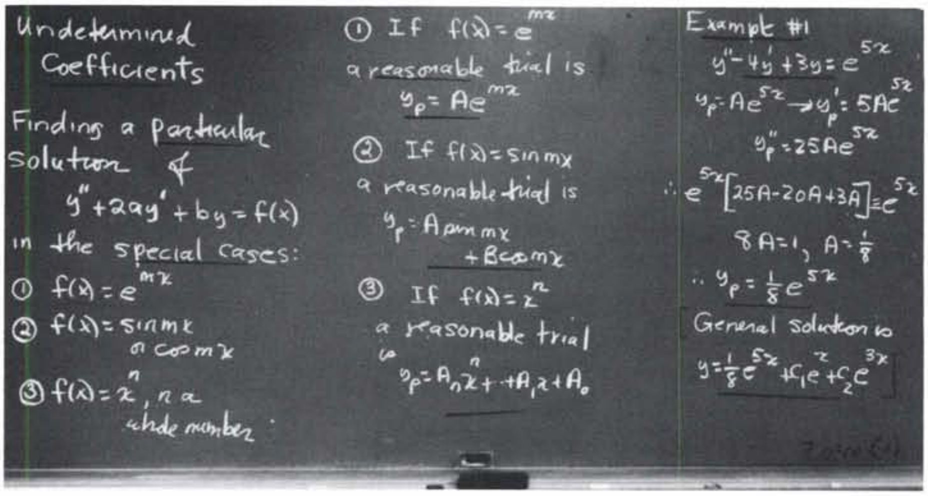
Unit 6: The Method of Undetermined Coefficients

1. Overview

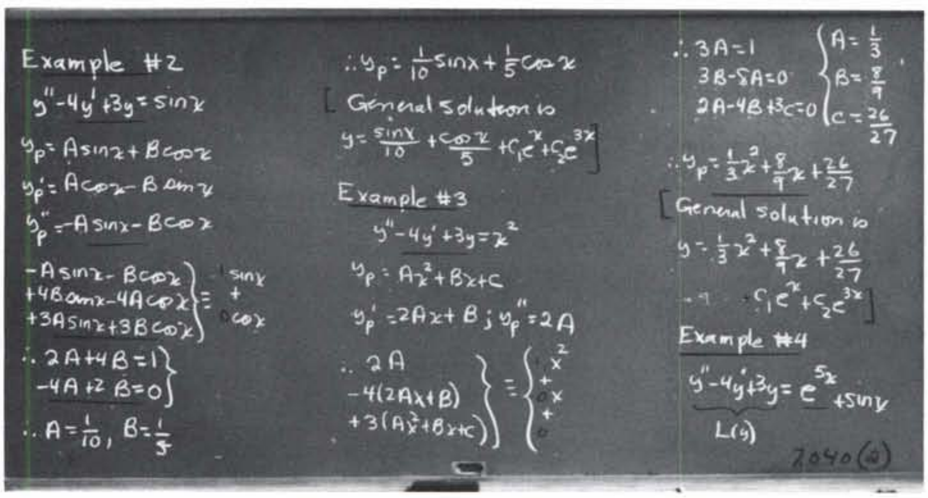
We now know how to find the general solution of $L(y) = 0$ in the case that $L(y)$ has constant coefficients. Thus, in terms of the general theory, if $L(y) = f(x)$ has constant coefficients and $f \neq 0$, then we need only find a particular solution of this equation in order to obtain the general solution of the equation (since the general solution is obtained by adding the particular solution to the general solution of the reduced equation).

The purpose of this unit is to present a rather nice technique for finding a particular solution - a technique which only applies if (1) we have constant coefficients and (2) $f(x)$ has a very special form. What this special form is will be discussed as part of the exercises, but a general insight will be supplied in the lecture.

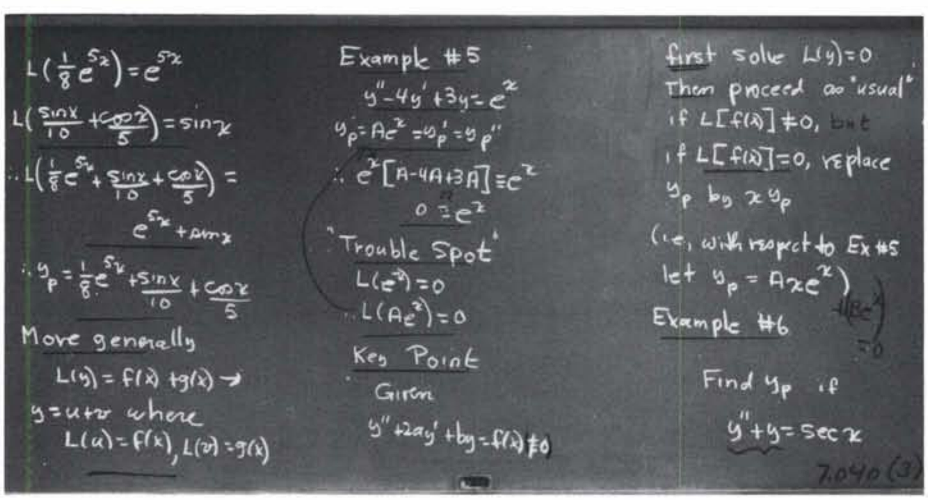
2. Lecture 2.040



a.



b.



c.

3. Do the Exercises. [Exercise 2.6.1(L) sets the tone for the entire set of exercises. Its many parts are meant to highlight the various finepoints which arise in the study of the method of undetermined coefficients. It is strongly advised that you work this exercise in great detail since it emphasizes the various techniques which may occur in different cases.]

4. (Optional) Read: Thomas, Section 20.12.
This section may be omitted without disruption of our central theme. The beauty of the material discussed in the section, however, is important from two points of view. First of all, it presents some practical applications of second order linear differential equations; and for this reason alone may be interesting to many students. The other reason is that these applications involve equations with constant coefficients. Even if you elect not to study this section too diligently, it is probably a good idea to skim it. The point is that for the past few units we have been concentrating on equations with constant coefficients even though the more general case involves variable coefficients. This section supplies some proof to the assertion that the special case of constant coefficients has wide-spread application.

5. Exercises:

2.6.1(L)

a. 1. Let $f(x) = e^{3x}$. Show that every derivative $f^{(n)}(x)$, is a (constant) multiple of $f(x)$.

2. Let $f(x) = xe^{3x}$. Show that every derivative $f^{(n)}(x)$, is a linear combination of $f(x)$ and $f'(x)$. That is, for each n there exist constants a_n and b_n such that $f^{(n)}(x) = a_n f(x) + b_n f'(x)$.

b. Use the method of undetermined coefficients to find a particular solution of $y'' - 8y' + 7y = e^{3x}$. Then write down the general solution of this equation.

c. Use the method of undetermined coefficients to help find the general solution of $y'' - 8y' + 7y = xe^{3x}$.

d. Use the method of undetermined coefficients to help find the general solution of $y'' - 8y' + 7y = e^{7x}$.

2.6.2 (optional)

This exercise explains why the method of undetermined coefficients requires that the right side of the equation be a linear combination of terms of the form $x^k e^{\alpha x}\sin \beta x$ and $x^k e^{\alpha x}\cos \beta x$. The exercise is not too difficult to solve and the solution is not very lengthy so it may be worthwhile to at least glance at the solution.

Suppose f(x) is analytic (recall that this means that f and all its derivatives exist) and that $S = \{f, f', f'', \ldots f^{(n)}, \ldots\}$. Suppose it is also known that there exists a finite subset T of S such that each element in S is a linear combination of the elements in T. In plainer English, we are supposing that every derivative of f is a linear combination of the first m derivatives for a suitably chosen m. Describe f.

2.6.3

Find the general solution of $y'' - 6y' + 9y = f(x)$ in each of the following cases.

a. $f(x) = e^{4x}$

b. $f(x) = \sin 3x$

c. $f(x) = xe^x$

d. $f(x) = e^{3x}$

2.6.4

Use the superposition principle (i.e., the fact that L is linear) and the answers to (a) and (b) of the previous exercise to find the general solution of $y'' - 6y' + 9y = 3e^{4x} + \sin 3x$.

2.6.5

Find the general solution of $y'' + 3y' + 2y = x^2e^x$.

2.6.6

Find the solution of $y''' - y' = e^x$ subject to the conditions that when $x=0$; $y=1$, $y'=3/2$, and $y''=4$.

2.6.7(L)

a. Make the substitution $z = \ln x$ to convert the equation

$$x^3y''' + xy' - y = 0 \quad (x > 0)$$

into a linear equation with constant coefficients in which y is expressed as a function of z.

b. Use your answer in (a) to determine the general solution of

$$x^3y''' + xy' - y = 0 \ (x > 0).$$

c. Apply undetermined coefficients to the proper equation to help find the general solution of

$$x^3y''' + xy' - y = x \ln x \ (x > 0) .$$