Unit 2: The Double Sum as an Iterated Integral

5.2.1(L)

a. In Block 3, we showed that if g(x) was defined by

$$\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$$

and if $\alpha(x)$ and $\beta(x)$ were differentiable functions of x, then g was also differentiable, and in fact,

$$g'(x) = \int_{\alpha(x)}^{\beta(x)} f_x(x,y)\,dy + f\big(x,\beta(x)\big)\frac{d\beta}{dx} - f\big(x,\alpha(x)\big)\frac{d\alpha}{dx}. \qquad (1)$$

The main point of equation (1), for our immediate purpose, is that since g is differentiable, it is automatically continuous. Hence,

$$\int_a^b g(x)\,dx$$

does exist [and is equal to G(b) - G(a) where G'(x) = g(x)].

In other words,

$$\int_a^b \left[\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right]dx = \int_a^b g(x)\,dx = G(b) - G(a)$$

where G' = g.

Thus, the idea of viewing the anti-derivative in the form of the "definite indefinite integral" as we did in Part 1 of our course is also valid in the present situation.

b. Consider the cylinder whose base is the given region R in the xy-plane and whose top is the surface z = f(x,y). In the previous unit, we showed that the limit of the double infinite sum, denoted by $\iint_R f(x,y)\,dA$, was the volume of the given solid.

5.2.1(L) continued

On the other hand, in our treatment of differentiation of integrals in Block 3, we showed that

$$\int_{\alpha(x_o)}^{\beta(x_o)} f(x_o y)\,dy$$

was the cross-sectional area of the intersection of the solid with the plane $x = x_o$. Thus,

$$\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy = A(x),$$

is the cross-sectional area of the solid. Since the volume of the solid is $\int_a^b A(x)\,dx$, we have that

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right] dx = \int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dydx. \quad (2)$$

Note

One often uses (2) as an identity. That is, we acquire the habit of denoting $\iint_R f(x,y)\,dA$ by $\int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dydx$. In this sense, it becomes confusing as to whether $\int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dydx$ represents a double limit or an iterated integral. To avoid this possible confusion, many authors use $\int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dydx$ when they mean the double limit and they introduce the notation

$$\int_a^b dx \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$$

5.2.1(L) continued

to mean the iterated integral

$$\int_a^b \left[\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right] dx.$$

The point is that <u>when equation (2) holds</u>, the numerical value of

$$\int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\,dx$$

is the same for either interpretation.

c. We simply observe that

$$\int_{\gamma(y_o)}^{\delta(y_o)} f(x,y_o)\,dx = A(y_o)$$

is the cross-section of our solid when it is cut by the plane $y = y_o$. Hence, the volume of our solid is given by

$$\int_c^d A(y)\,dy = \int_c^d \left[\int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx\right] dy.$$

Combining this with the result of equation (2), we have

$$\int_a^b \left[\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right] dx = \int_c^d \left[\int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx\right] dy$$

since both represent the volume of the solid, i.e.,

$$\iint_R f(x,y)\,dA.$$

5.2.1(L) continued

Note

Part (c) supplies us with a geometric argument, at least in the case that $f(x,y) \geqslant 0$, of the Fundamental Theorem, which we elect to state as follows:

Fundamental Theorem

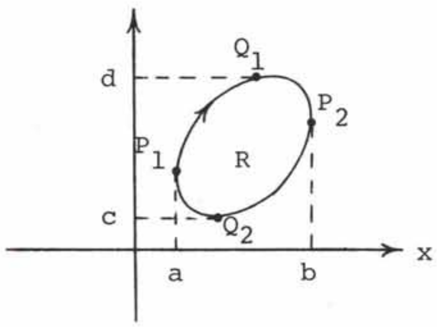
Suppose that the region R can be described in the two ways:

$R = \{(x,y): a \leqslant x \leqslant b,\ \alpha(x) \leqslant y \leqslant \beta(x)\}$

and

$R = \{(x,y): \gamma(y) \leqslant x \leqslant \delta(y),\ c \leqslant y \leqslant d\}$.

Pictorially (for example),



$$\left.\begin{array}{l} Q_1P_2Q_2 \text{ is } x = \delta(y) \\ Q_1P_1Q_2 \text{ is } x = \gamma(y) \end{array}\right\}$$

$$\left.\begin{array}{l} P_1Q_1P_2 \text{ is } y = \beta(x) \\ P_1Q_2P_2 \text{ is } y = \alpha(x) \end{array}\right\}$$

Then if $\alpha(x)$, $\beta(x)$, $\gamma(y)$, $\delta(y)$ are <u>differentiable</u> and f is <u>continuous</u> on R, it follows that

$$\int_a^b \left[\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right] dx = \int_c^d \left[\int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx\right] dy$$

[and both denote the value of $\iint\limits_R f(x,y)\,dA$].

In concluding this exercise, it is only fair to point out that the fundamental theorem is true for far less stringent conditions on the shape of the region R (i.e., as we have stated it, the

5.2.1(L) continued

fundamental theorem gives sufficient but not necessary restrictions on R). While a precise refinement of the theorem to include the most general shape of R is very difficult (but the refinement is described in most texts on advanced calculus), the fact remains that the theorem is valid for virtually any region R which one might expect to encounter in the "real world."

Thus, without belaboring the point, we shall assume that we are dealing only with those regions R for which the theorem is true. Among other things, if R is a simple closed (bounded) region, even if the boundary is not "smooth," the theorem is valid.

It is also true that f need not even be continuous on R provided that it is not "too discontinuous" on R. [This statement will be refined as a note on Exercise 5.2.5(L).] In summary, however, we should note that the existence of the limit denoted by

$$\iint_R f(x,y)\,dA$$

depends on both the region R and the function f.

5.2.2

a. The iterated integral

$$\int_0^3 \left[\int_0^{x^3} x^2y\ dy\right]dx$$

is equal to

$$\int_0^3 \left[\frac{1}{2}\,x^2y^2\Big|_{y=0}^{x^3}\right]dx$$

$$=\int_0^3 \left[\frac{1}{2}\,x^2(x^3)^2 - 0\right]dx$$

$$= \frac{1}{2}\int_0^3 x^8dx = \frac{1}{18}\,x^9\ \Big|_0^3 = \frac{3^9}{18} = \frac{3^7}{2} = \frac{2187}{2}.$$

5.2.2 continued

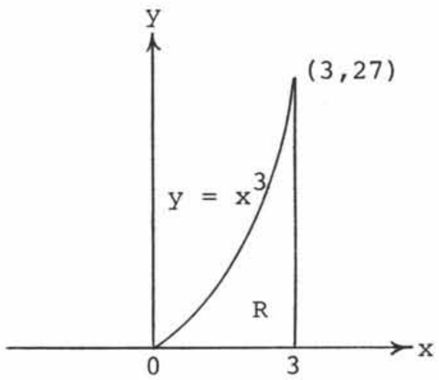
b. From our discussion in the previous exercise

$$\int_0^3 \left[\int_0^{x^3} x^2 y \, dy\right] dx = \iint_R x^2 y \, dA$$

where

$R = \{(x,y): 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant x^3\}.$

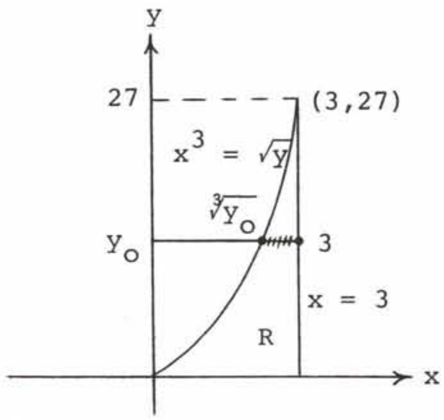
Hence, viewed as a region in the xy-plane, R is given by



Hence, a second description of R is given by

$\{(x,y): \sqrt[3]{y} \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 27\}.$

That is, for (x,y) to be in R, for a fixed of y, x varies continuously from the curve $x = \sqrt[3]{y}$ [i.e., $y = x^3$] to the curve $x = 3$; and y can be chosen anywhere from 0 to 27. Again pictorially,



5.2.2 continued

c. In the notation of the previous exercise; $a = 0$, $b = 3$, $c = 0$, $d = 27$,

$\alpha(x) = 0$, $\beta(x) = x^3$, $\gamma(y) = \sqrt[3]{y}$, $\delta(y) = 3$, and $f(x,y) = x^2y$.

In any event, we have

$$\int_0^3 \left[\int_0^{x^3} x^2y\ dy\right]dx = \int_0^{27}\left[\int_{\sqrt[3]{y}}^3 x^2y\ dx\right]dy.$$

As a check

$$\int_0^{27}\left[\int_{\sqrt[3]{y}}^3 x^2y\ dx\right]dy$$

$$= \int_0^{27}\left[\frac{1}{3}x^3y\ \Big|_{x=\sqrt[3]{y}}^3\ dy\right]$$

$$= \int_0^{27}\left[\frac{1}{3}(3)^3y - \frac{1}{3}(\sqrt[3]{y})^3y\right]dy$$

$$= \int_0^{27}(9y - \frac{1}{3}y^2)dy$$

$$= \frac{9}{2}y^2 - \frac{1}{9}y^3\Big|_0^{27}$$

$$= \frac{9}{2}(27)^2 - \frac{1}{9}(27)^3$$

$$= (27)^2\left[\frac{9}{2} - \frac{1}{9}(27)\right]$$

$$= (27)^2\left[\frac{3}{2}\right] = \frac{3^7}{2} = \frac{2187}{2},$$

which agrees with our answer to (a).

5.2.2 continued

d. Again by the Fundamental Theorem, the iterated integral may be viewed as the volume of the solid which is the cylinder whose base is the region R in the xy-plane and whose top is the surface $z = x^2y$.

It is also the mass of the thin plate whose shape is the region R and whose density at $(x,y) \in R$ is x^2y.

5.2.3

$\int_a^b \int_{\alpha(x)}^{\beta(x)} dydx$ is the special case of $\int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)dydx$ where $f(x,y) \equiv 1$.

Hence, if we let R denote the region

$\{(x,y): a \leqslant x \leqslant b,\ \alpha(x) \leqslant y \leqslant \beta(x)\}$

$\int_a^b \int_{\alpha(x)}^{\beta(x)} dydx$ is the volume of the cylinder whose base is the region R in the xy-plane and whose top is the plane (parallel to the xy-plane) $z = 1$.

We know, however, that the volume of such a cylinder is also given by the product of the area of its base and its altitude; in this case, $A_R \times 1$, or A_R. In other words, the <u>volume</u> of the cylinder <u>numerically</u>* equals the area of R.

5.2.4(L)

a. Treating x as a constant (parameter), we observe that $\int_0^1 \frac{(x - y)dy}{(x + y)^3}$ is well defined [and, in fact, is equal to $F(x,1) - F(x,0)$ where $F_y(x,y) = \frac{x - y}{(x + y)^3}$] provided we can be sure that the integrand

*We say "numerically" because conceptually a volume is not the same as an area.

5.2.4(L) continued

$\frac{x - y}{(x + y)^3}$ is continuous. Since the integrand is the quotient of two polynomials, it is continuous unless the denominator is zero. In particular, $(x + y)^3 = 0 \leftrightarrow x + y = 0 \leftrightarrow \underline{x = -y}$.

Now, since the limits of integration tell us that $0 \leqslant y \leqslant 1$, we see that $\int_0^1 \frac{(x - y)dy}{(x + y)^3}$ is well-defined provided $x \notin [-1,0]$. That is, if g is defined by

$$g(x) = \int_0^1 \frac{(x - y)dy}{(x + y)^3}$$

then

$$\text{dom } g = (-\infty,-1) \cup (0,\infty)$$

(or, $R - [-1,0]$ where R = set of real numbers).

b. Assuming that $x \notin [-1,0]$, we compute g(x) by using integration by parts. In other words, with

$$g(x) = \int_0^1 \frac{(x - y)dy}{(x + y)^3} \tag{1}$$

we let $u = x - y$ and $dv = \frac{dy}{(x + y)^3}$ (recalling that x is being treated as a constant), whereupon

$$du = -dy \quad \text{and} \quad v = \frac{-1}{2(x + y)^2}.$$

Therefore, from (1), we obtain

5.2.4(L) continued

$$g(x) = \left.\frac{-(x - y)}{2(x + y)^2}\right|_{y=0}^{1} - \frac{1}{2}\int_0^1 \frac{dy}{(x + y)^{+2}}$$

$$= \left.\frac{-(x - y)}{2(x + y)^2}\right|_{y=0}^{1} + \left.\frac{1}{2(x + y)}\right|_{y=0}^{1}$$

$$= \left[\frac{-(x - y)}{2(x + y)^2} + \frac{1}{2(x + y)}\right]^1$$

$$= \left.\frac{-(x - y) + (x + y)}{2(x + y)^2}\right|_{y=0}^{1}$$

$$= \left.\frac{y}{(x + y)^2}\right|_{y=0}^{1}. \tag{2}$$

From (2), it follows that

$$g(x) = \frac{1}{(x + 1)^2}. \tag{3}$$

c. $\int_0^1 \left[\int_0^1 \frac{(x - y)dy}{(x + y)^3}\right]dx = \int_0^1 g(x)dx$ <u>except</u> that $g(x)$ is not defined at $x = 0$. In other words, the validity of equation (3) required that $x \notin [-1,0]$, while $\int_0^1 g(x)dx$ indicates that $0 \leqslant x \leqslant 1$, so that $\int_0^1 g(x)dx$ is an improper integral, undefined at $x = 0$. Accordingly, as is the usual case when we talk about improper integrals,

$$\int_0^1 g(x)dx \text{ means } \lim_{b\to 0} \int_b^1 g(x)dx.$$

5.2.4(L) continued

Thus, from (3), we have

$$\int_0^1 g(x)dx = \lim_{b\to 0}\int_b^1 g(x)dx$$

$$= \lim_{b\to 0}\int_b^1 \frac{dx}{(x+1)^2}$$

$$= \lim_{b\to 0}\left[\frac{-1}{x+1}\Big|_b^1\right]$$

$$= \lim_{b\to 0}\left[\frac{-1}{1+1} + \frac{1}{b+1}\right].$$

Therefore,

$$\int_0^1 \left[\int_0^1 \frac{(x-y)dy}{(x+y)^3}\right]dx = \int_0^1 g(x)dx$$

$$= \lim_{b\to 0}\left[-\frac{1}{2} + \frac{1}{b+1}\right]$$

$$= -\frac{1}{2} + 1$$

$$= \frac{1}{2}. \qquad (4)$$

d. To evaluate $\int_0^1 \left[\int_0^1 \frac{(x-y)dx}{(x+y)^3}\right]dy$, we let $h(y) = \int_0^1 \frac{(x-y)dx}{(x+y)^3}$ where $0 \not\leqslant y \leqslant 1$ [i.e., we mimic our procedure of the previous parts only using $h(y)$ rather than $g(x)$; and we restrict the domain of h to $0 \not\leqslant y \leqslant 1$ rather than to $(-\infty,-1) \cup (0,\infty)$ since $\int_0^1 h(y)dy$ implies that we are interested only in those values of y such that $0 \leqslant y \leqslant 1$].

5.2.4(L) continued

Again, integrating by parts, but treating y as a constant, we obtain

$$u = x - y,\ dv = \frac{dx}{(x + y)^3};$$

hence,

$$du = dx \quad \text{and} \quad v = \frac{-1}{2(x + y)^2}.$$

Therefore,

$$h(y) = \left.\frac{-(x - y)}{2(x + y)^2}\right|_{x=0}^{1} + \int_0^1 \frac{dx}{2(x + y)^2}$$

$$= \left[\frac{-(x - y)}{2(x + y)^2} - \frac{1}{2(x + y)}\right]_{x=0}^{1}$$

$$= \left.\frac{-(x - y) - (x + y)}{2(x + y)^2}\right|_{x=0}^{1}$$

$$= -\left.\frac{x}{(x + y)^2}\right|_{x=0}^{1}$$

$$= -\frac{1}{(1 + y)^2}$$

Therefore,

5.2.4(L) continued

$$\int_0^1 \left[\int_0^1 \frac{(x - y)dx}{(x + y)^3}\right]dy = \int_0^1 h(y)dy = \lim_{b\to 0} \int_b^1 \frac{-dy}{(1 + y)^2}$$

$$= \lim_{b\to 0} \left(\frac{1}{1 + y}\Bigg|_{y=b}^{1}\right)$$

$$= \lim_{b\to 0} \left[\frac{1}{1 + 1} - \frac{1}{1 + b}\right]$$

$$= \frac{1}{2} - 1$$

$$= -\frac{1}{2}. \tag{5}$$

e. Comparing (4) and (5), we see that

$$\int_0^1 \left[\int_0^1 \frac{(x - y)dy}{(x + y)^3}\right]dx = \frac{1}{2} = -\int_0^1 \left[\int_0^1 \frac{(x - y)dx}{(x + y)^3}\right]dy.$$

This does not contradict the fundamental theorem since the fundamental theorem requires that $f(x,y)$ be <u>continuous</u> on $R = \{(x,y): 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ if we are to conclude that

$$\int_0^1 \left[\int_0^1 f(x,y)dy\right]dx = \int_0^1 \left[\int_0^1 f(x,y)dx\right]dy.$$

In this example $f(x,y) = \dfrac{x - y}{(x + y)^3}$ and this is <u>not</u> continuous at $(0,0) \in R$. That is, $f(0,0) = \dfrac{0 - 0}{(0 + 0)^3} = \dfrac{0}{0}$ and this is undefined; so that $(0,0) \notin \text{dom } f$, while $(0,0) \in R$.

In any event, this example should prove the point that, even in apparently-simple cases, we cannot arbitrarily replace dydx by dxdy without changing our answer.

5.2.5(L)

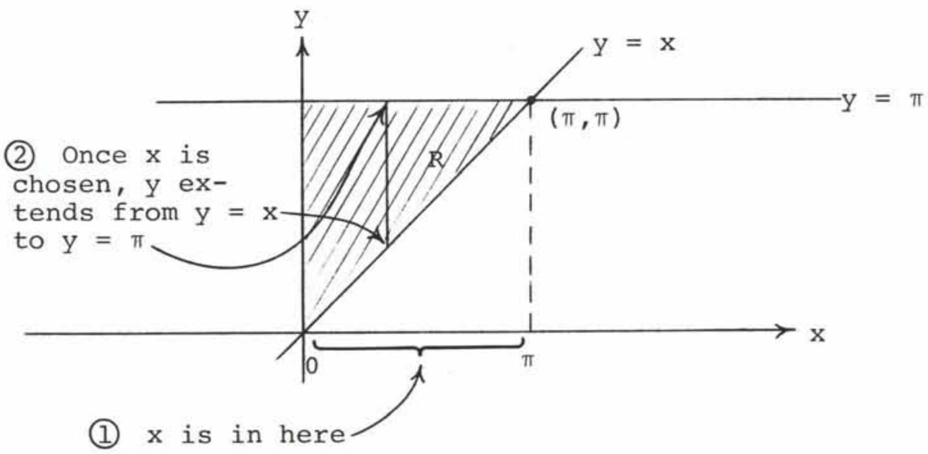
From a computationally-mechanical point of view, this problem is not too difficult to solve. We have

$$\int_0^{\pi}\left[\int_x^{\pi} \frac{\sin y}{y}\, dy\right]dx = \int_0^{\pi}\left[\int_0^{y} \frac{\sin y}{y}\, dx\right]dy. \qquad (1)$$

Again, we obtained our new limits of integration pictorially. That is, the region R described by the limits of integration in the integral on the left side of equation (1) is the set

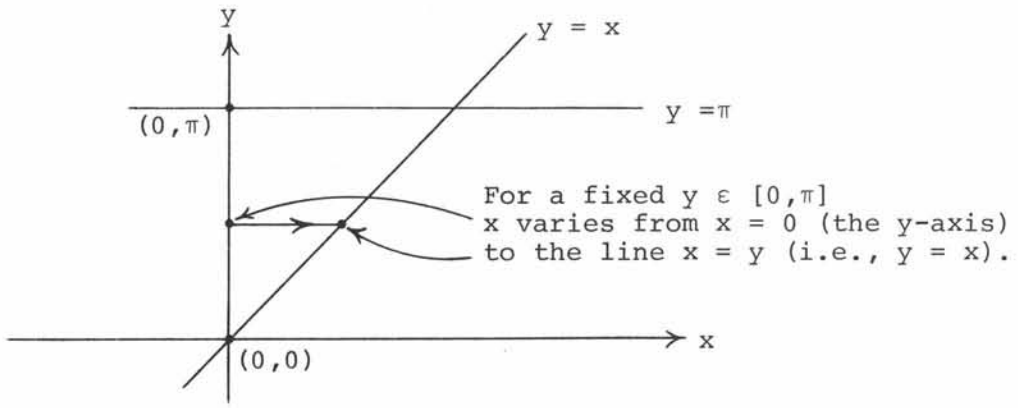
$R = \{(x,y): 0 \leqslant x \leqslant \pi,\ x \leqslant y \leqslant \pi\}.$

This in turn says that for a fixed $x \in [0,\pi]$, y varies continuously from $y = x$ to $y = \pi$. Hence,



From this diagram, we see that R may also be obtained by picking $y \in [0,\pi]$ and then letting x vary from $x = 0$ to $x = y$. That is,

5.2.5(L) continued



and this accounts for equation (1).

At any rate,

$$\int_0^\pi \left[\int_0^y \frac{\sin y}{y}\,dx\right]dy = \int_0^\pi \frac{x \sin y}{y}\Big|_{x=0}^{y} dy$$

$$= \int_0^\pi \left\{\frac{y \sin y}{y} - \frac{0 \sin y}{y}\right\}dy \qquad (2)$$

$$= \int_0^\pi \sin y\,dy \qquad (3)$$

$$= 2,$$

so, again from equation (1),

$$\int_0^\pi \left[\int_x^\pi \frac{\sin y}{y}\,dy\right]dx = 2.$$

Thus, what we have shown here (similar to a result obtained in the lecture) is that in this problem a certain integrand which is difficult to handle with the given order of integration (i.e., $\int_a^b \frac{\sin y}{y}\,dy$ cannot be handled by "ordinary" techniques in terms

5.2.5(L) continued

of an anti-derivative), is easy to handle if the order of integration is reversed.

Both this exercise and the one used in the lecture are examples of the special case in which $f(x,y)$ is a function of either only x or only y. In other words, even if we know of no function $H(y)$ such that $H'(y) = h(y)$, the fact remains that if y is treated as a parameter then $\int h(y)\,dx = xh(y)$. Thus, if $f(x,y) = h(y)$, we have

$$\int_a^b \left[\int_{\gamma(x)}^{\delta(x)} f(x,y)\,dy\right]dx = \int_a^b \left[\int_{\gamma(x)}^{\delta(x)} h(y)\,dy\right]dx$$

$$= \int_c^d \left[\int_{\alpha(y)}^{\beta(y)} h(y)\,dx\right]dy$$

$$= \int_c^d xh(y)\Big|_{x=\alpha(y)}^{\beta(y)} dy$$

$$= \int_c^d [\beta(y) - \alpha(y)]h(y)\,dy \qquad (4)$$

and it may just happen (as in this exercise) that $[\beta(y) - \alpha(y)]$ is an "integrating factor." That is, we <u>may</u> be able to find a function $g(y)$ such that

$$g'(y) = [\beta(y) - \alpha(y)]h(y)$$

even though we could not find $H(y)$ such that $H'(y) = h(y)$.

Should this be the case, then (4) yields

$$\int_a^b \left[\int_{\gamma(x)}^{\delta(x)} h(y)\,dy\right]dx = g(d) - g(c).$$

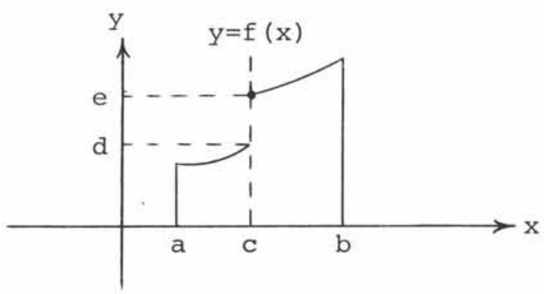
Be this as it may, we have an equally important, different reason for stressing this exercise, and this reason is the subject of the following note.

5.2.5(L) continued

Note:

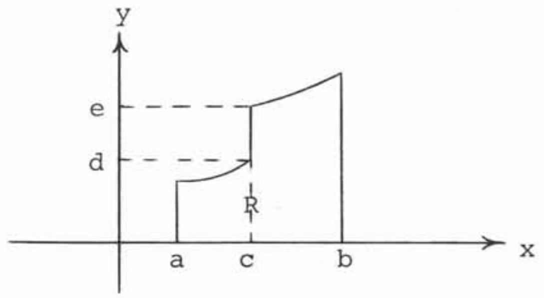
In our discussion of The Fundamental Theorem, we talked about the case in which f(x,y) was continuous on R. At that time, we did not want to cloud the basic issue with still other fine points, but at this time we wish to point out that just as in the case of functions of a single real variable, the condition that f be continuous on R is too strong (that is, it is a sufficient but not necessary condition) if all we desire is that f be integrable on R. In fact, all that is required is that f be bounded on R (that is, we must be very wary of what happens if f "blows up" at one or more points of R) and that once this is insured, f must not have "too many" points of discontinuity on R.

More precisely, one usually introduces the concept of a set of content (measure) zero (which can be generalized into a discussion of Lebesgue Measure); the vague term "too many" becomes replaced by the condition that the points at which the bounded function f is discontinuous on R is a set of content zero. From a semi-intuitive point of view, the easiest way to view a set of content zero is as a set which has zero area. For example, if f(x,y) is discontinuous on some line or curve in R, the volume of the region is not affected by the deletion of this curve since the curve has no thickness. In the calculus of a single variable, this is equivalent to studying the area under a curve which has finite discontinuities on a set of content zero. Namely,



① $\int_a^b f(x)dx$ may still be viewed as an area even though f is discontinuous at x = c.

5.2.5(L) continued



② We may "add" the set of points (line segment) $\{(x,y): x = c, d \leqslant y \leqslant e\}$ without changing the area.

At any rate, at least in the two-dimensional case, we may generalize the fundamental theorem by including $\iint\limits_R f(x,y)dA$ where f is bounded on R [i.e., f(x,y) is finite for all $(x,y) \in R$] and f is continuous on R except at most on a set of points of content zero.

Applying this to Exercise 5.2.4(L), notice that f(x,y) was not bounded on R. Namely, if $f(x,y) = \dfrac{x - y}{(x + y)^3}$ and $R = \{(x,y): 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ then f(0,0) is undefined, but more importantly $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \infty$ so that f is unbounded in any neighborhood of (0,0). More specifically, if we let $(x,y)\to(0,0)$ along the line L, y = mx, we obtain

$$f(x,y) = f(x,mx) = \frac{x - mx}{(x + mx)^3} = \left[\frac{1 - m}{(1 + m)^3}\right]\frac{1}{x^2}.$$

Hence, on L

$$\lim_{x\to 0} f(x,y) = \left[\frac{1 - m}{(1 + m)^3}\right]\left[\frac{1}{\lim\limits_{x\to 0} x^2}\right]$$

$$= \infty, \text{ [unless } m = 1\text{, in which case } f(x,y) \equiv 0 \text{ on L]}.$$

5.2.5(L) continued

Thus, the fundamental theorem did not apply in that case, even though f was 0 discontinuous only at the single point (0,0).

In the present exercise, $f(x,y) = \frac{\sin y}{y}$ and this is discontinuous when $y = 0$ [i.e., undefined since $f(x,0) = \frac{0}{0}$]. However, f(x,y) is bounded for all points (x,y); the only non-obvious case being at (0,0), but $\lim_{y\to 0} \frac{\sin y}{y} = 1$, so all is well. Moreover, the set of points <u>in R</u> for which $y = 0$ consists of the single point (0,0), and this set clearly has content zero.

Thus, the fundamental theorem applies in this exercise even though f is discontinuous at (0,0) while it didn't apply in the previous exercise when f was discontinuous at (0,0).

5.2.6

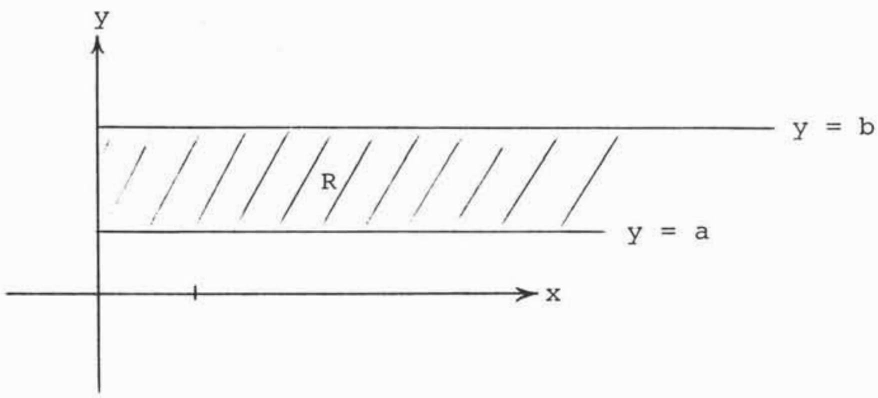
$$\int_a^b e^{-xy}dy = -\frac{1}{x}e^{-xy}\Big|_{y=a}^{y=b} = -\frac{1}{x}e^{-bx} - \left[-\frac{1}{x}e^{-ax}\right] = \frac{e^{-ax} - e^{-bx}}{x}.$$

(Keep in mind that x is being treated as a <u>constant</u>.)

We then have

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}dx = \int_0^\infty \left[\int_a^b e^{-xy}dy\right]dx. \qquad (1)$$

Our region R is given by



5.2.6 continued

(Technically speaking, R is bounded on the right by $x = c$, say, and we take the limit as $c \to \infty$; i.e.,

$$\int_0^\infty f(x)\,dx = \lim_{c\to\infty} \int_0^c f(x)\,dx.)$$

Therefore,

$$R = \{(x,y): a \leqslant y \leqslant b,\ 0 \leqslant x < \infty\},$$

so by (1),

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \int_a^b \left[\int_0^\infty e^{-xy}dx\right]dy$$

$$= \int_a^b -\frac{1}{y}\, e^{-xy} \Big|_{x=0}^{\infty} dy$$

$$= \int_a^b \left(-\frac{1}{y}\, e^{-\infty} + \frac{1}{y}\right) dy$$

or

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \int_a^b \frac{dy}{y}$$

$$= \ln|b| - \ln|a|$$

$$= \ln b - \ln a \text{ (since } 0 < a < b)$$

$$= \ln \frac{b}{a}$$

and we see from this example that a single integral is sometimes evaluated by converting it to an appropriate double integral.

5.2.6 continued

Note

$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$ also causes trouble when $x = 0$. Thus, if we wished a precise formulization of this problem, we should view $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$ as

$$\lim_{\varepsilon\to 0} \int_\varepsilon^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \lim_{c\to\infty}\left[\lim_{\varepsilon\to 0} \int_\varepsilon^c \frac{e^{-ax} - e^{-bx}}{x}\,dx\right]$$

and proceed accordingly.