Unit 8: Fourier Series

3.8.1(L)

The fact that sin mx cos nx, for all choices of the constants m and n, is an <u>odd</u>* function guarantees that $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$. Thus, each member of the sine family is orthogonal to each member of the cosine family.

To show that each member of the sine family is orthogonal to every <u>other</u> member of the sine family, we must show that $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0$ <u>whenever $n \neq m$</u> (when n = m, we are "dotting" sin nx with itself).

If we recall the trigonometric identities

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$

then we see at once that

$$\cos (A - B) - \cos (A + B) = 2 \sin A \sin B, \text{ or}$$

$$\sin A \sin B = \frac{1}{2} [\cos (A - B) - \cos (A + B)]. \quad (1)$$

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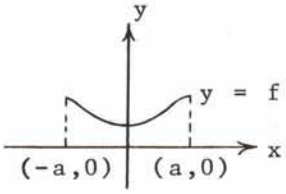
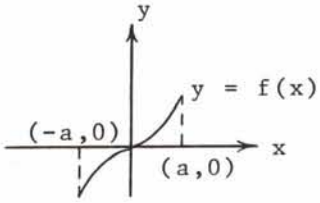
*Recall that f(x) is odd means f(x) = -f(-x) while f(x) is even means that f(x) = f(-x). In terms of area, it is easy to see that if f(x) is even then $\int_{-a}^{a} f(x)dx = 2 \int_{0}^{a} f(x)\,dx$ while if f(x) is odd, then $\int_{-a}^{a} f(x)dx = 0$, i.e.



f(x) even



f(x) odd

3.8.1(L) continued

Similarly,

$$\cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]. \tag{2}$$

We may now use (1) with A = mx and B = nx to obtain

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx^* = \int_{-\pi}^{\pi} \frac{1}{2} [\cos(mx - nx) - \cos(mx + nx)]dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m - n)x - \cos(m + n)x]dx.$$

Hence

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{\sin(m - n)x}{2(m - n)} \Bigg]_{x=-\pi}^{\pi} - \frac{\sin(m + n)x}{m + n} \Bigg]_{x=-\pi}^{\pi}. \tag{3}$$

Since m and n are integers, so also are m + n and m - n; and since the sine of all integral multiples of $\pm\pi$ is 0, we conclude that the right side of (3) is 0.

Hence, for $m \neq n$,

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0. \tag{4}$$

Note #1

Notice that the first term on the right side of (3) contains m - n as a factor of the denominator. Hence, since division by zero is "taboo," equation (3) is invalid when m = n. Since (4) is prefaced by $m \neq n$, this does not affect our equation (4). If we let m = n, then

*Since sin mx sin nx is an even function [i.e. sin m(-x) sin n(-x) = sin(-mx) sin(-nx) = (-sin mx)(-sin nx) = sin mx sin nx] you may replace $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx$ by $2\int_{0}^{\pi} \sin mx \sin nx \, dx$, if you prefer.

3.8.1(L) continued

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \int_{-\pi}^{\pi} \sin^2 nx \, dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2nx}{2} \, dx$$

$$= \frac{x}{2}\Big]_{-\pi}^{\pi} - \frac{\sin 2nx}{4n}\Big]_{-\pi}^{\pi} = \frac{\pi}{2} - \left(\frac{-\pi}{2}\right) + 0 = \pi. \qquad (5)$$

Thus, for the sake of completeness, we have

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases} \qquad (6)$$

Note #2

The fact that n appears as a factor of the denominator in one term on the right side of (5) warns us to beware of the validity of (5) when $n = 0$. Now it turns out that $n = 0$ is of no concern to us here since $\sin nx = 0$ when $n = 0$. Indeed, this is why the family $\{\sin nx\}$ begins with $n = 1$ rather than $n = 0$. On the other hand, when $n = 0$, $\cos nx = 1$. Hence, when we consider later the situation $\int_{-\pi}^{\pi} \cos^2 nx \, dx$, it may be wise to remember $n = 0$ as a special case.

Note #3

Some authors, because they prefer orthonormal to orthogonal, introduce the weighting factor $\frac{1}{\sqrt{\pi}}$ and replace $\sin nx$ by $\frac{1}{\sqrt{\pi}} \sin nx$. The reason for this is contained in equation (5). Namely, the factor $\frac{1}{\sqrt{\pi}}$ does not alter the orthogonality of the sine family, but it does make the dot product of any member of the sine family with itself equal to 1. That is, from (5), we conclude that

$$\int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin mx \frac{1}{\sqrt{\pi}} \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin mx \sin ndx \, dx$$

$$= \begin{cases} \frac{1}{\pi}\, 0 = 0, & \text{if } m \neq n \\ \frac{1}{\pi}\, \pi = 1, & \text{if } m = n \end{cases}$$

3.8.1(L) continued

For our purposes, we prefer to stress the orthogonal property rather than to introduce $\frac{1}{\sqrt{\pi}}$ as a weighting factor. As we shall see, this has no bearing on the general techniques which we employ in this unit.

All that remains now is to look at

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx \text{ when } m \neq n.$$

From equation (2), we have

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m + n)x + \cos(m - n)x]dx$$

$$= \frac{\sin(m + n)x}{2(m + n)}\Bigg|_{x=-\pi}^{\pi} + \frac{\sin(m - n)x}{2(m - n)}\Bigg|_{x=-\pi}^{\pi}. \qquad (7)$$

Since $\pm\sin(m \pm n)\pi = 0$, we conclude from (7) that

$$n \neq m \rightarrow \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0. \qquad (8)$$

All that remains to tie together any loose ends is to compute $\int_{-\pi}^{\pi} \cos^2 nx \, dx$ (i.e. the case $m = n$), even though this is not asked for explicitly in this exercise.

Since $\cos^2 nx = \frac{1 + \cos 2nx}{2}$, we have that

$$\int_{-\pi}^{\pi} \cos^2 nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos 2nx)dx$$

$$= \frac{1}{2} [x - \frac{\sin 2nx}{2n}\Bigg|_{x=-\pi}^{\pi} \qquad (9)$$

$$= \pi,$$

3.8.1(L) continued

except (review Note #2) that equation (9) is not valid when n = 0. In the special case n = 0, we have

$$\int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} dx = 2\pi.$$

In summary then, for any whole numbers, m and n

$$\int_{-\pi}^{\pi} \sin^2 nx\, dx = \begin{cases} 0, & \text{if } n = 0 \\ \pi, & \text{if } n \neq 0 \end{cases} \tag{10}$$

$$\int_{-\pi}^{\pi} \cos^2 nx\, dx = \begin{cases} 2\pi, & \text{if } n = 0 \\ \pi, & \text{if } n \neq 0 \end{cases} \tag{11}$$

$$\left.\begin{array}{l} \displaystyle\int_{-\pi}^{\pi} \sin nx \cos mx\, dx = 0 \quad \text{(even if } n = m) \\ \\ \displaystyle\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \int_{-\pi}^{\pi} \cos nx \cos mx\, dx = 0, \text{ when } n \neq m \end{array}\right\} \tag{12}$$

In particular, if f·g means $\int_{-\pi}^{\pi} f(x)g(x)dx$, then the family {1,cos x,cos 2x,...,cos nx,...,sin x,sin 2x,...,sin nx,...} is orthogonal. If we want an orthonormal family, then we see that from (10) and (11), our family of functions should be

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \ldots, \frac{1}{\sqrt{\pi}} \cos nx, \ldots, \frac{1}{\sqrt{\pi}} \sin x, \ldots, \frac{1}{\sqrt{\pi}} \sin nx, \ldots\right\}.$$

3.8.2(L)

What we have is the assumption that

$$f(x) = a_o + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \tag{1}$$

3.8.2(L) continued

Since $1 \cdot \cos nx = 1 \cdot \sin nx = 0$ for $n > 0$ (where $1 \cdot \cos nx$ means $\int_{-\pi}^{\pi} 1 \cos nx\, dx$ etc.) we may use this orthogonal property to find a_0. Namely, we integrate both sides of (1) from $-\pi$ to π, to obtain

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\right]dx. \tag{2}$$

Assuming that we are permitted to interchange the order of integration and summation (which, as we know for infinite sums, is not always valid), equation (2) yields

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} \underbrace{\int_{-\pi}^{\pi} a_n \cos nx\, dx}_{=\,0} + \sum_{n=1}^{\infty} \underbrace{\int_{-\pi}^{\pi} b_n \sin nx\, dx}_{=\,0}$$

Hence

$$\int_{-\pi}^{\pi} f(x)dx = 2\pi a_0,$$

or,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx. \tag{3}$$

To find a_k for $k \neq 0$, we multiply both sides of (1) by $\cos kx$ and integrate from $-\pi$ to π, to obtain

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos kx\, dx &= \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\right]\cos kx\, dx \\ &= \int_{-\pi}^{\pi} a_0 \cos kx\, dx + \int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx \cos kx\right)dx + \\ &\quad + \int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx \cos kx\right)dx. \end{aligned} \tag{4}$$

3.8.2(L) continued

Again, under the assumption that we may interchange the order of the infinite sum and the integration, we deduce from (4) that

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = a_o \underbrace{\int_{-\pi}^{\pi} \cos kx\,dx}_{=\,0} + \sum_{n=1}^{\infty} a_n \underbrace{\left(\int_{-\pi}^{\pi} \cos nx \cos kx\,dx\right)}_{\substack{0,\text{ when } n \neq k \\ \pi,\text{ when } n = k}}$$

$$+ \sum_{n=1}^{\infty} b_n \underbrace{\left(\int_{-\pi}^{\pi} \sin nx \cos kx\,dx\right)}_{0,\text{ for all } n}$$

Hence,

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = 0 + a_k\pi + 0 + 0 = \pi a_k.$$

Therefore, for $k = 1,2,3,\ldots$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos kx\,dx. \qquad (5)$$

To find b_k, we multiply both sides of (1) by $\sin kx$ and integrate from $-\pi$ to π under the assumption that we are permitted to interchange the order of the infinite sum and the integration. This yields

$$\int_{-\pi}^{\pi} f(x)\sin kx\,dx = a_o \underbrace{\int_{-\pi}^{\pi} \sin kx\,dx}_{=\,0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi} \cos nx \sin kx\,dx}_{=\,0}$$

$$+ \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi} \sin nx \sin kx\,dx}_{\substack{0,\text{ if } n \neq k \\ \pi,\text{ if } n = k}}$$

3.8.2(L) continued

Hence,

$$\int_{-\pi}^{\pi} f(x)\sin kx\,dx = b_k\pi,$$

or

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx. \tag{6}$$

Note #1

We used k rather than n simply to avoid confusing a particular term with the general term. That is, we did not want n to have two different meanings in the same problem. With this in mind, (5) and (6) may be rewritten as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (n \neq 0) \tag{5'}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad (n = 1,2,3,\ldots) \tag{6'}$$

Note #2

Notice that we do not say that

$$\begin{aligned} f(x) &= a_o + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} f(x)\cos nx\,dx\right)\cos nx + \\ &\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right)\sin nx. \end{aligned} \tag{7}$$

3.8.2(L) continued

Rather it is F(x) which is defined by the right side of (7). In other words, the right side of (7) was derived under the not-always-valid assumption that we were justified in interchanging the order of the integration and the infinite sum. In summary, then

$$f(x) \sim \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-\pi}^{\pi} f(x)\cos nx\, dx\right)\cos nx + \left(\int_{-\pi}^{\pi} f(x)\sin nx\, dx\right)\sin nx\right]}_{= F(x);}$$

and F(x) is called the Fourier representation of f(x).

3.8.3(L)

In the previous exercise, we had you compute the Fourier coefficients of a function f(x), pointing out that the Fourier series need not actually equal the function. At the same time, recall that we pointed out in our lecture that if f(x) is piecewise-smooth on $[-\pi,\pi]$, then the Fourier series, F(x), of f(x) converges to f(x) everywhere in the interval where f(x) is continuous and to the average of the jump wherever f(x) has a jump-discontinuity.

In this exercise (as well as the next two), we continue our drill in deriving Fourier coefficients, but at the same time, we choose f(x) to be piecewise-smooth so that we can see how the Fourier series actually converges to the function in this case.

a. Given that

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases} \qquad (1)$$

3.8.3(L) continued

we have that

(i) $$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx^{*} \qquad (2)$$

$$= \int_{-\pi}^{0} -1\, dx + \int_{0}^{\pi} 1\, dx$$

$$= \int_{0}^{-\pi} dx + \int_{0}^{\pi} dx$$

$$= -\pi + \pi$$

$$= 0. \qquad (3)$$

(ii) $$\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \int_{-\pi}^{0} -\cos nx\, dx + \int_{0}^{\pi} \cos nx\, dx \quad (n \neq 0)$$

$$= \int_{0}^{-\pi} \cos nx\, dx + \int_{0}^{\pi} \cos nx\, dx$$

$$= \frac{\sin nx}{n}\Bigg|_{x=0}^{-\pi} + \frac{\sin nx}{n}\Bigg|_{x=0}^{\pi}$$

$$= 0. \qquad (4)$$

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*We rewrite $\int_{-\pi}^{\pi} f(x)dx$ in the indicated way as the sum of two integrals to take into account that f behaves one way on $0 < x < \pi$ and another way on $-\pi < x < 0$.

3.8.3(L) continued

(iii) $$\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \int_{-\pi}^{0} -\sin nx\, dx + \int_{0}^{\pi} \sin nx\, dx \quad (n \neq 0)$$

$$= \int_{0}^{-\pi} \sin nx\, dx + \int_{0}^{\pi} \sin nx\, dx$$

$$= \frac{-\cos nx}{n}\Bigg]_{x=0}^{-\pi} - \frac{\cos nx}{n}\Bigg|_{x=0}^{\pi}$$

$$= \left(\frac{-\cos(-\pi n)}{n} - \left[\frac{-\cos 0}{n}\right]\right) - \left(\frac{\cos \pi n}{n} - \frac{\cos 0}{n}\right)$$

$$= \frac{-\cos \pi n}{n} + \frac{1}{n} - \frac{\cos \pi n}{n} + \frac{1}{n}$$

$$= \frac{2}{n} - \frac{2\cos \pi n}{n}. \qquad (5)$$

We observe that for even values of n, $\cos \pi n = 1$ while for odd values of n, $\cos \pi n = -1$. Hence,

$$\frac{2}{n} - \frac{2\cos \pi n}{n} = \begin{cases} \frac{2}{n} - \frac{2}{n}, \text{ when n is even} \\ \frac{2}{n} - \left(\frac{-2}{n}\right) = \frac{4}{n}, \text{ when n is odd} \end{cases}$$

Hence, we conclude from (5) that

$$\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \begin{cases} 0, \text{ if n is even} \\ \frac{4}{n}, \text{ if n is odd} \end{cases} \qquad (6)$$

Now using the results of our previous exercise, we have that the Fourier series representation of f(x) is given by

$$F(x) = a_o + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad (7)$$

3.8.3(L) continued

where

$$a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx \quad (n \neq 0)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx.$$

Thus, from (3), (4), and (6), we conclude that

$$a_o = \frac{1}{2\pi}(0) = 0$$

$$a_n = \frac{1}{\pi}(0) = 0 \ (n \neq 0)$$

and

$$b_n = \begin{cases} \frac{1}{\pi}\, 0 = 0, \text{ if n is even} \\ \frac{1}{\pi}\left(\frac{4}{n}\right) = \frac{4}{\pi n}, \text{ if n is odd} \end{cases}$$

Hence, (7) yields

$$F(x) = 0 + 0 + \sum_{n\ odd} \frac{4}{\pi n}\sin nx \tag{8}$$

$$= \frac{4}{\pi}\sum_{n\ odd}\frac{\sin nx}{n} \tag{8.1}$$

$$= \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin(2n+1)x}{2n+1} + \ldots\right) \tag{8.2}$$

$$= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}. \tag{8.3}$$

3.8.3(L) continued

Note #1

Notice that our Fourier series contains only sine terms. This happened because f(x) in this exercise is an odd function.

In other words, since the product of an odd function and an even function is odd, f(x) cos nx is odd because f(x) is odd while cos nx is even.

Consequently, $\int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0$.

In a similar way, $\int_{-\pi}^{\pi} f(x)\sin nx \, dx = 0$ when f(x) is even. Thus, the Fourier series representation of f(x) will be purely a sine series when f(x) is odd; and purely a cosine series when f(x) is even. In fact, in deriving equations (3), (4), and (6), we could have telescoped a few steps by recognizing at once that since f was odd, $\int_{-\pi}^{\pi} f(x)\cos nx \, dx = 0$, while $\int_{-\pi}^{\pi} f(x) \sin nx \, dx =$

$2\int_{0}^{\pi} f(x)\sin nx \, dx$.

Note #2

Since $\frac{f(x) + f(-x)}{2}$ is always an even function and $\frac{f(x) - f(-x)}{2}$ is always an odd function, the identity

$$f(x) = \underbrace{\left[\frac{f(x) + f(-x)}{2}\right]}_{\text{even}} + \underbrace{\left[\frac{f(x) - f(-x)}{2}\right]}_{\text{odd}}$$

tells us that we may always decompose the construction of a Fourier series into two problems, one involving a sine series and the other, a cosine series.

b. Notice that our general theory tells us that at any x between $-\pi$ and π for which f is continuous $F(x) = f(x)$. In particular, $x = \frac{\pi}{2}$ is such a point. Now, by definition of f, when $x = \frac{\pi}{2}$, $f(x) = 1$ (since $0 < \frac{\pi}{2} < \pi$). Hence, $F(\frac{\pi}{2}) = 1$, and we conclude from equation (8.3) that

3.8.3(L) continued

$$1 = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\frac{\pi}{2}}{2n+1}. \tag{9}$$

Now odd multiples of $\frac{\pi}{2}$ have the property that their sine is either 1 or -1. More specifically

$$\sin\frac{\pi}{2} = \sin\frac{5\pi}{2} = \sin\frac{9\pi}{2} = \ldots = 1$$

while

$$\sin\frac{3\pi}{2} = \sin\frac{7\pi}{2} = \sin\frac{11\pi}{2} = \ldots = -1.$$

More succinctly,

$$\sin(2n+1)\frac{\pi}{2} = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

Hence, (9) may be rewritten as

$$1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right),$$

from which it follows that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \frac{\pi}{4}. \tag{10}$$

<u>Note #3</u>

There are certainly other ways of deducing that $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4}$. Indeed, the beauty of (10), at least from one point of view, is that there are methods of deriving the result without reference to Fourier series. Consequently, (10) serves as a reinforcement of the theory which states that $f(x) = F(x)$ wherever f is continuous.

3.8.3(L) continued

Note #4

In this same context, notice that $f(0^-) = -1$ and $f(0^+) = 1$. Hence, the general theory predicts that

$$F(0) = \frac{f(0^+) + f(0^-)}{2} = \frac{1 + (-1)}{2} = 0.$$

As a check of this result, we see that with $x = 0$ in equation (8) [or (8.1),(8.2),(8.3)], we obtain

$$F(0) = 0.$$

Note #5

To get a better idea of what we mean by the Fourier series converging to f(x) "in the large" it may be helpful to graph

$$y = \frac{4}{\pi}\sum_{n=0}^{k} \frac{\sin(2n+1)x}{2n+1} \qquad (11)$$

for $k = 0$ and $k = 1$ ($k = 1$ and $k = 3$ were sketched in the lecture). [Notice that (11) represents the kth partial sum of the series defined by equation (8.3).]

When $k = 0$, we have

$$y = \frac{4}{\pi}\sin x,$$

from which we have

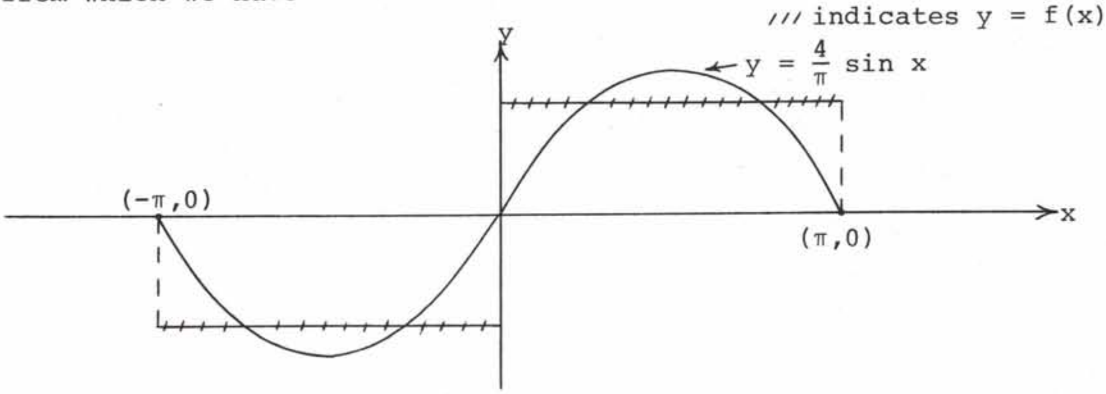


Figure 1

3.8.3(L) continued

Observe that $y = \frac{4}{\pi} \sin x$ does not look too much like $y = f(x)$. In terms of our remark in the lecture (proved in Exercise 3.8.6) about least square approximations what is true is that if we let

$$g(x) = a_o + a_1 \cos x + b_1 \sin x$$

then

$$\int_{-\pi}^{\pi} [f(x) - g(x)]^2 dx$$

is minimum when $a_o = a_1 = 0$ and $b_1 = \frac{4}{\pi}$.

With $k = 1$, we have

$$y = \frac{4}{\pi} [\sin x + \frac{1}{3} \sin 3x]. \tag{12}$$

To graph (12), we have

$$y' = \frac{4}{\pi} [\cos x + \cos 3x] \tag{13}$$

$$y'' = \frac{4}{\pi} [- \sin x - 3 \sin 3x]. \tag{14}$$

A method for determining sin 3x and cos 3x in terms of sin x and cos x is given by DeMoivre's Theorem. Namely,

$$(\cos x + i \sin x)^3 = \cos 3x + i \sin 3x, \tag{15}$$

from which we obtain

$$\cos^3 x + 3 \cos^2 x (i \sin x) + 3 \cos x (i \sin x)^2 + (i \sin x)^3 =$$

$$\cos 3x + i \sin 3x.$$

Hence,

$$(\cos^3 x - 3 \cos x \sin^2 x) + i(3 \cos^2 x \sin x - \sin^3 x) =$$

$$\cos 3x + i \sin 3x. \tag{16}$$

3.8.3(L) continued

Equating the real parts in (16) and equating the imaginary part, we conclude that

$$\cos 3x = \cos^3 x - 3 \cos x \sin^2 x$$

and

$$\sin 3x = 3 \cos^2 x \sin x - \sin^3 x.$$

Putting these results into (12), (13), and (14), we obtain

$$y = \frac{4}{\pi} [\sin x + \cos^2 x \sin x - \frac{1}{3} \sin^3 x]$$

$$= \frac{4}{\pi} [\sin x + (1 - \sin^2 x)\sin x - \frac{1}{3} \sin^3 x]$$

$$= \frac{4}{\pi} [2 \sin x - \frac{4}{3} \sin^3 x] \qquad (17)$$

$$y' = \frac{4}{\pi} [\cos x + \cos^3 x - 3 \cos x \sin^2 x]$$

$$= \frac{4}{\pi} [\cos x + \cos^3 x - 3 \cos x (1 - \cos^2 x)]$$

$$= \frac{4}{\pi} [4 \cos^3 x - 2 \cos x] \qquad (18)$$

etc.

From (18), we conclude that

$$y' = 0 \leftrightarrow 4 \cos^3 x - 2 \cos x = 0$$

$$\leftrightarrow 2 \cos x (2 \cos^2 x - 1) = 0$$

$$\leftrightarrow \cos x = 0 \quad \text{or} \quad \cos x = \pm\sqrt{\frac{1}{2}}.$$

When $\cos x = 0$, $\sin x = \pm 1$; so that (17) yields

3.8.3(L) continued

$$y = \frac{4}{\pi}\left[\pm 2 \mp \frac{4}{3}\right]$$

$$= \frac{4}{\pi}\left(2 - \frac{4}{3}\right) \quad \text{or} \quad \frac{4}{\pi}\left(-2 + \frac{4}{3}\right)$$

$$= \frac{\pm 4}{\pi}\left(\frac{2}{3}\right)$$

$$= \frac{\pm 8}{3\pi} \approx 0.85$$

When $\cos x = \pm\sqrt{\frac{1}{2}} = \frac{\pm 1}{\sqrt{2}}$, $\sin x = \frac{\pm 1}{\sqrt{2}}$. Hence, (17) yields

$$y = \frac{4}{\pi}\left[2\left(\frac{\pm 1}{\sqrt{2}}\right) - \frac{4}{3}\left(\frac{\pm 1}{\sqrt{2}}\right)^3\right]$$

$$= \pm\frac{4}{\pi}\left(\frac{2}{\sqrt{2}} - \frac{2}{3\sqrt{2}}\right)$$

$$= \pm\frac{4}{\pi}\left(\frac{6 - 2}{3\sqrt{2}}\right)$$

$$= \pm\frac{16}{3\pi\sqrt{2}}$$

$$= \pm\frac{8\sqrt{2}}{3\pi} \approx \pm 1.2.$$

Pictorially,

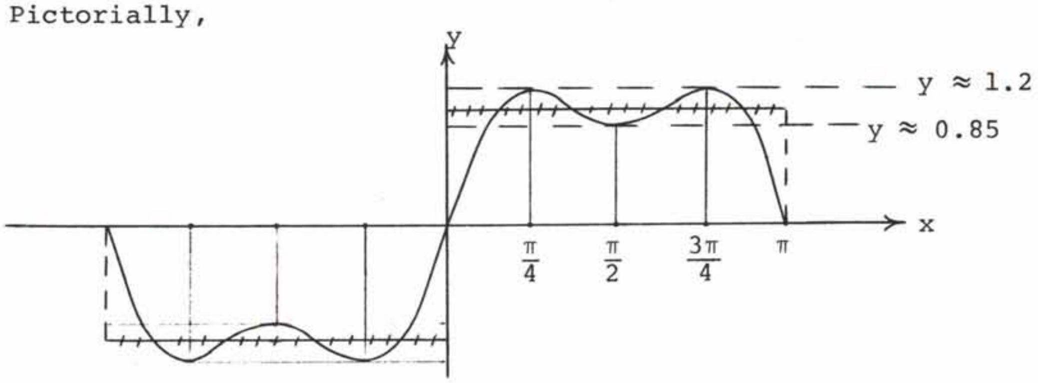


Figure 2

3.8.3(L) continued

Again, in terms of least mean squares, if

$$h(x) = a_o + a_1\cos x + a_2\cos 2x + a_3\cos 3x + b_1\sin x + b_2\sin 2x + b_3\sin 3x$$

then

$$\int_{-\pi}^{\pi} [f(x) - h(x)]^2 dx$$

is minimized when $a_o = a_1 = a_2 = a_3 = b_2 = 0$; $b_1 = \frac{4}{\pi}$ and $b_3 = \frac{4}{3\pi}$ [i.e. these are the values of the Fourier coefficients as determined from (12)].

Note #6

$$F(x) = \lim_{k\to\infty} \frac{4}{\pi} \sum_{n=0}^{k} \frac{\sin(2n + 1)x}{2n + 1}$$

so that by the major theorem, $y = F(x)$ is the same curve as $y = f(x)$, except that $F(0) = 0$ while $f(0)$ is undefined.

Again, pictorially,

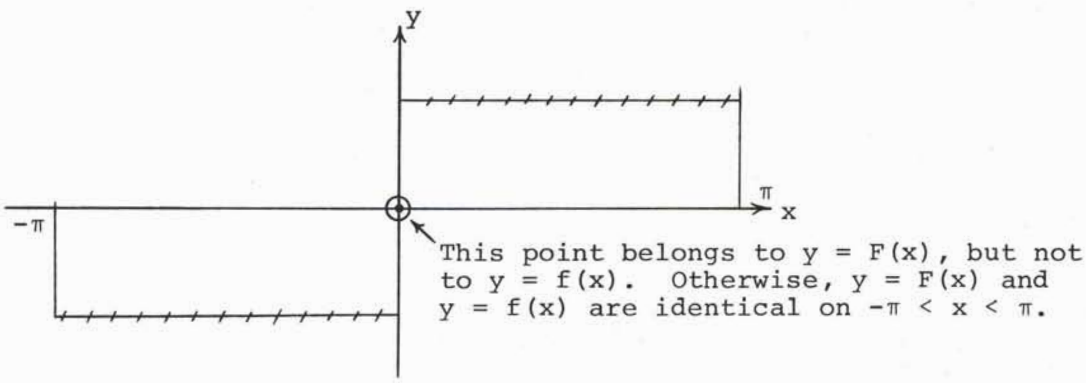


Figure 3

Note #7

By its very definition, $F(x)$ is periodic of period 2π since cos nx and sin nx are each periodic of period 2π. Thus, the graph

3.8.3(L) continued

$y = F(x)$ with F as defined in equation (8.3) is defined on the entire x-axis, not just on the interval $-\pi < x < \pi$. For this reason we often view $f(x)$ as being extended periodically, but this point is usually clear from context.

3.8.4

a. $f(x) = |x|$ for $-\pi \leqslant x \leqslant \pi$ means that

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant \pi \\ -x, & -\pi \leqslant x < 0 \end{cases}$$

Hence,

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} -xdx + \int_{0}^{\pi} xdx$$

$$= 2\int_{0}^{\pi} xdx$$

$$= x^2\Big]_0^{\pi}$$

$$= \pi^2.$$

Consequently,

$$a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{\pi}{2}. \qquad (1)$$

Similarly, since $f(x)$ is even,

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx \quad (n \neq 0)$$

3.8.4 continued

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx. \tag{2}$$

Using integration by parts (or tables, etc.) let $u = x$, $dv = \cos nx\, dx$. Then $du = dx$ and $v = \frac{\sin nx}{n}$. Hence,

$$\int_0^{\pi} x \cos nx\, dx = \frac{x \sin nx}{n}\Bigg|_{x=0}^{\pi} - \frac{1}{n}\int_0^{\pi} \sin nx\, dx$$

$$= 0 + \frac{\cos nx}{n^2}\Bigg|_{x=0}^{\pi}$$

$$= \frac{\cos \pi n}{n^2} - \frac{1}{n^2}. \tag{3}$$

Now when n is even, $\cos \pi n = 1$; while when n is odd, $\cos \pi n = -1$. Hence,

$$\frac{\cos \pi n}{n^2} - \frac{1}{n^2} = \begin{cases} \frac{1}{n^2} - \frac{1}{n^2} = 0, \text{ when n is even} \\ -\frac{1}{n^2} - \frac{1}{n^2} = \frac{-2}{n^2}, \text{ when n is odd} \end{cases}$$

Therefore,

$$\int_0^{\pi} x \cos nx\, dx = \begin{cases} 0, \text{ n even} \\ \frac{-2}{n^2}, \text{ n odd} \end{cases}$$

Consequently, we see from (2) that with $n > 0$,

$$a_n = \begin{cases} \frac{2}{\pi}(0) = 0, \text{ n even} \\ \frac{2}{\pi}\left(\frac{-2}{n^2}\right) = \frac{-4}{\pi n^2}, \text{ n odd} \end{cases} \tag{4}$$

3.8.4 continued

Finally, since f(x) is even, f(x)sin nx is odd; hence,

$$\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0,$$

and this in turn means that

$$b_n = 0, \text{ for } n = 1,2,3,\ldots . \tag{5}$$

From (1), (4), and (5), we conclude that

$$F(x) = a_o + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi}{2} + \sum_{n \text{ odd}} -\frac{4}{\pi n^2} \cos nx + 0$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nx}{n^2}$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}. \tag{6}$$

b. Since $f(x) = |x|$ implies that f is piecewise smooth on $[-\pi,\pi]$, we conclude that $F(x) = f(x)$ except at those points at which f is discontinuous; but since f is everywhere continuous on $[-\pi,\pi]$, $f(x) \equiv F(x)$ on $[-\pi,\pi]$.

Hence, if we let $x = \pi$ in (6) and observe that $f(\pi) = |\pi| = \pi$, we obtain

$$\pi = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi}{(2n+1)^2}. \tag{7}$$

3.8.4 continued

Since $\cos(2n+1)\pi = -1$ for all n (i.e. all odd multiples of π have their cosine equal to -1), we conclude from (7) that

$$\pi = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{-1}{(2n+1)^2}$$

or

$$\pi = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \tag{8}$$

From (8), it follows that

$$\frac{\pi}{2} = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},$$

or

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi}{4}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8}.$$

In other words,

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots + \frac{1}{(2n+1)^2} + \ldots = \frac{\pi^2}{8}.$$

3.8.5(L)

a. Since

$$f(x) = \begin{cases} 0, & -\pi < x \leqslant 0 \\ x^2, & 0 \leqslant x < \pi \end{cases}$$

it follows that

3.8.5(L) continued

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx$$

$$= \int_{-\pi}^{0} 0\ dx + \int_{0}^{\pi} x^2 dx$$

$$= 0 + \frac{1}{3} x^3 \Big|_{x=0}^{\pi}$$

$$= \frac{\pi^3}{3}.$$

Hence,

$$a_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi} \left[\frac{\pi^3}{3}\right]$$

$$= \frac{\pi^2}{6}. \qquad (1)$$

We also have that

$$\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \int_{-\pi}^{0} 0 \cos nx\ dx + \int_{0}^{\pi} x^2 \cos nx\ dx \quad (n \neq 0)$$

$$= \int_{0}^{\pi} x^2 \cos nx\ dx.$$

Hence, for $n \neq 0$,

3.8.5(L) continued

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x^2\cos nx\,dx. \qquad (2)$$

Aside:

Letting $u = x^2$, $dv = \cos nx$, we have $du = 2x dx$, $v = \frac{\sin nx}{n}$. Hence,

$$\int_0^{\pi} x^2\cos nx\,dx = \underbrace{\frac{x^2\sin nx}{n}\Big|_{x=0}^{\pi}}_{0} - 2\int_0^{\pi} \frac{x\sin nx\,dx}{n}$$

$$= -\frac{2}{n}\int_0^{\pi} x\sin nx\,dx. \qquad (3)$$

Again, using parts with $u = x$ and $dv = \sin nx\,dx$, we have $du = dx$ and $v = -\frac{1}{n}\cos nx$. Hence

$$\int_0^{\pi} x\sin nx\,dx = \frac{-x}{n}\cos nx\Big|_{x=0}^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx$$

$$= \frac{-\pi}{n}\underbrace{\cos \pi n}_{=(-1)^n} + \underbrace{\frac{1}{n^2}\sin nx\Big|_{x=0}^{\pi}}_{=0}$$

$$= \frac{(-1)^{n+1}\pi}{n}. \qquad (4)$$

Combining (3) and (4), we have

3.8.5(L) continued

$$\int_0^\pi x^2 \cos nx\, dx = \frac{-2}{n^2} (-1)^{n+1}\pi,$$

so that from (2),

$$n \neq 0 \rightarrow a_n = \frac{-2}{n^2} (-1)^{n+1}$$

$$= \frac{2(-1)^1(-1)^{n+1}}{n^2}$$

$$= \frac{2(-1)^{n+2}}{n^2},$$

or, since $(-1)^{n+2} = (-1)^n$,

$$a_n = \frac{2(-1)^n}{n^2}. \tag{5}$$

Similarly,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi} \int_0^\pi x^2 \sin nx\, dx. \tag{6}$$

Again, by parts, we can show that

$$\int_0^\pi x^2 \sin nx\, dx = \left. \frac{(2 - n^2x^2)\cos nx + 2xn \sin nx}{n^3} \right|_{x=0}^{\pi}$$

$$= \frac{(2 - n^2\pi^2)\cos \pi n - 2}{n^3}$$

$$= \frac{(2 - n^2\pi^2)(-1)^n - 2}{n^3}$$

3.8.5(L) continued

$$= \frac{[2(-1)^n - 2] - n^2\pi^2(-1)^n}{n^3}$$

$$= \frac{2[(-1)^n - 1]}{n^3} - \frac{\pi^2(-1)^n}{n}$$

$$= \begin{cases} -\frac{\pi^2}{n}, \text{ when n is even} \\ \frac{-4}{n^3} + \frac{\pi^2}{n} = \frac{\pi^2 n^2 - 4}{n^3}, \text{ when n is odd} \end{cases}$$

Hence, from (6),

$$b_n = \begin{cases} -\frac{\pi}{n}, \text{ when n is even} \\ \frac{\pi^2 n^2 - 4}{n^3 \pi}, \text{ when n is odd} \end{cases} \qquad (7)$$

Combining (1), (5), and (7), we have

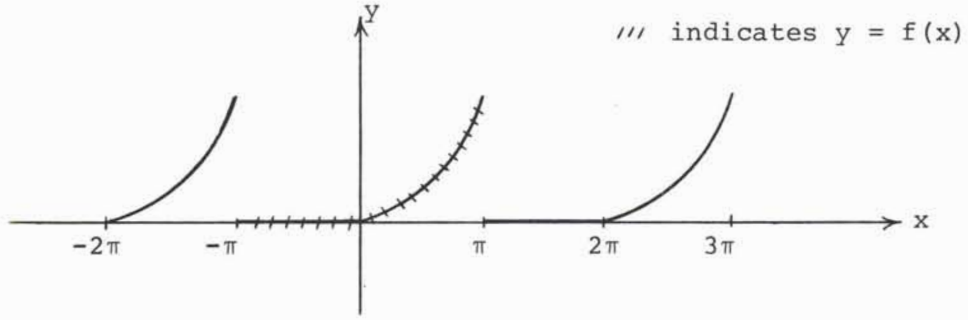
$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos nx + \sum_{n \text{ even}} \frac{-\pi}{n} \sin nx +$$

$$+ \sum_{n \text{ odd}} \frac{\pi^2 n^2 - 4}{n^3 \pi} \sin nx \qquad (8)$$

b. Here we see the difference between F and f in terms of periodicity. Notice that f is not defined either at $-\pi$ or π, but we may think of a new function $\overline{f}$ that is obtained by reproducing f with period 2π.

In terms of a picture, we have

3.8.5(L) continued



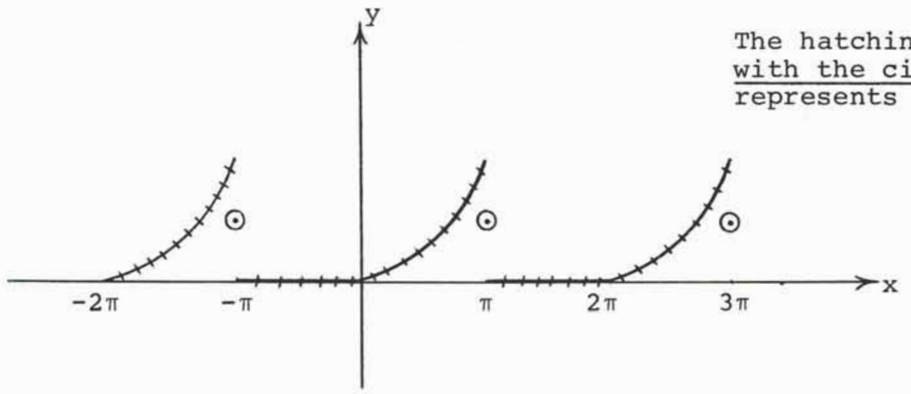
From the picture, we see that

$$F(\pi) = \frac{\bar{f}(\pi^+) + \bar{f}(\pi^-)}{2} = \frac{0 + \pi^2}{2}$$

or

$$F(\pi) = \frac{\pi^2}{2}.$$

In other words, the graph y = F(x) is given by



The hatching, together with the circled points, represents y = F(x).

c. If we now return to (8) and recall that on $(-\pi,\pi)$ $f(x) \equiv F(x)$ but that $F(\pi) = \frac{\pi^2}{2}$; and if we observe that $\sin n\pi = 0$, we obtain from (8) that

$$\underbrace{F(\pi)}_{\frac{\pi^2}{2}} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos n\pi + 0.$$

3.8.5(L) continued

Hence,

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} (-1)^n,$$

or, since $(-1)^n(-1)^n = (-1)^{2n} = 1$,

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}.$$

Hence,

$$\frac{\pi^2}{3} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

or

$$\frac{1}{6} \pi^2 = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

That is,

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} + \ldots = \frac{\pi^2}{6}.$$

Note

This was our first exercise in finding the Fourier series of f(x) in which f(x) was neither even nor odd. For this reason, our solution contained both sine and cosine terms. If we want a way of seeing what portion of the answer comes from the sine terms and what part from the cosine terms, we need only observe that the cosine terms come from the even part of f while the sine terms come from the odd part of f. By the even part of f, we mean $\frac{f(x) + f(-x)}{2}$, and by the odd part of f, we mean $\frac{f(x) - f(-x)}{2}$.

3.8.5(L) continued

Applied to this particular exercise, we have

$$f[\ ] = \begin{cases} 0, & \pi < [\ ] \leqslant 0 \\ [\ ]^2, & 0 \leqslant [\ ] < \pi \end{cases}$$

Hence,

$$f(-x) = \begin{cases} 0, & -\pi < -x \leqslant 0 \\ (-x)^2, & 0 \leqslant -x < \pi \end{cases}$$

That is

$$f(-x) = \begin{cases} 0, & \pi > x \geqslant 0 \\ x^2, & 0 \geqslant x > -\pi \end{cases}$$

or

$$f(-x) = \begin{cases} x^2, & -\pi < x \leqslant 0 \\ 0, & 0 \leqslant x < \pi \end{cases}$$

Coupling this with the fact that

$$f(x) = \begin{cases} 0, & -\pi < x \leqslant 0 \\ x^2, & 0 \leqslant x < \pi \end{cases}$$

we see that

$$f(x) + f(-x) = \begin{cases} 0 + x^2, & -\pi < x \leqslant 0 \\ x^2 + 0, & 0 \leqslant x < \pi \end{cases}$$

while

3.8.5(L) continued

$$f(x) - f(-x) = \begin{cases} 0 - x^2, & -\pi < x \leqslant 0 \\ x^2 - 0, & 0 \leqslant x < \pi \end{cases}$$

In other words,

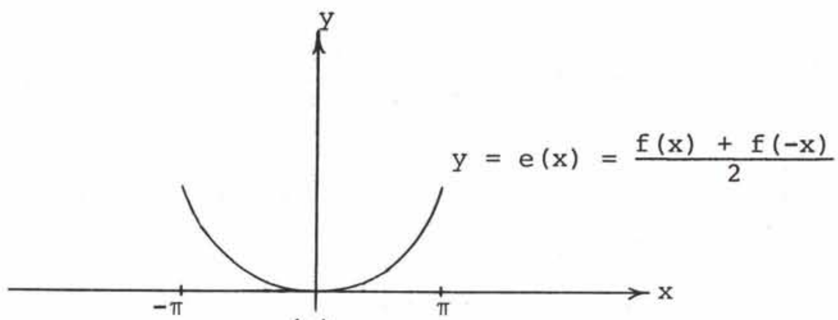
$$\frac{f(x) + f(-x)}{2} = \frac{1}{2} x^2$$

and this is the even part of f(x), which is represented by the cosine terms in (8).

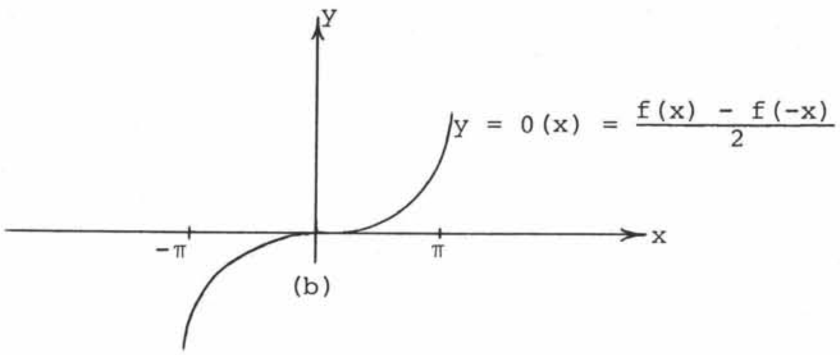
On the other hand, the sine terms in (8) represent

$$\frac{f(x) - f(-x)}{2} = \begin{cases} -\frac{1}{2} x^2, & -\pi < x \leqslant 0 \\ \frac{1}{2} x^2, & 0 \leqslant x < \pi \end{cases}$$

Pictorially,

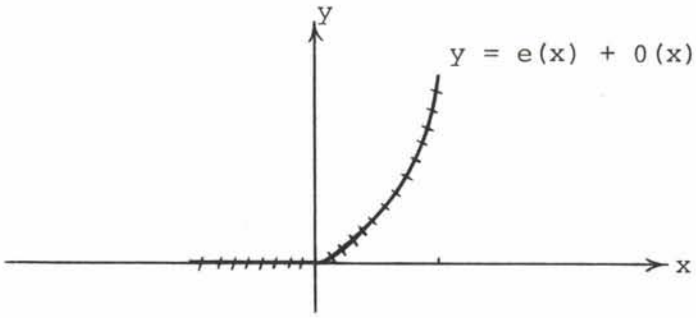


(a)



(b)

3.8.5 (L) continued



(c)

3.8.6 (Optional)

$$\int_a^b (f - t_n)^2 dx = \int_a^b (f^2 - 2t_n f + t_n^2) dx$$

$$= \int_a^b f^2 dx - 2\int_a^b t_n f\ dx + \int_a^b t_n^2\ dx$$

$$= \int_a^b f^2 dx - 2\int_a^b \left(\sum_{k=0}^{n} \gamma_k \phi_k\right) f\ dx + \int_a^b t_n^2\ dx$$

$$= \int_a^b f^2 dx - 2\sum_{k=0}^{n} \gamma_k \underbrace{\left(\int_a^b f\phi_k dx\right)}_{= c_k} + \int_a^b t_n^2\ dx$$

$$= \int_a^b f^2 dx - 2\sum_{k=0}^{n} \gamma_k c_k + \underbrace{\int_a^b \left(\sum_{k=0}^{n} \gamma_k \phi_k\right)^2 dx}$$

$$\underbrace{\int_a^b \sum \gamma_k{}^2 \phi_k{}^2\ dx} + \int_a^b \sum_{k \neq j} \gamma_k \gamma_j \underbrace{\phi_k \phi_j}_{\substack{= 0 \text{ by} \\ \text{orthogonality}}} dx$$

$$\sum_{k=0}^{n} \gamma_k{}^2 \underbrace{\int_a^b \phi_k{}^2\ dx}_{= 1 \text{ by orthonormal property of } \{\phi_n\}}$$

3.8.6 continued

Hence

$$\int_a^b (f - t_n)^2 dx = \int_a^b f^2 dx - 2\sum_{k=0}^n \gamma_k c_k + \sum_{k=0}^n \gamma_k^2$$

$$= \int_a^b f^2 dx + \sum_{k=0}^n c_k^2 - 2\sum_{k=0}^n \gamma_k c_k + \sum_{k=0}^n \gamma_k^2 - \sum_{k=0}^n c_k^2$$

[i.e. we add and subtract $\sum c_k^2$ to get a perfect square]

$$= \int_a^b f^2 dx + \sum_{k=0}^n (c_k^2 - 2\gamma_k c_k + \gamma_k^2) - \sum_{k=0}^n c_k^2$$

$$= \int_a^b f^2 dx - \sum_{k=0}^n c_k^2 + \sum_{k=0}^n \underbrace{(c_k - \gamma_k)^2}_{\geqslant 0;\ = 0 \leftrightarrow c_k = \gamma_k} \qquad (5)$$

Since f and c_k are fixed, we see from (5) that

$$\int_a^b (f - t_n)^2 dx$$

is minimized $\leftrightarrow \gamma_k = c_k$ for $k = 0,1,\ldots,n$.

<u>Note #1</u>

If we let $\gamma_k = c_k$ in (5), we conclude that

$$\int_a^b f^2 dx - \sum_{k=0}^n c_k^2 = \int_a^b (f - t_n)^2 dx \geqslant 0,$$

so that

3.8.6 continued

$$\sum_{k=0}^{n} c_k{}^2 \leqslant \int_a^b f^2(x)\,dx. \qquad (6)$$

Since $\int_a^b f^2(x)dx$ is a finite (positive) number, we see from (6) that $\sum_{k=0}^{n} c_k{}^2$ is a bounded positive series; hence, it converges. In particular

$$\lim_{n\to\infty} c_n = 0,$$

and this at least shows that the Fourier coefficients do at least get small.

Note #2

If f happens to be continuous, then one can prove

(i) If $\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$ $(n = 0,1,2,3,\ldots)$ then $f(x) \equiv 0$. Consequently,

(ii) If f and g are both continuous and have the same Fourier series, then $f(x) \equiv g(x)$.

(iii) Moreover, when f is piecewise smooth

$$\lim_{n\to\infty} \int_0^{2\pi} (f - s_n)^2 dx = 0$$

so that the error does get squeezed to zero.

3.8.7

The key step in proving the orthogonality of {sin nx, cos mx} lay in the fact that $\sin(m \pm n)x \Big|_{x=-\pi}^{\pi} = 0$. If we now replace $[-\pi,\pi]$ by $[-p,p]$, the important computation would involve $\sin(m \pm n)x \Big|_{x=-p}^{p}$.

The problem is that $\sin(m \pm n)p$ need not be zero.

If we observe that $\sin(m \pm n)\pi$ is zero, it is easy to conjecture that it would have been nice had a factor of $\frac{\pi}{p}$ been introduced in the expression $\sin(m \pm n)p$. That is

$$\sin \frac{\pi}{p}(m \pm n)p = \sin(m \pm n)\pi = 0.$$

This suggests that if we want the Fourier series for f(x) on the interval $[-p,p]$, then the series should have the form

$$f(x) \sim a_o + \sum_{n=1}^{\infty} a_n \cos \frac{\pi n x}{p} + \sum_{n=1}^{\infty} b_n \sin \frac{\pi n x}{p}. \tag{1}$$

A direct check, similar to the one used in Exercise 3.8.1, shows that

$$\left\{1, \cos \frac{\pi x}{p}, \cos \frac{2\pi x}{p}, \ldots, \cos \frac{\pi n x}{p}, \sin \frac{\pi x}{p}, \ldots, \sin \frac{\pi n x}{p}, \ldots\right\}$$

is orthogonal on $[-p,p]$.

Moreover,

$$\int_{-p}^{p} dx = 2p \tag{2}$$

$$\begin{aligned}\int_{-p}^{p} \cos^2 \frac{\pi n x}{p}\,dx &= \frac{1}{2}\int_{-p}^{p}\left(1 + \cos \frac{2\pi n x}{p}\right)dx \\ &= \frac{1}{2}\left(x + \frac{p}{2\pi n}\sin \frac{2\pi n x}{p}\right)\Bigg|_{x=-p}^{p} \\ &= \frac{1}{2}(2p + 0) \\ &= p.\end{aligned} \tag{3}$$

3.8.7 continued

Similarly,

$$\int_{-p}^{p} \sin^2 \frac{\pi nx}{p}\, dx = \frac{1}{2}\int_{-p}^{p}\left[1 - \cos\left(\frac{2\pi nx}{p}\right)\right] dx$$

$$= p. \qquad (4)$$

From (2), (3), and (4), we see that p replaces π when the interval switches from $[-\pi,\pi]$ to $[-p,p]$. That is, if f(x) is integrable on $[-p,p]$, the Fourier series of f(x), F(x) is given by

$$F(x) = a_o + \sum_{n=1}^{\infty} a_n \cos\frac{\pi nx}{p} + \sum_{n=1}^{\infty} b_n \sin\frac{\pi nx}{p} \qquad (5)$$

where

$$a_o = \frac{1}{2p}\int_{-p}^{p} f(x)dx \qquad (6)$$

$$a_n \text{ (for } n \neq 0) = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{\pi nx}{p}\, dx \qquad (7)$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{\pi nx}{p}\, dx. \qquad (8)$$

With this general background, we see that our present exercise is the special case with p = 1 and f(x) = x. Under these conditions, we deduce from (6), (7), and (8) that

$$a_o = a_1 = \ldots = a_n = \ldots = 0$$

since f(x) is an odd function [hence, $f(x)\cos\frac{\pi nx}{p}$ is also odd],

$$b_n = \int_{-1}^{1} x \sin \pi nx\, dx.$$

3.8.7 continued

Letting $u = x$ and $dv = \sin \pi nx\, dx$, we have that $u = dx$ and $v = \frac{-\cos \pi nx}{\pi n}$. Hence,

$$\int_{-1}^{1} x \sin \pi nx\, dx = \left.\frac{-x \cos \pi nx}{\pi n}\right|_{x=-1}^{1} + \frac{1}{\pi n}\int_{-1}^{1} \cos \pi nx\, dx$$

$$= \frac{-2 \cos \pi n}{\pi n} + \frac{1}{\pi^2 n^2} \underbrace{\sin \pi nx \Big|_{x=-1}^{1}}_{0}$$

$$= \frac{-2(-1)^n}{\pi n}$$

$$= \frac{2(-1)(-1)^n}{\pi n}$$

$$= \frac{2(-1)^{n+1}}{\pi n}. \qquad (9)$$

Putting these results into (5) yields

$$F(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin \pi nx$$

or

$$F(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin \pi nx}{n} \qquad (10)$$

$$= \frac{2}{\pi}\left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \frac{\sin 4\pi x}{4} + \ldots\right). \qquad (10')$$

Note #1

As a partial check of (10'), we know that if $-1 < x < 1$, $f(x) = F(x)$. Hence, letting $x = \frac{1}{2}$ in (10') yields

3.8.7 continued

$$F\left(\tfrac{1}{2}\right) = f\left(\tfrac{1}{2}\right) = \frac{1}{2} = \frac{2}{\pi}\left(\overbrace{\sin\frac{\pi}{2}}^{=1} - \frac{\overbrace{\sin\pi}^{=0}}{2} + \frac{\overbrace{\sin\frac{3\pi}{2}}^{-1}}{3} - \frac{\overbrace{\sin 2\pi}^{=0}}{4} + \frac{\overbrace{\sin\frac{5\pi}{2}}^{=1}}{5} \cdots\right)$$

or

$$\frac{1}{2} = \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right),$$

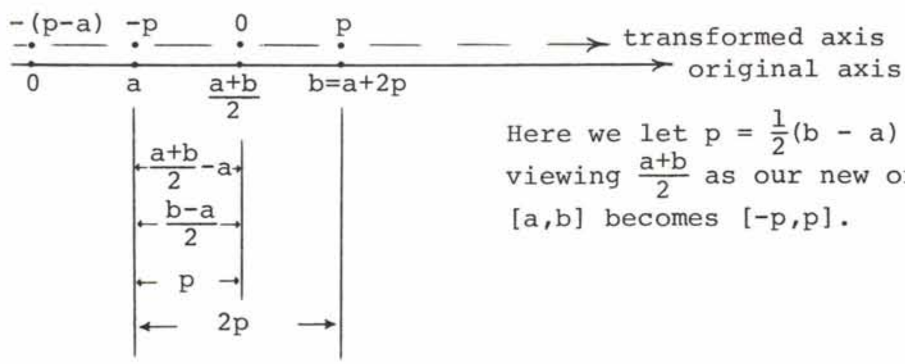
or

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

and this checks with the result of Exercise 3.8.3(L).

<u>Note #2</u>

The result of this exercise can be generalized to any interval [a,b], not just intervals which are symmetric with respect to $x = 0$. Without going into detail here, the gist of the argument is that if $f(x)$ is integrable on [a,b] and $\{\phi_n(x)\}$ is orthogonal on [a,b], then we may shift our coordinates by letting $\frac{a+b}{2}$ (i.e. the midpoint of [a,b]) serve as our new origin. Pictorially,



Here we let $p = \frac{1}{2}(b - a)$ so that viewing $\frac{a+b}{2}$ as our new origin, [a,b] becomes [-p,p].

3.8.7 continued

To find the Fourier series of f(x) on [a,b], we need only replace -p by a, p by b, and the period 2p by b - a. Leaving the details to the interested reader, we then find that equations (5) through (8) are amended by:

If f(x) is integrable on [a,b], then the Fourier series, F(x) of f(x) is given by

$$F(x) = a_o + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{b-a}\right)^* + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{b-a}\right)^*$$

where

$$a_o = \frac{1}{b-a}\int_a^b f(x)dx$$

$$a_n \ (n > 0) = \frac{1}{\frac{b-a}{2}} \int_a^b f(x)\cos\left(\frac{2\pi nx}{b-a}\right)dx$$

$$= \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2\pi nx}{b-a}\right)dx$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{2\pi nx}{b-a}\right)dx$$

Note #3

As an example, suppose we try to find the Fourier series of the function of period 1 defined by

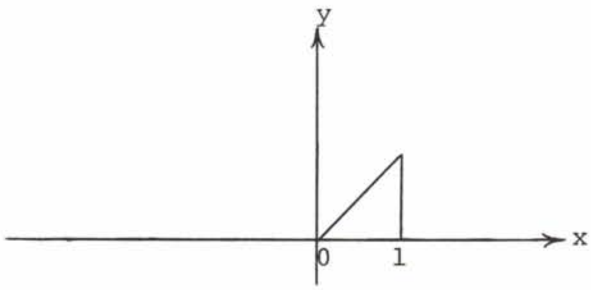
$f(x) = x,\ 0 \leqslant x < 1.$

Pictorially,

---

*I.e., $\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{\pi nx}{p} = \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{\pi nx}{\frac{b-a}{2}} = \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{2\pi nx}{b-a}.$

3.8.7 continued



We have that

$$F(x) = a_o + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{b - a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{b - a}\right)$$

where

$$a_o = \int_0^1 x\,dx \tag{11}$$

$$a_n = 2\int_0^1 x \cos 2\pi nx\,dx \tag{12}$$

$$b_n = 2\int_0^1 x \sin 2\pi nx\,dx \tag{13}$$

We then obtain

$$a_o = \frac{1}{2} x^2 \Big]_{x=0}^{1} = \frac{1}{2}. \tag{14}$$

By parts, we have that

$$\int x \sin 2\pi nx\,dx = \frac{-x \cos 2\pi nx}{2\pi n} + \frac{1}{2\pi n}\int \cos 2\pi nx\,dx$$

3.8.7 continued

and in a similar way

$$\int x \cos 2\pi nx \, dx = \frac{x \sin 2\pi nx}{2\pi n} - \frac{1}{2\pi n} \int \sin 2\pi nx \, dx.$$

Hence,

$$a_n \ (n \neq 0) = \underbrace{\left.\frac{x \sin 2\pi nx}{2\pi n}\right|_{x=0}^{1}}_{0} - \frac{1}{2\pi n} \underbrace{\int_0^1 \sin 2\pi nx \, dx}_{0}$$

$$= 0,$$

while

$$b_n = \left.\frac{-x \cos 2\pi nx}{2\pi n}\right|_{x=0}^{1} + \frac{1}{2\pi n} \underbrace{\int_0^1 \cos 2\pi nx \, dx}_{0}$$

$$= \frac{-\cos 2\pi n}{2\pi n}$$

$$= \frac{-1}{2\pi n}.$$

Hence,

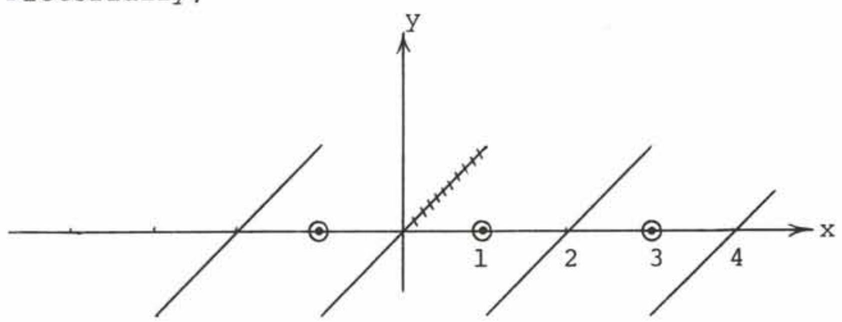
$$F(x) = \frac{1}{2} - \frac{1}{2\pi} \sum \frac{\sin (2\pi nx)}{n}. \qquad (15)$$

Note #4

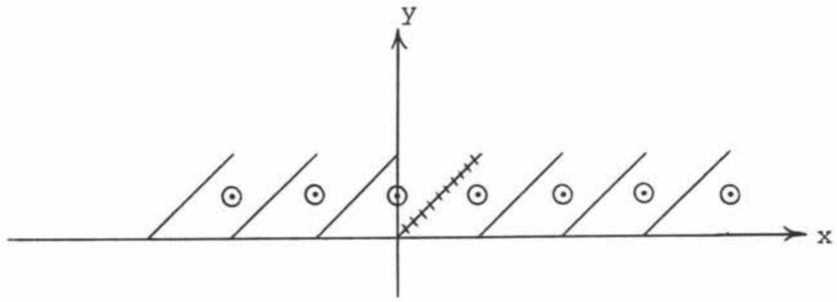
$\frac{1}{2} - \frac{1}{2\pi} \sum \frac{\sin (2\pi nx)}{n}$ (the result of Note #3) and equation (10') both converge to $f(x)$ on [0,1]. Notice, however, that they behave quite differently on [-1,0]. In other words, unlike in the case of power series, different Fourier series may express the same function on certain subintervals.

3.8.7 continued

Pictorially,



① y = F(x) with F as in (10').



② y = F(x) with F as in (15).

③ Both (10') and (15) agree on the hatched region.

In summary, if f(x) and g(x) are convergent power series and if f(x) = g(x) on some interval [a,b], then f and g are identical everywhere. That is, once f and g fit well on one interval, no matter how small the (non-zero) interval, then they fit well everywhere. On the other hand, there are many different Fourier series which fit the function f on a particular interval, but which are very different on other intervals.

Quiz

1. (a) Using our usual matrix coding system and the row-reduction technique, we have:

$$\begin{matrix} \underline{u_1} & \underline{u_2} & \underline{u_3} & \underline{u_4} & \underline{v_1} & \underline{v_2} & \underline{v_3} & \underline{v_4} \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 & 0 & 1 & 0 & 0 \\ 3 & 4 & 6 & 6 & 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 5 & 0 & 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & -3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 & -2 & 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 3 & 3 & -2 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{matrix} \underline{u_1} & \underline{u_2} & \underline{u_3} & \underline{u_4} & | & \underline{v_1} & \underline{v_2} & \underline{v_3} & \underline{v_4} \end{matrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 3 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \qquad (1)$$

From (1) we see at once that

1. continued

$$\left.\begin{aligned} u_1 &= - 2v_2 + v_3 \\ u_2 &= 3v_1 + 3v_2 - 2v_3 \\ u_3 &= -v_1 + v_3 - v_4 \\ u_4 &= -v_1 - v_2 + v_4 \end{aligned}\right\} \qquad (2)$$

<u>Check</u>

$$\begin{array}{rl} -2v_2 + v_3 = & -2u_1 - 4u_2 - 6u_3 - 6u_4 \\ & \underline{+3u_1 + 4u_2 + 6u_3 + 6u_4} \\ & u_1 \end{array}$$

$$\begin{array}{rl} 3v_1 + 3v_2 - 2v_3 = & 3u_1 + 3u_2 + 3u_3 + 3u_4 \\ & +3u_1 + 6u_2 + 9u_3 + 9u_4 \\ & \underline{-6u_1 - 8u_2 - 12u_3 - 12u_4} \\ & u_2 \end{array}$$

$$\begin{array}{rl} -v_1 + v_3 - v_4 = & -u_1 - u_2 - u_3 - u_4 \\ & +3u_1 + 4u_2 + 6u_3 + 6u_4 \\ & \underline{-2u_1 - 3u_2 - 4u_3 - 5u_4} \\ & u_3 \end{array}$$

$$\begin{array}{rl} -v_1 - v_2 + v_4 = & -u_1 - u_2 - u_3 - u_4 \\ & -u_1 - 2u_2 - 3u_3 - 3u_4 \\ & \underline{+2u_1 + 3u_2 + 4u_3 + 5u_4} \\ & u_4 \end{array}$$

(b) We have that

$w = 4u_1 + 3u_2 + 2u_3 + u_4$,

so that from (2),

$$\begin{aligned} w &= 4(-2v_2 + v_3) + 3(3v_1 + 3v_2 - 2v_3) + 2(-v_1 + v_3 - v_4) + (-v_1 - v_2 + v_4) \\ &= 6v_1 - v_4 \\ &= (6,0,0,-1). \end{aligned}$$

1. continued

<u>Check</u>

$$\begin{array}{rl} 6v_1 - v_4 = & 6u_1 + 6u_2 + 6u_3 + 6u_4 \\ & \underline{-2u_1 - 3u_2 - 4u_3 - 5u_4} \\ & 4u_1 + 3u_2 + 2u_3 + u_4 = w \end{array}$$

2. (a) Technically speaking, we do not need to use the augmented matrix technique to do this part of the exercise. However, the augmented matrix technique is useful in parts (b) and (c) so we might as well introduce it in part (a). We, thus, have:

$$\begin{array}{cccccccc} \underline{u_1} & \underline{u_2} & \underline{u_3} & \underline{u_4} & \underline{w_1} & \underline{w_2} & \underline{w_3} & \underline{w_4} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -4 & 3 & 0 & 1 & 0 & 0 \\ 1 & 8 & -5 & 6 & 0 & 0 & 1 & 0 \\ 1 & -4 & 7 & -6 & 0 & 0 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & -3 & 3 & 1 & 1 & 0 & 0 \\ 0 & 6 & -6 & 6 & -1 & 0 & 1 & 0 \\ 0 & -6 & 6 & -6 & -1 & 0 & 0 & 1 \end{bmatrix} \qquad (1)$$

To continue our row-reduction of (1) we might observe that the last two rows are related to the second since $(0,-6,6,-6) = -(0,6,-6,6) = -2(0,3,-3,3)$; but we are not always this fortunate in our use of inspection. The more general approach is to rewrite (1) in a form in which each element of the second column is divisible by 6. To this end, we may multiply the first row by 3 and the second row by 2 to obtain

$$\begin{bmatrix} 3 & 6 & 3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 6 & -6 & 6 & 2 & 2 & 0 & 0 \\ 0 & 6 & -6 & 6 & -1 & 0 & 1 & 0 \\ 0 & -6 & 6 & -6 & -1 & 0 & 0 & 1 \end{bmatrix} \sim$$

2. continued

$$\begin{bmatrix} 3 & 0 & 9 & -6 & 1 & -2 & 0 & 0 \\ 0 & 6 & -6 & 6 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{matrix} \underline{u_1} & \underline{u_2} & \underline{u_3} & \underline{u_4} & \underline{w_1} & \underline{w_2} & \underline{w_3} & \underline{w_4} \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -2 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 \\ 0 & 1 & -1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \end{bmatrix} \qquad (2)$$

From (2) we see that W is spanned by β_1 and β_2 where

$$\left.\begin{aligned} \beta_1 &= u_1 + 3u_3 - 2u_4 \\ \beta_2 &= u_2 - u_3 + u_4 \end{aligned}\right\} \qquad (3)$$

where the reduced-echelon form of (2) guarantees us that $\{\beta_1,\beta_2\}$ is linearly independent.

Since β_1 and β_2 are linearly independent and span W they are a basis for W. Hence:

$\dim W = 2.$

(b) Looking at the last two rows of (2) we see at once that

$$0 = -3w_1 - 2w_2 + w_3$$

and

$$0 = w_1 + 2w_2 + w_4.$$

Therefore,

$$\left.\begin{aligned} w_3 &= 3w_1 + 2w_2 \\ w_4 &= -w_1 - 2w_2 \end{aligned}\right\} \qquad (4)$$

2. continued

(c) There are several ways of tackling this part of the problem. Perhaps the most efficient way is to utilize equation (3). From the echelon form of (3) we see that the only linear combination of β_1 and β_2 that can have the form $3u_1 + 5u_2 + xu_3 + yu_4$ is

$$3\beta_1 + 5\beta_2. \tag{5}$$

(Quite in general, if $w \epsilon W$, the u_1-component of w must be the coefficient of β_1 and the u_2-component of w must be the coefficient of β_2).

From (3) we see that

$$\begin{aligned} 3\beta_1 + 5\beta_2 &= 3(u_1 + 3u_3 - 2u_4) + 5(u_2 - u_3 + u_4) \\ &= 3u_1 + 5u_2 + 4u_3 - u_4. \end{aligned} \tag{6}$$

From (6) we see that

$x = 4$ and $y = -1$.

3. (a) From the previous exercise, we know that $\beta_1 = u_1 + 3u_3 - 2u_4$ and $\beta_2 = u_2 - u_3 + u_4$ span W. Hence, to find the space spanned by β_1, β_2 and the three given vectors, we may use the following matrix technique:

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 5 & -3 \\ 3 & 2 & 7 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \tag{1}$$

3. continued

The right side of (1) tells us that $\alpha_1 = u_1 + 4u_4$, $\alpha_2 = u_2 - u_4$, and $\alpha_3 = u_3 - 2u_4$ form a basis for U + W. Hence,

$\dim(U + W) = 3.$

(b) $\gamma = (x_1,x_2,x_3,x_4)\varepsilon U \leftrightarrow \gamma = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$

$$= x_1(u_1 + 4u_4) + x_2(u_2 - u_4) + x_3(u_3 - 2u_4)$$

$$= x_1u_1 + x_2u_2 + x_3u_3 + (4x_1 - x_2 - 2x_3)u_4$$

$$\leftrightarrow x_4 = 4x_1 - x_2 - 2x_3 \qquad (2)$$

Now from the previous exercise

$\gamma = (x_1,x_2,x_3,x_4)\varepsilon W \leftrightarrow \gamma = x_1\beta_1 + x_2\beta_2$

$$= x_1(u_1 + 3u_3 - 2u_4) + x_2(u_2 - u_3 + u_4)$$

$$= x_1u_1 + x_2u_2 + (3x_1 - x_2)u_3 + (x_2 - 2x_1)u_4$$

$$\leftrightarrow \begin{cases} x_3 = 3x_1 - x_2 \\ x_4 = x_2 - x_1 \end{cases} \qquad (3)$$

Replacing x_3 and x_4 in (2) by their values in (3), we obtain:

$x_2 - 2x_1 = 4x_1 - x_2 - 6x_1 + 2x_2$

or

$0 = 0.$ (4)

What (4) indicates is that once x_3 and x_4 satisfy (3), they automatically satisfy (4). In other words, in this particular example, we have the special case in which

$W \subset U.$

Hence,

$W \cap U = W$

or

$\dim(W \cap U) = \dim W = 2$ (from the previous exercise).

(c) We already know that

$\dim W = 2$

$\dim(U \cap W) = 2$

$\dim(U + W) = 3$

Since $W \subset U$, $\dim U = \dim U + W = 3$ and the result follows. We could have shown more explicitly that $\dim U = 3$. Namely,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 5 & -3 \\ 3 & 2 & 7 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 3 & -5 \\ 0 & -1 & 4 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & -4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

or $u = [\gamma_1, \gamma_2, \gamma_3]$ where

$$\begin{cases} \gamma_1 = u_1 + 4u_4 \\ \gamma_2 = u_2 - u_4 \\ \gamma_3 = u_3 - 2u_4 \end{cases}$$

Notice that $\gamma_1, \gamma_2, \gamma_3$ are simply the $\alpha_1, \alpha_2, \alpha_3$ of part (a) since $U = U + W$.

Hence, $\dim U = 3$.

4. We have by linearity that

$T(x\vec{i} + y\vec{j} + z\vec{k}) = xT(\vec{i}) + yT(\vec{j}) + zT(\vec{k})$.

Hence, from the given values for $T(\vec{i})$, $T(\vec{j})$, and $T(\vec{k})$ we conclude that

4. continued

$$T(x\vec{i} + y\vec{j} + z\vec{k}) = x(2\vec{i} + \vec{j} + \vec{k}) + y(-\vec{i} + 2\vec{j} + 7\vec{k}) + z(\vec{i} - \vec{j} - 4\vec{k})$$

$$= (2x - y + z)\vec{i} + (x + 2y - z)\vec{j} + (x + 7y - 4z)\vec{k} \qquad (1)$$

(a) From (1) we see that

$$T(x\vec{i} + y\vec{j} + z\vec{k}) = \vec{0} \leftrightarrow$$

$$\left.\begin{aligned} 2x - y + z &= 0 \\ x + 2y - z &= 0 \\ x + 7y - 4z &= 0 \end{aligned}\right\}^{*} \qquad (2)$$

Solving (2), either by matrix methods or otherwise, yields

$$\left.\begin{aligned} x + 2y - z &= 0 \\ x + 7y - 4z &= 0 \\ 2x - y + z &= 0 \end{aligned}\right\} \sim \left.\begin{aligned} x + 2y - z &= 0 \\ 5y - 3z &= 0 \\ -5y + 3z &= 0 \end{aligned}\right\} \sim \left.\begin{aligned} x + 2y - z &= 0 \\ 5y - 3z &= 0 \end{aligned}\right\} \sim$$

$$\left.\begin{aligned} 5x + 10y - 5z &= 0 \\ 10y - 6z &= 0 \end{aligned}\right\} \sim \left\{\begin{aligned} 5x + z &= 0 \\ 5y - 3z &= 0 \end{aligned}\right. \sim \left\{\begin{aligned} x &= -\frac{z}{5} \\ y &= \frac{3z}{5} \end{aligned}\right. \qquad (3)$$

From (3) we see that the null space of T is given by those vectors of the form

$$\left(-\frac{z}{5}, \frac{3z}{5}, z\right)$$

---

*Recall that since $\{\vec{i},\vec{j},\vec{k}\}$ is a linearly independent set, $a\vec{i} + b\vec{j} + c\vec{k} = \vec{0} \leftrightarrow a = b = c = 0$.

4. continued

or

$(-z,3z,5z)$.

In particular, letting $z = -1$, we see that

$(1,-3,-5)$

is a basis for the null space, N, of T.

<u>Check</u>

$T(1,-3,-5) = T(\vec{i}) - 3T(\vec{j}) - 5T(\vec{k})$

$= (2\vec{i} + \vec{j} + \vec{k}) + (3\vec{i} - 6\vec{j} - 21\vec{k}) + (-5\vec{i} + 5\vec{j} + 20\vec{k}) = \vec{0}.$

Hence,

$\dim N = 1.$ (4)

(b) To find the space spanned by $T(\vec{i})$, $T(\vec{j})$, and $T(\vec{k})$, we need only row-reduce the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 7 \\ 1 & -1 & -4 \end{bmatrix},$$

and we see that

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 7 \\ 1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -4 \\ -1 & 2 & 7 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the image of T is the space for which $\vec{\alpha}_1 = \vec{i} + 7\vec{k}$ and $\vec{\alpha}_2 = \vec{j} + 3\vec{k}$ is a basis.

In particular, $\dim [T(E^3)] = 2$.

4. continued

Geometric Interpretation

T maps 3-space onto the plane determined by $\vec{\alpha}_1$ and $\vec{\alpha}_2$. The "loss of dimension" occurs because the line $\begin{cases} x=t \\ y=-3t \\ z=-5t \end{cases}$ is mapped into the origin.

5. We have:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 3 & 3 & 4 \\ 3 & 4 & 5 & 4 & 5 \\ 4 & 5 & 4 & 4 & 4 \end{vmatrix} = \begin{vmatrix} \overset{+}{1} & 1 & 1 & 1 & 1 \\ \overset{-}{0} & 1 & 1 & 2 & 2 \\ \overset{-}{0} & 1 & 1 & 1 & 2 \\ \overset{-}{0} & 1 & 2 & 1 & 2 \\ \overset{-}{0} & 1 & 0 & 0 & 0 \end{vmatrix} \qquad (1)$$

Expanding (1) with respect to its first column we obtain:

$$\begin{vmatrix} \overset{+}{1} & 1 & 2 & 2 \\ \overset{-}{1} & 1 & 1 & 2 \\ \overset{+}{1} & 2 & 1 & 2 \\ \overset{-}{1} & 0 & 0 & 0 \end{vmatrix} \qquad (2)$$

We now expand (2) along the last row to obtain

$$-\begin{vmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix}$$

and this equals

$$-\begin{vmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & -3 & -2 \end{vmatrix}. \qquad (3)$$

Expanding (3) down its first column, we obtain

5. continued

$$-\begin{vmatrix} -1 & 0 \\ -3 & -2 \end{vmatrix} = -[(-1)(-2) - (-3)(0)]$$

$$= -2.$$

Hence,

det A = -2.

6. We know that the values of c come from the equation:

det(A - cI) = 0

where A is the matrix of coefficients and I in this case is the 3 by 3 identity matrix.

Since

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$$

and

$$cI = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

we see that we must have

$$\begin{vmatrix} 3-c & 2 & 2 \\ 1 & 2-c & 2 \\ -1 & -1 & -c \end{vmatrix} = 0$$

or

$$(3-c)[-c(2-c) + 2] - 2[1(-c) - (-1)2] + 2[1(-1) - (-1)(2-c)] = 0.$$

Collecting terms we obtain,

6. continued

$$-c^3 + 5c^2 - 8c + 4 = 0. \tag{1}$$

Noticing that $c = 1$ is a root of the above equation, we may divide the left side of the equation by $c - 1$ to conclude that

$$-c^3 + 5c^2 - 8c + 4 = -(c - 1)(c^2 - 4c + 4).$$

Hence, we conclude from (1) that

$$c = 1 \quad \text{or} \quad c = 2.$$

Letting $\vec{X} = [x,y,z]$ where $v = xu_1 + yu_2 + zu_3$, we see that with $c = 1$ we obtain

$$T(v) = v$$

or in matrix notation

$$\vec{X}\,A = \vec{X}.$$

That is,

$$[x\ y\ z]\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix} = [x\ y\ z]. \tag{2}$$

[Had we desired to write v as a column vector, we would have $A^T \overset{\downarrow}{X} = \overset{\downarrow}{X}$, but this will yield the same answer as the one we shall obtain from (2).]

Equation (2) implies

$$\left.\begin{aligned} 3x + y - z &= x \\ 2x + 2y - z &= y \\ 2x + 2y &= z \end{aligned}\right\}$$

or,

6. continued

$$\left.\begin{array}{l} 2x + y - z = 0 \\ 2x + y - z = 0 \\ 2x + 2y - z = 0 \end{array}\right\} . \qquad (3)$$

System (3) reduces to

$$\left.\begin{array}{l} -2x - y + z = 0 \\ 2x + 2y - z = 0 \end{array}\right\} . \qquad (4)$$

Adding both equations in (4) yields $y = 0$.

With $y = 0$, equations (4) imply $z = 2x$. Hence, $T(x,y,z) = (x,y,z) \leftrightarrow y = 0$ and $z = 2x$.

In other words, every vector v of the form

$$(x,0,2x) = xu_1 + 2xu_3$$

has the property $T(v) = v$.

In particular, letting $x = 1$, we see that $u_1 + 2u_3 = (1,0,2)$ is one such vector. Therefore, letting $\alpha_1 = u_1 + 2u_3$ we have that

$$T(v) = v \leftrightarrow v = x\alpha_1.$$

If we let $c = 2$, we obtain

$$\vec{X} A = 2\vec{X}$$

or

$$[x\ y\ z] \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix} = 2[x\ y\ z].$$

Hence,

6. continued

$$\left.\begin{aligned} 3x + y - z &= 2x \\ 2x + 2y - z &= 2y \\ 2x + 2y &= 2z \end{aligned}\right\}.$$

Hence,

$$\left.\begin{aligned} x + y - z &= 0 \\ 2x - z &= 0 \\ 2x + 2y - 2z &= 0 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} x + y - z &= 0 \\ 2x - z &= 0 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} x + y - z &= 0 \\ - 2y + z &= 0 \end{aligned}\right\}. \qquad (5)$$

The second equation in (5) says that $z = 2y$ and putting this in the first equation yields $x + y - 2y = 0$ or $x = y$.

In other words, for an arbitrary value of y

$T(y,y,2y) = 2(y,y,2y)$.

Letting $y = 1$, we obtain that

$T(1,1,2) = 2(1,1,2) = (2,2,4)$.

Summary

Let $\alpha_1 = (1,0,2)$ and $\alpha_2 = (1,1,2)$. Then $T(v) = v \leftrightarrow v$ is a scalar multiple of α_1 and $T(v) = 2v \leftrightarrow v$ is a scalar multiple of α_2. Otherwise,

6. continued

$T(v) = cv \quad (c \neq 0)$

can be solved $\leftrightarrow$ $v = 0$.

7. The component of u_2 orthogonal to u_1 is given by

$$u_2^* = u_2 - \frac{u_1 \cdot u_2}{u_1 \cdot u_1} u_1$$

$$= (-1,0,7) - \left[\frac{(3,0,4) \cdot (-1,0,7)}{(3,0,4) \cdot (3,0,4)}\right] (3,0,4)$$

$$= (-1,0,7) - \left[\frac{-3 + 28}{9 + 16}\right] (3,0,4)$$

$$= (-1,0,7) - (3,0,4)$$

$$= (-4,0,3).$$

Next letting $u_1^* = (3,0,4)$ and $u_2^* = (-4,0,3)$ we replace u_3 by

$$u_3^* = u_3 - \left[\frac{u_3 \cdot u_1^*}{u_1^* \cdot u_1^*}\right] u_1^* - \left[\frac{u_3 \cdot u_2^*}{u_2^* \cdot u_2^*}\right] u_2^*$$

$$= (2,9,11) - \left[\frac{(2,9,11)\cdot(3,0,4)}{(3,0,4)\cdot(3,0,4)}\right](3,0,4) - \left[\frac{(2,9,11)\cdot(-4,0,3)}{(-4,0,3)\cdot(-4,0,3)}\right](-4,0,3)$$

$$= (2,9,11) - \left[\frac{6 + 0 + 44}{9 + 0 + 16}\right](3,0,4) - \left[\frac{-8 + 0 + 33}{16 + 0 + 9}\right](-4,0,3)$$

$$= (2,9,11) - 2(3,0,4) - (-4,0,3)$$

$$= (0,9,0).$$

Hence, an orthogonal basis is

$$u_1^* = (3,0,4),\ u_2^* = (-4,0,3),\ u_3^* = (0,9,0).$$

Therefore, an orthonormal basis is given by

$$v_1 = \frac{u_1^*}{|u_1^*|},\ v_2 = \frac{u_2^*}{|u_2^*|},\ v_3 = \frac{u_3^*}{|u_3^*|},$$

7. continued

or

$v_1 = (\frac{3}{5}, 0, \frac{4}{5})$

$v_2 = (-\frac{4}{5}, 0, \frac{3}{5})$

$v_3 = (0,1,0).$