Unit 5: The Total Differential Revisited

4.5.1

Since f is continuously differentiable in a neighborhood of (a,b), we have that

$$f(a + h, b + k) - f(a,b) = f_x(a,b)h + f_y(a,b)k + e_1h + e_2k$$

where e_1 and e_2 both approach zero as h and k approach zero. This, in turn, says that $e_1h + e_2k$ goes to zero "faster" than either h or k (that is, $e_1h + e_2k$ is a higher order infinitesimal), and this is what we mean when we say that near (a,b), we may view $\Delta w = f(a + h, b + k) - f(a,b)$ as a linear combination of h and k, i.e., $f_x(a,b)h + f_y(a,b)k$.

In still other words, in a sufficiently small neighborhood of (a,b), the error involved in neglecting $e_1h + e_2k$ is negligible both from a practical point of view as well as from a theoretical point of view, and accordingly, we may think in terms of saying that Δw "behaves like"

$$f_x(a,b)h + f_y(a,b)k. \qquad (1)$$

If we observe in (1) that h is what we usually denote by Δx and k is what we usually denote as Δy, we see that expression (1) is simply what we have previously called Δw_{tan} (or in higher dimensions, Δw_{lin}). That is, Δw_{tan} is exactly equal to (1) and it is Δw which is approximately equal to Δw_{tan}.

a. Given that $w = f(x,y) = x^2 - y^2$, we have

$$f_x(x,y) = 2x$$
$$f_y(x,y) = -2y\ .$$

Hence

$$f_x(3,2) = 6$$
$$f_y(3,2) = -4\ .$$

4.5.1 continued

Therefore,

$$f(3 + h, 2 + k) - f(3,2) \approx 6h - 4k \tag{2}$$

if (3 + h, 2 + k) is sufficiently close to (3,2).

In particular f(3.001, 1.99) has the form f(3 + h, 2 + k) with h = 0.001 and k = -0.01. Therefore, (2) becomes

$$\begin{aligned} f(3.001, 1.99) - f(3,2) \approx 6(0.001) - 4(-0.01) &= 0.006 + 0.04 \\ &= 0.046. \end{aligned} \tag{3}$$

Since $f(3,2) = 3^2 - 2^2 = 5$, equation (3) yields

$$f(3.001, 1.99) \approx 5 + 0.046 = 5.046. \tag{4}$$

An exact computation of f(3.001, 1.99) yields f(3.001, 1.99)

$$\begin{aligned} &= (3.001)^2 - (1.99)^2 \\ &= 9.006001 - 3.9601 \\ &= 5.045901. \end{aligned} \tag{5}$$

A comparison of (4) and (5) shows the percentage error in the approximation is

$$\frac{|5.045901 - 5.046|}{5.045901} \times 100$$

or

$$\frac{0.000099 \times 100}{5.045901}.$$

Therefore the error is about 0.002%.

b. $$f(7,5) = 7^2 - 5^2 = 24 . \tag{6}$$

Since f(7,5) = f(3 + 4, 2 + 3), equation (2) would yield

$$f(7,5) - f(3,2) = 6(4) - 4(3).$$

4.5.1 continued

Therefore,

$$f(7,5) \approx f(3,2) + 24 - 12 = 5 + 24 - 12 = 17. \tag{7}$$

If we compare (6) and (7) we see that the percentage error in our approximation is $\frac{7}{24} \times 100$ or nearly 30%.

The point is that (7,5) is "far enough away from" (3,2) so that the error term $e_1h + e_2k$ is no longer negligible. Notice that $f_x(3,2)h + f_y(3,2)k$ is still exactly equal to Δw_{tan}, but that Δw_{tan} is not a good approximation for $\Delta w[= f(7,5) - f(3,2)]$.

c. $$\begin{aligned} f(3 + h, 2 + k) - f(3,2) &= (3 + h)^2 - (2 + k)^2 - 5 \\ &= 9 + 6h + h^2 - 4 - 4k - k^2 - 5 \\ &= (6h - 4k) + (h^2 - k^2). \end{aligned} \tag{8}$$

Since $f_x(3,2)h + f_y(3,2)k = 6h - 4k$, equation (8) becomes

$$f(3 + h, 2 + k) - f(3,2) = f_x(3,2)h + f_y(3,2)k + (h^2 - k^2). \tag{9}$$

We can now write $h^2 - k^2$ in the form $e_1h + e_2k$ as follows:

$$\begin{aligned} h^2 - k^2 &= (h - k)(h + k) \\ &= (h - k)h + (h - k)k . \end{aligned} \tag{10}$$

Letting $e_1 = e_2 = h - k$ we see that e_1 and e_2 approach zero as h and k approach zero. Thus, using the result of (10) in (9), we have

$$f(3 + h, 2 + k) = f(3,2) + f_x(3,2)h + f_y(3,2)k + e_1h + e_2k$$

where e_1 and e_2 approach zero as h and k approach zero since $e_1 = e_2 = h - k$.

With respect to part (a) we had $h = 0.001$ and $k = -0.01$. Hence $h - k = 0.001 - (-0.01) = 0.011$.

Therefore,

4.5.1 continued

$$e_1h + e_2k = (h - k)h + (h - k)k$$

$$= (0.011)(0.001) + (0.011)(-0.01)$$

$$= 0.000011 - 0.00011$$

$$= -0.000099$$

which checks with the computation 5.045901 - 5.046 of part (a).

Notice that the determination of e_1 and e_2 is not unique. For example, equation (8) could have been written as

$$f(3 + h, 2 + k) - f(3,2) = (6h - 4k) + h(h) + (-k)k, \qquad (1')$$

in which case we could let $e_1 = h$ and $e_2 = -k$. [This is actually more straight-forward than the choice of e_1 and e_2 from equation (10), but the former method shows that e_1(and e_2) <u>may</u> depend on <u>both</u> h and k.]

What is important is the fact that e_1 and e_2 approach zero as h and k approach zero. It is not too important (except for computational case in making certain approximations) otherwise how e_1 and e_2 are expressed in terms of h and k.

4.5.2

The main aim of this exercise is to emphasize why we always specify "continuously differentiable" when we talk about differentials.

Notice that we have already, in previous units, done the computations required for this exercise. By way of review, we have shown that $w_x(0,0) = w_y(0,0) = 0$. Namely, for example,

$$w_x(0,0) = \lim_{\Delta x \to 0} \left[\frac{w(\ x,0) - w(0,0)}{\Delta x} \right]$$

4.5.2 continued

$$= \lim_{x \to 0} \left[\frac{\frac{2\ \Delta x(0)}{\Delta x^2 + 0^2} - 0}{x} \right]$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$

$$= 0\ .$$

Hence $w_x(0,0)\ dx + w_y(0,0)dy$ is well-defined. The crucial point, however, is that this is not a good approximation for Δw in any neighborhood of $(0,0)$, no matter how small the neighborhood!

In particular

$$w_x(0,0)dx + w_y(0,0)dy \equiv 0 \qquad (1)$$

while

$$\Delta w = w(x,y) - w(0,0) \qquad [(x,y) \neq (0,0)]$$

$$= \frac{2xy}{x^2 + y^2}\ . \qquad (2)$$

Introducing polar cordinates, (2) becomes

$$\Delta w = \frac{2r^2 \sin\theta \cos\theta}{r^2}$$

and since $r \neq 0$ [i.e., $(x,y) \neq (0,0)$]

$$\Delta w = 2 \sin\theta \cos\theta$$
$$= \sin 2\theta. \qquad (3)$$

From (3) we see that in any neighborhood of $(0,0)$ Δw takes on all values from -1 to 1 inclusively, while from (1) we see that $w_x(0,0)dx + w_y(0,0)dy$ is always zho.

In other words if we let $dw = w_x(0,0)dx + w_y(0,0)dy$, dw is not a reasonable approximation for Δw. For this reason we refrain

4.5.2 continued

from using the notation dw unless w is a continuously differentiable function of x and y.

As a final computational check, note that "continuously differentiable" means that w_x and w_y not only exist but they are also continuous.

If we compute $w_x(x,y)$ at any point $(x,y) \neq (0,0)$, we have

$$w_x(x,y) = \frac{(x^2 + y^2)2y - 2xy(2x)}{(x^2 + y^2)^2}$$

$$= \frac{2y^3 - 2x^2y}{(x^2 + y^2)^2} \quad . \tag{4}$$

If we apply polar coordinates to (4), we see that

$$w_x(x,y) = \frac{2r^3\sin^3\theta - 2r^2\cos^2\theta\ r\sin\theta}{r^4}$$

$$= \frac{2(\sin^3\theta - \sin\theta\cos^2\theta)}{r} \quad . \tag{5}$$

From (5) we see that

$$\lim_{(x,y)\to(0,0)} w_x(x,y) = \lim_{r\to 0} \frac{2(\sin^3\theta - \sin\theta\cos^2\theta)}{r} \quad . \tag{6}$$

Therefore, unless $\sin^3\theta - \sin\theta\cos^2\theta = 0$, equation (6) reveals that

$$\lim_{(x,y)\to(0,0)} w_x(x,y) = \infty \neq w(0,0) \ [= 0].$$

In summary, in this exercise w is not continuously differentiable in any neighborhood of (0,0), and it turned out that $w_x(0,0)\,dx + w_y(0,0)\,dy$ was not a good approximation for Δw.

4.5.3

Given that

$$\left.\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \end{aligned}\right\} \tag{1}$$

our definition of differentials yields

$$\left.\begin{aligned} du &= 2xdx - 2ydy \\ dv &= 2ydx + 2xdy \end{aligned}\right\} \tag{2}$$

We may now solve (2) for dx and dy in terms of du and dv to obtain

$$xdu + ydv = (2x^2 + 2y^2)dx$$

and

$$-ydu + xdv = (2x^2 + 2y^2)dy$$

so that

$$\left.\begin{aligned} dx &= \frac{x}{2x^2 + 2y^2}\,du + \frac{y}{2x^2 + 2y^2}\,dv \\ &\text{and} \\ dy &= \frac{-y}{2x^2 + 2y^2}\,du + \frac{x}{2x^2 + 2y^2}\,dv \end{aligned}\right\} \tag{3}$$

Assuming that x and y are continuous differentiable functions of u and v, we also know that

$$\left.\begin{aligned} dx &= x_u du + x_v dv \\ &\text{and} \\ dy &= y_u du + y_v dv \end{aligned}\right\} \tag{4}$$

Then, since du and dv are independent variables, we know that $M_1du + N_1dv \equiv M_2du + N_2dv \leftrightarrow M_1 \equiv M_2$ and $N_1 \equiv N_2$. Therefore, equating the expressions for dx and dy in (3) and (4) we may

4.5.3 continued

conclude that

$$x_u = \frac{x}{2(x^2 + y^2)} \quad , \quad x_v = \frac{y}{2(x^2 + y^2)}$$

$$y_u = \frac{-y}{2(x^2 + y^2)} \quad , \quad y_v = \frac{x}{2(x^2 + y^2)}$$

4.5.4(L)

First of all, let us observe that

$$\underline{f}(x,y) = (x^2 - y^2, 2xy) \tag{1}$$

is another way of writing

$$\left.\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \end{aligned}\right\} . \tag{2}$$

That is, we may think of $\underline{f}(x,y)$ as being the 2-tuple (u,v) [since $\underline{F}:E^2 \to E^2$] and then by (1), since $(u,v) = (x^2 - y^2, 2xy)$ it follows from the definition of equality that $u = x^2 - y^2$ and $v = 2xy$.

a. This part of the exercise is designed merely to help make sure that you understand the language implied by (1) and (2).

(1) From (1)

$$\underline{f}(3,2) = [3^2 - 2^2, 2(3)(2)]$$

$$= (5,12).$$

(2) Again from (1)

$$\underline{f}(3.001, 1.99) = [(3.001)^2 - (1.99)^2, 2(3.001)(1.99)]$$

and, leaving the computational details to the reader, this means

4.5.4(L) continued

$$\underline{f}(3.001,\ 1.99) = (5.045901,\ 11.94398) \tag{3}$$

(3) This is a generalization of (2). Namely, from (1)

$$\underline{f}(3 + h,\ 2 + k) = [(3 + h)^2 - (2 + k)^2,\ 2(3 + h)(2 + k)]$$

$$= (9 + 6h + h^2 - 4 - 4k - k^2,$$

$$12 + 4h + 6k + 2hk)$$

$$= (5 + 6h - 4k + h^2 - k^2,\ 12 + 4h + 6k + 2hk). \tag{4}$$

As a partial check of equation (4), let us observe that equation (3) is the special case $h = 0.001$, $k = -0.01$ since $3.001 = 3 + 0.001$ and $1.99 = 2 + (-0.01)$. Putting these values of h and k into (4) yields

$$\underline{f}(3.001,\ 1.99) = [5 + 6(0.001) - 4(-0.01) + (0.001)^2 - (-0.01)^2,$$

$$12 + 4(0.001) + 6(-0.01) + 2(0.001)(-0.01)]$$

$$= (5 + 0.006 + 0.04 + 0.000001 - 0.0001,$$

$$12 + 0.004 - 0.06 - 0.00002)$$

$$= (5.045901,\ 11.94398),$$

which checks with (3).

b. Here we start to come to grips with what linear algebra is all about using the language of differentials discussed in the previous unit, equations (2) yield

$$\left.\begin{aligned} du &= 2x\,dx - 2y\,dy \\ dv &= 2y\,dx + 2x\,dy \end{aligned}\right\} \tag{5}$$

In another form (5) says that

4.5.4(L) continued

$$\left.\begin{aligned}\Delta u_{tan} &= 2x\Delta x - 2y\Delta y\\ \Delta v_{tan} &= 2y\Delta x + 2x\Delta y\end{aligned}\right\} \quad . \tag{5'}$$

In particular, when $x = 3$ and $y = 2$, we see from (5) that

$$\left.\begin{aligned}du &= 6\Delta x - 4\Delta y\\ dv &= 4\Delta x + 6\Delta y\end{aligned}\right\} \quad . \tag{6}$$

The point we are stressing in this part of the exercise is that <u>sufficiently close</u> to (3.2) du is a good approximation for Δu and dv is a good approximation for Δv. Hence, (and again letting $h = \Delta x$ and $k = \Delta y$)

$$\underline{f}(3 + h, 2 + k) = (5 + \Delta u, 12 + \Delta v) \quad . \tag{7}$$

[That is, $\underline{f}(3,2) = (5,12)$ and changing the input of $\underline{f}$ to $(3 + h, 2 + k)$ means that the new output has the form $(5 + \Delta u, 12 + \Delta v)$.]

If we now make the assumptions that $du \approx \Delta u$ and $dv \approx \Delta v$, we may substitute (5) [or (5') into (7) to obtain

$$\underline{f}(3 + h, 2 + k) \approx (5 + 6\,\Delta x - 4\Delta y, 12 + 4\,\Delta x + 6\,\Delta y)$$

$$= (5 + 6h - 4k, 12 + 4h + 6k). \tag{8}$$

If we compare (8) with (4) we see that the error in our u-coordinate is $h^2 - k^2$ (which certainly goes to zero "rapidly" as h and k approach zero), while the error in our v-coordinate is 2hk (which also goes to zero rapidly).

Thus, <u>near</u> (3,2) the mapping (function), $\underline{f}:E^2 \rightarrow E^2$ defined by (1), may be replaced by the <u>linear</u> mapping ,

$\underline{g}:E^2 \rightarrow E^2$, defined by (8).

That is, $\underline{g}(3 + h, 2 + k) = (5 + 6h - 4k, 12 + 4h + 6k)$ so that

4.5.4(L) continued

$$\Delta \underline{g} = \underline{g}(3 + h, 2 + k) - \underline{g}(3,2)$$

$$= (5 + 6h - 4k, 12 + 4h + 6k) - (5,12)$$

$$= (6h - 4k, 4h + 6k). \tag{9}$$

If we write (9) in the form

$$\Delta \underline{g} = (\Delta u, \Delta v)$$

we see that

$$\left.\begin{aligned} \Delta u &= 6\Delta x - 4\Delta y \\ \Delta v &= 4\Delta x + 6\Delta y \end{aligned}\right\} . \tag{10}$$

Notice in (10) that these are no longer approximations. These are the exact values for Δx and Δy if we are dealing with $\underline{g}$. It is only an approximation when we replace $\underline{f}$ by $\underline{g}$.

The important point is that near (3,2), the values of $\underline{f}(3 + h, 2 + k)$ and $\underline{g}(3 + h, 2 + k)$ are "about the same". The advantage of using $\underline{g}$ rather than $\underline{f}$ lies in the linear properties of $\underline{g}$.

At any rate,

$$\underline{g}(3 + h, 2 + k) = (5 + 6h - 4k, 12 + 4h + 6k)$$

so that

$$\begin{aligned} \underline{g}(3.001, 1.99) &= [5 + 6(0.001) - 4(-0.01), 12 + 4(0.001) + \\ &\qquad 6(-0.01)] \\ &= (5.046, 11.944). \end{aligned} \tag{11}$$

Comparing (11) with (3) shows that $\underline{g}$ is indeed a good approximation for $\underline{f}$ at (3.001, 1.99). In fact, using the language

4.5.4(L) continued

of the Euclidean metric, we have

$$\|\underline{f}(3.001, 1.99) - \underline{g}(3.001, 1.99)\| = \|(5.045901, 11.94898) - (5.046, 11.944)\|$$

$$= \sqrt{(5.045901 - 5.046)^2 +} \sqrt{(11.94398 - 11.944)^2}$$

$$\approx \sqrt{1 \times 10^{-8}}$$

$$\approx 10^{-4}. \qquad (12)$$

As we shall see in the next exercise, one does not need a geometric interpretation for replacing $\underline{f}:E^n \to E^n$ by $\underline{g}:E^n \to E^n$, but in the cases n = 1 or 2 (especially n = 1), there is an interesting geometric interpretation which we shall discuss in the following note.

A Note on Exercise 4.5.4(L)

In the special case r = 1, $\underline{f}:E^n \to E^n$ reduced to a function of a single variable, for which we usually used the notation $y = f(x)$.

The idea was that if f was differentiable* at x = c, then $y = f'(c)x + c_1$ was a good approximation for $y = f(x)$ in a sufficiently small neighborhood of x = c. In terms of a graph, all we were saying is that near x = c [i.e., near the point (c,f(c)] we could replace the curve by the line

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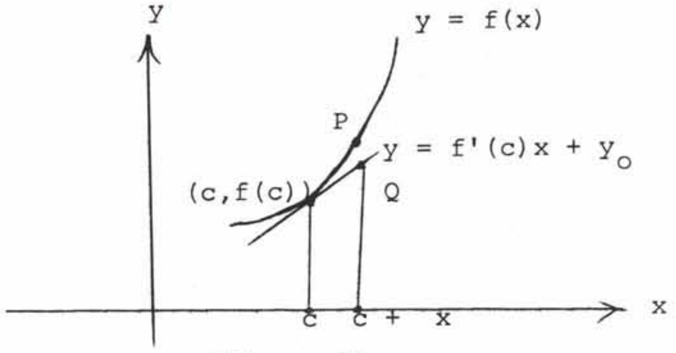
*Notice that in the case r = 1 we talked about f being differentiable at x = c. We did not invoke the more stringent condition that f be continuously differentiable at x = c, which would have meant that f' be continuous at x = c. The reason for this, quite simply, is that f' exists at x = c it is automatically continuous there. Pictorially, since the curve is smooth in a neighborhood of (c,f(c)) the slope can be made as nearly equal to f'(c) at a point P on y = f(x) just be choosing P sufficiently close to (c, f(c)).

4.5.4(L) continued

tangent to the curve at $(c,f(c))$. That is



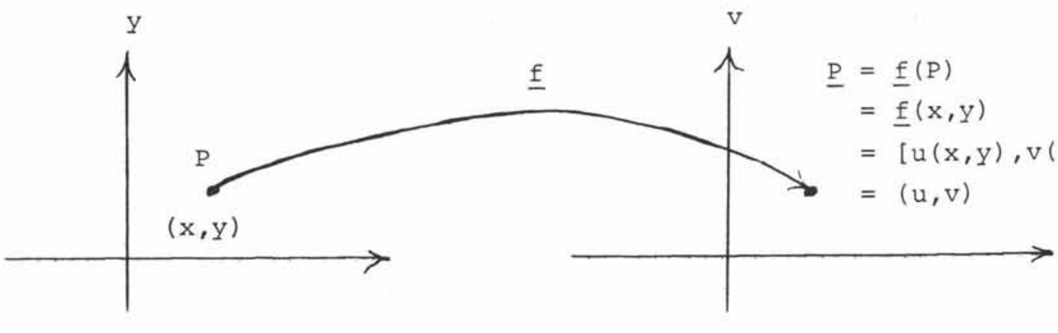
The distance between P and Q goes to zero faster than the distance between c and $c + \Delta x$ as $\Delta x \to 0$.

(Figure 1)

In terms of the language of Exercise 4.5.4 (6), $y = f'(c)x + y_o$ is the <u>linear</u> (straight line) function $\underline{g}:E^n \to E^n$ in this case.

In the case $n = 2$ we may view $\underline{f}:E^2 \to E^2$ as a mapping of the plane into itself. To avoid confusion we refer to the domain of $\underline{f}$ as the xy-plane and to the range of $\underline{f}$ as the uv-plane. While this might seem strange, notice that we have already done something like this in the case $n = 1$. Namely when $f:E \to E$ the domain of f and the range of f are the same (namely, E). Pictorially this means that both the domain and range of f are the number line. Yet to avoid confusion, we refer to the domain as the <u>x-axis</u> while the range is called the <u>y-axis</u>.

At any rate we have

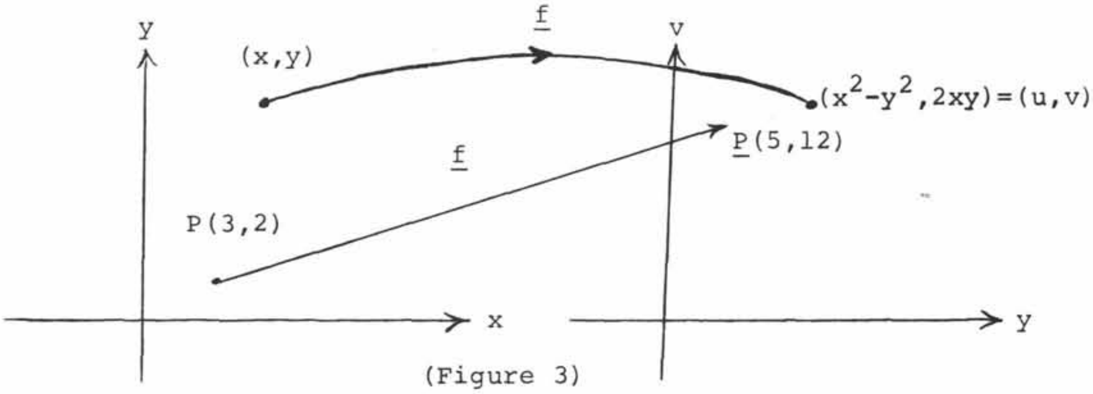


$$\begin{aligned}\underline{P} &= \underline{f}(P)\\ &= \underline{f}(x,y)\\ &= [u(x,y),v(x,y)]\\ &= (u,v)\end{aligned}$$

(Figure 2)

In terms of the specific $\underline{f}$ and P of Exercise 4.5.4(L) we have

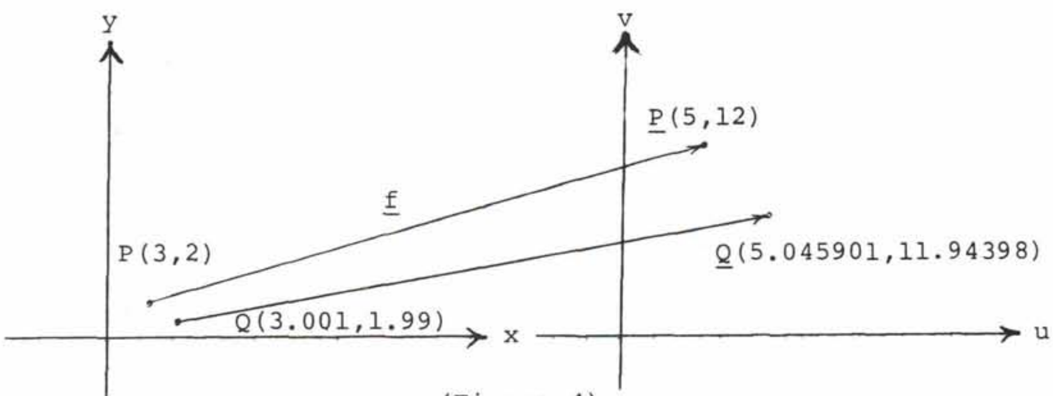
4.5.4(L) continued



(Figure 3)

If we let Q denote the point (3.001, 1.99) in the xy-plane then $\underline{f}(Q) = \underline{Q}(5.045901, 11.94398)$. If we were to add this information to Figure 3, drawn to scale, the points P and Q would seem to coincide since both would lie within the dot which denotes P.

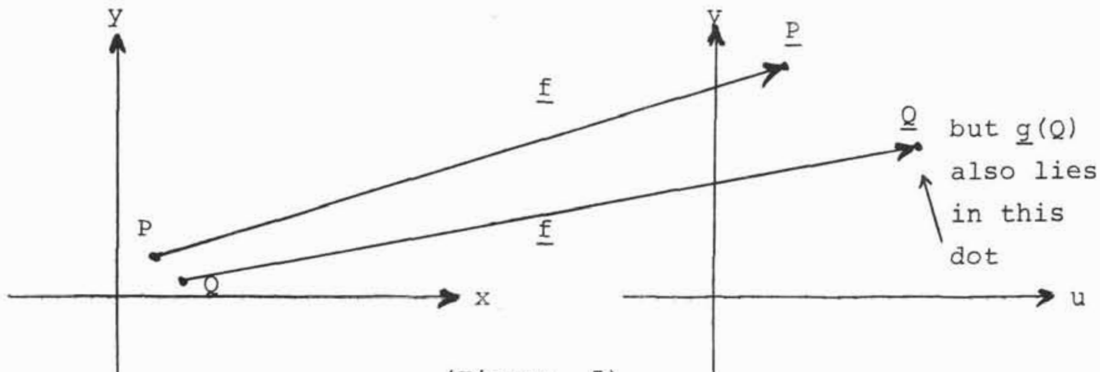
A similar result pertains to $\underline{P}$ and $\underline{Q}$. So let us distort the graph a bit so that all four points are clearly labeled.



(Figure 4)

According to equation (12) of Exercise 4.5.4(L), if we draw a circle centered at $\underline{Q}$ with radius $\sqrt{1 \times 10^{-8}} \approx 10^{-4}$, $\underline{g}(3.001, 1.99)$ would lie within this circle. Clearly, if drawn to scale, this circle would be encompassed by the dot that names $\underline{Q}$. That is, at least geometrically speaking, we would conclude that $\underline{f}(3.001, 1.99) = \underline{g}(3.001, 1.99)$, since we could not distinguish between these two points in the uv-plane. Again pictorially,

4.5.4(L) continued



(Figure 5)

Mapping the xy-plane into the uv-plane is not as geometrically "pleasant" as mapping the x-axis into the y-axis. That is, we may place the y-axis at right angles to the x-axis so that the mapping may be viewed as a plane curve. But in this context, mapping E^2 into E^2 would require a 4-dimensional graph since both the domain and range of $\underline{f}$ are 2-dimensional.

We shall, little by little, talk more about mappings of the xy-plane into the uv-plane as we continue with our course.

4.5.5

a. Our mapping in this case is equivalent to the change of variables

$$\left.\begin{aligned} u_1 &= x_1^{\,2} + x_2^{\,2} + x_3^{\,2} + x_4^{\,4} \\ u_2 &= x_1x_2x_3x_3 \\ u_3 &= x_1^{\,3} + x_2^{\,3} + x_3x_4 \\ u_4 &= x_1^{\,2} + x_2x_3 + x_1x_4^{\,2} \end{aligned}\right\} . \qquad (1)$$

Using differentials, (1) yields

$$\left.\begin{aligned} du_1 &= 2x_1dx_1 + 2x_2dx_2 + 2x_3dx_3 + 2x_4dx_4 \\ du_2 &= x_2x_3x_4dx_1 + x_1x_3x_4dx_2 + x_1x_2x_4dx_3 + x_1x_2x_3dx_4 \\ du_3 &= 3x_1^{\,2}dx_1 + 3x_2^{\,2}dx_2 + x_4dx_3 + x_3dx_4 \\ du_4 &= (2x_1 + x_4^{\,2})dx_1 + x_3dx_2 + x_2dx_3 + 2x_1x_4dx_4 \end{aligned}\right\} . \qquad (2)$$

4.5.5 continued

If we now let $x_1 = x_2 = x_3 = x_4 = 1$, the system (2) becomes

$$\left.\begin{aligned} du_1 &= 2dx_1 + 2dx_2 + 2dx_3 + 2dx_4 \\ du_2 &= dx_1 + dx_2 + dx_3 + dx_4 \\ du_3 &= 3dx_1 + 3dx_2 + dx_3 + dx_4 \\ du_4 &= 3dx_1 + dx_2 + dx_3 + 2dx_4 \end{aligned}\right\}. \qquad (3)$$

Now

$\underline{f}(1,1,1,1) = (u_1, u_2, u_3, u_4)$, so by (1),

$\underline{f}(1,1,1,1) = (4,1,3,3)$.

Therefore

$$\underline{f}(1.001, 1.001, 1.001, 1.001) = (4 + \Delta u_1, 1 + \Delta u_2, 3 + \Delta u_3, 3 + \Delta u_4). \qquad (4)$$

Therefore, if we assume that $du_1 \approx \Delta u_1$, $du_2 \approx \Delta u_2$, $du_3 \approx \Delta u_3$, and $du_4 \approx \Delta u_4$, we may compute du_1, du_2, du_3 and du_4 from (3) with $\Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x_4 = 0.001$ and then replace Δu_1, Δu_2, Δu_3 and Δu_4 in (4) with these values of du_1, du_2, du_3, and du_4.

In more detail, from (3)

$$\left.\begin{aligned} du_1 &= (2 + 2 + 2 + 2)(0.001) = 0.008 \\ du_2 &= (1 + 1 + 1 + 1)(0.001) = 0.004 \\ du_3 &= (3 + 3 + 1 + 1)(0.001) = 0.008 \\ du_4 &= (3 + 1 + 1 + 2)(0.001) = 0.007 \end{aligned}\right\} \qquad (5)$$

Hence, from (4),

$$\underline{f}(1.001, 1.001, 1.001, 1.001) = (4 + \Delta u_1, 1 + \Delta u_2, 3 + \Delta u_3, 3 + \Delta u_4)$$

4.5.5 continued

$\approx (4 + du_1, 1 + du_2, 3 + du_3, 3 + du_4)$

$\approx (4.008, 1.004, 3.008, 3.007)$. (6)

[where $(4 + du_1, 1 + du_2, 3 + du_3, 3 + du_4) = (4.008, 1.004, 3.008, 3.007)$.

The approximation refers to $\underline{f}(1.001, 1.001, 1.001, 1.001)$].

b. From (1) with $x_1 = x_2 = x_3 = x_4 = 1.001$, we have

$u_1 = (1.00)^2 + (1.001)^2 + (1.001)^2 + (1.001)^2$

$= 4(1.001)^2$

$= 4(1.002001)$

$= 4.008004$

$u_2 = (1.001)(1.001)(1.001)(1.001)$

$= [(1.001)^2]^2$

$= (1.002001)^2$

$= 1.004006004001$

$u_3 = (1.001)^3 + (1.001)^3 + (1.001)(1.001)$

$= 2(1.001)^3 + (1.001)^2$

$= (1.001)^2[2(1.001) + 1]$

$= (1.002001)(3.002)$

$= 3.008007002$

4.5.5 continued

$$u_4 = (1.001)^2 + (1.001)(1.001) + (1.001)(1.001)^2$$
$$= 2(1.001)^2 + (1.001)^3$$
$$= (1.001)^2(2 + 1.001)$$
$$= (1.002001)(3.001)$$
$$= 3.007005001,$$

so that

$$\underline{f}\,(1.001, 1.001, 1.001, 1.001) = (u_1, y_1, u_3, u_4)$$
$$= (4.008004, 1.004006004001,$$
$$3.008007002, 3.007005001). \quad (7)$$

If we define $\underline{g}:E^4 \to E^4$ by

$$\underline{g}(1 + \Delta x_1, 1 + \Delta x_2, 1 + \Delta x_3, 1 + \Delta x_4) = (4 + du_1, 1 + du_2,$$
$$3 + du_3, 3 + du_4)$$

equation (6) yields

$$\underline{g}(1.001, 1.001, 1.001, 1.001) = (4.008, 1.004, 3.008, 3.007)$$

whereupon from (7) we obtain

$$\underline{f}(1.001, 1.001, 1.001, 1.001) - \underline{g}(1.001, 1.001, 1.001, 1.001) =$$

$$\sqrt{(4.009004 - 4.008)^2 + (1.004006004001 - 1.004)^2 + (3.008007002 - 3.008)^2 + (3.007005001 - 3.007)^2}$$

$$\sqrt{(0.000004)^2 + (0.000006004001)^2 + (0.000007002)^2 + (0.000005001)^2} \approx$$

$$\sqrt{(4 \times 10^{-6})^2 + (6 \times 10^{-6})^2 + (7 \times 10^{-6})^2 + (5 \times 10^{-6})^2} =$$

$$\sqrt{(16 + 36 + 49 + 25) \times 10^{-12}} =$$

$$\sqrt{126 \times 10^{-12}} \approx 11 \times 10^{-6} \approx 10^{-5}$$

4.5.6(L)

Our aim here is to shed some light on inverse functions and to show how we may use differentials in studying this topic.

In Exercise 4.5.4(L) we talked about

$\underline{f}:E^2 \to E^2$

where $\underline{f}(x,y) = (x^2 - y^2, 2xy)$

and we investigated the value of $\underline{f}(a + h, b + k)$ knowing the value of $\underline{f}(a,b)$. In part (a) of this exercise we are tackling the inverse problem. Namely, knowing that $\underline{f}(3,2) = (5,12)$, we would like to find the point (x,y) near $(3,2)$ such that $\underline{f}(x,y) = (5.00052, 12.00026)$. To this end we have:

a. We know that $\underline{f}(3,2) = (5,12)$ and that

$$\underline{f}(3 + \Delta x, 2 + \Delta y) = (5 + \Delta u, 12 + \Delta v). \tag{1}$$

From the given information in this exercise we have that

$$\Delta u = 0.00052 \tag{2}$$

while

$$\Delta v = 0.0026 . \tag{3}$$

What we would like to know is what values of Δx and Δy in equation (1) would produce this change in u and v.

What we already know in this problem [see, for example, equation (6) in Exercise 4.5.4(b)] is that

$$\left.\begin{aligned} du &= 6\,dx - 4\,dy \\ dv &= 4\,dx + 6\,dy \end{aligned}\right\} . \tag{4}$$

From a purely mechanical point of view, we can solve the system of equations (4) to express dx and dy as linear combinations of du and dv. In particular

4.5.6(L) continued

$$\left.\begin{aligned} 3\,du &= 18\,dx - 12\,dy \\ 2\,dv &= 8\,dx + 12\,dy \end{aligned}\right\}$$

or

$$dx = \frac{3}{26}\,du + \frac{1}{13}\,dv \quad . \tag{5}$$

Similarly

$$\begin{aligned} -2\,du &= -12\,dx + 8dy \\ 3\,dv &= 12\,dx + 18\,dy \end{aligned}$$

so that

$$dy = -\frac{1}{13}\,du + \frac{3}{26}\,dv \quad . \tag{6}$$

Now, if we let $du = 0.00052$ and $dv = 0.00026$, equations (5) and (6) quickly yield

$$dx = \frac{3}{26}\,(0.00052) + \frac{1}{13}\,(0.00026)$$

$$= 0.00006 + 0.00002$$

$$= 0.00008 \tag{7}$$

while

$$dy = -\frac{1}{13}\,(0.00052) + \frac{3}{26}\,(0.00026)$$

$$= -\,0.00004 + 0.00003$$

$$= -\,0.00001. \tag{8}$$

If we then assume that $\Delta x \approx dx$ and $\Delta y \approx dy$ then (7) and (8) yield

$$3 + \Delta x \approx 3 + dx = 3 + 0.00008 = 3.00008 \tag{9}$$

and

$$2 + \Delta y \approx 2 + dy = 2 - 0.00001 = 1.99999\ . \tag{10}$$

4.5.6(L) continued

In other words,

$\underline{f}(3.00008, 1.99999) \approx (5.00052, 12.00026)$.

To check the accuracy of our approximation, we have by the definition of $\underline{f}$ that

$$\underline{f}(3.00008, 1.99999) = [(3.00008)^2 - (1.99999)^2, \\ 2(3.00008)(1.99999)].$$

Hence

$$\begin{aligned}\underline{f}(3.00008, 1.99999) &= (9.0004800064 - 3.9999600001, \\ &\quad 12.0002599984) \\ &= (5.0005200063, 12.0002599984). \qquad (11)\end{aligned}$$

Notice that what we wanted was the 2-tuple (x,y) such that

$$\underline{f}(x,y) = (5.00052, 12.00026). \qquad (12)$$

Comparing (11) and (12) we see that while (x,y) is not exactly $(3.00008, 1.99999)$, the approximation

$(3.00008, 1.99999) \approx (x,y)$

is extremely accurate.

To summarize what we did in this exercise, we replaced the function $\underline{f}:E^2 \to E^2$ by $\underline{g}:E^2 \to E^2$.

As described in Exercise 4.5.4(L), $\underline{g}$ is the linear function

$\underline{g}(3 + \Delta x, 2 + \Delta y) = (5 + 6\ \Delta x - 4\ \Delta y, 12 + 4\ \Delta x + 6\ \Delta y)$.

We then computed $\underline{g}^{-1}(5.00052, 12.00026)$ and used this to approximate the desired solution, $\underline{f}^{-1}(5.00052, 12.00026)$.

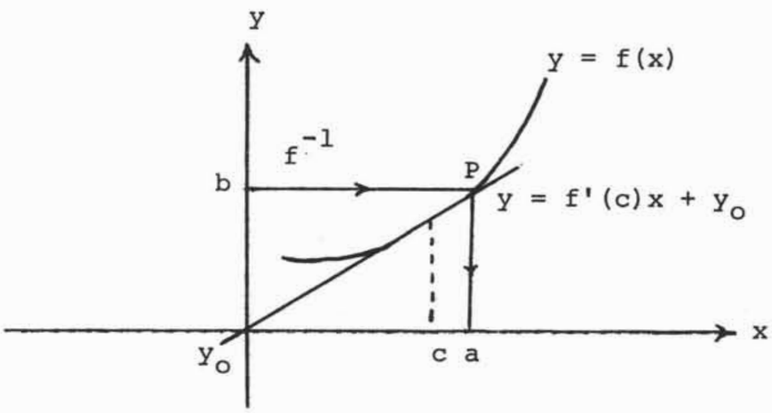
This type of substitution is perhaps the best intuitive way to think of the mapping

4.5.6(L) continued

$\underline{f}: E^n \to E^n$.

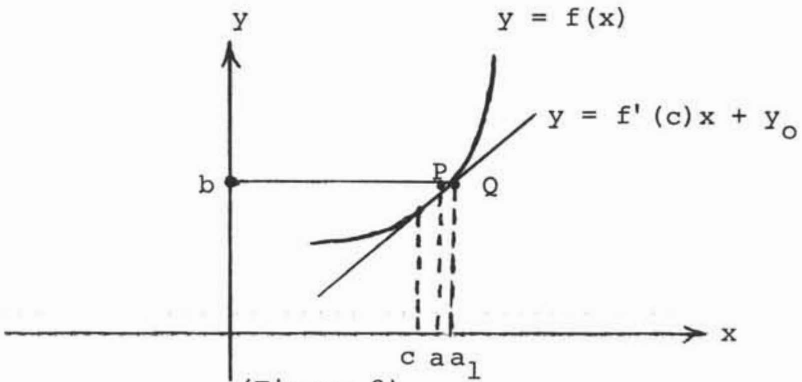
Namely, we replace $\underline{f}$ by its linear counterpart $\underline{g}$ in the neighborhood of some specified point, and let the answers obtained in terms of $\underline{g}$ serve as the approximation for the corresponding result involving $\underline{f}$.

Again, the case n = 1 is the easiest to see pictorially. In this case, we have:



(Figure 1)

In Figure 1, given b we want to find a such that f(a) = b. Graphically we draw the line y = b to meet the curve y = f(x) at P(a,f(a)) and the x-coordinate of P is the required value for a. Now it would be easier (arithmetically, not pictorially) to solve for a if y = f(x) were replaced by the linear equation $y = f'(c)x + y_o$. Pictorially, this would involve letting the line y = b intersect the <u>line</u> $y = f'(c)x + y_o$ at Q and approximating a by the x-coordinate of Q. That is

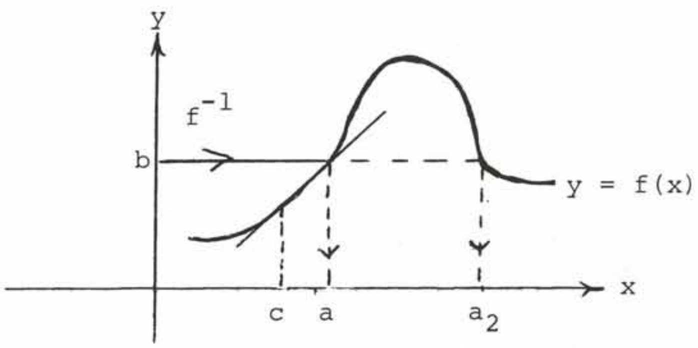


(Figure 2)

4.5.6(L) continued

Our claim is that if P is close to $(c,f(c))$ then a_1 is close to a.

Notice, however, (as we stressed in part 1 of our course) that in finding $f^{-1}(b)$ we want to use the fact that we want our answer to be near $x = c$. For example, f need not be 1-1 globally (i.e., for the entire domain) such as



(Figure 3)

That is, both $f(a)$ and $f(a_2)$ equal b, but we choose $x = a$ since it is "near" $x = c$.

In this same vein, we are assuming in this exercise that the domain of f is being restricted to a sufficiently small neighborhood of (3,2) so that in this neighborhood f is 1-1. (The conditions under which we can be sure such a neighborhood exists will be discussed in the next unit.)

Note:

We may generalize this exercise by picking the point (x_0, y_0) rather than (3,2). Then from equation (5) of Exercise 4.5.4 we have

$$\left.\begin{aligned} du &= 2x_0dx - 2y_0dy \\ dv &= 2y_0dx + 2x_0dy \end{aligned}\right\} . \qquad (13)$$

We may then solve (13) for dx and dy in terms of du and dv. That is,

4.5.6(L) continued

$$\left.\begin{aligned} x_o du &= 2x_o^2 dx - 2x_o y_o dy \\ y_o dv &= 2y_o^2 dx + 2x_o y_o dy \end{aligned}\right\} .$$

Hence

$$dx = \frac{x_o}{2(x_o^2 + y_o^2)} du + \frac{y_o}{2(x_o^2 + y_o^2)} dv . \tag{14}$$

Similarly,

$$dy = \frac{-y_o}{2(x_o^2 + y_o^2)} du + \frac{x_o}{2(x_o^2 + y_o^2)} dv \tag{15}$$

<u>except</u> that (14) and (15) are undefined when $2(x_o^2 + y_o^2) = 0$. This can only happen if $x_o = y_o = 0$, or when $(x_o,y_o) = (0,0)$. In other words, at least from a mechanical point of view, the technique used in this exercise applies for all (x_o, y_o) in E^2 except (0,0).

What does this mean pictorially? First of all, notice that since $\underline{f}(x,y) = (x^2 - y^2, 2xy)$, $\underline{f}(0,0) = (0,0)$. Now look at <u>any</u> neighborhood of (0,0) in the xy-plane. Let (h,k) and (-h,-k) be points in this neighborhood, other than (0,0). Then

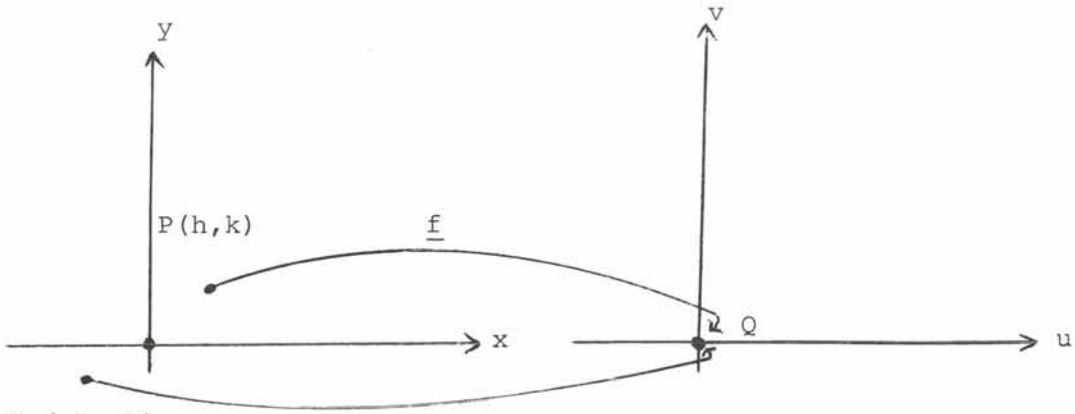
$$\underline{f}(h,k) = (h^2 - k^2, 2hk)$$

$$\underline{f}(-h, -k) = [(-h)^2 - (-k)^2, 2(-h)(-k)] = (h^2 - k^2, 2hk).$$

Therefore

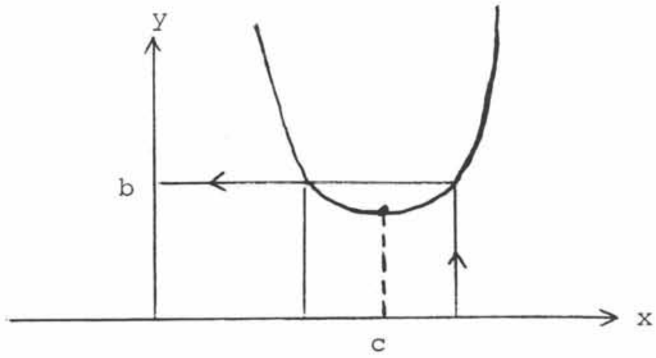
$\underline{f}(h,k) = \underline{f}(-h, -k)$ even though $h \neq -h$ and $k \neq -k$. That is, $\underline{f}$ is not 1-1 in <u>any</u> neighborhood of (0,0).

4.5.6(L) continued



Given $\underline{f}(P) = Q$ where $P = (h,k)$, let $P_1 = (-h, -k)$; then $\underline{f}(P_1) = Q$. That is, there is no neighborhood of (0,0) for which $\underline{f}$ is 1-1.

This happens in the case n = 1 as well. Namely suppose $f'(c) = o$. For example



There is no neighborhood of x = c in which f is 1-1.

Generalizing still further suppose $\underline{f}(x,y) = [u(x,y), v(x,y)]$ where u and v are continuously differentiable functions of x and y.

then

$$\begin{cases} du = u_x(x_o,y_o)dx + u_y(x_o,y_o)dy \\ dv = v_x(x_o,y_o)dx + v_y(x_o,y_o)dy. \end{cases} \tag{16}$$

We can then solve for dx and dy in terms of du and dv <u>unless</u>

4.5.6(L) continued

$u_x(x_0,y_0)v_y(x_0,y_0) - u_y(x_0,y_0)v_x(x_0,y_0) \neq 0$, that is, unless

$$\det\begin{bmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{bmatrix} = 0 \; . \tag{17}$$

For example, if we multiply the top equation in (16) by $v_y(x_0,y_0)$ and the bottom equation by $-u_y(x_0,y_0)$, we obtain

$$\left.\begin{aligned} v_y(x_0,y_0)\,du &= u_x(x_0,y_0)v_y(x_0,y_0)\,dx + u_y(x_0,y_0)v_y(x_0,y_0)\,dy \\ -u_y(x_0,y_0)\,dv &= -u_y(x_0,y_0)v_x(x_0,y_0)\,dx - u_y(x_0,y_0)v_y(x_0,y_0)\,dy \end{aligned}\right\}$$

whereupon

$$v_y(x_0,y_0)\,du - u_y(x_0,y_0)\,dv = [u_x(x_0,y_0)v_y(x_0,y_0)v_x(x_0,y_0)]\,dx. \tag{18}$$

Thus, as long as $u_x(x_0,y_0)v_y(x_0,y_0) - u_y(x_0,y_0)v_x(x_0,y_0) \neq 0$, we may divide both sides of (18) by it to obtain an expression for dx in terms of du and dv.

The coefficients of dx and dy in (16) [which, by the way, are constants once (x_0,y_0) is specified] form an important matrix called the <u>Jacobian Matrix</u> of <u>u and v with respect to x and y</u>. The Jacobian Matrix will be explored in somewhat more detail in the next unit.

4.5.7(L)

$u = \frac{1}{2}x^2 + y^2 + z^2$

$v = 3x^2y + 4z^3$

$w = 4xyz.$

Therefore,

4.5.7(L) continued

$$\left.\begin{aligned} du &= xdx + 2ydy + 2zdz \\ dv &= 6xydx + 3x^2dy + 12z^2dz \\ dw &= 4yzdx + 4xzdy + 4xydz \end{aligned}\right\} . \qquad (1)$$

In particular if x = y = z = 1, equations (1) become

$$\left.\begin{aligned} du &= dx + 2dy + 2dz \\ dv &= 6dx + 3dy + 12dz \\ dw &= 4dx + 4dy + 4dz \end{aligned}\right\} . \qquad (2)$$

Using our augmented matrix technique, we may solve equations (2) for dx, dy, and dz in terms of du, dv, and dw as follows:

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 6 & 3 & 12 & 0 & 1 & 0 \\ 4 & 4 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -9 & 0 & -6 & 1 & 0 \\ 0 & -4 & -4 & -4 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 18 & 36 & 36 & 18 & 0 & 0 \\ 0 & -36 & 0 & -26 & 4 & 0 \\ 0 & 36 & 36 & 36 & 0 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 18 & 0 & 36 & -6 & 4 & 0 \\ 0 & -36 & 0 & -24 & 4 & 0 \\ 0 & 0 & 36 & 12 & 4 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 18 & 0 & 0 & -18 & 0 & 9 \\ 0 & -36 & 0 & -24 & 4 & 0 \\ 0 & 0 & 36 & 12 & 4 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1/2 \\ 0 & 1 & 0 & 2/3 & -1/9 & 0 \\ 0 & 0 & 1 & 1/3 & 1/9 & -1/4 \end{bmatrix} .$$

Therefore

$$\left.\begin{aligned} dx &= -du + \tfrac{1}{2}\,dw \\ dy &= \tfrac{2}{3}\,du - \tfrac{1}{9}\,dv \end{aligned}\right\} \qquad (3)$$

4.5.7(L) continued

$$\left.dz = \frac{1}{3}\,du + \frac{1}{9}\,dv - \frac{1}{4}\,dw\right\} \qquad (3)$$

Again, assuming that x,y, and z can be expressed in terms of the independent variables u, v, and w, we know that

$$\left.\begin{aligned} dx &= x_u du + x_v dv + x_w dw \\ dy &= y_u du + y_v dv + y_w dw \\ dz &= z_u du + z_v dv + z_w dw \end{aligned}\right\}. \qquad (4)$$

Comparing (3) and (4) we have

$$\left.\begin{aligned} &x_u = -1,\ x_v = 0,\ x_w = \frac{1}{2} \\ &y_u = \frac{2}{3},\ y_v = -\frac{1}{9},\ y_w = 0 \\ &z_u = \frac{1}{3},\ z_v = \frac{1}{9},\ z_w = -\frac{1}{4} \end{aligned}\right\}. \qquad (5)$$

From our original equations, when x = y = z = 1 we have that

$$u = \frac{1}{2} + 1 + 1 = \frac{5}{2}$$
$$v = 3 + 4 = 7$$
$$w = 4.$$

So that (5) really is read

$$\left.\begin{aligned} &x_u(\tfrac{5}{2}, 7, 4) = -1,\ x_v(\tfrac{5}{2}, 7, 4) = 0,\ x_w(\tfrac{5}{2}, 7, 4) = \tfrac{1}{2} \\ &y_u(\tfrac{5}{2}, 7, 4) = \tfrac{2}{3},\ y_v(\tfrac{5}{2}, 7, 4) = -\tfrac{1}{9},\ y_w(\tfrac{5}{2}, 7, 4) = 0 \\ &z_u(\tfrac{5}{2}, 7, 4) = \tfrac{1}{3},\ z_v(\tfrac{5}{2}, 7, 4) = \tfrac{1}{9},\ z_w(\tfrac{5}{2}, 7, 4) = -\tfrac{1}{4} \end{aligned}\right\} \qquad (6)$$

4.5.8(L)

The main aim of this exercise is to show the difference between a local inverse and a global inverse.

4.5.8(L) continued

a. Given the mapping $\underline{f}:E^2 \to E^2$ defined by $\underline{f}(x,y) = (e^x \sin y, e^x \cos y)$, we have

$$u = e^x \sin y$$

$$v = e^x \cos y \ .$$

Hence

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix}$$

$$= -e^{2x}\sin^2 y - e^{2x}\cos^2 y$$

$$= -e^{2x}(\sin^2 y + \cos^2 y)$$

$$= -e^{2x} \ . \qquad (1)$$

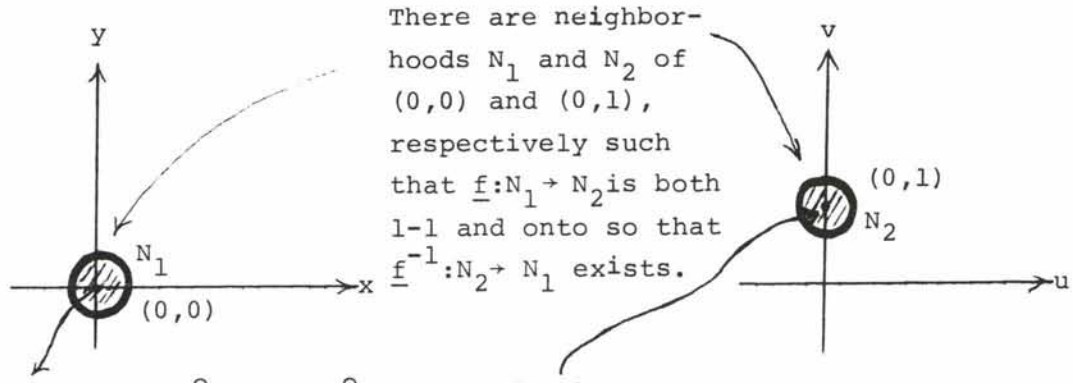
This is essentially equivalent to saying that $du = e^x \sin y\, dx + e^x \cos y dy$, $dv = e^x \cos y dy - e^x \sin y dy$ and solving for dx and dy in terms of du and dv, just as we did in the previous exercises, but $e^{2x} \underset{\neq}{>} 0$ for all real values of x.

Therefore

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| \neq 0 \quad \text{for all } (x,y).$$

Notice, however, that the validity of our approach requires that Δx and Δy be sufficiently small so that $\Delta u \approx \Delta u_{tan}$, etc. Thus our results are "<u>local</u>" which means that for any point (x,y) in the xy-plane there is a <u>sufficiently small</u> neighborhood N of (x,y) such that $\underline{f}$ is 1-1 <u>on</u> <u>N</u>. That is, <u>locally</u> $\underline{f}^{-1}$ exists at each point (u,v) in the uv-plane. Pictorially,

4.5.8(L) continued



$\underline{f}(0,0) = (e^0 \sin 0,\ e^0 \cos 0) = (0,1)$

(Figure 1)

Moreover the results in Figure 1 extend to the treatment of any pair of points (x_0,y_0) in the xy-plane and $(e^{x_0}\sin y_0,\ e^{x_0}\cos y_0)$ in the uv-plane.

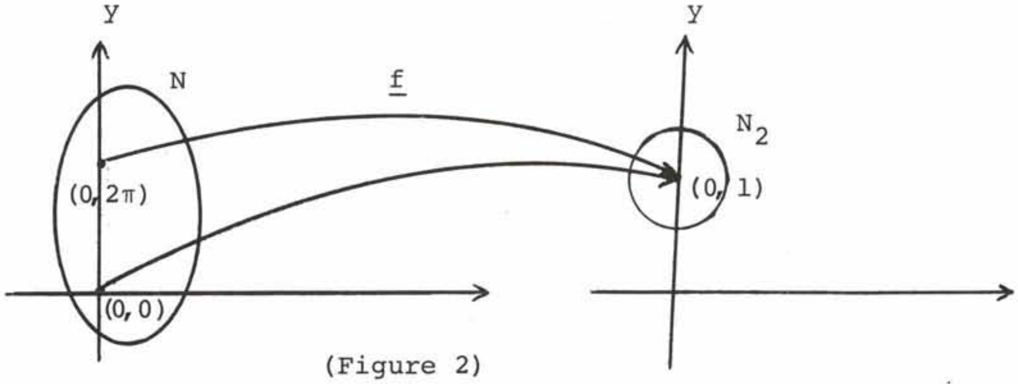
b. Since $\sin(y_0 + 2\pi) = \sin y_0$ and $\cos(y_0 + 2\pi) = \cos y_0$, we have that $\underline{f}(x_0, y_0)\ \ \underline{f}(x_0, y_0 + 2\pi)$.

That is,

$$\begin{aligned}\underline{f}(x_0, y_0) &= (e^{x_0}\sin y_0,\ e^{x_0}\cos y_0)\\ &= (e^{x_0}\sin[y_0 + 2\pi],\ e^{x_0}\cos[y_0 + 2\pi])\\ &= \underline{f}(x_0, y_0 + 2\pi). \qquad (2)\end{aligned}$$

Since $(x_0,y_0) \neq (x_0,y_0 + 2\pi)$, equation (2) shows that $\underline{f}$ is not 1-1 globally. In terms of Figure 1, what this means is the following, We were able to find sufficiently small neighborhoods N_1 or (0,0) in the xy-plane and N_2 of (0,1) in the uv-plane such that $\underline{f}:N_1 \to N_2$ was 1-1 and onto. But if we now choose a neighborhood of (0,0) which is not sufficiently small then $\underline{f}$ need not be 1-1 on this new neighborhood. For example, utilizing equation (1), suppose we choose our neighborhood N of (0,0) to be large enough to include $(0, 2\pi)$. Then not only does $\underline{f}(0,0) = (0,1)$ but $\underline{f}(0,2\pi)$ also equals (0,1) [since $\underline{f}(0,2\pi)$ is equal to $(e^0\sin 2\pi,\ e^0\cos 2\pi) = (0,1)$]. Pictorially

4.5.8(L) continued



(Figure 2)

No matter how we shrunk N_2 we cannot have $\underline{f}^{-1}:N_2 \to N$ since $\underline{f}^{-1}(0,1)$ would equal $(0,0)$ and $(0,2\pi)$ contrary to our definition that $\underline{f}^{-1}$ is a single-valued function.

Thus, the fact that $\underline{f}:E^n \to E^n$ has a local inverse everywhere is not enough to guarantee that $\underline{f}$ has a global inverse.

Are there conditions under which we can be sure that $\underline{f}:E^n \to E^n$ has a global inverse?

In part 1 of our course we took care of the answer of $r = 1$. Namely if f was differentiable, the fact that $f'(x)$ was never zero guaranteed that f was 1-1. (I.e., the graph was either always rising or always falling.)

For the case $r = 2$ the following special result applies (again, stated without proof).

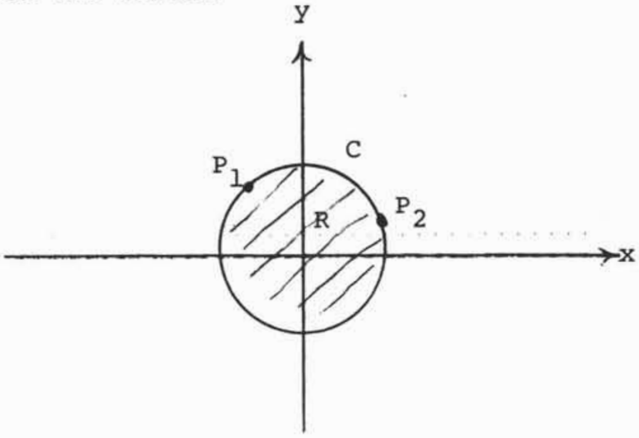
Theorem

Let $\underline{f}$ be a continuously differentiable function defined in a region R of E^2 such that the Jacobian determinant of $\underline{f}$ is never zero in R. Suppose further that C is a simple closed curve (i.e., a closed curve that never crosses itself and which can be drawn from start to finish without taking the pencil off the paper. For example  is a simple closed curve but  isn't) which, together with its interior, lies

4.5.8(L) continued

in R, and that $\underline{f}$ is 1-1 on C. Then $\underline{f}$(c) is also a simple closed curve <u>and</u> f is 1-1 on the domain C∪R (i.e., on R and its boundary).

In terms of an example, consider Figure 1 of part (a). Suppose we let C be the circle centered at (0,0) in the xy-plane with radius equal to 1 and let R denote the interior of the circle



Then, $\underline{f}'(\underline{x}) \neq 0$ for all $\underline{x} \in R$ since $\underline{f}'(\underline{x}) \neq 0$ for every $\underline{x}$. Moreover, for every pair of distinct points P_1 and P_2 on c, $f(P_1) \neq f(P_2)$ since the circle is too small to permit two points whose y-coordinates differ by 2π to exist on the circle. (I.e., the diameter of the circle is 2 which is less than 2π). In more detail, if $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ then $\underline{f}(P_1) = \underline{f}(P_2)$ implies that

$$(e^{x_1}\sin y_1, e^{x_1}\cos y_1) = (e^{x_2}\sin y_2, e^{x_2}\cos y_2), \text{ so that}$$

$$\left.\begin{aligned} e^{x_1}\sin y_1 &= e^{x_2}\sin y_2 \\ e^{x_1}\cos y_1 &= e^{x_2}\cos y_2 \end{aligned}\right\} . \qquad (3)$$

Squaring both equations in (3) and adding yields

$$e^{2x_1}(\sin^{2}y_1 + \cos^{2}y_1) = e^{2x_2}(\sin^2 y_2 + \cos^2 y_2)$$

or

$$e^{2x_1} = e^{2x_2} .$$

4.5.8(L) continued

But e^x is an increasing function (hence 1-1). Therefore

$$2x_1 = 2x_2$$

or

$$x_1 = x_2.$$

If we divide one equation in (3) by the other we obtain

$$\frac{\sin y_1}{\cos y_1} = \frac{\sin y_2}{\cos y_2}$$

or $\tan y_1 = \tan y_2$, etc.

At any rate, the theorem guarantees that $\underline{f}$ will map the circle C into a simple closed curve C' in the uv-plane, R will be mapped onto the interior of C', and if we let S denote C' and its interior, we have that $\underline{f}: R \cup C \to S$ is 1-1 and onto.

(Notice that the mechanics of computing $\underline{f}(C)$ etc. are not too simple.)