic* polynomial of degree n is zero if and only if $a_{n-1} = 0$. In particular as soon as $n > 1$ the trace of $x^n - a$ is zero since in this case all coefficients except the leading coefficient and the constant term are zero.

Therefore, by our previous remarks if we solve the polynomial equation, $x^n - a = 0$, where n is a natural number greater than 1, the sum of the roots of the equation must be zero since the sum of the roots is the negative of the coefficient of x^{n-1} (the trace of the polynomial) which is zero.

In particular, this accounts for why the sum of the four (complex) roots of 1 equals 0 as is the sum of the three (complex) cube roots of -8.

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* We stress "monic" since the polynomials

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_o \text{ and } x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \dots + \frac{a_1}{a_n}x + \frac{a_o}{a_n} \quad (a_n \neq 0)$$

are different when $a_n \neq 1$ even though they both have the same roots when equated to 0. The trace is the coefficient of x^{n-1} only when $a_n = 1$. However, as long as $a_n \neq 0$ notice that $a_{n-1} = 0$ if and only if $a_{n-1}/a_n = 0$. Hence, when we want to see whether the trace of a polynomial is 0 it is unnecessary to insist that the polynomial be monic.

1.2.6(L)

a. Suppose $Z = a + bi$. Then $\overline{Z} = a - bi$.

Hence,

$$Z + Z = 2a$$
$$= 2[\text{real part of } Z^*]$$
$$= 2\ \text{Re}(Z).$$

b. $Z = a + bi$ where $a = 3 + 2i$, $b = 4 + 5i$.

Hence,

$$\begin{aligned} Z &= (3 + 2i) + (4 + 5i)i \\ &= 3 + 2i + 4i - 5 \\ &= -2 + 6i. \end{aligned}$$

Therefore,

$$\overline{Z} = -2 - 6i$$

The key point here is that if $Z = a + bi$, $\overline{Z}$ is not $a - bi$ unless a and b are real. In the present example

$$\begin{aligned} a - bi &= (3 + 2i) - (4 + 5i)i \\ &= 3 + 2i - 4i + 5 \\ &= 8 - 2i \\ &\neq \overline{Z}. \end{aligned}$$

c. If $Z = a + bi$ where a and b are real then $\overline{Z} = a - bi$. Hence,

$$\begin{aligned} Z = \overline{Z} &\leftrightarrow a + bi = a - bi \\ &\leftrightarrow 2bi = 0 \\ &\leftrightarrow b = 0 \quad \leftrightarrow \quad \text{Im}(Z) = 0. \end{aligned}$$

---

*We abbreviate the real part of Z by Re(Z) and the imaginary part of Z by Im(Z). In particular if $Z = A + bi$ where a and b are real then $\text{Re}(Z) = a$ and $\text{Im}(Z) = b$.

1.2.6(L) continued

Hence,

$Z = \overline{Z} \leftrightarrow Z$ is a real number.

d. $Z_1 = a_1 + b_1 i \rightarrow \overline{Z}_1 = a_1 - b_1 i$

$Z_2 = a_2 + b_2 i \rightarrow \overline{Z}_2 = a_2 - b_2 i.$

Hence,

$$Z_1 + Z_2 = (a_1 + a_2) + (b_1 + b_2)i \tag{1}$$

while

$$\overline{Z}_1 + \overline{Z}_2 = (a_1 + a_2) - (b_1 + b_2)i. \tag{2}$$

But from (1),

$$\overline{Z_1 + Z_2} = (a_1 + a_2) - (b_1 + b_2)i \tag{3}$$

and comparing (2) and (3) we conclude that

$$\overline{Z_1 + Z_2} = \overline{Z}_1 + \overline{Z}_2$$

(i.e., the conjugate of a sum is the sum of the conjugates).

e. $Z_1 Z_2 = (a_1 + b_1 i)(a_2 + b_2 i)$

$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1).$

Hence,

$$\overline{Z_1 Z_2} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1). \tag{1}$$

On the other hand,

$\overline{Z}_1 = a_1 - b_1 i$ and $\overline{Z}_2 = a_2 - b_2 i.$

Hence,

1.2.6(L) continued

$$\overline{z}_1\overline{z}_2 = (a_1 - b_1 i)(a_2 - b_2 i)$$

$$= (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1). \qquad (2)$$

Comparing (1) and (2) we see that the conjugate of a product is the product of the conjugates.

1.2.7

a. Since $z = a + bi$, $\overline{z} = a - bi$. Hence $\overline{(\overline{z})} = a + bi$. Therefore $\overline{(\overline{z})} = z$. (i.e., the conjugate of a complex number is the complex number.)

b. If $\overline{z}_1 = \overline{z}_2$ then the conjugates of $\overline{z}_1$ and $\overline{z}_2$ must also be equal. That is,

$$\overline{\overline{z}}_1 = \overline{\overline{z}}_2$$

But by part (a) this means $z_1 = z_2$.

(This is the converse of saying that equal numbers have equal conjugates. Namely it says if the conjugates are equal then the numbers are equal.)

1.2.8

Given that z satisfies

$$a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_o = 0 \qquad (1)$$

it follows that

$$\overline{a_nz^n} + \overline{a_{n-1}z^{n-1}} + \ldots + \overline{a_1z} + \overline{a_o} = \overline{0}. \qquad (2)$$

Now, from the two previous exercises we know that

1. $\overline{0} = 0$ (since 0 is a real number).

2. $[\overline{(z)^k}] = \overline{z \ldots z} = \overline{z} \ldots \overline{z} = (\overline{z})^k$

1.2.8 continued

(since the conjugate of a product is the product of the conjugates)

3. $\overline{a_k z^k} = \overline{a}_k \cdot \overline{z}^k = \overline{a}_k (\overline{z})^k$

4. $\overline{a_n z^n + \ldots + a_1 z + a_o} = \overline{a_n z^n} + \ldots + \overline{a_1 z} + \overline{a}_o$

(since the conjugate of a sum is the sum of the conjugates).

Putting these results into (2) we see that

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_o = 0$$

implies that

$$\overline{a}_n (\overline{z})^n + \overline{a}_{n-1} (\overline{z})^{n-1} + \ldots + \overline{a}_1 z + \overline{a}_o = 0. \tag{3}$$

Notice that equation (3) follows inescapably from equation (1) even if the coefficients a_o, a_1, ..., a_{n-1}, and a_n are not real.

If, however, a_o, a_1,..., and a_n are each real it follows that $a_o = \overline{a}_o$, $a_1 = \overline{a}_1$,..., and $a_n = \overline{a}_n$; so that equation (3) becomes

$$a_n (\overline{z})^n + a_{n-1} (\overline{z})^{n-1} + \ldots + a_1 \overline{z} + a_o = 0. \tag{4}$$

Equation (4) says that $\overline{z}$ is a root of

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o = 0. \tag{5}$$

In other words, if z is a root of (5), so also is $\overline{z}$.

Notice that this result requires that the coefficients be real. For example,

$$x^2 - 3ix - 2 = 0 \tag{6}$$

has i and $2i$ as roots since $x^2 - 3ix - 2 = (x - i)(x - 2i)$.

But $-i$, which is the complex conjugate of i, is not a root of (6).

1.2.8 continued

This is not a contradiction of the result proved in this exercise since our result was valid only for polynomial equations with real coefficient of x while here the coefficient of x is -3i which is not real.

1.2.9 (optional)

a. Letting $Z_1 = a_1 + b_1 i$, $Z_2 = a_2 + b_2 i$, and $Z_3 = a_3 + b_3 i$, we have

$$\begin{aligned} Z_1(Z_2 + Z_3) &= (a_1 + b_1 i)[(a_2 + b_2 i) + (a_3 + b_3 i)] \\ &= (a_1 + b_1 i)[(a_2 + a_3) + (b_2 + b_3)i] \\ &= a_1(a_2 + a_3) - b_1(b_2 + b_3) + i[b_1(a_2 + a_3) \\ &\qquad + a_1(b_2 + b_3)] \\ &= (a_1a_2 + a_1a_3 - b_1b_2 - b_1b_3) + (b_1a_2 + b_1a_3 \\ &\qquad + a_1b_2 + a_1b_3)i. \qquad (1) \end{aligned}$$

On the other hand,

$$Z_1Z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$$

$$Z_1Z_3 = (a_1 + b_1 i)(a_3 + b_3 i) = (a_1a_3 - b_1b_3) + i(a_3b_1 + a_1b_3).$$

Hence,

$$\begin{aligned} Z_1Z_2 + Z_1Z_3 &= [(a_1a_2 - b_1b_2) + (a_1a_3 - b_1b_3)] + i[(a_1b_2 + a_2b_1) \\ &\qquad + (a_3b_1 + a_1b_3) \\ &= (a_1a_2 + a_1a_3 - b_1b_2 - b_1b_3) + (a_1b_2 + a_2b_1 \\ &\qquad + a_3b_1 + a_1b_3)i \qquad (2) \end{aligned}$$

and since the a's and b's are real numbers we see that the right sides of (1) and (2) are equal, from which it follows that

1.2.9 continued

$$Z_1(Z_2 + Z_3) = Z_1Z_2 + Z_1Z_3. \tag{3}$$

Equation (3) establishes that the distributive rule applies to complex numbers. In a similar way we can prove that the rules of closure, associativity, and commutativity, etc. for the real numbers apply also to the complex numbers.

b. If $Z = a + bi \neq 0$ then

$$\frac{1}{Z} = \frac{1}{a + bi}$$

$$= \frac{a - bi}{(a + bi)(a - bi)}$$

$$= \frac{a - bi}{a^2 + b^2}.$$

Hence,

$$Z\left(\frac{1}{Z}\right) = (a + bi)\left[\frac{a - bi}{a^2 + b^2}\right] = \frac{a^2 + b^2}{a^2 + b^2} = 1 \text{ (since } a^2 + b^2 \neq 0).$$

c. We need only mimic the proof in the real case since the rules used there are valid here. In particular, if $Z_1 \neq 0$ then [by part (b)] there exists a number $\frac{1}{Z}$, such that $\frac{1}{Z_1} Z_1 = 1$. Hence,

$$Z_1Z_2 = 0 \quad \rightarrow$$

$$\frac{1}{Z_1}(Z_1Z_2) = 0 \rightarrow$$

$$\left(\frac{1}{Z_1} Z_1\right) Z_2 = 0 \rightarrow$$

$$1\ Z_2 = 0 \quad \rightarrow$$

$$Z_2 = 0.$$

d. Again, mimicking the real case since the structure is the same, we have

1.2.9 continued

$$
\begin{aligned}
0Z &= Z0 \\
&= Z(0 + 0) \\
&= Z0 + Z0.
\end{aligned}
$$

Hence, $Z0 = Z0 + Z0$. Therefore, by "cancellation" $Z0 = 0$.

While this exercise may not seem too significant it has some far reaching effects in certain types of applications. Namely as we shall see in the next unit it is not always convenient to express complex numbers in the form a + bi where a and b are real (by the way, this form is called the Cartesian form and will be explained more fully in the next unit). There are times when one might want to use other symbolism (numerals) to denote complex numbers (in the next unit, we discuss the <u>polar form</u> of complex numbers).

The idea is that we often want our results to depend only on the fact that we are dealing with complex numbers, not on the particular system of numerals we are using to denote the complex numbers. What we have done in this exercise is to show that the complex <u>numbers</u> obey certain rules (or equivalently, have certain properties) that predict certain results, without reference to any system of numerals.

A more sophisticated question (and one that is a bit too abstract for us to solve here) is that if there isn't at least one rule that the complex numbers obey that isn't obeyed by the real numbers (i.e., unless the two structures have at least one different rule) how can we tell the two structures apart algebraically? Obviously, we couldn't. Yet we know that the complex numbers are a different system than are the reals. Hence, there must be at least one rule for the complex numbers which isn't obeyed by the real numbers. We shall try to answer this more satisfactorily in the next unit, but for now what we are trying to say is that when we extend a number system we extend all of the rules of the old system. Since our only theorems are those results that follow inescapably from the rules, it means that if the two sets of rules are the same so also will be the set of valid theorems in each structure.

1.2.9 continued

Thus, for an extension to be algebraically meaningful, the extended system must have at least one new rule which is not obeyed by the original system.

Unit 3: Complex Numbers From a Geometric Point of View

1.3.1(L)

Our aim in this exercise is to show how we may give the complex numbers a real interpretation simply by viewing the xy-plane as the Argand Diagram. In essence this technique simply allows us to view planar vectors as complex numbers.

Given the point P(a,b) in the xy-plane, we may view $\vec{OP}$ as a + bi. Multiplying by i yields ai - b = -b + ai. Our claim is that the vector $\vec{OQ}$, where Q is the point (-b,a), is the 90° rotation of $\vec{OP}$. That is,

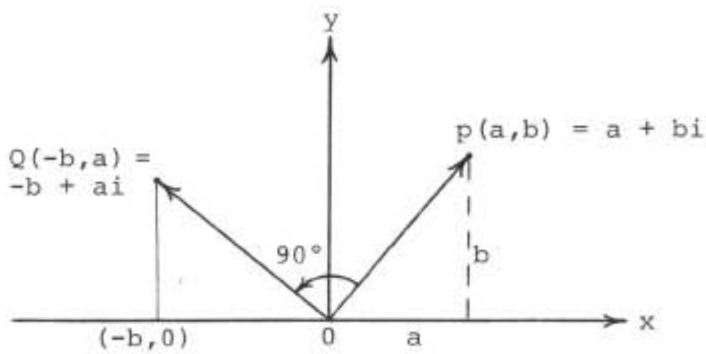


Figure 1

While the result indicated in Figure 1 is not hard to come by, the fact is that it is even easier to understand when we introduce polar coordinates. Namely, as a vector in the xy-plane, i has magnitude 1 and argument 90°. When we multiply complex numbers, we obtain the product by multiplying the magnitudes and adding the arguments.

Thus, multiplying (a + bi) by i preserves the magnitude of (a + bi) [since the magnitude of i is 1] and adds 90° to the argument of a + bi. That is, i(a + bi) is a 90° rotation of a + bi.

But, as shown in Figure 1, a + bi may be identified with the point (a,b) in the xy-plane. Since (a,b) could denote any point in the plane, we see that multiplication by i is equivalent to a 90° rotation of the plane.

1.3.1(L) continued

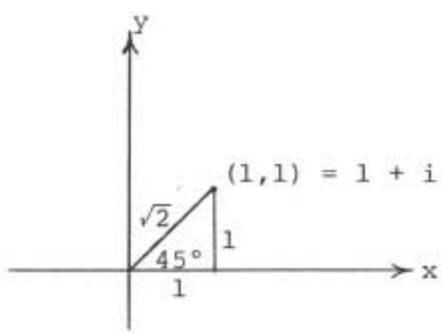
More specifically, given the point (a,b), we may find its image under this rotation by writing $(a,b) = a + bi$, and then computing

$$i(a + bi) = ia - b = -b + ai = (-b,a).$$

Hence, we have shown (even though other methods were available) that under the transformation of rotating the xy-plane by +90°, the point (a,b) is mapped into the point (-b,a); and that this transformation may be viewed as being obtained from the Argand Diagram by multiplying each complex number by i.

1.3.2

a. In the Argand Diagram, $1 + i$ is the point (1,1).



Hence, its magnitude is $\sqrt{2}$ and its argument is 45°.

Therefore, multiplying the vector $a + bi$ by $1 + i$ multiplies the magnitude of $a + bi$ by $\sqrt{2}$ and increases the argument of $a + bi$ by 45°.

Thus, multiplication by $1 + i$ may be viewed as a +45° rotation of the xy-plane in which the distance of each point from the origin is multiplied by $\sqrt{2}$.

b. To find the image of (a,b) under this mapping, we look at

$$(a + bi)(1 + i) = a - b + (a + b)i$$

and since $(a - b) + (a + b)i$ denotes the point $(a - b, a + b)$, we see that this mapping sends (a,b) into the point $(a - b, a + b)$.

1.3.2 continued

c. (1,1) should be mapped into (0,2) under this mapping since its magnitude is $\sqrt{2}$ and its argument is 45°. [Hence, multiplying its magnitude by $\sqrt{2}$ yields 2 and adding 45° to its argument yields 90°, and this corresponds, in Cartesian coordinates, to (0,2)].

Algebraically, we see that (1,1) corresponds to $1 + i$ and $(1 + i)(1 + i) = (1 - 1) + 2i = 0 + 2i = (0,2)$; which checks with the geometric result.

1.3.3

Since multiplication of complex numbers is rather easy to describe in polar coordinates, let us write each of these numbers in polar form.

$z_1 = 1 + i$ therefore, $|z_1| = \sqrt{2}$, $\arg z_1 = 45°$

$z_2 = 1 + i$ therefore, $|z_2| = \sqrt{2}$, $\arg z_2 = 45°$

$z_3 = \frac{1}{2} + \frac{1}{2}\sqrt{3}i$ therefore, $|z_3| = 1$, $\arg z_3 = 60°$

Therefore,

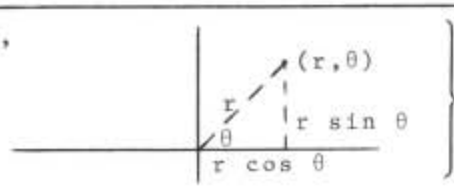
$$|z_1 z_2 z_3| = (\sqrt{2})(\sqrt{2})(1) = 2$$

$$\arg(z_1 z_2 z_3) = 45° + 45° + 60° = 150°.$$

Therefore, in polar form,

$$\begin{aligned} z_1 z_2 z_3 &= (2,150°) \\ &= 2 \cos 150° + i\, 2 \sin 150°* \\ &= -\sqrt{3} + i. \end{aligned}$$

*Pictorially,



Hence, in polar coordinates, (r,θ) corresponds to r cos θ + i r sin θ.

1.3.4

We look at DeMoivre's Theorem with $n = 5$. Namely,

$$(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta. \tag{1}$$

But, by the binomial theorem,

$$\begin{aligned}(\cos \theta + i \sin \theta)^5 &= \cos^5\theta + 5 \cos^4\theta\ (i \sin \theta) + 10 \cos^3\theta \\ &\quad (i \sin \theta)^2 + 10 \cos^2\theta\ (i \sin \theta)^3 \\ &\quad + 5 \cos \theta\ (i \sin \theta)^4 + (i \sin \theta)^5 \\ &= \cos^5\theta + 5 \cos^4\theta \sin \theta\ i - 10 \cos^3\theta \sin^2\theta \\ &\quad -10 \cos^2\theta \sin^3\theta\ \ i + 5 \cos \theta \sin^4\theta \\ &\quad + i \sin^5\theta \\ &= (\cos^5\theta - 10 \cos^3\theta \sin^2\theta + 5 \cos \theta \sin^4\theta) \\ &\quad + (5 \cos^4\theta \sin \theta - 10 \cos^2\theta \sin^3\theta \\ &\quad + \sin^5\theta) i. \end{aligned} \tag{2}$$

Replacing $(\cos \theta + i \sin \theta)^5$ in (1) by its value in (2) and recalling that two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal, we deduce from (1) that

$$\left.\begin{aligned}\cos 5\theta &= \cos^5\theta - 10 \cos^3\theta \sin^2\theta + 5 \cos \theta \sin^4\theta \\ \sin 5\theta &= 5 \cos^4\theta \sin \theta - 10 \cos^2\theta \sin^3\theta + \sin^5\theta\end{aligned}\right\} \tag{3}$$

Hopefully, the relative ease with which (3) was derived shows you the power of the complex number system to the real problem of deriving certain trigonometric identities.

1.3.5(L)

Algebraically, if $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$. Geometrically, $|z|$ is the distance from z to the origin. Our aim in this exercise is to show, once again, the power of the geometric interpretation.

1.3.5(L) continued

a. Let

$$z = x + iy.$$

Then

$$z - (1 + i) = (x + iy) - (1 + i)$$

$$= (x - 1) + (y - 1)i.$$

Hence,

$$|z - (1 + i)| = \sqrt{(x - 1)^2 + (y - 1)^2},$$

so that

$$|z - (1 + i)| = 2$$

implies that

$$(x - 1)^2 + (y - 1)^2 = 4$$

which is the circle of radius 2 centered at (1,1).

b. $|z - a|$ denotes the distance between z and a. Hence,

$$|z - (1 + i)| = 2$$

represents those points which are 2 units from $1 + i$.

Since $1 + i$ denotes the point (1,1), we see that our set consists of the circle centered at (1,1) with radius 2.

c. Since $|z - a|$ denotes the distance between z and a, we have that

$$\{z: |z - (1 + i)| < 2\}$$

is the interior of the circle c_1 centered at the point (1,1) in the Argand Diagram with radius 2.

On the other hand,

1.3.5(L) continued

$\{z: |z - 2i| > \frac{3}{2}\}$

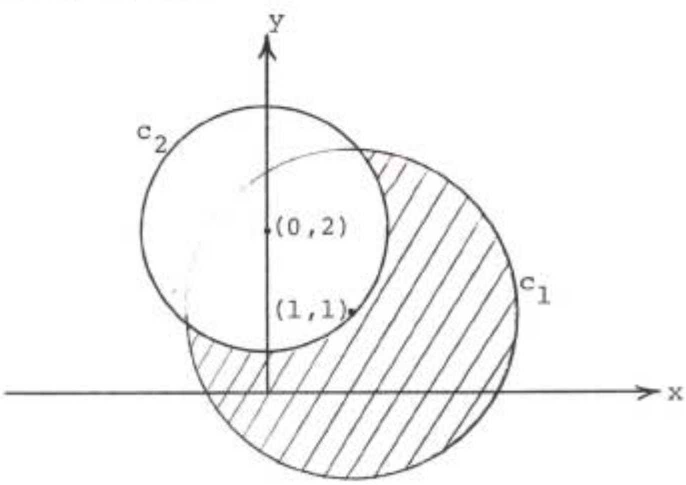
is the exterior of the circle c_2 centered at (0,2) [i.e., at 2i] with radius $\frac{3}{2}$.

Thus,

$\{z: |z - (1 + i)| < 2 \text{ and } |z - 2i| > \frac{3}{2}\}$

is the set of points inside c_1 but outside c_2.

Pictorially,



Notice that in theory, we have the same situation here as we had when we studied the absolute value function of a real variable. Our diagrams involve circles instead of line segments and our algebraic equations involve two unknowns (a real and imaginary part) instead of one unknown; but the general idea is the same.

Note

In the exercises of the previous unit, we mentioned that if a number system were extended then unless the extended system had certain properties that were different from the original system, we could not, algebraically, distinguish the extension from the original.

1.3.5(L) continued

The purpose of this note is to indicate one basic difference between the arithmetic of the complex numbers and the arithmetic of the real numbers. In the real number system, we saw that a given number was either positive, negative, or zero; and that this was equivalent to saying that given two real numbers x and y either $x > y$, $x < y$, or $x = y$.

In the complex number system, given the two complex numbers z_1 and z_2, it is certainly true that either $z_1 = z_2$ or $z_1 \neq z_2$. However, if $z_1 \neq z_2$, we cannot order z_1 and z_2 by size other than by magnitude.

Pictorially, what happens is that in the real case, the negative of a number is a point on the real axis located symmetrically with respect to the origin to the original number. Since our diagram is 1-dimensional it is very easy to keep track of all possibilities. On the other hand, in dealing with complex numbers, we use the 2-dimensional Argand Diagram. In this case it is still true that the point $-z_1$ is located symmetrically opposite the point z_1 with respect to the origin. Now, however, there are infinitely many points which are the same distance from the origin as a given point z_1 and this complicates things considerably.

We do not wish to pursue this idea further here but we do wish to point out that since the absolute value of a complex number is a (non-negative) real number, we may use results from the theory of real functions of a real variable whenever we are interested in results which involve the magnitude of a complex number. All that must be kept in mind is that there are infinitely many different complex numbers which have a given magnitude.

In summary, if r is any positive real number, then

$$\{z\colon |z| = r\} = \{r, -r\}$$

if our universe of discourse is the real numbers; but

$$\{z\colon |z| = r\}$$

has infinitely many members if our universe of discourse is the complex numbers.

1.3.5(L) continued

In essence, then, the "labels," positive and negative, are applicable to complex numbers in the sense that we can talk about $-z$; but not applicable as an aid in ordering the complex numbers (other than by magnitude).

1.3.6

a. Letting $z = x + iy$, we have

$$z - (1 + i) = x + iy - (1 + i)$$
$$= (x - 1) + i(y - 1)$$

while

$$z - (3 + 2i) = (x + iy) - (3 + 2i)$$
$$= (x - 3) + i(y - 2).$$

Hence,

$$|z - (1 + i)| = |z - (3 + 2i)|$$

implies that

$$|(x - 1) + i(y - 1)| = |(x - 3) + i(y - 2)|,$$

or

$$(x - 1)^2 + (y - 1)^2 = (x - 3)^2 + (y - 2)^2.$$

Therefore,

$$x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 - 6x + 9 + y^2 - 4y + 4$$

or

$$4x + 2y = 11.$$

1.3.6 continued

Thus,

$$S = \{z: |z - (1 + i)| = |z - (3 + 2i)|\}$$

is the line

$$4x + 2y = 11.$$

b. Since $|z - (1 + i)|$ is the distance between z and $1 + i$ while $|z - (3 + 2i)|$ is the distance between z and $3 + 2i$, we see that S is the set of points equidistant from (1,1) [i.e., $1 + i$] and (3,2) [i.e., $3 + 2i$].

Thus, S is the perpendicular bisector of the line which joins (1,1) and (3,2); and it is easy to verify that the equation of this line is $4x + 2y = 11$.

1.3.7(L)

Note

This exercise is perhaps one of the most significant points in the invention of the complex numbers. It is the exercise that lies at the foundation of why the complex number system is algebraically closed. The fact that the nth root of a complex number is always a complex number (or numbers) is the last loophole that had to be closed when we tried to solve polynomial equations.

For example, when our universe of discourse was the natural numbers, we could not solve the equation $x + 4 = 3$ because the solution required that we compute $3 - 4$ which is not a natural number. In other words, since the natural numbers were not closed with respect to subtraction and since we solve addition-type equations by subtraction, it was possible that polynomial equations with natural numbers as coefficients would not have natural numbers as roots.

By the same token, when we had the integers we could not solve the equation $2x - 3 = 0$ because the integers were not closed with respect to division. For example, in the present instance, a root involved determining the number $3 \div 2$ which is not an integer.

1.3.7(L) continued

Finally, even when the ultimate real number system was developed, we could not be guaranteed that the nth root of a real number was real. In particular, any even root (in particular, the square root) of a negative could not be a real number.

What we are showing in this exercise, however, is that any nth root of any complex number is a complex number and this means that (except for division by 0) the complex numbers are closed with respect to addition, subtraction, multiplication, division, and extracting roots.

Why is this so significant? The point is that in solving an equation, we must perform the indicated arithmetic operations and their inverses (which was called transposing in the "old" days). Now in a polynomial equation we use the four basic operations of arithmetic as well as raising to integral powers. The inverse of the four operations of arithmetic are these operations themselves (i.e., subtraction is the inverse of addition and addition is the inverse of subtraction, etc.), but the inverse of raising to powers is extracting roots AND THIS IS THE OPERATION THAT WE COULD NOT GUARANTEE TO BE CLOSED WITH RESPECT TO THE REAL NUMBERS. With respect to the complex numbers, however, this operation is closed; and accordingly, it should not be too surprising that we can conclude that a polynomial equation of degree n with complex coefficients has n complex roots (counting repetitive roots). For this reason, if our purpose in extending the number was to ensure that polynomial equations had roots, we no longer need to extend the number system.

Returning to the specifics of this exercise, assume $z = \sqrt[3]{i}$. Therefore,

$$z^3 = i. \tag{1}$$

If we introduce polar coordinates and let $z = (r,\theta)$, equation (1) becomes

$$(r,\theta)^3 = (1,90°) \tag{2}$$

and since $(r,\theta)^n = (r^n,n\theta)$, equation (2) implies

1.3.7(L) continued

$$(r^3, 3\theta) = (1, 90°). \tag{3}$$

Since r is a non-negative real number, $r^3 = 1$ implies that $\underline{r = 1}$ and if $3\theta = 90°$, then $\theta = 30°$.

Hence, $(1,30°)$ is a cube root of i. As a quick check,

$$(1,30°)^3 = \left(1^3, 3(30°)\right) = (1,90°) = i,$$

and in Cartesian form

$$\begin{aligned}(1,30°) &= \cos 30 + i \sin 30 \\ &= \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i\end{aligned}$$

and

$$\begin{aligned}\left(\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i\right)^3 &= \tfrac{1}{8}(\sqrt{3} + i)^3 \\ &= \tfrac{1}{8}\left[(\sqrt{3})^3 + 3(\sqrt{3})^2 i + 3\sqrt{3}i^2 + i^3\right] \\ &= \tfrac{1}{8}[3\sqrt{3} + 9i - 3\sqrt{3} - i] \\ &= \tfrac{1}{8}[8i] \\ &= i.\end{aligned}$$

The method for determining the other two cube roots of i hinges on the fact that as θ ranges between 0° and 360°, 3θ varies between 0° and 1080°. Hence, we must use the fact that i is named not only by $(1,90°)$ but also by $(1,450°)$ and $(1,810°)$; so that while in any case $r = 1$, 3θ may equal 450° and 810° as well as 90°. This leads to $\theta = 150°$ and $\theta = 270°$.

Thus,

$$(1,150°) = \cos 150° + i \sin 150° = -\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i$$

1.3.7(L) continued

and

$(1,270°) = \cos 270° + i \sin 270° = -i$

are the other two cube roots of i.

Again the check is easiest in polar coordinates since then

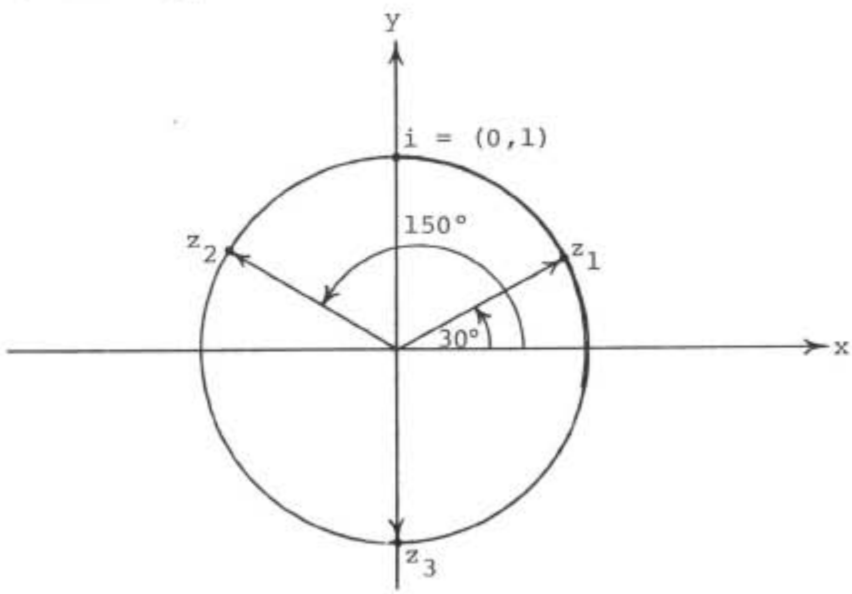
$(1,150°)^3 = (1^3,450°) = (1,450°) = (1,90°) = i$

and

$(1,270°)^3 = (1^3,810°) = (1,810°) = (1,90°) = i.$

[In Cartesian form, $(-i)^3 = (-1)^3(i^3) = -(-i) = i.$]

Pictorially,



$\sqrt[3]{i} = \{z_1,z_2,z_3\}.$

1.3.8

a. $(r,\theta)^8 = 1 \rightarrow$

$(r^8,8\theta) = 1.$ (1)

1.3.8 continued

$$1 = (1,0) = (1,2\pi) = (1,4\pi) = (1,6\pi) = (1,8\pi) = (1,10\pi) = (1,12\pi)$$

$$= (1,14\pi);$$

so from (1), we conclude that

$$r = 1$$

and

$$8\theta = 0,\ 2\pi,\ \ldots,\ 14\pi^*$$

so

$$\theta = 0,\ \frac{\pi}{4},\ \frac{\pi}{2},\ \frac{3\pi}{4},\ \pi,\ \frac{5\pi}{4},\ 3\pi,\ \text{and}\ \frac{7\pi}{4}.$$

In other words, the eight eighth roots of 1 are given in polar coordinates by

$$(1,0),\ (1,\tfrac{\pi}{4}),\ (1,\tfrac{\pi}{2}),\ (1,\tfrac{3\pi}{4}),\ (1,\pi),\ (1,\tfrac{5\pi}{4}),\ (1,3\pi),\ \text{and}\ (1,\tfrac{7\pi}{4}).$$

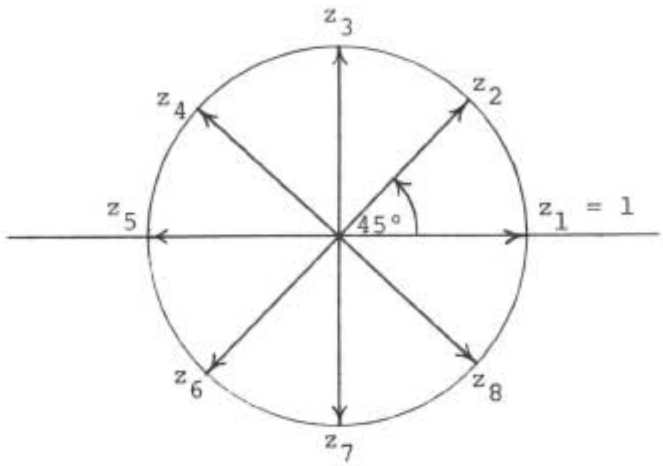
In Cartesian form, these are

$$1,\ \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}i,\ i,\ -\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}i,\ -1,\ -\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\sqrt{2}i,\ -i,\ \text{and}\ \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\sqrt{2}i.$$

Pictorially,

*We could continue and let $8\theta = 16\pi$ but then $\theta = 2\pi$ which yields the same point as does $\theta = 0$. The key idea is that we want θ to name the argument of the root and in this context values of the argument which vary by multiples of 2π (or 360°) name the same root.

1.3.8 continued



b. $z_2 = (1,45°)$.

Therefore,

$z_2{}^2 = (1,45°)^2 = (1,90°) = i = z_3$

$z_2{}^3 = (1,135°) = z_4$

$z_2{}^4 = (1,180°) = -1 = z_5$

$z_2{}^5 = (1,225°) = z_6$

$z_2{}^6 = (1,270°) = z_7$

$z_2{}^7 = (1,315°) = z_8$

$z_2{}^8 = (1,360°) = z_1$.

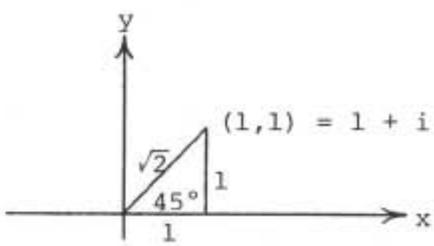
This means that the set $\{z: z^8 = 1\}$ is given very conveniently by $\left\{z_2, z_2{}^2, z_2{}^3, z_2{}^4, z_2{}^5, z_2{}^6, z_2{}^7, z_2{}^8\right\}$ where $z_2 = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i$.

1.3.9

In polar form,

$$1 + i = (\sqrt{2}, 45°).$$

Namely,



Now,

$$(\sqrt{2}, 45°) = (\sqrt{2}, 405°) = (\sqrt{2}, 765°).$$

Hence,

$$(r, \theta)^3 = 1 + i$$

implies

$$(r^3, 3\theta) = (\sqrt{2}, 45°) \quad \text{or} \quad (\sqrt{2}, 405°) \quad \text{or} \quad (\sqrt{2}, 765°).$$

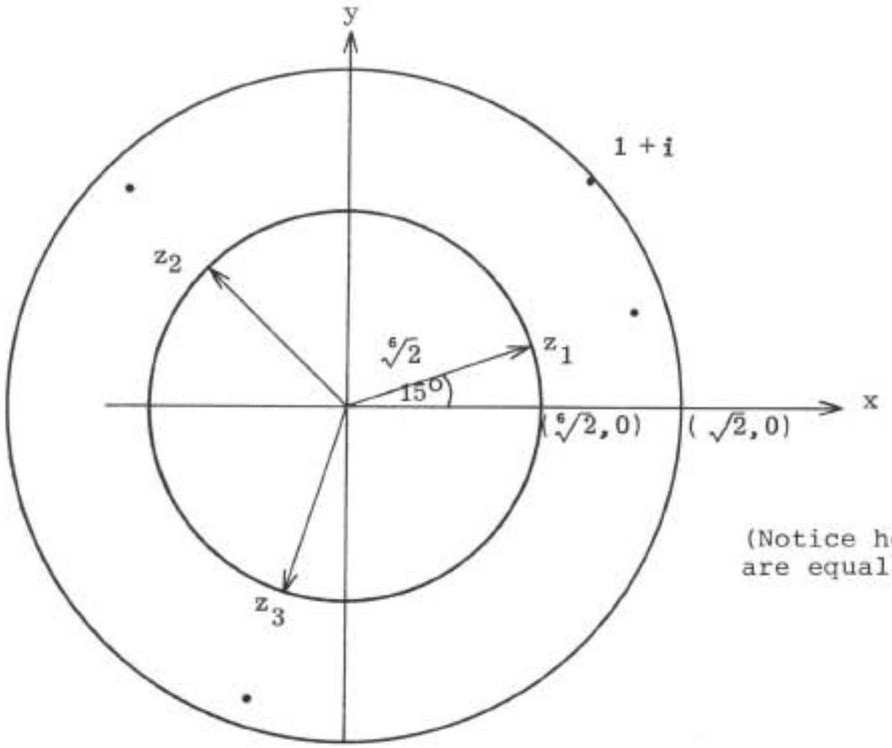
Hence, in any case, $r^3 = \sqrt{2}$ $(r > 0)$ so that $r = \sqrt[6]{2}$ while $\theta = 15°$, $135°$, or $255°$.

Thus, in polar form,

$$\sqrt[3]{1 + i} = \underbrace{(\sqrt[6]{2}, 15°)}_{z_1}, \underbrace{(\sqrt[6]{2}, 135°)}_{z_2}, \underbrace{(\sqrt[6]{2}, 255°)}_{z_3}.$$

Pictorially,

1.3.9 continued



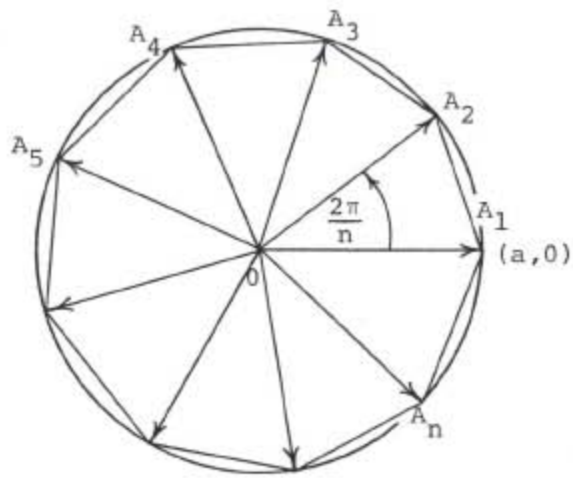
(Notice how the roots are equally spaced.)

1.3.10 (Optional)

Here we wish to present a practical application of the complex numbers to the real world. To this end, notice first of all, that the given problem exists (and is even solvable) had complex numbers never been invented.

By using complex numbers, however, watch how quickly the problem falls apart! We begin by agreeing to view the xy-plane as the Argand Diagram. We then view the circumscribed circle as being centered at (0,0) with one of the vertices at (a,0). Pictorially,

1.3.10 continued



We have let n = 9 in this diagram simply to get a concrete diagram, but this is not crucial to our general discussion.

Figure 1

Each of the vectors $\vec{OA}_1, \ldots, \vec{OA}_n$ viewed as a complex number is a solution (root) of the equation

$$z^n = a^n \tag{1}$$

or

$$z^n - a^n = 0. \tag{1'}$$

Namely, the magnitude of $\vec{OA}_2$, for examply, is $\underline{a}$ and its argument is $\frac{2\pi}{n}$. Hence, in polar form, it is given by $(a, \frac{2\pi}{n})$; and

$$\left(a, \frac{2\pi}{n}\right)^n = \left[a^n, n\left(\frac{2\pi}{n}\right)\right] = (a^n, 2\pi) = (a^n, 0) = a^n.$$

On the other hand, the sum of these roots must be 0 since this sum is the coefficient of z^{n-1} in equation (1').

Hence, viewed as complex numbers in the Argand Diagram, we see that the sum of $\vec{OP}_1, \vec{OP}_2, \ldots, \vec{OP}_n$ must be zero.

In summary, we have exhibited an example wherein an analytic application of complex numbers solves a real problem in a rather elegant manner.

1.3.11 (Optional)

Here again we want to show the realness of the complex numbers and we are again using a geometric illustration. The key point is that the vector $a\vec{i} + b\vec{j}$ in the xy-plane becomes the complex number a + bi when we view the xy-plane as the Argand Diagram. Thus, the rule for multiplying two vectors in the xy-plane given in this exercise is nothing more than a word-for-word translation of the rule for multiplying complex numbers - provided only that we identify the xy-plane with the Argand Diagram.

Since we know that multiplication for the complex numbers obeys the commutative, associative, and distributive rules; it follows that this form of vector multiplication obeys the same structure. The actual details which could be used to check this result are left as unofficial exercises for the interested student. It will be noted in the check that the proofs are word-for-word the same as for the complex number case; which is as it should be since one is just a different geometrical interpretation of the other.

In this same context, it is easy to explain geometrically what this new type of vector multiplication means. Namely, the product of two vectors multiplied in this way is the vector in the xy-plane whose magnitude is the product of the magnitude of the two given vectors; and the angle this vector makes with the positive x-axis is the sum of the angles that each of the two given vectors makes with the positive x-axis.

At this stage of the game, it is not important whether there are any nice physical examples in which this type of multiplication of vectors is important. What is important is that we have applied the theory of complex numbers to induce a real vector operation; and that this particular type of vector multiplication is precisely what is needed if we wish planar vectors to continue to be an accurate geometric model for the complex numbers.

Unit 4: Complex Functions of a Complex Variable

1.4.1(L)

Clearly each element of S is a complex number since both cos t and sin t are real for all $0 \leq t \leq \pi$. Thus, S exists without any reference to a picture. The point is, however, that if we use the Argand diagram, we view x + iy as the point z[= (x,y)] in the xy-plane.

In other words if we identify the position vector R with the complex number z, we see that the "graph" of S (by which we mean the set of points in the Argand diagram which represents S) is the curve whose vector equation is

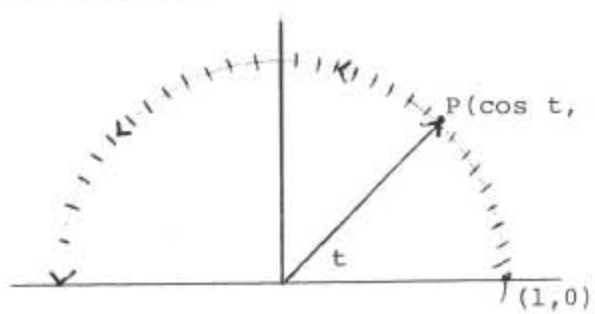
$$R(t) = \cos t\, i + \sin t\, j,\ 0 \leq t \leq \pi. \tag{1}$$

This, as we already know from our study of vectors, is the curve whose parametric form is

$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\}\ 0 \leq t \leq \pi \tag{1'}$$

which we recognize as the upper half of the unit circle centered at the origin.

Pictorially,



There is a 1-1 correspondence between complex numbers in S and points on the above semi-circle. The correspondence is defined by (cos t, sin t) $\leftrightarrow$ cos t + i sin t.

1.4.1(L) continued

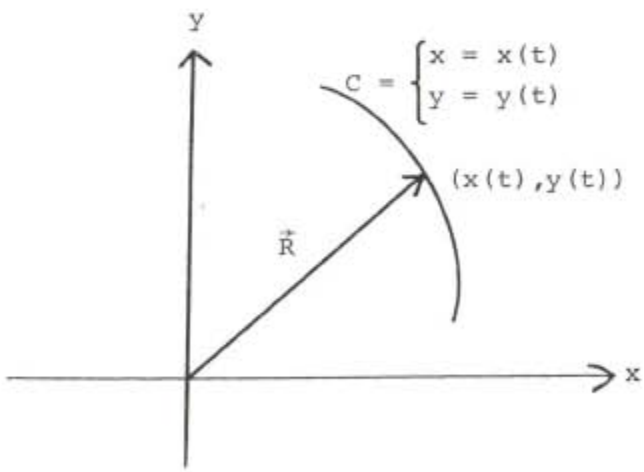
More generally, every curve in the z-plane has the equation of the form

$$z = x(t) + y(t)i \tag{2}$$

(we say more about this in Exercise 1.4.9) and because the Argand diagram has the structure of a 2-dimensional vector space, we see that equation (2) is equivalent to the vector function of a scalar variable, defined by

$$\vec{R}(t) = x(t)\ \vec{i} + Y(t)\ \vec{j}.$$

Summarized pictorially



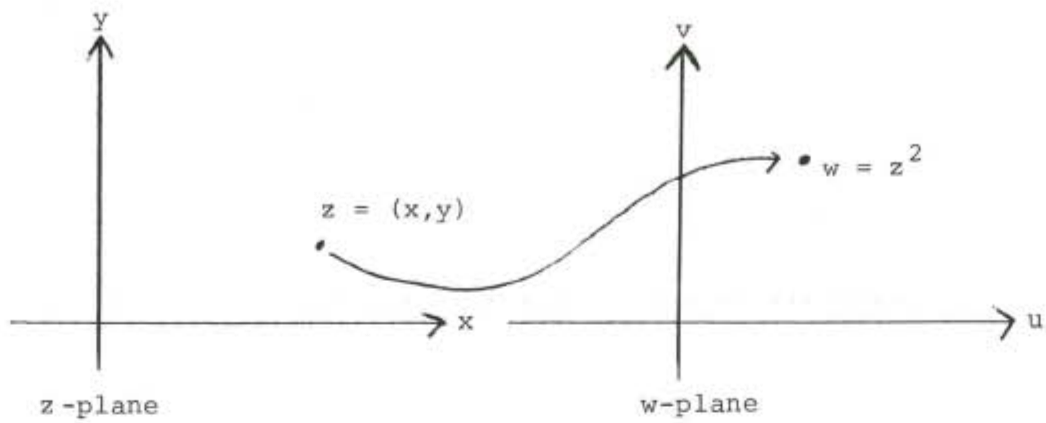
1. In vector form, C is given by $\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j}$.
2. In the Argand diagram $\vec{R}$ represents $\vec{z}$, and C is then the set of complex numbers, $\{z: z = x(t) + iy(t)\}$.

b. Let w denote the image of z with respect to f. In this case $w = z^2$. Since both z and w are complex, f is actually a mapping of a 2-dimensional vector space (the z-plane) into a 2-dimensional vector space (the w-plane).

If we now identify the z-plane with the xy-plane and the w-plane with the uv-plane, we see that $w = z^2$ actually is equivalent to mapping the xy-plane into the uv-plane (a topic we have already studied fairly thoroughly).

1.4.1(L) continued

Pictorially



More specifically, if $z = x + iy$ then $z^2 = (x^2 - y^2) + i\,2xy$; so that

$$w = u + iv, \text{ with } \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases} \qquad (3)$$

Notice that we have already discussed the mapping given by (3) in Blocks 3 and 4 of Part 2.

Of course, we have something "going for us" now that we didn't have then. Namely, we are now able to view mappings of the xy-plane into the uv-plane (a concept which certainly exists independently of the invention of complex numbers) as complex valued functions of a complex variable which map the z-plane into the w-plane.

With this interpretation, we are now able to discuss a vector product that was undefined before (although with hindsight we could have gone back to Blocks 2, 3, and 4 of Part 2 and invented the vector product which corresponds to the product of two complex numbers) and we may conclude that z^2 is the complex number whose magnitude is the square of the magnitude of z and whose argument is twice the argument of z.

In particular, then, since each point in S has unit magnitude, its image under the squaring function also has unit magnitude.

1.4.1(L) continued

Moreover, since the argument of the image is twice the argument of the point, we see that since the arguments of the points in S vary from 0 to 180°, the arguments of the images range from 0° to 360°. In summary, then, the mapping $w = z^2$ carries the set S into the whole unit circle centered at the origin. In particular, the point (r,θ) maps onto $(r,2\theta)$.

Here we see, as an important aside, how the theory of mapping the complex plane into the complex plane gives us new insight to real mappings. In particular, with respect to equation (3) we now have that this mapping, in terms of what it means to multiply complex numbers, is easy to explain pictorially, Specifically, the image of a given point in the xy-plane is found by doubling the argument of the point (vector) and squaring its magnitude.

Again we hasten to point out that we could have invented the product of two vectors to be the vector in the same plane equivalent to the product of the two given vectors as complex numbers. That is,

$$(a\vec{1} + b\vec{j})(c\vec{1} + d\vec{j}) = (ac - bd)\vec{1} + (bc + ad)\vec{j}.$$

But notice how much more natural this definition becomes in terms of the language of complex numbers.

In other words, one major real application of the theory of complex functions of a complex variable is to the real problem of mapping the xy-plane into the uv-plane. These problems can be tackled without reference to the complex numbers, but a knowledge of the complex numbers gives us a considerable amount of "neat" notation which is helpful in obtaining results fairly quickly.

As a final observation, let us observe that as a function f has the same structure (but a different domain) whether we write $f(x) = x^2$ or $f(z) = z^2$. In either case we have a function machine in which the output is the square of the input. The big difference is from the geometrical point of view. In the

1.4.1(L) continued

expression $f(x) = x^2$ se may view both the domain and the image of f as being 1-dimensional (since x is assumed to be real). Accordingly, we may graph the function in the 2-dimensional xy-plane in terms of the curve $y = x^2$.

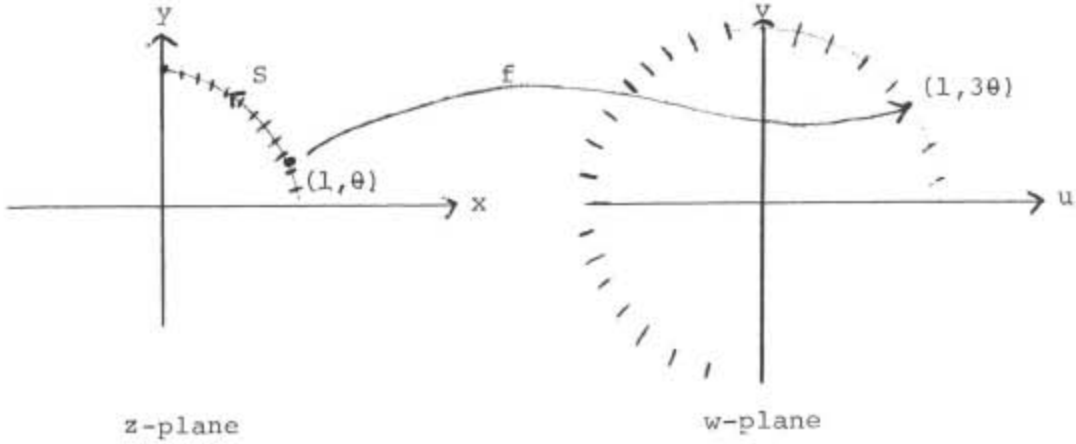
On the other hand, in the expression $w = f(z) = z^2$, the domain and the image of f must be 2-dimensional since neither z nor z^2 is required to be real. Thus, we would require a 4-dimensional space to graph this function if we wanted a graph which was the analog of the graph $y = f(x)$. Since we cannot, in the usual geometric sense, draw a 4-dimensional space, our geometric interpretation must involve viewing the z-plane (the domain of f) as being mapped into the w-plane (the range of f).

1.4.2

a. If we look at z as being the point (r,θ) in the z-plane, then $z^3 = (r,\theta)^3 = (r^3, 3\theta)$. Thus, under f each point in the z-plane is mapped into the point (ρ,ϕ) in the w-plane where $\rho = r^3$ and $\phi = 3\theta$ [i.e., the mapping cubes the magnitude and triples the argument].

In particular the point $(1,\theta)$ where $0 \leq \theta \leq 90°$ is mapped onto $(1^3, 3\theta) = (1, 3\theta)$ and since $0° \leq \theta \leq 90°$, $0° \leq \theta \leq 270°$. Thus, the first quadrant S of the unit circle is mapped onto the first three quadrants of the unit circle.

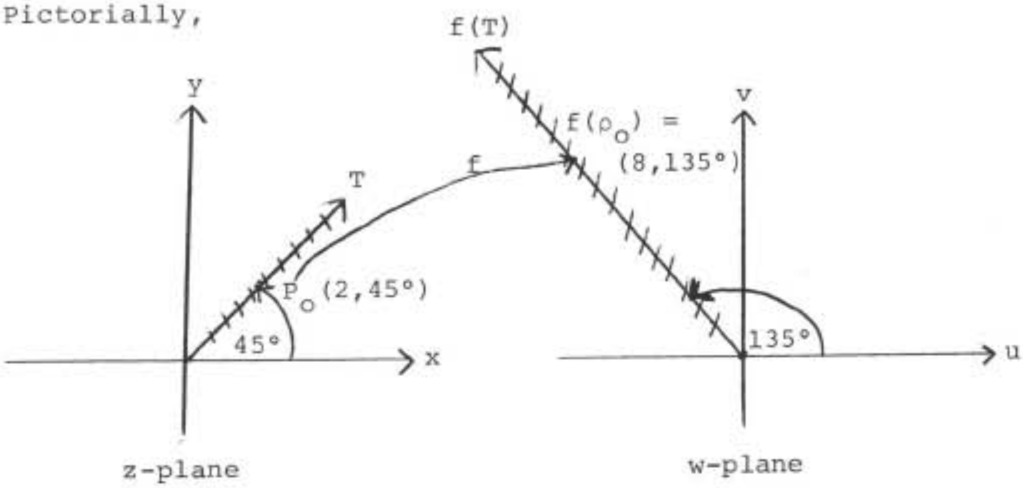
Again pictorially,



1.4.2 continued

By the same token, each point in T, written in polar coordinates, has the form (r, 45°). [If the line extended into the third quadrant, the points on this part would be represented as (r,225°).] Hence "cubing" such a point yields (r^3, 135°). In other words, the mapping defined by $f(z) = z^3$ maps the ray $\theta = 45°$ onto the ray $\theta = 135°$ in such a way that the image of each point has the cube of the magnitude of the point.

Pictorially,



b. $$\begin{aligned} f(z) &= z^3 \\ &= (x+iy)^3 \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 \\ &= x^3 + 3x^2yi - 3xy^2 - iy^3 \\ &= (x^3 - 3xy^2) + i(3x^2y - y^3). \end{aligned}$$

Hence,

$$\left.\begin{aligned} u &= x^3 - 3xy^2 \\ v &= 3x^2y - y^3 \end{aligned}\right\} \quad . \qquad (1)$$

Again, by way of review, equation (1) defines a real mapping of 2-space into 2-space, but from our knowledge of complex variables, we know that the rather cumbersome system (1) is equivalent to mapping each point (vector) in the xy-plane into the point whose magnitude is the cube of the given magnitude and whose argument is triple that of the given argument.

1.4.3

a. $z = x + iy \rightarrow 2z = 2x + i2y$. Therefore, $w = 2z = 2x + i2y$. Letting u denote the real part of w and v the imaginary part we have

$$\left.\begin{array}{l} u = 2x \\ v = 2y \end{array}\right\} .$$

b. $w = f(z) = \bar{z} \rightarrow w = x - iy$. Hence,

$$\left.\begin{array}{l} u = x \\ v = -y \end{array}\right\} .$$

c. $f(z) = |z| \rightarrow$

$f(z) = |x + iy| \rightarrow$

$f(z) = \sqrt{x^2 + y^2} \rightarrow$

$f(z) = \sqrt{x^2 + y^2} + 0i \rightarrow$

$$u = \sqrt{x^2 + y^2}$$
$$v = 0$$

d. $f(z) = z^2 + 2z + i$

$$= (x + iy)^2 + 2(x + iy) + i$$

$$= x^2 - y^2 + i2xy + 2x + i2y + i$$

$$= (x^2 - y^2 + 2x) + (2xy + 2y + 1)i.$$

Hence,

$$\left.\begin{array}{l} u = x^2 - y^2 + 2x \\ v = 2xy + 2y + 1 \end{array}\right\}$$

e. $f(z) = \frac{1}{z}$

$$= \frac{1}{x + iy}$$

$$= \frac{x - iy}{(x + iy)(x - iy)} = \frac{x}{x^2 + y^2} + i\left[\frac{-y}{x^2 + y^2}\right].$$

1.4.3 continued

Hence,

$$\left.\begin{aligned} u &= \frac{x}{x^2 + y^2} \\ y &= \frac{-y}{x^2 + y^2} \end{aligned}\right\} \qquad \begin{matrix} x^2 + y^2 \neq 0 \\ (\text{since } z \neq 0) \end{matrix}$$

Again, as a reminder, this problem shows us that we may view the mapping

$$\left.\begin{aligned} u &= 2x \\ v &= 2y \end{aligned}\right\}$$

as $f(z) = 2z$

$$\left.\begin{aligned} u &= x \\ v &= -y \end{aligned}\right\}$$

as $f(z) = \overline{z}$;

$$\left.\begin{aligned} u &= x^2 - y^2 + 2x \\ v &= 2xy + 2y + 1 \end{aligned}\right\}$$

as $f(z) = z^2 + 2z + i$; and

$$u = \frac{x}{x^2 + y^2}$$

$$v = \frac{-y}{x^2 + y^2}$$

as $f(z) = \frac{1}{z}$.

1.4.4(L)

Our main aim in this exercise is to get a better feeling for the "reality" of complex functions of a complex variable. Parts (b) and (c) are concerned with extending the analogs of $f(x) = x + c$ and $f(x) = cx$ where c and x are real numbers to $f(z) = z + c$ and $f(z) = cz$ where c and z are now complex numbers. As we shall

1.4.4(L) continued

see, the algebra of these functions is the same as that of their real analogs, but the geometric interpretation is a bit more sophisticated (the result of both our domain and image space being 2-dimensional rather than 1-dimensional). In part (a) we want to emphasize the fact that what looks like a new function to us is really an old function that we handled in a very real situation. In particular,

a. Recall in our treatment of the double integral that when we wanted to reverse the order of integration, the technique was geometrically expressed by the mapping of the xy-plane into the uv-plane given by

$$\left.\begin{aligned} u &= y \\ v &= x \end{aligned}\right\}. \tag{1}$$

It should be clear that we do not have to know anything about complex numbers to talk about the mapping defined by equation (1). If, however, we want to view the mapping as being from the z-plane into the w-plane, our procedure is to write (1) in the form u + iv, which in this case means that we study the complex function of a complex variable defined by

$$\begin{aligned} f(z) &= x + i(-y) \\ &= x - iy. \end{aligned} \tag{2}$$

If we now recall that z is x + iy, we see that x - iy is by definition $\bar{z}$. Thus, (2) becomes

$$f(z) = \bar{z}. \tag{3}$$

Of course we arrived at (3) rather inversely to the wording of the exercise in which we were to begin with (3) and derive (1). Our purpose for doing this was simply to start the exercise emphasizing the relationship between complex functions of a complex variable and real mappings. Had we begun with (3), we would have merely reversed our steps to obtain:

1.4.4(L) continued

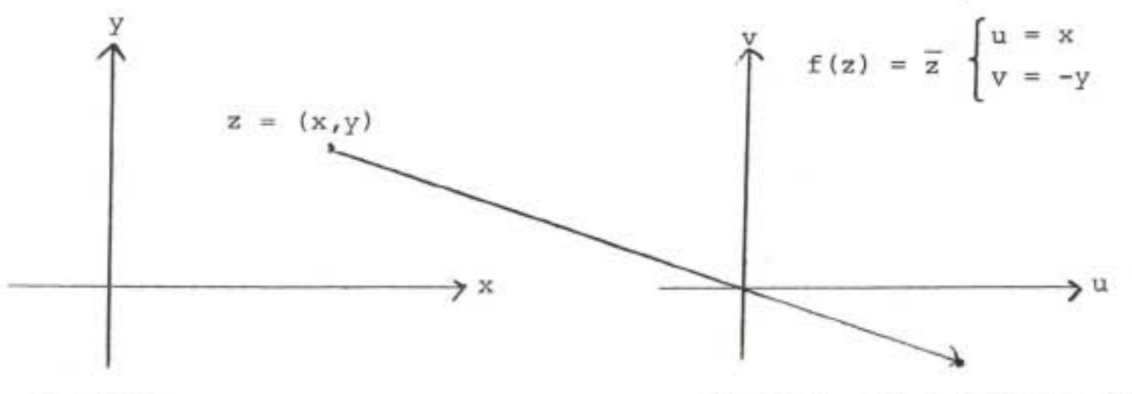
$$f(z) = \overline{z}$$
$$= \overline{x + iy}$$
$$= x - iy \tag{4}$$

and from the real and imaginary parts of f(z), we would have concluded that the graph of f was equivalent to the mapping defined by

$$\begin{cases} u = x \\ v = -y. \end{cases}$$

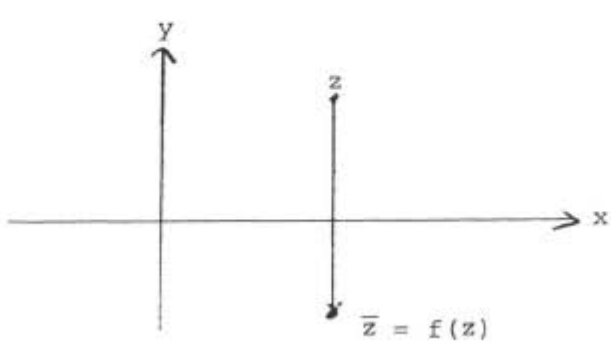
This mapping is equivalent to reflecting the xy-plane about the x-axis (i.e., we leave x alone and change the sign of y.

Pictorially,



(Figure 1)

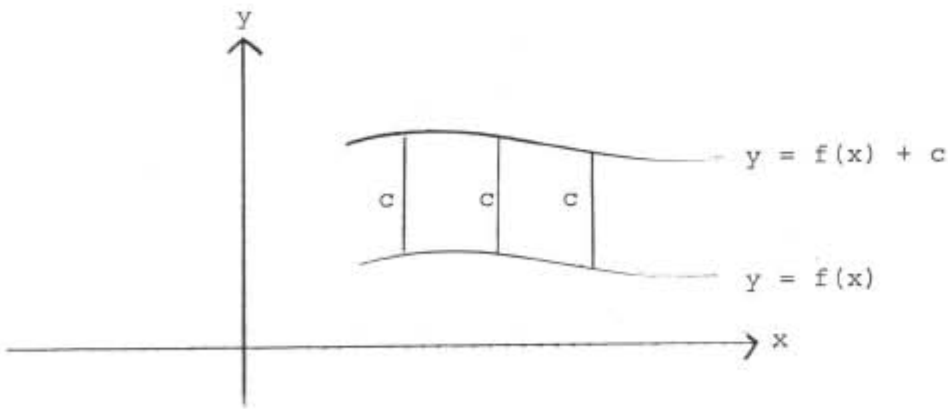
But since the w-plane is a replica of the z-plane we may superimpose the two planes in Figure 1 to obtain



1.4.4(L) continued

Thus, the effect of f on set S in the Argand diagram is to produce the mirror image of S with respect to the x-axis. In particular if S is any closed region, $f(S) \cong S$ (i.e., S and its image have the same size and shape).

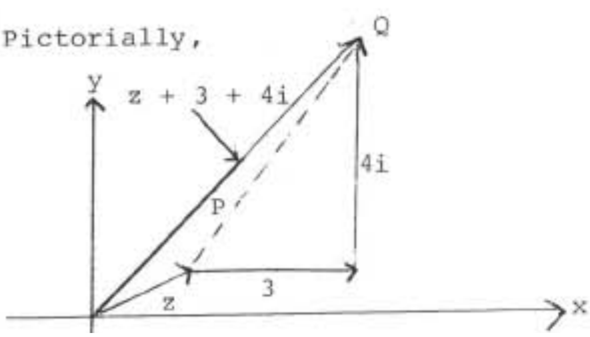
b. In the real case, we saw that the graph of $f(x) + c$ just "raised" each point of the curve $y = f(x)$ by an amount c. In particular the graph of $f(x) = x + c$, was obtained by lifting each point on the line $y = x$ by c units. Pictorially,



Now, given $f(z) = z + c$, we see that in the Argand diagram this sum must be interpreted as a vector sum. As a vector the complex number c is written as $c_1\vec{i} + c_2\vec{j}$ (where we are assuming that $c = c_1 + c_2 i$). Letting c denote $c_1 i + c_2 j$, we see that adding c to z is equivalent to displacing z by an amount equal to the magnitude of c in the direction of c.

For example, the mapping defined by $f(z) = z + 3 + 4i$ maps the point z into the point 5 units from z in the direction $\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$.

Pictorially,

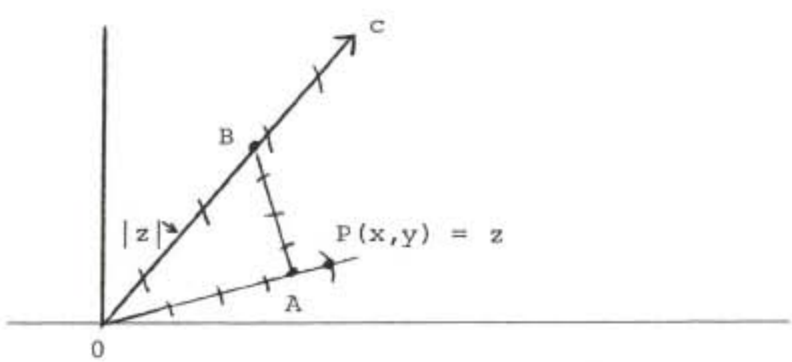


1.4.4(L) continued

Geometrically, adding 3 + 4i onto z shifts (translates) P to Q. That is, P is translated 5 units in the direction $3\vec{i} + 4\vec{j}$.

c. Here we invoke the fact that we have a very convenient way of multiplying complex numbers using polar coordinates. In particular if $c = (r_0, \theta_0)$ then cz has as its magnitude r_0 times the magnitude of z and as its argument θ_0 plus the argument of z. In other words we obtain the image of z by rotating the vector z through θ_0 degrees and increasing its magnitude by a factor of r_0.

By way of an example, if $f(z) = (3 + 4i)z$, then the image of a given number z is obtained by rotating z through an angle equal to arc tan $\frac{4}{3}$ and replacing the magnitude of z by 5 times its value. Pictorially,



1. We pick any point on OP.
2. We erect a perpendicular to OA and locate B on OA such that $\overline{AB} = \frac{4}{3}\,\vec{OA}$. Therefore, $\tan \measuredangle AOB = \frac{4}{3}$.
3. We mark off the length $\overline{OP}$ (i.e., $|z|$) 5 times along OB.
4. $\vec{OC}$ then denotes $(3 + 4i)\,\vec{OP} = (3 + 4i)z$.

1.4.4(L) continued

As a very interesting special case, notice that if the magnitude of c is 1 then the mapping $f(z) = cz$ simply rotates z through an angle equal to the argument of c(i.e., the magnitude is preserved because c has unit magnitude).

If we let θ denote the argument of c, the fact that c is of unit magnitude means that $c = \cos\theta + i\sin\theta$.

Hence,

$$\begin{aligned} cz &= (\cos\theta + i\sin\theta)(x + iy) \\ &= x\cos\theta - y\sin\theta + i(x\sin\theta + y\cos\theta), \end{aligned}$$

and as discussed in our earlier exercises, this is equivalent to the real mapping

$$\begin{aligned} u &= x\cos\theta - y\sin\theta \\ v &= x\sin\theta + y\cos\theta . \end{aligned} \qquad (1)$$

Thus, comparing this result with our polar coordinate interpretation, we see that the mapping defined by equation (1) is equivalent to rotating the xy-plane through $\theta°$.

Hopefully, this shows us still another way in which complex numbers have a real interpretation. By the way, in the special case that c is real, the argument of c is either 0° or 180°, depending upon whether c is positive or negative. Notice then that in this case the result checks with the usual result in the real case; i.e., multiplying by (real) c leaves the direction alone, changes the magnitude by a factor of $|c|$ and preserves the sense if $c > 0$, reverses the sense if $c < 0$.

As a final note on this exercise notice that the linear mapping defined by $f(z) = c_1 z + c_2$ where both c_1 and c_2 are <u>complex</u>-valued constants maps lines through the origin into lines through the origin; and circles centered at the origin into circles centered at the origin. Namely, the mappping f(z) is a rotation (accompanied by a uniform magnification factor equal to c_1) followed by a translation. Under a rotation, lines

1.4.4(L) continued

through the origin remain lines through the origin, and circles centered at the origin remain circles centered at the origin.

Notice also that the algebra of inverting this type of function is word-for-word the same as in the real case since the structural rules are the same. Namely, if $w = c_1z + c_2$ ($c_1 \neq 0$) then

$$z = \frac{w - c_2}{c_1}$$

etc.

In other words, the algebra remains the same, but the geometric interpretation is elevated by a dimension of sophistication (so to speak).

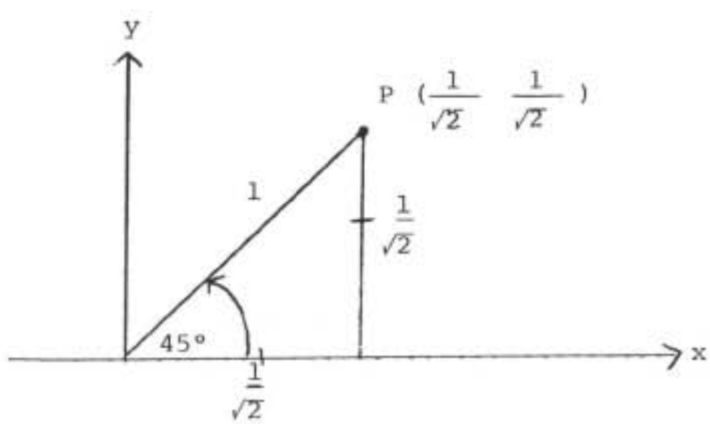
1.4.5

a. Here we have $f(z) = c_1z + c_2$ where $c_1 = \frac{1 + i}{\sqrt{2}}$ and $c_2 = i$.

By the result of the previous exercise c_1z rotates z through an angle equal to the argument of c_1 and multiples the magnitude of z by $|c_1|$. In our case, $|c_1| = 1$ [i.e.,

$$\sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2}]$$

while the argument of c_1 is 45°.



Hence, c_1z is a 45° rotation of the z-plane. Then since

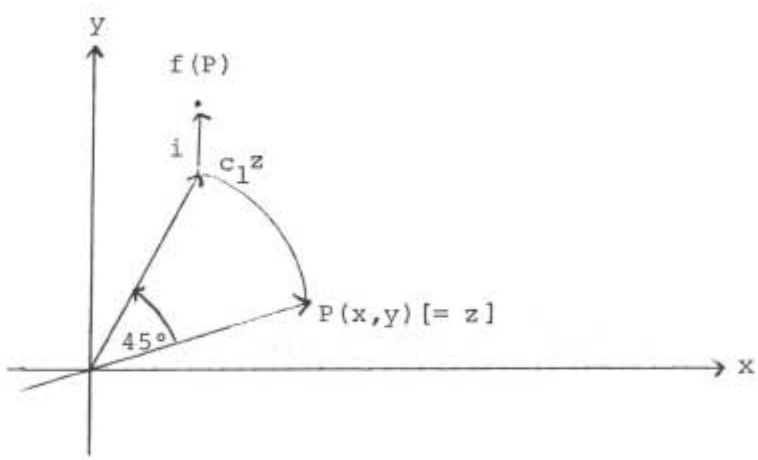
1.4.5 continued

$c_1z + i$ "translates" c_1z an amount equivalent to the vector $\vec{j}$ (i.e., adding i raised the point by 1 unit; we see that

$$f(z) = \left(\frac{1 + i}{\sqrt{2}}\right)z + i$$

is equivalent to rotating each point in the plane through 45° and then raising it 1 unit).

Pictorially,



Rotate P through 45° and then lift it (i.e., move it parallel to the y-axis) one unit.

b. $\left(\frac{1 + i}{\sqrt{2}}\right)z + i$

$$= \frac{(1 + i)(x + iy)}{\sqrt{2}} + i$$

$$= \frac{(x - y) + i(x + y)}{\sqrt{2}} + i$$

$$= \frac{(x - y) + i(x + y) + \sqrt{2}\, i}{\sqrt{2}}$$

$$= \left(\frac{x - y}{\sqrt{2}}\right) + i\left(\frac{x + y + \sqrt{2}}{\sqrt{2}}\right).$$

1.4.5 continued

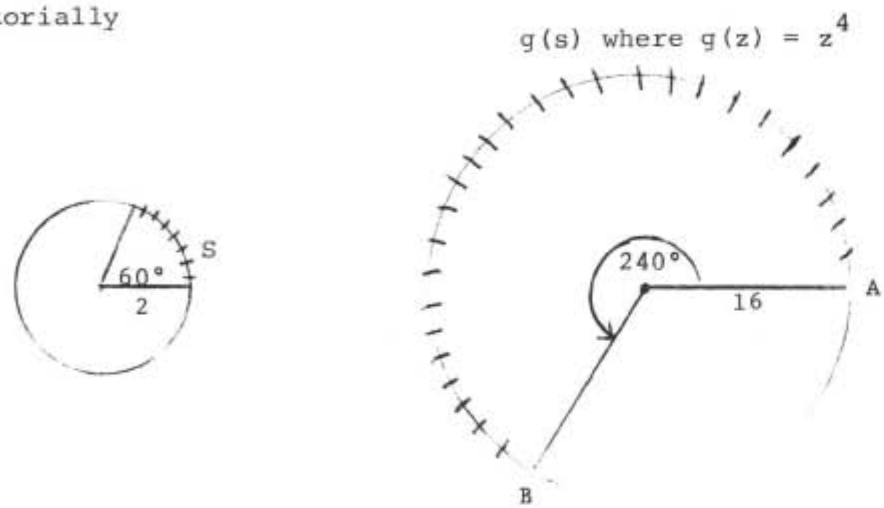
Hence in Cartesian form, the mapping is given by

$$\left.\begin{aligned} u &= \frac{1}{\sqrt{2}}(x - y) \\ v &= \frac{1}{\sqrt{2}}(x + y + \sqrt{2}\,) \end{aligned}\right\}$$

1.4.6

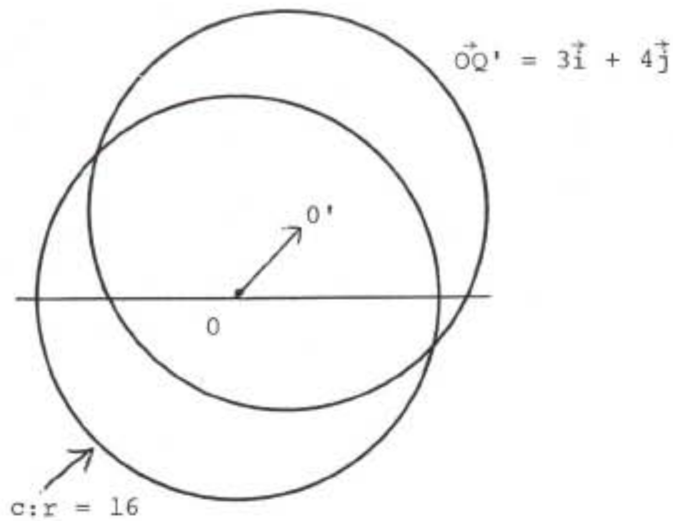
z^4 has magnitude 16 if z has magnitude 2, and the argument of z^4 is four times the argument of z. Hence as z traces the portion of the circle r = 2 between $\theta = 0°$ and $\theta = 60°$, f(z) traces the portion of the circle r = 16 between $\theta = 0°$ and $\theta = 240°$.

Pictorially

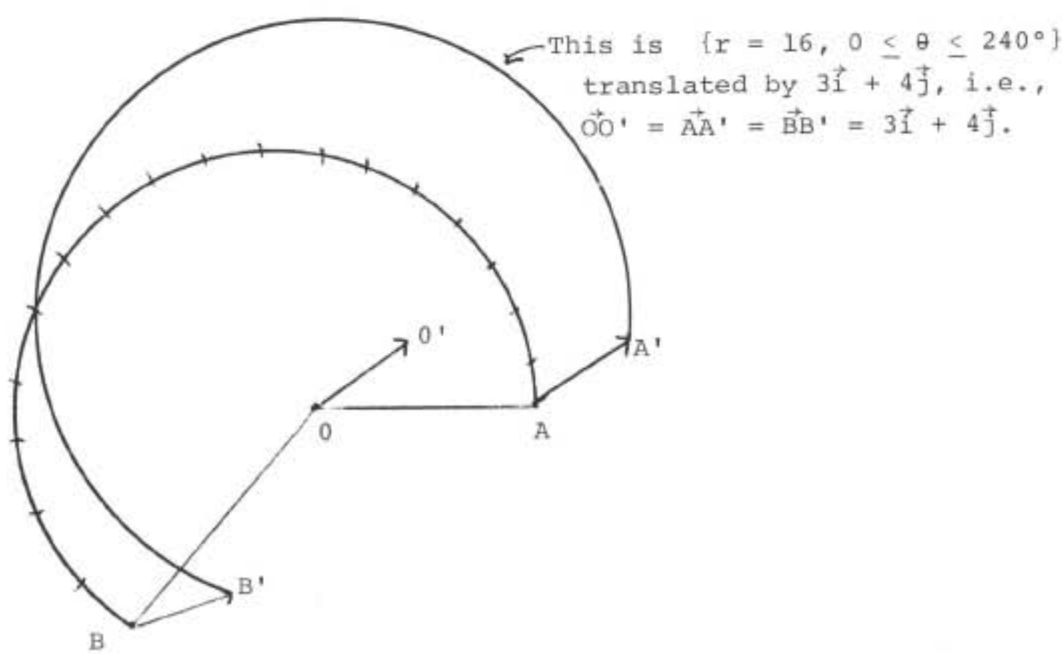


Finally, adding 3 + 4i translates each point 5 units in the direction $\frac{3}{5} + \frac{4}{5}$ i.

1.4.6 continued



Adding 3 + 4i to each point on the circle r = 16 translates the circle from center at 0 to center at 0'.



1.4.7(L)

Here again we see how our knowledge of vector calculus helps us here. Namely, it is natural, if only by mimicking, to define

$$\lim_{z \to c} f(z) = L$$

which means that, given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |z - c| < \delta \to |f(z) - L| < \varepsilon$.

The above definition makes sense even though z, c, and L need not be real since we are dealing only with absolute values - which are (non-negative) real numbers.

Moreover, from a pictorial point of view (i.e., in terms of the Argand diagram) the above definition is precisely the same as our limit definition when we dealt with vector functions of a vector variable.

Recall in that case we showed that the definition was equivalent to saying that if $\vec{f}(\vec{R}) = u(x,y)\vec{1} + v(x,y)\vec{j}$ and if $\vec{L} = L_1\vec{1} + L_2\vec{j}$, $\vec{c} = c_1\vec{1} + c_2\vec{j}$; then

$$\lim_{\vec{R} \to \vec{c}} \vec{f}(\vec{R}) = \vec{L}$$

was equivalent to

$$\lim_{(x,y) \to (c_1,c_2)} u(x,y) = L_1$$

<u>and</u>

$$\lim_{(x,y) \to (c_1,c_2)} v(x,y) = L_2.$$

Translated into the Argand diagram this says that if $c = c_1 + c_2i$ then

$$\lim_{z \to c} f(z) = L$$

1.4.7(L) continued

means

$$\lim_{(x,y)\to(c_1,c_2)} \operatorname{Re}[f(z)] = \operatorname{Re}(L)$$

and

$$\lim_{(x,y)\to(c_1,c_2)} \operatorname{Im}[f(z)] = \operatorname{Im}(L).$$

For example, in the given exercise

$$\begin{aligned} f(z) &= z^3 \\ &= (x + iy)^3 \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 \\ &= (x^3 - 3xy) + (3x^2y - y^3)i. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{z\to(1+i)} f(z) &= \lim_{(x,y)\to(1,1)} (x^3 - 3xy) + i \lim_{(x,y)\to(1,1)}(3x^2 y - y^3) \\ &= -2 + 2i. \end{aligned}$$

The key point is that using the Argand diagram model for the complex numbers we need not invent any new ideas to handle $\lim_{z\to c} f(z)$ if c is complex and f is complex-valued.

In particular, every limit theorem that was true in our study of vector functions of a vector variable remains true in our study of complex functions of a complex variable. More specifically, we may continue to use such results as the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, etc. Again, the main idea is that once we view

1.4.7(L) continued

complex numbers in the Argand diagram we cannot distinguish between complex numbers and planar vectors structurally. Thus, theorems for one model remain theorems for the other.

1.4.8

$(z + h)^2 = z^2 + 2zh + h^2$ (just as in the real case).

Hence, $(z + h)^2 - z^2 = 2zh + h^2$. Hence,

$$\frac{(z + h)^2 - z^2}{h} = \frac{2zh + h^2}{h} = \frac{h}{h}(2z + h)$$

$$= 2z + h, \text{ provided } h \neq 0.$$

Hence,

$$\lim_{h\to 0}\left[\frac{(z + h)^2 - z^2}{h}\right] = \lim_{h\to 0}\,[2z + h]$$

$$= \lim_{h\to 0} 2z + \lim_{h\to 0} h$$

$$= 2z.$$

Notice that this exercise seems to be the complex equivalent of finding $f'(x)$ when $f(x) = x^2$. This idea is the topic of the next unit.

1.4.9

Our main aim in this exercise is to show that the study of complex-valued functions of a single real variable was made when we studied the planar problem of vector functions of a scalar variable.

Namely, if we view z as $x + iy$, then the fact that z is a function of the scalar (real) variable t means that we may write

$$z(t) = x(t) + iy(t). \qquad (1)$$

1.4.9 continued

The critical point is that if we elect to use the Argand diagram as a geometric model, we see at once that equation (1) is <u>structurally equivalent</u> to the vector equation:

$$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j}. \tag{2}$$

In summary, the curve in the xy-plane defined by equation (2) is the "graph" of the complex numbers defined by equation (1). In other words, one way of visualizing a (continuous) complex function of a real variable is as a curve in the z-plane.

The main point is that since we may identify a complex function of a real variable with a vector function of a scalar variable, we may also assume that the calculus structure of vector functions of scalar variables is inherited by complex functions of real variables; and both parts (a) and (b) of this exercise are designed to illustrate this.

a. We assume here that $f'(t)$ has the usual meaning, except that f is now a vector function rather than a scalar function. The point is that had we been given the problem

$$\vec{R}(t) = t\vec{i} + t^2\vec{j} \tag{3}$$

we would have been able to conclude that

$$\vec{R}'(t) = \vec{i} + 2t\vec{j}. \tag{4}$$

Since equation (3) translates, in the Argand diagram, into

$$z = t + t^2 i \ [= f(t)],$$

it follows that $f'(t)$ must be the analog of equation (4), namely,

$$f'(t) = 1 + 2ti. \tag{5}$$

1.4.9 continued

More generally, then, in terms of the Argand diagram if $z = f(t)$ where f is a differentiable complex function of a real variable, we may view $z = f(t)$ as the curve $z = g(t) + h(t)\,i$ where g is the real part of f and h is the imaginary part of f. In this event, $f'(t)$ is a vector tangent to this curve with magnitude equal to

$$\sqrt{g'^2(t) + h'^2(t)}\ .$$

The key point is that the calculus here is a "carbon copy" of the calculus of vector functions of a scalar function.

b. If $R'(t) = t^2 i + e^{3t} j$, then we already know that

$$R(t) = \frac{1}{3}\,t^3 i + \frac{1}{3}\,e^{3t} j + c. \tag{6}$$

Translating the result (6) into the language of complex numbers we have that, if $f'(t) = t^2 + e^{3t} i$, then

$$f(t) = \frac{1}{3}\,t^3 + \frac{1}{3}\,e^{3t} i + c, \text{ where c is an arbitrary complex constant.} \tag{7}$$

If we now use the fact that $f(0) = 1 + i$, equation (7) becomes $1 + i = \frac{1}{3}\,i + c$ so that $c = 1 + \frac{2}{3}\,i$. Putting this result into (7), we have that $f(t) = \frac{1}{3}\,t^3 + \frac{1}{3}\,e^{3t} i + 1 + \frac{2}{3}\,i$, or

$$f(t) = (\tfrac{1}{3}t^3 + 1) + \tfrac{1}{3}(e^{3t} + 2)i.$$

In summary, we already know how to differentiate and integrate complex functions of a real variable because our previous knowledge of vector functions of scalar variables. In particular

1. If $z = x(t) + y(t)\,i$, then $\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}\,i$; and

2. If $z = x'(t) + y'(t)\,i$, then $\int z\,dt = x(t) + y(t)\,i + c$; where $x'(t) = \frac{dx(t)}{dt}$ and $y'(t) = \frac{dy(t)}{dt}$ and c is an arbitrary complex constant.

Thus, while complex functions of a real variable are important in our study of complex variables (e.g., as mentioned in Exercise 1.4.1, the "graph" of a set of complex numbers in the Argand

1.4.9 continued

diagram has this form), we do not devote much time to such a study since the main results are already available to us from our study of planar vectors.

1.4.10

The result of this exercise justifies why the study of real-valued functions of a complex variable is usually ignored from a calculus point of view*. Namely, assuming that the result of this exercise holds, we have that if $y = f(z)$ and if $\frac{dy}{dz}$ exists, then $\frac{dy}{dz} = 0$. This, in turn, implies that $f(z)$ is constant. Thus, if $f: c \to R$ such that f' exists, then $f(z)$ must be constant. In other words, unless $f(z) = \text{constant}$, $\frac{dy}{dz}$ $\left(= \frac{df}{dz}\right)$ fails to exist. Thus, the study of differentiable real functions of a complex variable is "short and sweet".

Now, turning to the specifics of this exercise, we must first define what we mean by f' in the case that f is a real-valued function of a complex variable. In terms of our usual approach in terms of structure, we define $\frac{dy}{dz} = f'(z)$ by

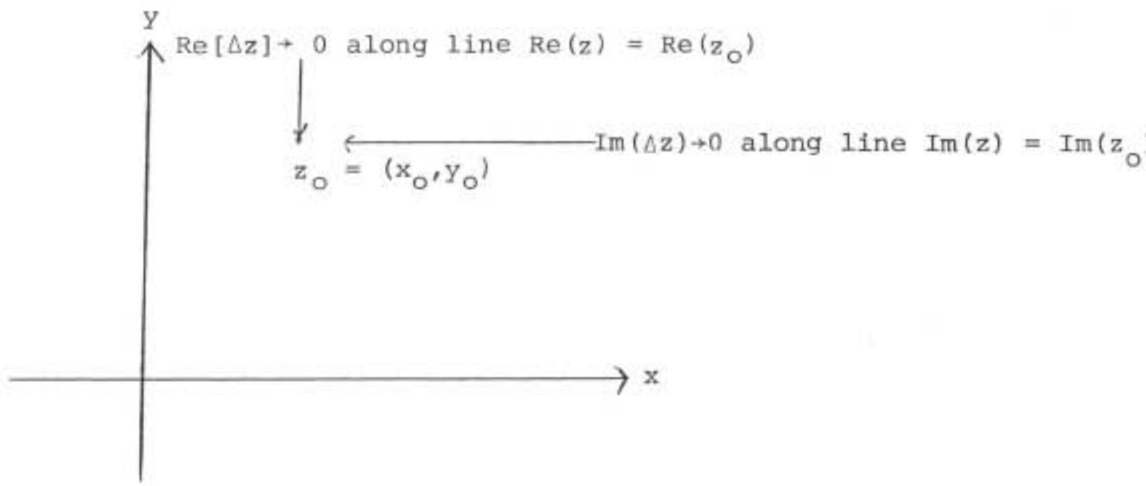
$$f'(z_0) = \lim_{\Delta z \to 0} \left[\frac{f(z_0 + \Delta z) - f(z_0)}{z}\right] \tag{1}$$

provided that the limit exists. Since z, and hence Δz, is complex, it means that there are many paths by which Δz may apprcach 0. One such path is the one defined by the change in the imaginary part of Δz being 0; and another, by the change of the real part of Δz being 0.

*We hasten to stress "calculus" lest you erroneously be led to believe that such functions are unimportant in all respects. For example, the absolute value of a complex variable is extremely important and this is an example of a real-valued function of a complex variable. That is, if z is complex and $f(z) = |z|$ then the range of f is the non-negative real numbers.

1.4.10 continued

In terms of the Argand diagram we have,



Algebraically speaking, we are saying that if $z = x + iy$ then $\Delta z = \Delta x + i\Delta y$; and we are looking at Δz in one case with $\Delta y = 0$ and in the other with $\Delta x = 0$.

The key point is that numerator in the bracketed expression in equation (1) must be real since f is given to be real valued.

Thus, with $\Delta y = 0$,

$$\frac{f(z_o + \Delta z) - f(z_o)}{\Delta z}$$

is equal to

$$\frac{f(x_o + \Delta x, y_o) - f(x_o, y_o)}{\Delta x}$$

in which case, f', if it exists must be given by

$$f'(z_o) = \lim_{\Delta x \to 0} \left[\frac{f(x_o + \Delta x, y_o) - f(x_o, y_o)}{\Delta x}\right] = \left.\frac{\partial f}{\partial x}\right|_{(x_o,y_o)} \qquad (2)$$

1.4.10 continued

Similarly with $\Delta x = 0$, equation (1) becomes

$$f'(z_0) = \lim_{\Delta y \to 0} \left[\frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y}\right]$$

$$= \lim_{\Delta y \to 0} \left[\frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{i\Delta y}\right]$$

$$= \frac{1}{i} \lim_{\Delta y \to 0} \left[\frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}\right]$$

$$= \frac{1}{i} \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}$$

$$= -i \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} . \tag{3}$$

Since the existence of the limit in (1) means that the value of $f'(z_0)$ must be independent of the direction in which $z \to 0$, we may equate the values of $f'(z_0)$ found in (2) and (3) to conclude

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = -i \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}$$

or

$$f_x(x_0, y_0) + 0i = 0 + i \left[- f_y(x_0, y_0)\right]. \tag{4}$$

1.4.10 continued

Equating the real and imaginary parts in the equality given by (4), we conclude that

$$f_x(x_0,y_0) = 0 \text{ and } f_y(x_0,y_0) = 0. \tag{5}$$

Finally, since $(x_0,y_0) = z_0$ was an arbitrary point (number) in the domain of f we may conclude from equation (5) that

$$f_x(x,y) \equiv f_y(x,y) \equiv 0 \tag{6}$$

and from our knowledge of real-valued functions of several (two) real variables*, we may conclude that

$$f(x,y) = \text{constant.} \qquad (\text{i.e., } df \equiv 0) \tag{7}$$

Then since $f(x,y)$ is simply the geometric equivalent of $f(z)$, we conclude that $f(z)$ = (real) constant.

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* Notice that we have identified $f(z)$ with $f(x,y)$ by viewing z as the point (x,y) in the Argand diagram. Since f is real-valued it follows that $f(x,y)$ is a real function of the real variables x and y. Consequently the statement given in (6) is independent of our knowing anything about complex numbers (although the derivation of (6) came from our treatment of the complex numbers). Accordingly (7) is merely a reaffirmation that if $df \equiv 0dx + 0dy$ then $f(x,y)$ is constant.