Unit 9: The Laplace Transform, Part 1

2.9.1(L)

a. Here we stress the linearity of the Laplace transform. We begin with the knowledge (derived in the lecture) that

$$\mathcal{L}(e^{at}) = \frac{1}{s - a} \quad (s > a). \tag{1}$$

We then write cosh bt as $\frac{1}{2}(e^{bt} + e^{-bt})$ to obtain

$$\begin{aligned}\mathcal{L}(\cosh bt) &= \mathcal{L}\left[\frac{1}{2}(e^{bt} + e^{-bt})\right] \\ &= \frac{1}{2}\mathcal{L}(e^{bt} + e^{-bt}) \\ &= \frac{1}{2}\left[\mathcal{L}(e^{bt}) + \mathcal{L}(e^{-bt})\right].\end{aligned} \tag{2}$$

Letting a equal b and then -b in (1) yields

$$\mathcal{L}(e^{bt}) = \frac{1}{s - b} \quad (s > b)$$

and

$$\mathcal{L}(e^{-bt}) = \frac{1}{s - (-b)} = \frac{1}{s + b} \quad (s > -b)$$

so that (2) becomes

$$\mathcal{L}(\cosh bt) = \frac{1}{2}\left[\frac{1}{s - b} + \frac{1}{s + b}\right] \text{ (where } |s| > |b|\text{)}.$$

In other words,

$$\begin{aligned}\mathcal{L}(\cosh bt) &= \frac{1}{2}\left[\frac{(s + b) + (s - b)}{(s - b)(s + b)}\right] \\ &= \frac{s}{s^2 - b^2}.\end{aligned} \tag{3}$$

2.9.1(L) continued

b. Here we illustrate the so-called "shifting theorem." Namely, if $\mathcal{L}(f(t)) = \overline{f}(s)$ then $\mathcal{L}[e^{at}f(t)] = \overline{f}(s - a)$. That is, multiplying $f(t)$ by e^{at} shifts the value of s by an amount a [i.e., to $(s - a)$]. The proof is not difficult. Specifically, let

$$\overline{f}(s) = \int_0^\infty e^{-st}f(t)dt \tag{4}$$

then

$$\overline{f}(s - a) = \int_0^\infty e^{-(s - a)t}f(t)dt. \tag{5}$$

On the other hand,

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}[e^{at}f(t)]dt$$

$$= \int_0^\infty e^{-(s - a)t}f(t)dt. \tag{6}$$

Comparing (5) and (6), we obtain the desired result.

Note #1

Our approach, while valid, more or less presupposes we know the desired result [otherwise, why would we have known to deduce (5) from (4)]. In real life, we might have been more likely to first compute $\mathcal{L}[e^{at}f(t)]$, thus obtaining (6), and then seeing how this was related to $\overline{f}(s)$.

Note #2

As indicated in the lecture and as we shall see in later exercises, one often starts with the Laplace transform and then tries to find the function. This is the problem of finding inverse transforms. What part (b) tells us is that if $\mathcal{L}(g(t)) = \overline{f}(s - a)$ then $g(t) = e^{at}f(t)$. This is illustrated in part (c).

2.9.1(L) continued

c. From Lerch's Theorem and part (a), we know that if

$$\bar{f}(s) = \frac{s}{s^2 - b^2} \tag{7}$$

then

$$f(t) = \cosh bt.$$

Now, given that

$$\bar{g}(s) = \frac{s - a}{(s - a)^2 - b^2} \tag{8}$$

we see from (7) that

$$\bar{g}(s) = \bar{f}(s - a). \tag{9}$$

Applying part (b) to (7), we see that

$$\begin{aligned}\bar{f}(s - a) &= \mathcal{L}[e^{at} f(t)] \\ &= \mathcal{L}[e^{at} \cosh bt], \qquad (10)\end{aligned}$$

and since $\bar{f}(s - a) = \bar{g}(s)$, we conclude from (10) that

$$\bar{g}(s) = \mathcal{L}[e^{at} \cosh bt],$$

or

$$g(t) = e^{at} \cosh bt$$

i.e.,

$$\underbrace{\mathcal{L}^{-1}\bar{g}(s)}_{g(t)} = \underbrace{\mathcal{L}^{-1}(\mathcal{L}}_{\text{these cancel}}[e^{at} \cosh bt]).$$

2.9.1(L) continued

Note

As stated in the lecture, Lerch's theorem says that if f and g are continuous and if f and g have the same Laplace transform then f and g are equal. We should keep in mind that the value of a definite integral does not depend on whether we introduce some finite jump discontinuities in the integrand. That is, we identify the definite integral with an area and the area of a region does not depend on changing the heights of some isolated points. Thus, if the condition that f and g are both continuous is removed, Lerch's theorem states that two functions which have the same transform cannot differ on any interval of positive length (i.e., isolated jump discontinuities involve changes on intervals of zero length, namely a point is an interval of zero length).

To keep things simple, however, we have added the condition that all functions under consideration are continuous. By means of an example, if we define f by $f(t) = 1$ for all t, and g by $g(t) = 1$ unless $t = 0,1,2,3$, and 4 at which points $g(t) = 100$; then the Laplace transform of both f and g is given by $\frac{1}{s}$. When we deal with integrals, it is, in a sense, artificial to distinguish between f and g since both f and g lead to the same area under the curve.

2.9.2

a. $$\begin{aligned}\mathcal{L}[\sinh 3t] &= \mathcal{L}\left[\frac{1}{2}(e^{3t} - e^{-3t})\right] \\ &= \frac{1}{2}\left[\mathcal{L}(e^{3t}) - \mathcal{L}(e^{-3t})\right] \\ &= \frac{1}{2}\left[\frac{1}{s-3} - \frac{1}{s+3}\right] \\ &= \frac{1}{2}\left[\frac{s+3-(s-3)}{s^2-9}\right] \\ &= \frac{3}{s^2-9}. \qquad (1)\end{aligned}$$

2.9.2 continued

b. By the "shifting theorem" of part (b) in the previous exercise, we conclude from (1) that

$$\mathcal{L}[e^{4t} \sinh 3t] = \frac{3}{(s - 4)^2 - 9}. \tag{2}$$

We are given that

$$\bar{g}(s) = \mathcal{L}[g(t)] = \frac{3}{(s - 4)^2 - 9}. \tag{3}$$

Comparing (2) and (3), we conclude that

$$\mathcal{L}[g(t)] = \mathcal{L}[e^{4t} \sinh 3t].$$

Hence, by Lerch's Theorem,

$$g(t) = e^{4t} \sinh 3t.$$

2.9.3(L)

a. One way to compute $\mathcal{L}(\cos bx)$ is to use the definition to conclude that

$$\mathcal{L}(\cos bx) = \int_0^\infty e^{-sx} \cos bx\, dx^* \tag{1}$$

whereupon we could "bludgeon out" the result by integrating the right side of (1) by parts etc.

A more sophisticated way, which also illustrates yet another real application of complex numbers is to observe that

*Notice the switch from t to x. We have done this deliberately in order to emphasize the fact that $\mathcal{L}[f(t)]$ does not depend on whether we use t or x. Namely, $\mathcal{L}[f(t)]$ equals $\int_0^\infty e^{-st} f(t)dt$, in which t is a dummy variable. That is, the integral is a function of <u>s</u>, not of t! For this reason, many authors write $\mathcal{L}(f)$ rather than $\mathcal{L}(f(t))$.

2.9.3(L) continued

$$\cos bx = \frac{1}{2}\left(e^{ibx} + e^{-ibx}\right)$$

$$= \cosh (ibx). \tag{2}$$

Now in Exercise 2.9.1, we showed that

$$\mathcal{L}[\cosh bx] = \frac{s}{s^2 - b^2} \tag{3}$$

where b was assumed to be real. The key point is that structurally our definition of $\mathcal{L}[f(t)]$ makes good sense even if $f(t)$ happens to be a complex function of a real variable. (In that case, we would write $f(t)$ as $g_1(t) + ig_2(t)$ where g_1 and g_2 are real and apply the previous theory to g_1 and g_2 separately.)
In particular, assuming that (3) holds even when b is not real, we may replace b in (3) by ib to obtain

$$\mathcal{L}[\cosh (ibx)] = \frac{s}{s^2 - (ib)^2}$$

$$= \frac{s}{s^2 + b^2}. \tag{4}$$

Hence, from (2)

$$\mathcal{L}(\cos bx) = \frac{s}{s^2 + b^2}. \tag{5}$$

b. From (5), we know that

$$\mathcal{L}[\cos 3x] = \frac{s}{s^2 + 9} \tag{6}$$

and from the previous exercise, we know that

$$\mathcal{L}[\sinh 3x] = \frac{3}{s^2 - 9}. \tag{7}$$

Hence, from (6) and (7), we conclude that

2.9.3(L) continued

$$\mathscr{L}[\cos 3x] + \mathscr{L}[\sinh 3x] = \frac{s}{s^2 + 9} + \frac{3}{s^2 - 9},$$

so that by the linearity of $\mathscr{L}$,

$$\mathscr{L}[\cos 3x + \sinh 3x] = \frac{s}{s^2 + 9} + \frac{3}{s^2 - 9}.$$

Therefore,

$$f(x) = \cos 3x + \sinh 3x. \qquad (8)$$

[We shall revisit result (8) in the next exercise.]

Since our text does not discuss the Laplace transform, we are including a short table of functions and their transforms. (This table is shown on the following page.) Some of these results will be derived in the exercises (in this unit and the next) and some of them have already been discussed in the lecture. Other results will be introduced in the next unit. Our main aim here is to present the results so that you may make reference to them as needed. Keep in mind that the table is used in two ways. On the one hand, we may start with f and then look up $\bar{f}$ and on other occasions, we start with $\bar{f}$ and then look up f.

2.9.4(L)

We first observe that our denominator may be written in the form $(s - a)^2 + b^2$ simply by observing that

$$s^2 - 4s + 20 = (s^2 - 4s + 4) + 16$$

$$= (s - 2)^2 + 4^2.$$

Using our short list of Laplace transforms given on the following page, we notice that $\frac{s}{s^2 + 4^2}$ would be the Laplace transform of cos 4t; hence, by the "shifting theorem," $\frac{(s - 2)}{(s - 2)^2 + 4^2}$ is the Laplace transform of $e^{2t} \cos 4t$.

2.9.4(L) continued

	Function	Transform		Function	Transform
(1)	$f(t)$	$\bar{f}(s) = \mathscr{L}(f) = \int_0^\infty e^{-st} f(t)dt$	(7)	1	$\frac{1}{s}$
(2)	$af(t) + bg(t)$	$a\bar{f}(s) + b\bar{g}(s)$	(8)	$\sin at$	$\frac{a}{s^2 + a^2}$
(3)	$f^{(n)}(t)$	$s^n\bar{f}(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$	(9)	$\cos at$	$\frac{s}{s^2 + a^2}$
(4)	$e^{at}f(t)$	$\bar{f}(s - a)$	(10)	$\sinh at$	$\frac{a}{s^2 - a^2}$
(5)	$t^n f(t)$	$(-1)^n \bar{f}^{(n)}(s)$	(11)	$\cosh at$	$\frac{s}{s^2 - a^2}$
(6)	$\int_0^t f(x)dx$	$\frac{1}{s} f(s)$			

Similarly, $\frac{4}{s^2 + 4^2}$ is the Laplace transform of sin 4t, so again by the shifting theorem, $\frac{4}{(s - 2)^2 + 4^2}$ is the Laplace transform of $e^{2t}\sin 4t$.

With this as foresight, we now proceed as follows:

$$\begin{aligned}
\mathscr{L}[f(t)] &= \frac{2s + 3}{(s - 2)^2 + 16} \\
&= \frac{2(s - 2) + 7}{(s - 2)^2 + 16} \\
&= 2\left[\frac{(s - 2)}{(s - 2)^2 + 16}\right] + \frac{7}{4}\left[\frac{4}{(s - 2)^2 + 16}\right] \\
&= 2\mathscr{L}(e^{2t}\cos 4t) + \frac{7}{4}\mathscr{L}(e^{2t}\sin 4t) \\
&= \mathscr{L}(2e^{2t}\cos 4t) + \mathscr{L}(\tfrac{7}{4} e^{2t}\sin 4t) \\
&= \mathscr{L}(2e^{2t}\cos 4t + \tfrac{7}{4} e^{2t}\sin 4t),
\end{aligned}$$

2.9.4(L) continued

so that by Lerch's theorem,

$$f(t) = 2e^{2t}\cos 4t + \frac{7}{4} e^{2t}\sin 4t.$$

2.9.5

a. $$\mathscr{L}(f(t)) = \frac{1}{s(s+1)}$$

$$= \frac{1}{s} - \frac{1}{s+1}$$

$$= \mathscr{L}(1) - \mathscr{L}(e^{-t})$$

$$= \mathscr{L}(1 - e^{-t}).$$

Hence, by Lerch's theorem,

$$f(t) = 1 - e^{-t}.$$

b. $$\mathscr{L}(f(t)) = \frac{1}{s^2 + 4s + 29}$$

$$= \frac{1}{s^2 + 4s + 4 + 25}$$

$$= \frac{1}{(s+2)^2 + 5^2}. \qquad (1)$$

Then, since $\mathscr{L}^{-1}\left(\frac{5}{s^2 + 5^2}\right) = \sin 5t$, we may use the shifting theorem to conclude that

$$\mathscr{L}\left[\frac{5}{(s+2)^2 + 5^2}\right] = e^{-2t}\sin 5t. \qquad (2)$$

Using (2) in (1) yields

2.9.5 continued

$$\mathcal{L}\left(f(t)\right) = \frac{1}{5}\left[\frac{5}{(s+2)^2 + 25}\right]$$

$$= \frac{1}{5}\mathcal{L}(e^{-2t}\sin 5t).$$

Therefore,

$$\mathcal{L}\left(f(t)\right) = \mathcal{L}\left(\frac{1}{5}e^{-2t}\sin 5t\right);$$

whence,

$$f(t) = \frac{1}{5}e^{-2t}\sin 5t.$$

c. $$\frac{1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \qquad (3)$$

$$= \frac{A(s+2)^2 + Bs(s+2) + Cs}{s(s+2)^2}.$$

Hence,

$$A(s+2)^2 + Bs(s+2) + Cs \equiv 1.$$

That is,

$$(A+B)s^2 + (4A + 2B + C)s + 4A \equiv 1.$$

Therefore,

$$\left.\begin{aligned} A + B &= 0 \\ 4A + 2B + C &= 0 \\ 4A &= 1 \end{aligned}\right\}.$$

Consequently,

$A = \frac{1}{4}$, $B = -\frac{1}{4}$, and $C = -\frac{1}{2}$.

2.9.5 continued

Thus, (3) yields

$$\frac{1}{s(s+2)^2} = \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{1}{s+2}\right) - \frac{1}{2}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{4}\mathcal{L}(1) - \frac{1}{4}\mathcal{L}(e^{-2t}) - \frac{1}{2}\mathcal{L}\left(\frac{te^{-2t}}{1!}\right).$$

Accordingly

$$\frac{1}{s(s+2)^2} = \mathcal{L}\left(\frac{1}{4}\right) + \mathcal{L}\left(-\frac{1}{4}e^{-2t}\right) + \mathcal{L}\left(-\frac{1}{2}te^{-2t}\right)$$

$$= \mathcal{L}\left(\frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}\right),$$

so that

$$f(t) = \frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}.$$

d. Using partial fractions, we have

$$\frac{3s+1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}. \qquad (4)$$

Multiplying both sides of (4) by $s + 1$ and letting $s = -1$, we obtain

$$\frac{3(-1)+1}{(-1+2)(-1+3)} = A = \frac{-2}{(1)(2)}$$

or

$$A = -1.$$

Similarly, multiplying both sides of (4) by $s + 2$ and letting $s = -2$ yields

$$\frac{-5}{(-1)(1)} = B, \text{ or } B = 5.$$

2.9.5 continued

Now multiplying both sides of (4) by s + 3 and letting s = -3, we obtain

$$C = \frac{-8}{(-2)(-1)} = -4.$$

Putting these values of A, B, and C into (4) yields

$$\frac{3s + 1}{(s + 1)(s + 2)(s + 3)} = \frac{-1}{s + 1} + \frac{5}{s + 2} - \frac{4}{s + 3}.$$

Hence,

$$\mathcal{L}^{-1}\left[\frac{3s + 1}{(s + 1)(s + 2)(s + 3)}\right] = -\mathcal{L}^{-1}\left(\frac{1}{s + 1}\right) + 5\mathcal{L}^{-1}\left(\frac{1}{s + 2}\right) - 4\mathcal{L}^{-1}\left(\frac{1}{s + 3}\right)$$

$$= -e^{-t} + 5e^{-2t} - 4e^{-3t}.$$

That is,

$$\mathcal{L}[-e^{-t} + 5e^{-2t} - 4e^{-3t}] = \frac{3s + 1}{(s + 1)(s + 2)(s + 3)}.$$

2.9.6(L)

Given that

$$y'' + 2y' + y = e^{t}, \tag{1}$$

we have that

$$\mathcal{L}(y'' + 2y' + y) = \mathcal{L}(e^{t}). \tag{2}$$

By the linear properties of $\mathcal{L}$, (2) becomes

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(e^{t}),$$

or, since $\mathcal{L}(e^{t}) = \frac{1}{s - 1}$,

2.9.6(L) continued

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + \mathcal{L}(y) = \frac{1}{s-1}. \tag{3}$$

Now, as described in the lecture, we know that $\mathcal{L}(y')$ and $\mathcal{L}(y'')$ may be related to $\mathcal{L}(y)$ by

$$\mathcal{L}(y') = -y(0) + s\mathcal{L}(y) \tag{4}$$

and

$$\mathcal{L}(y'') = -y'(0) - sy(0) + s^2\mathcal{L}(y). \tag{5}$$

Putting (4) and (5) into (3) yields

$$-y'(0) - sy(0) + s^2\mathcal{L}(y) - 2y(0) + 2s\mathcal{L}(y) + \mathcal{L}(y) = \frac{1}{s-1}$$

or

$$(s^2 + 2s + 1)\mathcal{L}(y) = \frac{1}{s-1} + y'(0) + sy(0) + 2y(0). \tag{6}$$

Equation (6) holds for any choices of $y(0)$ and $y'(0)$, but in this exercise, we have chosen the initial conditions, $y(0) = y'(0) = 0$. Accordingly, (6) becomes

$$(s^2 + 2s + 1)\mathcal{L}(y) = \frac{1}{s-1},$$

or

$$\mathcal{L}(y) = \frac{1}{(s+1)^2(s-1)}^*. \tag{7}$$

We next invoke partial fractions and consider

$$\frac{1}{(s+1)^2(s-1)} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s-1}, \tag{8}$$

*Since the domain of $\bar{y}(s)$ is an interval of the form $s > a$, we may guarantee that our denominator never vanishes by choosing $s > 1$.

2.9.6(L) continued

from which we conclude that

$$A(s - 1) + B(s + 1)(s - 1) + C(s + 1)^2 \equiv 1.$$

Therefore,

$$(B + C)s^2 + (A + 2C)s - A - B + C \equiv 1,$$

so that

$$\left.\begin{aligned} B + C &= 0 \\ A + 2C &= 0 \\ -A - B + C &= 1 \end{aligned}\right\} . \qquad (9)$$

Solving (9) yields $C = \frac{1}{4}$, $B = -\frac{1}{4}$, $A = -\frac{1}{2}$, so that (8) becomes

$$\frac{1}{(s + 1)^2(s - 1)} = -\frac{1}{2}\left[\frac{1}{(s + 1)^2}\right] - \frac{1}{4}\left[\frac{1}{s + 1}\right] + \frac{1}{4}\left[\frac{1}{s - 1}\right].$$

Thus, (7) becomes

$$\mathcal{L}(y) = -\frac{1}{2}\left[\frac{1}{(s + 1)^2}\right] - \frac{1}{4}\left[\frac{1}{s + 1}\right] + \frac{1}{4}\left[\frac{1}{s - 1}\right]. \qquad (10)$$

Our tables reveal that

(i) $\mathcal{L}\left[\frac{te^{-t}}{1!}\right] = \frac{1}{(s + 1)^2}$ $\left[\text{i.e., } \mathcal{L}\big(tf(t)\big) = (-1)^n \frac{d\bar{f}}{ds} = (-1)\frac{d}{ds}\left(\frac{1}{s + 1}\right)\right]$

(ii) $\mathcal{L}(e^{-t}) = \frac{1}{s + 1}$

(iii) $\mathcal{L}(e^{t}) = \frac{1}{s - 1}$

Consequently, (10) becomes

2.9.6(L) continued

$$\mathscr{L}(y) = -\frac{1}{2}\mathscr{L}(te^{-t}) - \frac{1}{4}\mathscr{L}(e^{-t}) + \frac{1}{4}\mathscr{L}(e^{t})$$

$$= \mathscr{L}\left[-\frac{1}{2}te^{-t} - \frac{1}{4}e^{-t} + \frac{1}{4}e^{t}\right].$$

Hence,

$$y = -\frac{1}{2}te^{-t} - \frac{1}{4}e^{-t} + \frac{1}{4}e^{t}. \qquad (11)$$

Notice in this exercise that we could have solved things nicely by the undetermined coefficients. We elected to use Laplace transforms simply to solve a problem which we could easily check. Indeed, the general solution of $y'' + 2y' + y = 0$ is $y = c_1e^{-t} + c_2te^{-t}$ while a particular solution of $y'' + 2y' + y = e^t$ is $y = \frac{1}{4}e^t$. Hence, the general solution of $y'' + 2y' + y = 0$ is

$$y = c_1e^{-t} + c_2te^{-t} + \frac{1}{4}e^{t}. \qquad (12)$$

Hence,

$$y' = -c_1e^{-t} + c_2e^{-t} - c_2te^{-t} + \frac{1}{4}e^{t}. \qquad (13)$$

Using $y(0) = y'(0) = 0$ in (12) and (13) yields

$$\left.\begin{aligned} 0 &= c_1 + \frac{1}{4} \\ 0 &= -c_1 + c_2 + \frac{1}{4} \end{aligned}\right\}$$

so that

$$c_1 = -\frac{1}{4} \text{ and } c_2 = -\frac{1}{2},$$

whereupon (12) becomes

$$y = -\frac{1}{4}e^{-t} - \frac{1}{2}te^{-t} + \frac{1}{4}e^{t}$$

which agrees with (11).

2.9.7

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 2 \rightarrow$$

$$\mathcal{L}\left(\frac{d^2y}{dx^2}\right) + 2\mathcal{L}\left(\frac{dy}{dx}\right) + 2\mathcal{L}(y) = \mathcal{L}(2) \rightarrow$$

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + 2\mathcal{L}(y) = 2\mathcal{L}(1) = \frac{2}{s}. \qquad (1)$$

Now

$$\mathcal{L}(y'') = -y'(0) - sy(0) + s^2\overline{y}(s) \qquad (2)$$

and

$$\mathcal{L}(y') = -y(0) + s\overline{y}(s). \qquad (3)$$

Using (2) and (3) in (1), we obtain

$$-y'(0) - sy(0) + s^2\overline{y}(s) - 2y(0) + 2s\overline{y}(s) + 2\overline{y}(s) = \frac{2}{s},$$

and since $y(0) = 0$ and $y'(0) = 1$, this becomes

$$-1 + (s^2 + 2s + 2)\overline{y}(s) = \frac{2}{s}.$$

Hence,

$$(s^2 + 2s + 2)\overline{y}(s) = \frac{2}{s} + 1$$

$$= \frac{s + 2}{s}.$$

Therefore,

$$\overline{y}(s) = \frac{s + 2}{s(s^2 + 2s + 2)}. \qquad (4)$$

Using partial fractions, we have

2.9.7 continued

$$\bar{y}(s) = \frac{s + 2}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}. \qquad (5)$$

From (5), we have that

$$A(s^2 + 2s + 2) + s(Bs + C) \equiv s + 2.$$

Hence,

$$(A + B)s^2 + (2A + C)s + 2A \equiv s + 2.$$

Therefore,

$$\left.\begin{array}{l} A + B = 0 \\ 2A + C = 1 \\ 2A = 2 \end{array}\right\} \text{ or } \left\{\begin{array}{l} B = -1 \\ C = -1 \\ A = 1 \end{array}\right.$$

Thus, from (5)

$$\bar{y}(s) = \frac{1}{s} - \left[\frac{s + 1}{s^2 + 2s + 2}\right]$$

$$= \frac{1}{s} - \left[\frac{s + 1}{(s + 1)^2 + 1}\right]. \qquad (6)$$

Now we know that $\mathcal{L}(1) = \frac{1}{s}$ and we also know that $\mathcal{L}(\cos x) = \frac{s}{s^2 + 1}$. Hence, by the shifting theorem,

$$\mathcal{L}[e^{-x} \cos x] = \frac{s + 1}{(s + 1)^2 + 1}.$$

Consequently, (6) may be rewritten as

2.9.7 continued

$$y(x) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \left[\frac{s+1}{(s+1)^2+1}\right]\right\}$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]$$

$$= 1 - e^{-x}\cos x.$$

2.9.8

$$y''' - y' = e^{2t} \rightarrow$$

$$\mathcal{L}(y''') - \mathcal{L}(y') = \mathcal{L}(e^{2t}) \rightarrow$$

$$\mathcal{L}(y''') - \mathcal{L}(y') = \frac{1}{s-2}. \qquad (1)$$

Then, since

$$\mathcal{L}(y''') = s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0)$$

and

$$\mathcal{L}(y') = s\bar{y}(s) - y(0),$$

(1) becomes

$$s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0) - s\bar{y}(s) + y(0) = \frac{1}{s-2} \qquad (2)$$

But we are told that $y(0) = y'(0) = y''(0) = 0$, so (2) becomes

$$(s^3 - s)\bar{y}(s) = \frac{1}{s-2}$$

or

$$\bar{y}(s) = \frac{1}{(s-2)(s^3-s)}$$

$$= \frac{1}{(s-2)s(s+1)(s-1)}. \qquad (3)$$

2.9.8 continued

Use of partial fractions yields

$$\frac{1}{s(s-1)(s+1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s-2}, \tag{4}$$

Rather than use undetermined coefficients etc., in (4) we may "pick off" A, B, C, and D rather conveniently by multiplying both sides of (4) by s, s - 1, s + 1, or s - 2. For example, multiplying both sides of (4) by a (assuming, of course, that $s \neq 0$), we obtain

$$\frac{1}{(s-1)(s+1)(s-2)} = A + s\left[\frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s-2}\right]. \tag{5}$$

Now, letting s = 0 in (5) [i.e., we take the limit of both sides of (5) as $s \to 0$ since (5) was derived under the assumption that $s \neq 0$], we obtain

$$\frac{1}{(0-1)(0+1)(0-2)} = A$$

or

$$A = \frac{1}{2}.$$

Similarly

$$B = -\frac{1}{2}$$

$$C = -\frac{1}{6}$$

$$D = \frac{1}{6}$$

so that from (3) and (4), we conclude that

2.9.8 continued

$$\bar{y}(s) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right) - \frac{1}{6}\left(\frac{1}{s+1}\right) + \frac{1}{6}\left(\frac{1}{s-2}\right)$$

$$= \frac{1}{2}\mathcal{L}(1) - \frac{1}{2}\mathcal{L}(e^{t}) - \frac{1}{6}\mathcal{L}(e^{-t}) + \frac{1}{6}\mathcal{L}(e^{2t})$$

$$= \mathcal{L}\left(\frac{1}{2} - \frac{1}{2}e^{t} - \frac{1}{6}e^{-t} + \frac{1}{6}e^{2t}\right).$$

Therefore,

$$y(t) = \frac{1}{6}(e^{2t} - e^{-t} - 3e^{t} + 3).$$

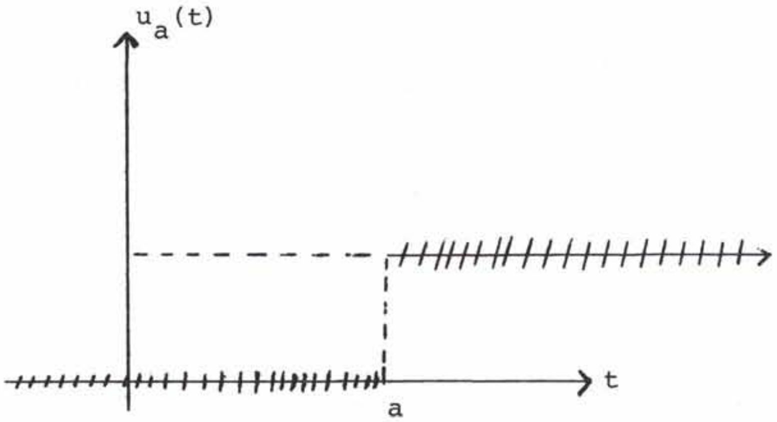
Unit 10: The Laplace Transform, Part 2

2.10.1

The unit step function, $u_a(t)$, is defined by

$$u_a(t) = \begin{cases} 0, & t \leq a \\ 1, & t > a \end{cases}$$

Pictorially



$u_a(t)$ is the factor we use if we want $g(t)$ "delayed" until $t = a$. Namely, if we let

$$f(t) = u_a(t)g(t - a)$$

then $f(t) = 0$, until $t > a$ since for $t \leq a$, $u_a(t) = 0$; and for $t > a$, $f(t) = g(t - a)$ since for $t > a$, $u_a(t) = 1$. In summary,

$$f(t) = u_a(t)g(t - a)$$

means

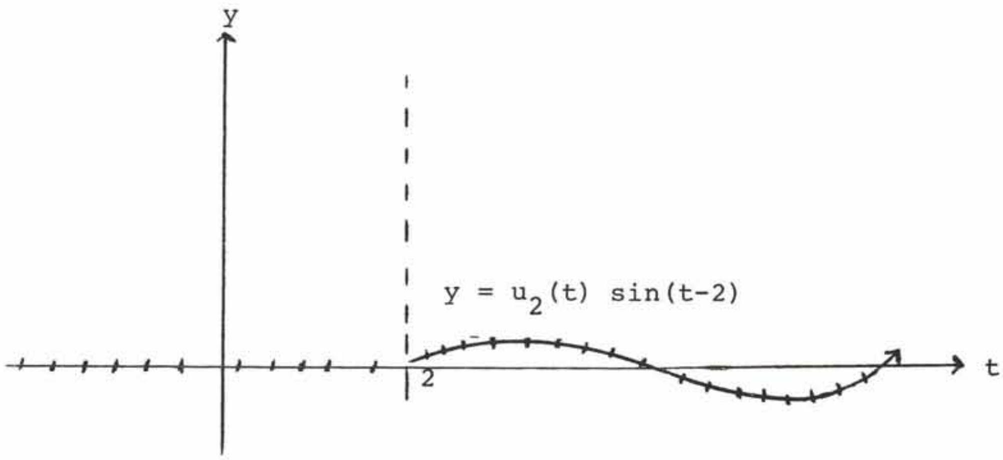
$$f(t) = \begin{cases} 0 & \text{if } t \leq a \\ g(t - a) & \text{if } t > a \end{cases}$$

2.10.1 continued

For example, if $g(t) = \sin t$ is delayed until $t = 2$, we would represent this function as

$$f(t) = u_2(t)\sin(t - 2).$$

Pictorially,



$y = u_2(t)\sin(t - 2)$ is the curve $y = \sin t$ shifted to begin at $t = 2$, and is 0 prior to $t \geq 2$.

a. If $f(t) = u_a(t)$, then

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st}\, u_a(t)\,dt$$

and since $u_a(t) = 0$ for $t \leq a$ and 1 for $t > a$, we have that

$$\mathcal{L}(f(t)) = \int_0^a e^{-st}\, u_a(t)\,dt + \int_a^\infty e^{-st} u_a(t)\,dt$$

$$= 0 + \int_a^\infty e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\Big]_{t=a}^{\infty}$$

$$= -\frac{1}{s}e^{-\infty} - \left(-\frac{1}{s}e^{-as}\right).$$

2.10.1 continued

Hence,

$$\mathcal{L}[u_a(t)] = \frac{1}{s}\, e^{-as}.$$

b. The result of (a) generalizes as follows

$$[u_a(t)f(t - a)] = \int_0^\infty e^{-st}\, u_a(t)f(t - a)dt$$

$$= \int_0^a e^{-st}\underbrace{u_a(t)}_{=0}f(t - a)dt + \int_a^\infty e^{-st}\underbrace{u_a(t)}_{=1}f(t-a)dt$$

$$= \int_a^\infty e^{-st}\, f(t - a)dt. \qquad (1)$$

Now, to put (1) in terms of the more familiar $\mathcal{L}(f)$, we make the change of variables $x = t - a$ in the integral in (1) to obtain

$$\mathcal{L}[u_a(t)f(t - a)] = \int_0^\infty e^{-s(x + a)}f(x)dx$$

$$= \int_0^\infty e^{-sa}\, e^{-sx}f(x)dx$$

$$= e^{-as}\int_0^\infty e^{-sx}\, f(x)dx$$

$$= e^{-as}\,\mathcal{L}[f(t)] \qquad (2)$$

$$= e^{-as}\,\bar{f}(s).$$

In other words, delaying $f(t)$ until $t = a$ yields a new function whose transform is $e^{-as}\,\bar{f}(s)$.

c. To emphasize inverse transforms we write (2) in the form

$$\mathcal{L}^{-1}[e^{-as}f(s)] = u_a(t)f(t - a). \qquad (3)$$

2.10.1 continued

Thus,

$$\mathcal{L}^{-1}\left[e^{-3s}\left(\frac{1}{s^2 + 4s + 5}\right)\right] = u_3(t)f(t - 3) \qquad (4)$$

where

$$\mathcal{L}(f(t) = \frac{1}{s^2 + 4s + 5}$$

$$= \frac{1}{(s + 2)^2 + 1}$$

$$= \mathcal{L}(e^{-2t} \sin t).$$

Hence,

$$f(t) = e^{-2t}\sin t,$$

so that

$$f(t - 3) = e^{-2(t - 3)}\sin(t - 3).$$

Thus, from (4) we conclude that

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 + 4s + 5}\right] = u_3(t)e^{-2(t - 3)}\sin (t - 3).$$

Caution

Our recipes get a bit confusing if we are not careful.
For example,

$$\mathcal{L}[e^{at}f(t)] = \bar{f}(s - a) \qquad (5)$$

but

$$\mathcal{L}[u_a(t)f(t - a)] = e^{-as}\bar{f}(s). \qquad (6)$$

The point is we must not confuse e^{at} $f(t)$ in (5) with $e^{-as}\bar{f}(s)$ in (6).

2.10.1 continued

Note:
In addition to the fact that the unit step function gives us a new application of computing inverse transforms, it should be again pointed out that the unit step function occurs independently of any application of the Laplace transform in the sense that many physical situations require the time delay of a given signal. For example, if the equation of the signal is $y = f(t)$, but we delay the start of the signal to the time $t = t_o$, then the new equation of the signal becomes

$$u_{t_o}(t)f(t) = \begin{cases} 0\ f(t), & t \leq t_o \\ 1\ f(t), & t > t_o \end{cases}$$

$$= \begin{cases} 0 & t \leq t_o \\ f(t) & t > t_o \end{cases}$$

2.10.2

Very often in mathematics, whether or not the Laplace transform is involved, we are called upon to deal with periodic functions. This occurs, obviously, when we are dealing with the circular functions; and it also occurs much more subtlely on the advanced level in the sense that many important applications involve trigonometric series (for example, Fourier series which are discussed in Chapter 18 of the text and which we shall touch upon in the next Block) rather than power series. For this reason, an analysis of periodic functions is important in its own right, but in the present exercise we limit our discussion to an investigation of the Laplace transforms of such functions.

a. To capitalize on the fact that $f(t) = f(t + p)$ for all t we write

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)dt$$

2.10.2 continued

$$= \int_0^p e^{-st} f(t)\,dt + \int_p^{2p} e^{-st} f(t)\,dt + \ldots + \int_{np}^{(n+1)p} e^{-st} f(t)\,dt + \ldots$$

$$= \sum_{n=0}^{\infty} \int_{np}^{(n+1)p} e^{-st} f(t)\,dt. \qquad (1)$$

We now let $t = x + np$ in (1) to obtain

$$\mathcal{L}[f(t)] = \sum_{n=0}^{\infty} \int_0^p e^{-s(x+np)} f(x + np)\,dx$$

$$= \sum_{n=0}^{\infty} e^{-snp} \int_0^p e^{-sx} f(x + np)\,dx,$$

or since $f(x + np) = f(x)$ [which is why we made the change of variable $t = x + np$],

$$\mathcal{L}[f(t)] = \sum_{n=0}^{\infty} (e^{-ps})^n \int_0^p e^{-sx} f(x)\,dx. \qquad (2)$$

But for $|u| < 1$,

$$\sum_{n=0}^{\infty} u^n = \frac{1}{1 - u}$$

and since $p > 0$, $e^{-ps} < 1$, so that

$$\sum_{n=0}^{\infty} (e^{-ps})^n = \frac{1}{1 - e^{-ps}} .$$

Thus, from (2) we see that if f is of exponential order and periodic with period $p > 0$, then

$$[f(t)] = \frac{\int_0^p e^{-st} f(t)\,dt}{1 - e^{-ps}} .$$

b. Let f(t) be periodic of period $p = 2$ where

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < 2 \end{cases}$$

2.10.2 continued

Then

$$\int_0^p e^{-st}\, f(t)\,dt = \int_0^2 e^{-st} f(t)\,dt = \int_0^1 e^{-st} f(t)\,dt + \underbrace{\int_1^2 e^{-st} f(t)\,dt}_{=\,0}$$

$$= \int_0^1 e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\Big]_{t=0}^{1}$$

$$= -\frac{1}{s}e^{-s} + \frac{1}{s}$$

$$= \frac{1 - e^{-s}}{s}.$$

Hence,

$$\mathcal{L}[f(t)] = \frac{\left[\frac{1 - e^{-s}}{s}\right]}{1 - e^{-ps}}$$

$$= \frac{\left[\frac{1 - e^{-s}}{s}\right]}{1 - e^{-2s}}$$

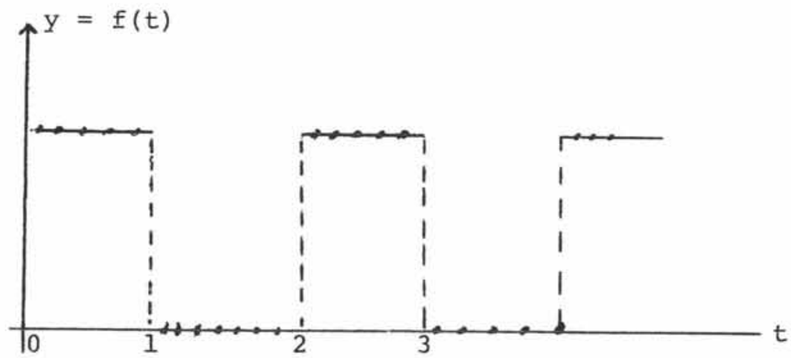
$$= \frac{1 - e^{-s}}{s(1 - e^{-2s})}$$

$$= \frac{1 - e^{-s}}{s(1 - e^{-s})(1 + e^{-s})}$$

$$= \frac{1}{s(1 + e^{-s})}.$$

2.10.2 continued

Pictorially f is given by:



2.10.3

In this exercise we try to show how the appearance of certain expressions that arise in a particular study force us to investigate certain aspects of a topic that we may not otherwise have elected to study; and that when such a need arises we often must draw on knowledge that was acquired previously in our study, but which at the time might not have seemed too important. It is this latter aspect that is very important in the learning process since one often learns to appreciate a result when it is used as means toward an end rather than as an end in itself.

The problem that occurs in this problem is that of finding the Laplace transform of tf(t), once the Laplace transform of f(t) is known. This problem would arise, in particular, if we were studying linear equations with constant coefficients in the sense that the previously described method of undetermined coefficients requires that the right side of the equation have the form $t^n f(t)$ for certain special choices of f.

a. One way of trying to compute $\mathcal{L}(tf(t))$ once $\mathcal{L}(f(t))$ is known is to invoke the basic definition that

$$\mathcal{L}(tf(t)) = \int_0^\infty e^{-st}\, tf(t)\,dt \qquad (1)$$

and then try to express the right side of (1) in a way which

2.10.3 continued

helps us get at $\mathscr{L}(f(t))$. For example, we might try to integrate by parts letting $u = t$ and $dv = e^{-st}f(t)dt$; whence $du = dt$ and

$$f = \bar{f}(s) = \int_0^\infty e^{-st}f(t)\,dt.$$

However, if we remember that for suitably chosen functions $F(s,t)$ that

$$\frac{d}{ds}\int_0^\infty F(s,t)\,dt = \int_0^\infty \frac{\partial F(s,t)}{\partial s}\,dt^*$$

and that $e^{-st}f(t)dt$ is of this type if F is of exponential order, then

$$\frac{d}{ds}\int_0^\infty e^{-st}\,f(t)\,dt = \int_0^\infty \frac{\partial[e^{-st}f(t)\,dt]}{\partial s}$$

$$= \int_0^\infty -te^{-st}f(t)\,dt$$

$$= -\int_0^\infty e^{-st}[tf(t)]\,dt$$

$$= -\mathscr{L}[tf(t)]. \qquad (2)$$

*The theorem states that if $\partial F(s,t)/\partial s$ is piecewise continuous on $a < s \leq b$ for each t and if

$$\int_0^\infty F(s,t)\,dt \quad \text{and} \quad \int_0^\infty \frac{\partial F}{\partial s}\,dt$$

both converge uniformly, the above stated result holds. In particular, since $|F(t)| \leq Me^{\alpha t}$, etc. we may prove that

$$\int_0^\infty t^n e^{-st}F(t)\,dt$$

converges uniformly for $s > \alpha$. All we are doing in our present example is "doing what comes naturally" but using this footnote as a reminder that in improper integrals we cannot avoid a certain amount of "nasty" theory to justify the validity of our results.

2.10.3 continued

The left side of (2) may be written as

$$\bar{f}'(s) = \frac{d\bar{f}(s)}{ds},$$

so that from (2) we conclude that

$$\mathcal{L}[tf(t)] = -\bar{f}'(s) = -\frac{d\bar{f}(s)}{ds} = -\frac{d[\mathcal{L}(f(t))]}{ds}. \qquad (3)$$

Note:
We may now proceed inductively. For example,

$$\frac{d}{ds}\int_0^\infty -e^{-st}\,tf(t)\,dt = \int_0^\infty e^{-st}t^2f(t)\,dt. \qquad (4)$$

Recalling from (2) that

$$\int_0^\infty -e^{-st}\,tf(t)\,dt$$

is

$$\frac{d\bar{f}(s)}{ds},$$

and observing that

$$\int_0^\infty e^{-st}\,t^2f(t)\,dt = \mathcal{L}[t^2f(t)],$$

we see that (4) may be rewritten as

$$\mathcal{L}[t^2f(t)] = \frac{d^2\bar{f}(s)}{ds^2}.$$

This process may then be applied to

$$\int_0^\infty e^{-st}t^2f(t)\,dt,$$

etc., since each time we differentiate with respect to s, a factor of -t is introduced and we may conclude that for any positive integer n,

$$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n\bar{f}(s)}{ds^n} = (-1)^n\frac{d^n\mathcal{L}(f)}{ds^n}.$$

2.10.3 continued

b. We apply the result of part (a) toward the solution of the equation

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty = 0 \qquad (5)$$

subject to the initial conditional $y(0) = y_o$.*

Equation (5) is called the Bessel Equation of order 0 and the solution which obeys the specific initial condition $y(0) = 1$ is denoted by $J_o(t)$ and is called Bessel's function of the first kind of order zero. Bessel equations arise in many areas of applied mathematics.

At any rate, from (5) we conclude that

$$\mathcal{L}\left(t\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty\right) = \mathcal{L}(0),$$

or, since $\mathcal{L}$ is linear and $\mathcal{L}(0) = 0$, we have

$$\mathcal{L}\left(t\frac{d^2y}{dt^2}\right) + \mathcal{L}\left(\frac{dy}{dt}\right) + \mathcal{L}(ty) = 0. \qquad (6)$$

From (3), with $f(t) = d^2y(t)/dt^2$, we conclude that

$$\mathcal{L}\left[t\frac{d^2y(t)}{dt^2}\right] = -\frac{d\mathcal{L}[d^2y(t)/dt^2]}{ds}, \qquad (7)$$

and with $f(t) = y(t)$, (3) yields

$$\mathcal{L}[ty(t)] = -\frac{d\mathcal{L}[y(t)]}{ds} = -\frac{d\bar{y}(s)}{ds} = -\frac{d\bar{y}}{ds}. \qquad (8)$$

We may further simplify (7) by recalling that

$$\mathcal{L}\left[\frac{d^2y(t)}{dt^2}\right] = -y'(0) - sy(0) + \underbrace{s^2\bar{y}(s)}_{\text{product of two functions of s}}$$

*Letting $t = 0$ in (5) yields $dy/dt = 0$. Hence $y'(0)$ must equal 0 so that we have no choice in prescribing the value of $y'(0)$ other than to let it be 0.

2.10.3 continued

so that

$$\frac{d\mathcal{L}[d^2y(t)/dt^2]}{ds} = -y(0) + 2s\bar{y}(s) + s^2\frac{d\bar{y}(s)}{ds}$$

$$= -y(0) + 2s\bar{y}(s) + s^2\frac{d\bar{y}(s)}{ds}.$$

Hence, (7) becomes

$$\mathcal{L}[t\frac{d^2y}{dt^2}] = y(0) - 2s\bar{y}(s) - s^2\frac{d\bar{y}(s)}{ds}$$

and since we are given that $y(0) = y_o$, we may further conclude that

$$\mathcal{L}[t\frac{d^2y}{dt^2}] = y_o - 2s\bar{y} - s^2\frac{d\bar{y}}{ds}. \qquad (9)$$

Recalling next that

$$(\frac{dy}{dt}) = -y(0) + s\bar{y}(s) = -y_o + s\bar{y}, \qquad (10)$$

we see from (8), (9), (10) that (6) may be rewritten as

$$y_o - 2s\bar{y} - s^2\frac{d\bar{y}}{ds} - y_o + s\bar{y} - \frac{d\bar{y}}{ds} = 0$$

or

$$-s\bar{y} - (1 + s^2)\frac{d\bar{y}}{ds} = 0.$$

Therefore,

$$\frac{d\bar{y}}{ds} + \frac{s\bar{y}}{1 + s^2} = 0. \qquad (11)$$

An integrating factor for (11) is

$$e^{\int\frac{sds}{1+s^2}}$$

or

2.10.3 continued

$$e^{\frac{1}{2}\ln(1+s^2)} = \sqrt{1+s^2}\ ,$$

so that (11) may be written as

$$\frac{d(\sqrt{1+s^2}\ \bar{y})}{ds} = 0.$$

Hence

$$\bar{y}\sqrt{1+s^2} = c.$$

In other words

$$\bar{y}(s) = \frac{c}{\sqrt{1+s^2}}\ . \qquad (12)$$

Note:

It is interesting to note that the result given in (12) does not depend on the choice of y_o. It can be shown (but we don't prove it here) that if we require that $y(0) = 1$, so that $y(t) = J_o(t)$, then $\bar{y}(s) = 1/1 + s^2$. That is assuming that $\mathcal{L}^{-1}(1/\sqrt{1+s^2}) = J_o(t)$, then $\mathcal{L}^{-1}(c/\sqrt{1+s^2}) = cJ_o(t)$.* Thus, aside from any other properties of $J_o(t)$, it arises in the study of inverse transform when we try to find a function whose Laplace transform is a constant multiple of

$$\frac{1}{\sqrt{s^2+1}}\ .$$

*Even if we didn't know that

$$\mathcal{L}^{-1}(1/\sqrt{1+s^2}) = J_o(t),$$

we know from (12) that $\mathcal{L}(J_o(t) = c_1/\sqrt{1+s^2}$ for some number c_1. Then given any number $c(\neq 0)$, we may write it as $c_1(c/c_1)$ so that (12) yields

$$\bar{y}(s) = \frac{c}{c_1}\left(\frac{c_1}{\sqrt{1+s^2}}\right) = \frac{c}{c_1}\mathcal{L}^{-1}(J_o(t));\ \text{or}\ \mathcal{L}^{-1}\left(\frac{c}{\sqrt{1+s^2}}\right) =$$

constant multiple of $J_o(t)$.

2.10.4

In Exercise 6.4.7 in Part 1 of our course we defined the Gamma Function by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \quad (1)$$

The Gamma Function arises in the study of the Laplace Transform when we try to compute $\mathcal{L}(x^n)$. In particular, by definition

$$\mathcal{L}(x^n) = \int_0^\infty e^{-sx} x^n dx. \quad (2)$$

Comparing the integrals in (1) and (2), the substitution t = sx seems to suggest itself (i.e., we compare the powers of e in both integrals). This leads to

$$dx = \frac{dt}{s}$$

and

$$x^n = \left(\frac{t}{s} \right)^n$$

so that (2) becomes

$$\mathcal{L}(x^n) = \int_0^\infty e^{-t} \frac{t^n}{s^n} \frac{dt}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty t^n e^{-t} dt. \quad (3)$$

The integral in (3) is precisely $\Gamma(n + 1)$. [Namely, simply let $x - 1 = n$ in (1).] Therefore,

$$\mathcal{L}(x^n) = \frac{\Gamma(n + 1)}{s^{n + 1}}, \text{ provided } n > -1^*. \quad (4)$$

---

*In part 1 we showed that

$\int_0^\infty t^{x-1} e^{-t} dt$

converged only when $x > 0$. Hence, letting $x - 1 = n$ implies that $x > 0 \leftrightarrow n > -1$.

2.10.4 continued

In particular, if n is a whole number, $\Gamma(n + 1) = n!$; so that

$$\mathcal{L}(x^n) = \frac{n!}{s^{n+1}} \text{ if n is a whole number.} \tag{5}$$

Equation (5) could have been derived without recourse to the Gamma Function. The beauty of (4), however, lies in the fact that it applies for all real values of n provided only that $n > -1$. For this reason one often defines n! to mean $\Gamma(n + 1)$. In this way n! is now defined for all real $n > -1$ and agrees with the traditional definition of n! when n is a whole number.

2.10.5

a. We have by definition of Γ that

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx. \tag{1}$$

We now make the change of variables defined by $x = t^2$. Equation (1) then becomes

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-t^2} (t^2)^{-\frac{1}{2}} d(t^2)$$

$$= \int_0^\infty e^{-t^2} t^{-1}\, 2t\, dt$$

$$= 2\int_0^\infty e^{-t^2} dt,$$

and since

$\int_0^\infty e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}$, we conclude from (2) that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{3}$$

b. The fact that $\Gamma(n + 1) = n\,\Gamma(n)$ means that we can compute $\Gamma(3/2)$, $\Gamma(5/2)$, etc. in terms of $\Gamma(1/2)$. Thus,

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi}$$

2.10.5 continued

$$\Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\,\Gamma(\tfrac{3}{2})$$

$$= \tfrac{3}{2}\,(\tfrac{1}{2}\sqrt{\pi})$$

$$= \tfrac{3}{4}\sqrt{\pi}\ .$$

Quite in general, if one knows $\Gamma(x_o)$ for each x_o such that $1 \leq x_o < 2$, one can compute $\Gamma(x)$ for any x. For example, since $2 < \sqrt{7} < 3$, we have that

$$\Gamma(\sqrt{7}) = (\sqrt{7} - 1)\,\Gamma(\sqrt{7} - 1)$$

$$= (\sqrt{7} - 1)[(\sqrt{7} - 2)\Gamma(\sqrt{7} - 2)]$$

$$= (\sqrt{7} - 1)(\sqrt{7} - 2)\,\Gamma(\sqrt{7} - 2),$$

where $0 < \sqrt{7} - 2 < 1$.

On the other hand, if $0 \leq x < 1$, we may write $\Gamma(x + 1) = x\,\Gamma(x)$ in the form

$$\Gamma(x) = \frac{\Gamma(x + 1)}{x}\ .$$

Then, since $0 \leq x < 1$, it follows that $1 \leq x + 1 < 2$ so that

$$\Gamma(x) = \frac{\Gamma(x_o)}{x}$$

where

$$x_o = x + 1\ \varepsilon\,[1,2].$$

For this reason, when one refers to tables to find $\Gamma(x)$, he usually finds that the tables are computed only for $1 \leq x < 2$. From these values, he can find $\Gamma(x)$ for any $x > 0$ by applying the recurrance formula: $\Gamma(x + 1) = x\,\Gamma(x)$. In a way, this is similar to tables of logarithms wherein one tabulates $\log x_o$ for $1 \leq x_o < 10$ and then computes $\log x$ by writing $x = x_o(10)^n$ where $1 \leq x_o < 10$; whereupon $\log x = \log x_o + n$.

2.10.5 continued

We include a short table of values for $\Gamma(x)$ for selected values of x between 1.00 and 1.99, as well as a few examples which show how to compute $\Gamma(x)$ from the table when x is not between 1 and 2. The results are finally summarized in a graph of $\Gamma(x)$.

Table of Values for $\Gamma(x) = (x - 1)!$

x	.00	.03	.06	.09
1.0	1.0000	.9835	.9687	.9555
.1	.9514	.9399	.9298	.9209
.2	.9182	.9108	.9044	.8990
.3	.8975	.8934	.8902	.8879
.4	.8873	.8860	.8856	.8859
.5	.8862	.8876	.8896	.8924
.6	.8935	.8972	.9017	.9068
.7	.9086	.9147	.9214	.9288
.8	.9314	.9397	.9487	.9584
.9	.9618	.9724	.9837	.9958

Example #1

To find $\Gamma(3.73)$ we have

$$\Gamma(3.71) = (2.73)(1.73)\Gamma(1.73)$$

$$= (2.73)(1.73)(.9147)$$

$$\approx 4.320$$

Example #2

To find $\Gamma(0.03)$ we use

$$\Gamma(n + 1) = n\Gamma(n)$$

with $n = 0.03$.

This yields

$$\Gamma(1.03) = 0.03\Gamma(0.03),$$

2.10.5 continued

so from the table we conclude that

$$(0.03) = \frac{(1.03)}{0.03}$$

$$= \frac{0.9835}{0.03}$$

$$= 32.78.$$

Example #3

Compute $\Gamma(-1.07)$. Now we are in trouble! Namely, our definition of $\Gamma(x)$ to be

$$\int_0^\infty t^{x-1}e^{-t}dt$$

required that $x > 0$. Since $x = -1.07$, this condition, obviously, is not fulfilled.

To get around this problem we use an argument similar to the one which led us to define 0! to be equal to 1.

We say that we want the recurrence formula

$$\Gamma(x + 1) = x\ (x);\ \text{or},\ \Gamma(x) = \frac{\Gamma(x + 1)}{x}$$

to hold even when x is negative. Thus,

$$\Gamma(-1.07) = \frac{\Gamma(-0.07)}{-1.07}^{*}. \qquad (4)$$

In turn

$$\Gamma(-0.07) = \frac{\Gamma[-0.07 + 1]}{-0.07} = \frac{\Gamma(0.03)}{-0.07}. \qquad (5)$$

Combining (4) and (5), we conclude that

*Notice that -1.07 means -(1 + .07) = -1 - .07. Hence -1.07 + 1 = -.07. We should not confuse this with (-1).07 = -1 + .07 in which case -1.07 + 1 = +.07.

2.10.5 continued

$$\Gamma(-1.07) = \frac{1}{-1.07}\left[\frac{(0.03)}{-0.07}\right]$$

$$= \frac{\Gamma(0.03)}{0.0749}. \qquad (6)$$

From Example #2, $\Gamma(0.03) = 32.78$ so that (6) becomes

$$(-1.07) = \frac{32.78}{.0749} \approx 437.5 \qquad (7)$$

Looking at (7) we observe that $\Gamma(-1.07)$ is quite large. This is not a quirk. Indeed, if x is a negative integer, we see that the recurrance relation guarantees that (x) be $\pm\infty$. Namely,

$$(x) = \frac{\Gamma(x+1)}{x}$$

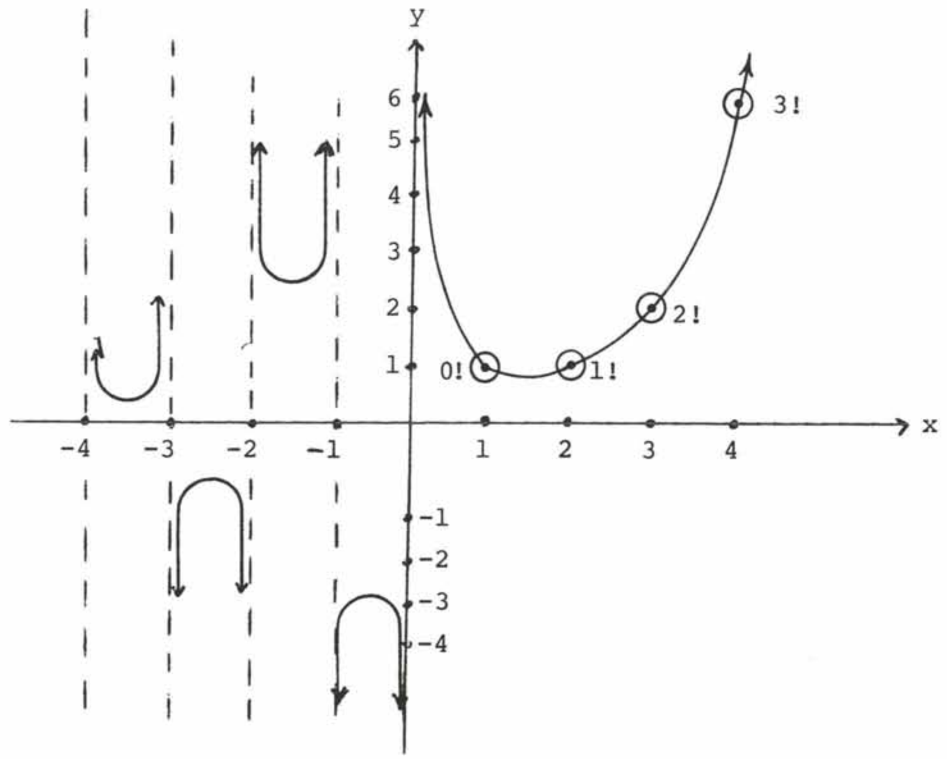
$$= \frac{1}{x}\ \frac{\Gamma(x+2)}{x+1}$$

$$= \frac{1}{x(x+1)}\ \frac{\Gamma(x+3)}{(x+2)}$$

and continuing in this way we eventually reach the stage where if x is a negative integer we obtain a 0 factor in our denominator. In fact $[x + (-x)] = 0$. In other words, $\Gamma(x)$, even when we use the recurrance formula, is not defined when x is a negative integer.

Without belaboring this point further, the graph of the Γ-function is given by

2.10.5 continued



Graph of $y = \Gamma(x) = (x - 1)!$

2.10.6

Our aim in this exercise is two-fold. On the one hand, we want to point out that

$$[\mathcal{L}(f)][\mathcal{L}(g)] \neq \mathcal{L}[fg].$$

In other words, linearity applies to addition and "scalar multiples" but not to products of two non-constant functions.

Secondly, we want to show how to compute h if we know that $\bar{h}(s) = \bar{f}(s)\bar{g}(s)$ where f and g are known. Such a result would be very helpful in certain problems involving inverse transforms.

a. Making use of dummy variables so that we may express the product as a double integral we have:

$$\mathcal{L}(f) \cdot \mathcal{L}(g) = \int_0^\infty e^{-sy} f(y)\,dy \int_0^\infty e^{-sx} g(x)\,dx$$

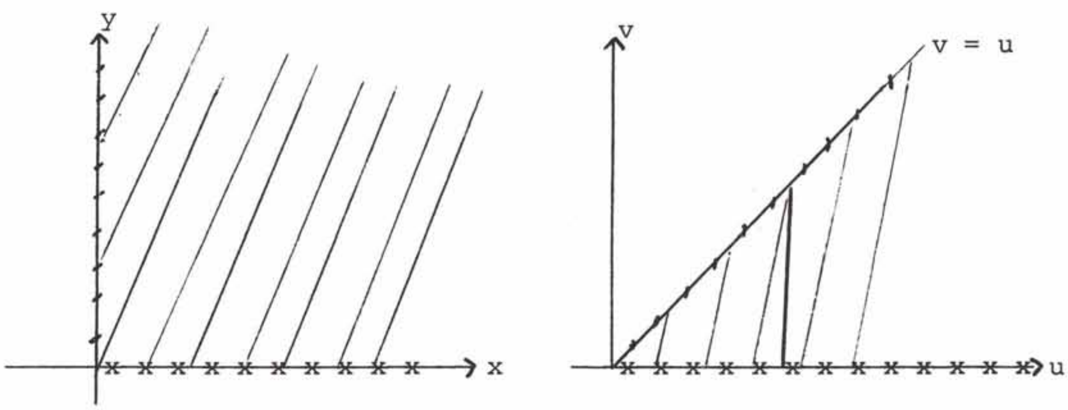
2.10.6 continued

$$= \int_0^\infty \int_0^\infty e^{-s(x+y)} f(y)g(x)\,dydx. \qquad (1)$$

Using a little hindsight, we sense that ultimately we want a factor of the form e^{-su} so that the proper form of a Laplace transform will be present. Observing that $e^{-s(x+y)}$ appears in (1), we try the change of variables

$$\left.\begin{aligned} u &= x + y \\ v &= y \end{aligned}\right\} \qquad (2)$$

Now the limits of integration in (1) indicate that we are integrating over the entire first quadrant of the xy-plane. The mapping (2) maps the first quadrant of the xy-plane into the region of the uv-plane shown below:



Namely, (2) may be rewritten as

$$\left.\begin{aligned} x &= u - v \\ y &= v \end{aligned}\right\} \qquad (3)$$

so that $x = 0$ implies $u - v = 0$ or $u = v$; and since $y \geq 0$, and $y = v$, we have that the positive y-axis (i.e., $x = 0$ and $y \geq 0$) maps onto $v = u$, $v \geq 0$.

Similarly, $y = 0$, $x \geq 0 \rightarrow v = 0$, $u \geq 0$; so the positive x-axis maps onto the positive u-axis.

Thus, in the uv-plane our limits of integration are given by that for a fixed u, v varies from 0 to u; and u may be chosen

2.10.6 continued

from 0 to ∞ . Moreover, from (3),

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

Hence, (1) becomes

$$\mathcal{L}(f)\mathcal{L}(g) = \int_0^\infty \int_0^u e^{-su} f(v)g(u - v)dv\,du$$

$$= \int_0^\infty [\int_0^u e^{-su} f(v)g(u - v)dv]du$$

$$= \int_0^\infty e^{-su}(\int_0^u f(v)g(u - v)dv)du$$

$$= \mathcal{L}[\int_0^u f(v)g(u - v)dv]. \qquad (4)$$

Equation (4) is called a convolution integral. More specifically, given f and g we define the convolution of f and g, written f*g by

$$f*g = \int_0^u f(v)g(u - v)dv.$$

Then (4) says:

$$\mathcal{L}[f*g] = \mathcal{L}(f)\mathcal{L}(g).$$

b. As usualy, we again pick a problem to which we already know the answer. Namely, if

$$\mathcal{L}(h) = \frac{1}{s(s - 1)},$$

then

$$\mathcal{L}(h) = \frac{1}{s - 1} - \frac{1}{s}$$

$$= \mathcal{L}(e^t) - \mathcal{L}(1)$$

$$= \mathcal{L}(e^t - 1).$$

2.10.6 continued

Hence

$$h(t) = e^t - 1.^* \qquad (5)$$

Using convolution we have

$$\mathcal{L}(h) = \frac{1}{s(s-1)}$$

$$= \left(\frac{1}{s}\right)\left(\frac{1}{s-1}\right)$$

$$= \underbrace{\mathcal{L}(1)}_{f(t)}\ \underbrace{\mathcal{L}(e^t)}_{g(t)}\ .$$

Hence, by convolution,

$$\mathcal{L}(h) = \mathcal{L}(1 * e^t)$$

$$= \mathcal{L}\left[\int_0^u 1 \cdot e^{u-v}\, dv\right]$$

$$= \mathcal{L}\left[\int_0^u e^{u-v} dv\right]$$

$$= \mathcal{L}\left[\ -e^{u-v} \Big|_{v=0}^{u}\ \right]$$

$$= \mathcal{L}[-e^{\circ} - (-e^u)]$$

$$= \mathcal{L}[e^u - 1].$$

Hence,

$$h(u) = e^u - 1$$

which, as discussed in our last footnote, agrees with (5).

---

*Again, keep in mind that t is not necessary. We could, for example, have written $\mathcal{L}(h) = \mathcal{L}(e^x) - \mathcal{L}(1)$; whence $h(x) = e^x - 1$. This is the same as (5); namely, $h([\ ]) = e^{[\ ]} - 1$. This is why we usually write $\mathcal{L}(h)$ rather than $\mathcal{L}(h(t))$.

2.10.7

a. $$f * g = \int_{v=0}^{u} f(v)g(u - v)dv. \qquad (1)$$

Now let $w = u - v$. Then

$$f * g = \int_{u}^{0} f(u - w)g(w)(-dw)$$
$$= \int_{0}^{u} f(u - w)g(w)dw. \qquad (2)$$

Since w is a dummy variable, we may replace it by v in (2) to obtain

$$f * g = \int_{0}^{u} f(u - v)g(v)dv$$
$$= \int_{0}^{u} g(v)f(u - v)dv$$
$$= g * f.$$

b. $$f * (g + h) = \int_{0}^{u} f(v)[g + h](u - v)dv$$
$$= \int_{0}^{u} f(v)[g(u - v) + h(u - v)]dv$$
$$= \int_{0}^{u} [f(v)g(u - v) + f(v)h(u - v)]dv$$
$$= \int_{0}^{u} f(v)g(u - v)dv + \int_{0}^{u} f(v)h(u - v)dv$$
$$= (f * g) + (f * k).$$

It may also be shown that convolution is associative. That is,

$$(f * g) * h = f * (g * h),$$

but we elect not to prove any further properties of the convolution integral since such proofs are peripheral to our immediate needs.

2.10.8

a. Given that

$$\mathcal{L}(h) = \frac{s}{(s-1)(s^2+1)} \tag{1}$$

we see from our tables (or otherwise) that

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1}$$

and

$$\mathcal{L}(e^t) = \frac{1}{s-1}.$$

We then rewrite (1) as

$$\begin{aligned}\mathcal{L}(h) &= \left(\frac{1}{s-1}\right)\left(\frac{s}{s^2+1}\right)\\ &= \mathcal{L}(e^t)\,\mathcal{L}(\cos t)\\ &= \mathcal{L}(e^t * \cos t)\\ &= \int_0^t e^x \cos(t-x)\,dx. \end{aligned} \tag{2}$$

Using tables, or else integrating by parts twice, (2) becomes

$$\mathcal{L}(h) = \frac{1}{2}(\sin t + e^t - \cos t). \tag{3}$$

b. Given that

$$\mathcal{L}(h) = \frac{1}{(s^2+1)^2}$$

we write

$$\begin{aligned}\mathcal{L}(h) &= \left[\frac{1}{s^2+1}\right]\left[\frac{1}{s^2+1}\right]\\ &= \mathcal{L}(\sin t)\,\mathcal{L}(\sin t)\\ &= \mathcal{L}(\sin t * \sin t).\end{aligned}$$

2.10.8 continued

Hence,

$$\mathcal{L}(h) = \mathcal{L}[\int_0^t \sin x \sin(t - x)dx]. \qquad (4)$$

Using the fact that $\sin A \sin B = \frac{1}{2}[\cos(A - B)]$, we see (4) may be rewritten as

$$\mathcal{L}(h) = \mathcal{L}[\frac{1}{2}\int_0^t [\cos(2x - t) - \cos t]dx].$$

Hence,

$$h(t) = \frac{1}{2}[\frac{1}{2}\sin(2x - t) - x\cos t \Big|_{x=0}^{t}$$

$$= \frac{1}{2}[\frac{1}{2}\sin t - t\cos t - \frac{1}{2}\sin(-t) + 0]$$

$$= \frac{1}{2}[\frac{1}{2}\sin t - t\cos t + \frac{1}{2}\sin t]$$

$$= \frac{1}{2}\sin t - \frac{1}{2}t\cos t. \qquad (5)$$

c. $$\frac{1}{(s - 1)(s^2 + 1)} = (\frac{1}{s - 1})(\frac{1}{s^2 + 1})$$

$$= \mathcal{L}(e^t)\mathcal{L}(\sin t)$$

$$= \mathcal{L}(e^t * \sin t)$$

$$= \mathcal{L}[\int_0^t e^x \sin(t - x)dx]$$

$$= \mathcal{L}[\frac{1}{2}(e^t - \sin t - \cos t)]$$

$$= \mathcal{L}(h(t)).$$

Hence,

$$h(t) = \frac{1}{2}(e^t - \sin t - \cos t). \qquad (6)$$

d. $$\frac{s}{s^2 + 1)^2} = (\frac{s}{s^2 + 1})(\frac{1}{s^2 + 1})$$

$$= \mathcal{L}(\cos t)\mathcal{L}(\sin t)$$

2.10.8 continued

$= \mathcal{L}(\cos t * \sin t)$

$= \mathcal{L}[\int_0^t \cos x \sin(t - x)dx]$

$= \mathcal{L}[\int_0^t \sin x \cos(t - x)dx]. \qquad (7)$

Since $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$, (7) becomes

$$\frac{s}{(s^2 + 1)^2} = \mathcal{L}[\frac{1}{2}\int_0^t [\sin t + \sin(2x - t)]dx]$$

$$= \mathcal{L}\{\frac{1}{2}[x \sin t - \frac{1}{2}\cos(2x - t)]\Big|_{x=0}^{t}\}$$

$$= \mathcal{L}\{\frac{1}{2}[t \sin t - \frac{1}{2}\cos t + \frac{1}{2}\cos t]\}$$

$$= \mathcal{L}\{\frac{1}{2} t \sin t\}$$

$$= \mathcal{L}(h(t)).$$

Hence,

$$h(t) = \frac{1}{2} t \sin t. \qquad (8)$$

e. Given the system

$$\left.\begin{aligned} \frac{dx}{dt} - y &= e^t \\ \frac{dy}{dt} + x &= \sin t \end{aligned}\right\} \qquad (9)$$

together with the initial condition $x(0) = 1$ and $y(0) = 0$; we take the Laplace transform of both sides of each equation in (9) to obtain

$$\left.\begin{aligned} \mathcal{L}(\frac{dx}{dt}) - \mathcal{L}(y) &= \mathcal{L}(e^t) \\ \mathcal{L}(\frac{dy}{dt}) + \mathcal{L}(x) &= \mathcal{L}(\sin t) \end{aligned}\right\}$$

or

2.10.8 continued

$$\left.\begin{aligned} s\bar{x}(s) - x(0) - \bar{y}(s) &= \frac{1}{s-1} \\ s\bar{y}(s) - y(0) + \bar{x}(s) &= \frac{1}{s^2+1} \end{aligned}\right\} \qquad (10)$$

Using our initial conditions and abbreviating $\bar{x}(s)$ and $\bar{y}(s)$ by $\bar{x}$ and $\bar{y}$ respectively, we may rewrite (10) as

$$\left.\begin{aligned} s\bar{x} - \bar{y} &= \frac{1}{s-1} + 1 \\ \bar{x} + sy &= \frac{1}{s^2+1} \end{aligned}\right\} \qquad (11)$$

To eliminate $\bar{y}$ from (11) we may multiply the top equation by s and then add the two equations. Thus:

$$s^2\bar{x} - s\bar{y} = \frac{s}{s-1} + s$$

$$\bar{x} + s\bar{y} = \frac{1}{s^2+1}$$

$$(s^2 + 1)\bar{x} = \frac{s}{s-1} + s + \frac{1}{s^2+1}.$$

Hence:

$$\bar{x} = \frac{s}{(s-1)(s^2+1)} + \frac{s}{s^2+1} + \frac{1}{(s^2+1)^2}. \qquad (12)$$

Then using (12) in the top equation of (11) we obtain

$$\bar{y} = \frac{-1}{(s-1)(s^2+1)} - \frac{1}{s^2+1} + \frac{s}{(s^2+1)^2}. \qquad (13)$$

We may now use the results of (a) and (b) to compute x(t) from (12); and the result of (c) and (d) to compute y(t) from (13). Namely, from (12) we have

$$x(t) = \frac{1}{2}(\sin t + e^t - \cos t) + \cos t + (\frac{1}{2}\sin t - \frac{1}{2} t \cos t)$$

$$= \frac{1}{2}(2 \sin t + e^t + \cos t - t \cos t); \qquad (14)$$

2.10.8 continued

(14)

and from (13)

$$y(t) = -\frac{1}{2}(e^t - \sin t - \cos t) - \sin t + \frac{1}{2} t \sin t$$

$$= \frac{1}{2}(-e^t - \sin t + \cos t + t \sin t). \qquad (15)$$

As a check we have from (14) and (15) that

$$\frac{dx}{dt} = \frac{1}{2}(2 \cos t + e^t - \sin t + t \sin t - \cos t)$$

$$= \frac{1}{2}(\cos t + e^t - \sin t + t \sin t)$$

and

$$\frac{dy}{dt} = \frac{1}{2}(-e^t - \cos t - \sin t + \sin t + t \cos t)$$

$$= \frac{1}{2}(-e^t - \cos t + t \cos t).$$

Thus,

$$\frac{dx}{dt} - y = \frac{1}{2}(\cos t + e^t - \sin t + t \sin t) - \frac{1}{2}(-e^t - \sin t + \cos t + t \sin t)$$

$$= e^t$$

and

$$\frac{dy}{dt} + x = \frac{1}{2}(-e^t - \cos t + t \cos t) + \frac{1}{2}(2 \sin t + e^t + \cos t - t \cos t)$$

$$= \sin t.$$

Moreover, the initial conditions are also obeyed since, from (14),

$$x(0) = \frac{1}{2}(0 + 1 + 1 - 0) = 1$$

2.10.8 continued

and from (15)

$$y(0) = \frac{1}{2}(-1 - 0 + 1 + 0) = 0.$$

As a final note on this exercise, observe that the check of our solution in (e) is part of the solution. Namely, we have shown that if the system has a solution, subject to the given initial conditions, then it must be the one given by equations (14) and (15). But until these equations are checked we do not know whether a solution exists.

The key caution in using Laplace transforms to solve simultaneous systems of linear equations is that while the method will yield the correct solution if it exists, it will yield an incorrect result if no correct result exists (and this happens when the presented initial conditions cannot be satisfied). Thus, in doubtful cases, one should always check that the solution obtained by the transform method obeys the given initial conditions.

For example, use of the transform method to solve the system

$$\left.\begin{aligned} &\frac{dx}{dt} + y = 0 \\ &\frac{d^2x}{dt^2} + \frac{dy}{dt} + y = e^t \end{aligned}\right\}$$

subject to the initial conditions $x(0) = 1$, $x'(0) = 0$, $y(0) = 0$, yields:

$$\left.\begin{aligned} &s\bar{x} + \bar{y} = 1 \\ &s^2\bar{x} + (s + 1)\,\bar{y} = s + \frac{1}{s - 1} \end{aligned}\right\}$$

Hence,

$$\left.\begin{aligned} &\bar{x} = \frac{2}{s} - \frac{1}{s - 1} \\ &\bar{y} = \frac{1}{s - 1} \end{aligned}\right\},$$

from which we conclude that

2.10.8 continued

$$\left.\begin{aligned} x &= 2 - e^t \\ y &= e^t \end{aligned}\right\}$$

Yet this cannot be the solution since it would imply that $y(0) = e^0 = 1$ rather than 0.

What is true is that the general solution of the given system is

$$\left.\begin{aligned} x &= c - e^t \\ y &= e^t \end{aligned}\right\} \qquad (16)$$

so that only the initial value of x is arbitrary. That is, (16) implies that $x'(0) = -e^0 = -1$ and $y(0) = 1$. As a result, we may arbitrarily prescribe $x(0)$ but unless we insist that $y(0) = 1$ and $x'(0) = -1$, the given system has no solution.

Based on the results of the exercises in this Unit together with appropriate applications of convolution,our table of Laplace transforms, given after the solution of Exercise 7.9.3, may be supplemented to include:

Function	Transform
(12) $u_a(t)f(t - a)$	$e^{-as}\,\bar{f}(s)$
(13) $\int_0^t f(t - u)g(u)\,du = \int_0^t f(u)g(t - u)\,du$	$\bar{f}(s)\bar{g}(s)$
(14) $\frac{1}{b - a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s + a)(s + b)}$
(15) $t \sin at$	$\frac{2as}{(s^2 + a^2)^2}$ $[= -\frac{d}{ds}\mathcal{L}(\sin at)]$
(16) $t \cos at$	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$
(17) $(\sin at)(\sinh at)$	$\frac{2a^2 s}{s^4 + 4a^4}$
(18) t^n $(n > -1)$	$\frac{\Gamma(n + 1)}{s^{n + 1}}$ $[= \frac{n!}{s^{n+1}}$, if n is an integer]

etc.

Quiz

1. (a) Separating variables, we have that

$$\frac{dy}{y} = 2x\,dx \tag{1}$$

provided $y \neq 0$.

Integrating (1) we have

$$\ln|y| = x^2 + C_1$$

or

$$|y| = e^{x^2 + C_1}$$

$$= e^{x^2} e^{C_1}. \tag{2}$$

Since C_1 is an arbitrary constant, e^{C_1} is an arbitrary positive constant. Hence, from (2) we conclude that

$$|y| = C_2 e^{x^2}, \text{ where } C_2 > 0.$$

Hence,

$$y = \pm C_2 e^{x^2}. \tag{3}$$

Letting $C_3 = \pm C_2$, we conclude from (3) that

$$y = C_3 e^{x^2}, \text{ where } C_3 \neq 0. \tag{4}$$

So far, the derivation of (4) required that $y \neq 0$. If $y = 0$, we see that (4) is satisfied with $C_3 = 0$. Hence, the general solution of $\frac{dy}{dx} = 2xy$ is given by

1. continued

$$y = C\ e^{x^2} \tag{5}$$

where C is an arbitrary (real) constant.

<u>Note</u>

If we let $f(x,y) = 2xy$, then both f_x and f_y are continuous. Therefore, by the fundamental theorem, one and only curve passes through a given point (x_o,y_o) and satisfies $\frac{dy}{dx} = 2xy$. This curve may be found from (5) by letting $y = y_o$ and $x = x_o$. That is,

$$y_o = C\ e^{x_o^2}$$

or

$$C = y_o\ e^{-x_o^2}.$$

Therefore,

$$y = \left(y_o\ e^{-x_o^2}\right)e^{x^2}$$

is the <u>only</u> curve which satisfies $\frac{dy}{dx} = 2xy$ and passes through (x_o,y_o). It is in this sense that (5) represents <u>the</u> general solution of $\frac{dy}{dx} = 2xy$. In other words, there is one and only one solution which passes through a given point, and this solution is a member of the family $y = C\ e^{x^2}$.

(b) We observe that the equation is linear and write it in the form

$$\frac{dy}{dx} - 2xy = e^{x^2}. \tag{6}$$

Letting $P(x) = -2x$, we see that

1. continued

$$e^{\int P(x)dx} = e^{\int -2x\,dx}$$

$$= e^{-x^2},$$

so that e^{-x^2} is an integrating factor of (6).

Multiplying both sides of (6) by e^{-x^2}, we obtain

$$e^{-x^2}\frac{dy}{dx} - 2xy\,e^{-x^2} = e^{-x^2}e^{x^2} = 1. \qquad (7)$$

We recognize the left side of (7) to be

$$\frac{d}{dx}\left(y\,e^{-x^2}\right)^*,$$

so that (7) becomes

$$\frac{d}{dx}\left(y\,e^{-x^2}\right) = 1. \qquad (8)$$

Integrating (8) yields

$$y\,e^{-x^2} = x + C,$$

or:

$$y = e^{x^2}(x + C). \qquad (9)$$

---

*Recall that whenever we use this technique, $e^{\int P(x)dx}$ is an integrating factor of $\frac{dy}{dx} + p(x)y = f(x)$; and that when we multiply both sides of this equation by $e^{\int P(x)dx}$, the left side becomes $\frac{d}{dx}\left(y\,e^{\int P(x)dx}\right)$.

1. continued

(c) The fact that not both x and y = 0, allows us to invoke the fundamental theorem to conclude that the equation does indeed have a general solution.

If we divide numerator and denominator of the right side of the equation by x^2 (by y^2 if x = 0), we obtain

$$\frac{dy}{dx} = \frac{2\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2}. \tag{10}$$

We let $v = \frac{y}{x}$ so that $y = vx$. Hence,

$$\frac{dy}{dx} = v + \frac{dv}{dx}\, x. \tag{11}$$

Replacing $\frac{dy}{dx}$ in (10) by its value in (11) and $\frac{y}{x}$ by v, we obtain from (10) that

$$v + x\,\frac{dv}{dx} = \frac{2v}{1 + v^2}$$

or

$$x\,\frac{dv}{dx} = \frac{2v}{1 + v^2} - v$$

$$= \frac{v - v^3}{1 + v^2}. \tag{12}$$

Separating variables in (12) we obtain

$$\frac{1 + v^2}{v - v^3} = \frac{dx}{x}$$

or

$$\frac{dx}{x} = \frac{1 + v^2}{v(1 + v)(1 - v)}\, dv. \tag{13}$$

1. continued

Using partial fractions, we may rewrite (13) as

$$\frac{dx}{x} = \left(\frac{1}{v} - \frac{1}{1 + v} + \frac{1}{1 - v}\right) dv. \tag{14}$$

[By way of review, the use of partial fractions requires that we look at the identity $\frac{1 + v^2}{v(1 + v)(1 - v)} \equiv \frac{A}{v} + \frac{B}{1 + v} + \frac{C}{1 - v}$. We solve for A by multiplying both sides of the identity by v and letting v = 0; for B by multiplying both sides of the identity by 1 + v and letting v = -1; and for C by multiplying both sides of the identity by 1 - v and letting v = 1.]

Integrating (14) we see that

$$\ell n|x| = \ell n|v| - \ell n|1 + v| - \ell n|1 - v| + C_1. \tag{15}$$

Then, in order to utilize logarithmic properties to simplify (15), we write C_1 as $\ell n|C|$, where C is any <u>non-zero</u> constant (i.e. $\ell n\ x$ is defined only if x > 0, and as x ranges over the positive reals, $\ell n\ x$ ranges over all the reals). We then obtain

$$\ell n|x| = \ell n|v| - \ell n|1 + v| - \ell n|1 - v| + \ell n|C|$$

$$= \ell n\left|\frac{Cv}{(1 + v)(1 - v)}\right|,$$

whence

$$|x| = \left|\frac{Cv}{(1 + v)(1 - v)}\right|,$$

or

$$x = \frac{\pm Cv}{1 - v^2}. \tag{16}$$

Since C is already an arbitrary non-zero constant (positive or negative) the ambiguous sign [where we have written our arbitrary constant as $\ell n\ C$ (C > 0) rather than as C to facilitate our logarithmic form of computation used in obtaining (15)].

1. continued

From (15) we conclude that

$$x = \frac{cv}{1 - v^2} \quad (c \neq 0),$$

and since $v = \frac{y}{x}$, we obtain

$$x = \frac{c\,\frac{y}{x}}{1 - \frac{y^2}{x^2}}$$

or

$$x = \frac{cxy}{x^2 - y^2}$$

or

$$1 = \frac{cy}{x^2 - y^2}$$

or

$$x^2 - y^2 = cy,\ c \neq 0. \qquad (17)$$

If $c = 0$, then (19) reduces to

$$y = \pm x$$

in which case

$$\frac{dy}{dx} = \pm 1$$

and

1. continued

$$\frac{2xy}{x^2 + y^2} = \frac{\pm 2x^2}{x^2 + y^2} = \pm 1 \text{ (if } x \neq 0\text{)},$$

so that (19) remains valid even if c = 0. Thus, our solution is the family (of hyperbolas)

$$x^2 - y^2 = cy.$$

2. (a) Given the 1-parameter family $f(x,y,c) = 0$, we find the envelope, if one exists, by solving the simultaneous system

$$\left.\begin{array}{l} f(x,y,c) = 0 \\ f_c(x,y,c) = 0 \end{array}\right\} . \qquad (1)$$

In this example, the fact that our family is

$$y = cx - 2c^2 \qquad (2)$$

means that

$$f(x,y,c) = y - cx + 2c^2,$$

and hence

$$f_c(x,y,c) = -x + 4c.$$

Therefore, $f_c(x,y,c) = 0 \leftrightarrow c = \frac{x}{4}$.

Letting $c = \frac{x}{4}$ in (2), we obtain

$$y = \frac{x^2}{4} - 2\left(\frac{x}{4}\right)^2,$$

or

2. continued

$$y = \frac{x^2}{8}. \tag{3}$$

Thus, $y = \frac{x^2}{8}$ is the envelope of the family of lines, $y = cx - 2c^2$.

(b) From $y = cx - 2c^2$, it follows that

$$\frac{dy}{dx} = c.$$

Replacing c by $\frac{dy}{dx}$ in (2), we obtain

$$y = x\,\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2. \tag{4}$$

[Recall that an equation of the type (4) is called a Clairant's equation.]

By its construction, we know that (4) is satisfied by the family $y = cx - 2c^2$. A trivial check shows that $y = \frac{x^2}{8}$ also satisfies (4). Namely,

$$y = \frac{x^2}{8} \rightarrow \frac{dy}{dx} = \frac{x}{4}.$$

Hence,

$$x\,\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2 = x\left(\frac{x}{4}\right) - 2\left(\frac{x}{4}\right)^2 = \frac{x^2}{8} = y.$$

(c) Notice that this equation is precisely the same as equation (4).

In order to apply our uniqueness of solution criterion, we would want to rewrite (4) in the form $\frac{dy}{dx} = g(x,y)$. To do this, we rewrite (4) as

2. continued

$$2\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$$

and use the quadratic formula to conclude

$$\frac{dy}{dx} = \frac{x \pm \sqrt{x^2 - 8y}}{4}.$$

That is,

$$\frac{dy}{dx} = \frac{x + \sqrt{x^2 - 8y}}{4} \qquad (5)$$

or

$$\frac{dy}{dx} = \frac{x - \sqrt{x^2 - 8y}}{4}. \qquad (5')$$

For either (5) or (5') we see that the equations have no (real) solution if $x^2 - 8y < 0$, i.e. if

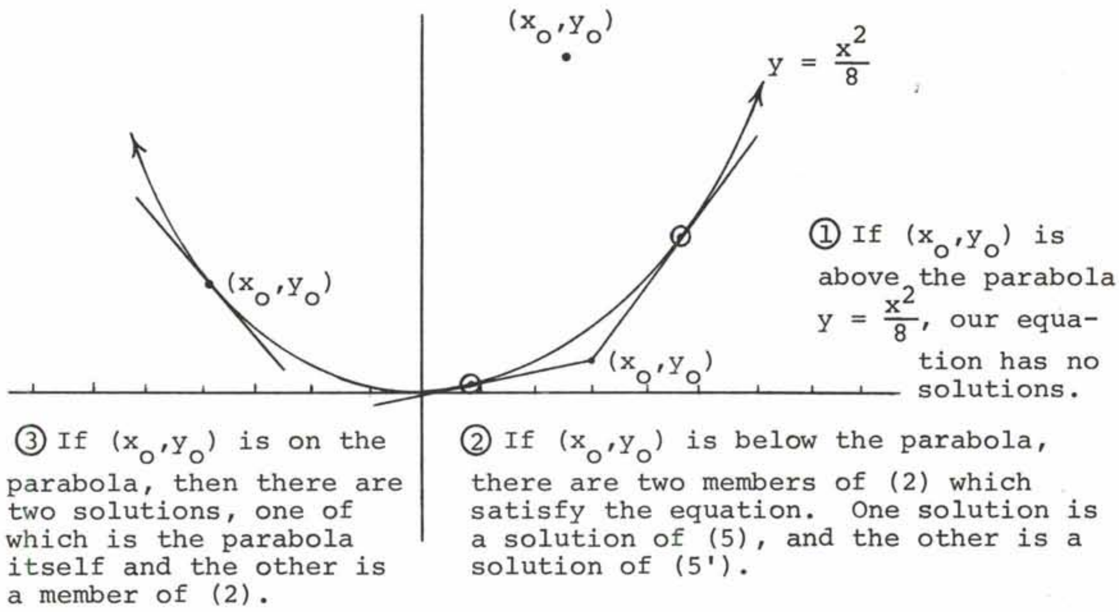
$$y > \frac{x^2}{8}.$$

If we let $g(x,y) = \frac{x \pm \sqrt{x^2 - 8y}}{4}$, we see that $g(x,y)$ is continuous if $x^2 - 8y \geqslant 0$; but $g_y(x,y)$ exists only if $x^2 - 8y > 0$ (i.e. when we compute $g_y(x,y)$, $\sqrt{x^2 - 8y}$ occurs in the denominator and this means that $x^2 - 8y$ must not equal zero). Thus, either (5) or (5') admit a general solution if and only if

$$y < \frac{x^2}{8}.$$

2. continued

Pictorially,



Applying this to the present exercise, we have that

(i) Since (4,2) implies $y = \frac{x^2}{8}$ (i.e. $y = 2$ and $\frac{x^2}{8} = 2$), we have that equation (4) is satisfied both by the parabola $y = \frac{x^2}{8}$ and the member of (2) in which c is determined by

$$2 = 4c - 2c^2$$

or

$$c = 1.$$

That is,

$$y = x - 2$$

is the other solution.

(ii) With $x = 4$ and $y = 3$, we have that $\frac{x^2}{8} = 2$, so that $y > \frac{x^2}{8}$. Consequently, our point is above the parabola $y = \frac{x^2}{8}$, and, as a result, we have no solutions in this case.

2. continued

(iii) Now we have that $y = 0$ and $\frac{x^2}{8} = 2$, so that $y < \frac{x^2}{8}$. We now have two solutions; one which satisfies (5) and the other (5'). In particular, if we let $y = 0$ and $x = 4$ in (2), we obtain

$$0 = 4c - 2c^2$$

so that $c = 0$ or $c = 2$.

In other words, our solutions are the lines

$y = 0$ (the x-axis)

and

$y = 2x - 8$.

[With reference to (5) and (5') notice that $x = 4$, $y = 0$ imply that $x^2 - 8y = 16$. Hence,

$$\frac{x + \sqrt{x^2 - 8y}}{4} = \frac{4 + 4}{4} = 2$$

while

$$\frac{x - \sqrt{x^2 - 8y}}{4} = \frac{4 - 4}{4} = 0.$$

That is, $y = 0$ is a solution of (5') and $y = 2x - 8$ is a solution of (5).]

3. Letting $y = e^{rx}$, we see that

$$y'' + 4y' - 21y = 0 \tag{1}$$

implies that

$$r^2 + 4r - 21 = 0,$$

so that

3. continued

$(r + 7)(r - 3) = 0.$

Hence, $r = -7$ or $r = 3$; whereupon the general solution of (1) is

$$y_h = c_1e^{-7x} + c_2e^{3x}. \qquad (2)$$

(a) As a trial solution of

$$y'' + 4y' - 21y = e^x \qquad (3)$$

we try

$$y_T = Ae^x. \qquad (4)$$

This leads to

$$y_T' = y_T'' = Ae^x,$$

and putting these results into (3) yields

$$Ae^x + 4Ae^x - 21Ae^x = e^x$$

or

$$-16Ae^x = e^x.$$

Therefore,

$$-16A = 1$$

or

$$A = -\frac{1}{16}.$$

Using this value of A in (4), we see that a particular solution of (3) is

3. continued

$$y_p = -\frac{1}{16} e^x. \tag{5}$$

Using (2) and (5) we have that the general solution of (2) is given by

$$y = y_h + y_p$$

or

$$y = c_1 e^{-7x} + c_2 e^{3x} - \frac{1}{16} e^x.$$

(b) Since the set of linearly independent derivatives of sin x contains only sin x and cos x, we try for a particular solution of

$$y'' + 4y' - 21y = \sin x \tag{6}$$

in the form

$$y_T = A \sin x + B \cos x. \tag{7}$$

This leads to

$$y_T' = A \cos x - B \sin x$$

and

$$y_T'' = -A \sin x - B \cos x.$$

Putting these results into (6) yields

$$(-A \sin x - B \cos x) + 4(A \cos x - B \sin x) - 21(A \sin x + B \cos x)$$

$$= \sin x$$

or

$$(-22A - 4B)\sin x + (4A - 22B)\cos x \equiv 1 \sin x + 0 \cos x.$$

3. continued

Hence,

$$\left.\begin{array}{l} -22A - 4B = 1 \\ 4A - 22B = 0 \end{array}\right\} . \tag{8}$$

Solving (8) yields

$$A = -\frac{11}{250} \text{ and } B = -\frac{1}{125},$$

so that (6) yields

$$y_p = -\frac{11}{250} \sin x - \frac{1}{125} \cos x. \tag{9}$$

Combining (2) and (9) yields the result that

$$y = c_1e^{-7x} + c_2e^{3x} - \frac{11}{250} \sin x - \frac{1}{125} \cos x$$

is the general solution of (6).

(c) From (5) and (9), we know that

$$L(-\frac{1}{16} e^x) = e^x$$

and

$$L(-\frac{11}{250} \sin x - \frac{1}{125} \cos x) = \sin x.$$

Hence,

$$\left.\begin{array}{l} 3\ L(-\frac{1}{16} e^x) = 3e^x \\ \text{and} \\ 5\ L(-\frac{11}{250} \sin x - \frac{1}{125} \cos x) = 5 \sin x \end{array}\right\} \tag{10}$$

3. continued

By linearity

$$3\ L\left(-\frac{1}{16}\,e^x\right) + 5\ L\left(-\frac{11}{250}\sin x - \frac{1}{125}\cos x\right) =$$

$$L\left(-\frac{3}{16}\,e^x\right) + L\left(-\frac{11}{50}\sin x - \frac{1}{25}\cos x\right) =$$

$$L\left(-\frac{3}{16}\,e^x - \frac{11}{50}\sin x - \frac{1}{25}\cos x\right).$$

Hence, from (10) we conclude that

$$L\left(-\frac{3}{16}\,e^x - \frac{11}{50}\sin x - \frac{1}{25}\cos x\right) = 3e^x + 5\sin x.$$

In other words,

$$y_p = -\frac{3}{16}\,e^x - \frac{11}{50}\sin x - \frac{1}{25}\cos x \qquad (11)$$

is a particular solution of

$$y'' + 4y' - 21y = 3e^x + 5\sin x. \qquad (12)$$

Therefore, the general solution of (12), from (2) and (11) is

$$y = c_1e^{-7x} + c_2e^{3x} - \frac{3}{16}\,e^x - \frac{11}{50}\sin x - \frac{1}{25}\cos x.$$

(d) The key point here is that we cannot obtain a trial solution from $y_T = Ae^{3x}$ since e^{3x} is a solution of the reduced equation (1). Since xe^{3x} is not a solution of (1) [since $\{e^{-7x}, e^{3x}, xe^{3x}\}$ is a linearly independent set], we may look for a trial solution of

$$y'' + 4y' - 21y = e^{3x} \qquad (13)$$

in the form

$$y_T = Axe^{3x}. \qquad (14)$$

3. continued

From (14) we have

$$y_T' = 3Axe^{3x} + Ae^{3x},$$

while

$$y_T'' = 9Axe^{3x} + 3Ae^{3x} + 3Ae^{3x}$$

$$= 9Axe^{3x} + 6Ae^{3x}.$$

Putting these results into (13) yields

$$9Axe^{3x} + 6Ae^{3x} + 4(3Axe^{3x} + Ae^{3x}) - 21Axe^{3x} \equiv e^{3x}$$

or

$$10Ae^{3x} \equiv e^{3x}.$$

Hence,

$$10A = 1$$

or

$$A = \frac{1}{10}.$$

From (14) we thus conclude that

$$y_p = \frac{1}{10} xe^{3x} \qquad (15)$$

is a particular solution of (13).

Again, using (2) and (15), we conclude that

$$y = c_1e^{-7x} + c_2e^{3x} + \frac{1}{10} xe^{3x}$$

is the general solution of (13).

4. Since $\frac{e^{-x}}{1 + x}$ does not have a finite set of linearly independent derivatives, we must use variation of parameters in this problem.

We begin by finding the general solution of the reduced equation,

$$y'' + 2y' + y = 0.$$

Letting $y = e^{rx}$, we see that

$$r^2 + 2r + 1 = 0$$

or

$$(r + 1)^2 = 0.$$

Thus, our only (repeated) root is $r = -1$. Accordingly,

$$y_h = c_1e^{-x} + c_2xe^{-x} \tag{1}$$

is the general solution of $y'' + 2y' + y = 0$.

To use variation of parameters, we have that

$$y = g_1(x)e^{-x} + g_2(x)xe^{-x} \tag{2}$$

is a particular solution of

$$y'' + 2y' + y = \frac{e^{-x}}{1 + x} \tag{3}$$

where

$$\left.\begin{aligned} g_1'(x)u_1(x) + g_2'(x)u_2(x) &= 0 \\ g_1'(x)u_1'(x) + g_2'(x)u_2(x) &= f(x) \end{aligned}\right\} \tag{4}$$

and

$$u_1(x) = e^{-x},\ u_2(x) = xe^{-x},\text{ and } f(x) = \frac{e^{-x}}{1 + x}. \tag{5}$$

4. continued

From (5)

$u_1'(x) = -e^{-x}$, $u_2'(x) = e^{-x} - xe^{-x}$,

so that (4) becomes

$$\left.\begin{aligned} g_1'(x)e^{-x} + g_2'(x)xe^{-x} &= 0 \\ -g_1'(x)e^{-x} + g_2'(x)[e^{-x} - xe^{-x}] &= \frac{e^{-x}}{1 + x} \end{aligned}\right\} \qquad (6)$$

Adding the two equations in (6) we obtain

$$g_2'(x)e^{-x} = \frac{e^{-x}}{1 + x}$$

or

$$g_2'(x) = \frac{1}{1 + x}. \qquad (7)$$

Using (7) in the first equation of (6) we obtain:

$$g_1'(x)e^{-x} + \frac{xe^{-x}}{1 + x} = 0,$$

so that

$$g_1'(x) = \frac{-x}{1 + x} = -1 + \frac{1}{1 + x}. \qquad (8)$$

Integrating (7) and (8) yields

$$\left.\begin{aligned} g_1(x) &= -x + \ln|1 + x| \\ g_2(x) &= \ln|1 + x| \end{aligned}\right\} \qquad (9)$$

and we have omitted constants of integration in (9) since we seek but one particular solution.

4. continued

At any rate, replacing $g_1(x)$ and $g_2(x)$ in (2) by their values in (9) yields that

$$y_p = (-x + \ln|1 + x|)e^{-x} + xe^{-x} \ln|1 + x| \tag{10}$$

is a particular solution of (3).

From (10) and (1) we have that

$$y = c_1e^{-x} + c_2xe^{-x} + e^{-x}(-x + \ln|1 + x| + x \ln|1 + x|) \tag{11}$$

is the general solution of (3).

Note

Equation (11) may be regrouped in the form

$$y = c_1e^{-x} + (c_2 - 1)xe^{-x} + e^{-x}(1 + x)\ln|1 + x|,$$

and since $c_2 - 1 = c_3$ is also arbitrary if c_2 is, we may write (11) as

$$y = c_1e^{-x} + c_3xe^{-x} + e^{-x}(1 + x)\ln|1 + x|.$$

5. Since the leading coefficient is 1 and the other coefficients ($-3x$ and -3) are analytic, the general solution of

$$y'' - 3xy' - 3y = 0 \tag{1}$$

can be found by the series technique.

We let

$$y = \sum_{n=0}^{\infty} a_nx^n$$

from which we obtain

5. continued

$$y' = \sum_{n=1}^{\infty} na_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Putting these results into (1) yields

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 3x \sum_{n=1}^{\infty} na_n x^{n-1} - 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} -3na_n x^n + \sum_{n=0}^{\infty} -3a_n x^n = 0. \qquad (2)$$

We make the exponent n in each of the three sums on the left side of (2) by rewriting

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

as

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

(i.e. we replace n by n + 2 everywhere within the summation and adjust for this by lowering the index of summation from n = 2 to n = 0).

5. continued

Equation (2) then becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} -3na_nx^n + \sum_{n=0}^{\infty} -3a_nx^n = 0. \qquad (3)$$

We finally "adjust" (3) by "splitting off" the first term in both the first and third summations on the left side of (3). This gives us a form in which each summation begins with n = 1.

In other words, we have:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = (0+2)(0+1)a_{0+2}x^0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$= 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

and

$$\sum_{n=0}^{\infty} -3a_nx^n = -3a_0 + \sum_{n=1}^{\infty} -3a_nx^n.$$

Hence, (3) becomes

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} -3na_nx^n - 3a_0 + \sum_{n=1}^{\infty} -3a_nx^n = 0$$

or

$$(2a_2 - 3a_0) + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - 3na_n - 3a_n]x^n \equiv 0.* \qquad (4)$$

*The validity of manipulating the summations in obtaining the left side of (4) hinges on the fact that we know there is a uniformly convergent power series solution, etc.

5. continued

From (4), using the fact that each coefficient on the left side must be 0 if the series is identically zero, we conclude that

$$2a_2 - 3a_0 = 0 \tag{5}$$

and that for each $n \geq 1$

$$(n+2)(n+1)a_{n+2} - 3na_n - 3a_n = 0$$

or

$$a_{n+2} = \frac{3(n+1)a_n}{(n+2)(n+1)}. \tag{6}$$

We now pick a_0 and a_1 at random, from which all other a_n's are then uniquely determined. Namely, from (5)

$$a_2 = \frac{3}{2} a_0$$

and from (6)

$$a_3 = \frac{3(2)a_1}{3(2)} = a_1$$

$$a_4 = \frac{3(3)a_2}{4(3)} = \frac{3}{4} a_2 = \frac{3}{4} (\frac{3}{2} a_0) = \frac{9}{8} a_o$$

$$a_5 = \frac{3(4)a_3}{5(4)} = \frac{3}{5} a_3 = \frac{3}{5} a_1$$

$$a_6 = \frac{3(5)a_4}{6(5)} = \frac{1}{2} a_4 = \frac{1}{2} (\frac{9}{8} a_0) = \frac{9}{16} a_0$$

$$a_7 = \frac{3(6)a_5}{7(6)} = \frac{3}{7} a_5 = \frac{3}{7} (\frac{3}{5} a_1) = \frac{9}{35} a_1, \text{ etc.}$$

Therefore,

5. continued

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$= a_0 + a_1 x + \frac{3}{2} a_0 x^2 + a_1 x^3 + \frac{9}{8} a_0 x^4 + \frac{3}{5} a_1 x^5 + \frac{9}{16} a_0 x^6 + \frac{9}{35} a_1 x^7 + \ldots$$

$$= a_0\left(1 + \frac{3}{2}x^2 + \frac{9}{8}x^4 + \frac{9}{16}x^6 + \ldots\right) + a_1\left(x + x^3 + \frac{3}{5}x^5 + \frac{9}{35}x^7 + \ldots\right). \quad (7)$$

Our initial conditions tell us that $y = 1$ when $x = 0$, so that (7) becomes

$$1 = a_0(1) + a_1(0).$$

Hence,

$$a_0 = 1. \quad (8)$$

Also from (7)

$$y' = a_0\left(0 + 3x + \frac{9}{2}x^3 + \ldots\right) + a_1\left(1 + 3x^2 + 3x^4 + \frac{9}{5}x^6 + \ldots\right) \quad (9)$$

and since we are given that $y' = 0$ when $x = 0$, we conclude from (9) that

$$0 = a_0(0 + 0 \ldots + 0 + \ldots) + a_1(1 + 0 + 0 + 0 + \ldots + 0 + \ldots)$$

or

$$a_1 = 0. \quad (10)$$

Using the results (8) and (10) in (7) we have that:

$$y = 1 + \frac{3}{2}x^2 + \frac{9}{8}x^4 + \frac{9}{16}x^6 + \ldots .$$

5. continued

Note

While it may not have seemed too obvious, we chose an example which could have been without series. Namely,

$$y'' - 3xy' - 3y = (y' - 3xy)'.$$

Hence, equation (1) may be written as

$$(y' - 3xy)' = 0$$

or

$$y' - 3xy = c_1. \qquad (11)$$

An integrating factor of (11) is

$$e^{\int -3x\,dx} = e^{-\frac{3}{2}x^2}.$$

That is, we may rewrite (11) as

$$e^{-\frac{3}{2}x^2}(y' - 3xy) = c_1 e^{-\frac{3}{2}x^2}$$

or

$$\frac{d}{dx}\left(y\, e^{-\frac{3}{2}x^2}\right) = c_1 e^{-\frac{3}{2}x^2}.$$

Hence,

$$y\, e^{-\frac{3}{2}x^2} = c_1 \int e^{-\frac{3}{2}x^2}\,dx + c_2$$

or

$$y = c_1 e^{\frac{3}{2}x^2} \int e^{-\frac{3}{2}x^2}\,dx + c_2 e^{\frac{3}{2}x^2}. \qquad (12)$$

5. continued

From (12) we obtain

$$y' = c_1 3xe^{\frac{3}{2}x^2}\int e^{-\frac{3}{2}x^2}dx + c_1 e^{\frac{3}{2}x^2}\ e^{-\frac{3}{2}x^2} + 3x\ c_2 e^{\frac{3}{2}x^2}$$

$$= 3x\ c_1 e^{\frac{3}{2}x^2}\int e^{-\frac{3}{2}x^2}dx + c_1 + 3x\ c_2 e^{\frac{3}{2}x^2}. \qquad (13)$$

Since $y' = 0$ when $x = 0$, we conclude from (13) that

$c_1 = 0$

whence (12) becomes

$$y = c_2 e^{\frac{3}{2}x^2}. \qquad (14)$$

Then, since $y = 1$ when $x = 0$, we conclude from (14) that

$$y = e^{\frac{3}{2}x^2}.$$

The point is that the previously-found solution turns out to be the series representation of $e^{\frac{3}{2}x^2}$. Thus, in this example, we can see that not only is our infinite series "meaningful" but also that it has the more "concrete" form $y = e^{\frac{3}{2}x^2}$.

6. Letting $x' = \frac{dx}{dt}$ and $x'' = \frac{d^2x}{dt^2}$, we have from

$$x'' + 2x' + 5x = 8 \sin t + 4 \cos t \qquad (1)$$

that

$$\mathcal{L}(x'' + 2x' + 5x) = \mathcal{L}(8 \sin t + 4 \cos t). \qquad (2)$$

6. continued

By linearity, (2) becomes

$$\mathcal{L}(x'') + 2\,\mathcal{L}(x') + 5\,\mathcal{L}(x) = 8\,\mathcal{L}(\sin t) + 4\,\mathcal{L}(\cos t). \qquad (3)$$

Now:

$$\mathcal{L}(\sin t) = \frac{1}{s^2 + 1} \qquad (4)$$

$$\mathcal{L}(\cos t) = \frac{s}{s^2 + 1} \qquad (5)$$

$$\mathcal{L}(x') = s\,\mathcal{L}(x) - x(0) \qquad (6)$$

$$\mathcal{L}(x'') = s^2\,\mathcal{L}(x) - s\,x(0) - x'(0) \qquad (7)$$

Since our initial conditions are $x(0) = 1$ and $x'(0) = 3$, equations (6) and (7) become

$$\mathcal{L}(x') = s\,\mathcal{L}(x) - 1 \qquad (6')$$

and

$$\mathcal{L}(x'') = s^2\,\mathcal{L}(x) - s - 3. \qquad (7')$$

Substituting (4), (5), (6'), and (7') into (3) yields

$$s^2\,\mathcal{L}(x) - s - 3 + 2s\,\mathcal{L}(x) - 2 + 5\,\mathcal{L}(x) = \frac{8}{s^2 + 1} + \frac{4s}{s^2 + 1}$$

or

$$(s^2 + 2s + 5)\,\mathcal{L}(x) = \frac{8 + 4s}{s^2 + 1} + s + 5$$

$$= \frac{s^3 + 5s^2 + 5s + 13}{s^2 + 1}.$$

Hence,

6. continued

$$\mathcal{L}(x) = \frac{s^3 + 5s^2 + 5s + 13}{(s^2 + 2s + 5)(s^2 + 1)},$$

or by using partial fractions

$$\mathcal{L}(x) = \frac{s + 3}{s^2 + 2s + 5} + \frac{2}{s^2 + 1}$$

$$= \frac{s + 3}{(s + 1)^2 + 4} + \frac{2}{s^2 + 1}. \qquad (8)$$

We notice that the tables have expressions of the form

$$\frac{s + a}{(s + a)^2 + b^2} \text{ and } \frac{a}{(s + a)^2 + b^2}.$$

More generally, we notice that if $\bar{f}(s)$ is in the tables then $\bar{f}(s + a)$ is the Laplace transform of $e^{-at}f(t)$. In this context, for example, since $\mathcal{L}(\cos 2t) = \frac{s}{s^2 + 4}$, it follows that $\mathcal{L}(e^{-t} \cos 2t) = \frac{s + 1}{(s + 1)^2 + 4}$.

With this in mind, we rewrite (8) as

$$\mathcal{L}(x) = \frac{s + 1}{(s + 1)^2 + 4} + \frac{2}{(s + 1)^2 + 4} + \frac{2}{s^2 + 1}$$

$$= \frac{s + 1}{(s + 1)^2 + 4} + \frac{2}{(s + 1)^2 + 2^2} + 2\left[\frac{1}{s^2 + 1}\right]$$

$$= \mathcal{L}(e^{-t} \cos 2t) + \mathcal{L}(e^{-t} \sin 2t) + 2\mathcal{L}(\sin t). \qquad (9)$$

By linearity, (9) may be rewritten as

$$\mathcal{L}(x) = \mathcal{L}(e^{-t} \cos 2t + e^{-t} \sin 2t + 2 \sin t). \qquad (10)$$

Finally, since $\mathcal{L}$ is 1-1 (Lerch's Theorem), we conclude from (10) that

6. continued

$x = e^{-t} \cos 2t + e^{-t} \sin 2t + 2 \sin t.$