Unit 2: Tangential and Normal Vectors

2.2.1(L)

The major aim of this part of the exercise is to help remove a bit of mystery as well as a computational sore spot that many students encounter in the formula

$$\vec{T} = \frac{d\vec{R}}{ds}. \tag{1}$$

What usually happens is that $\vec{R}$ is not expressed directly as a function of s (arclength). In many cases of a kinematics nature, for example, $\vec{R}$ will be expressed in terms of time (t).

For the sake of argument, let us suppose that $\vec{R}$ is expressed as a function of some scalar q, not necessarily either arclength or time. Then we have already seen Exercise 2.1.4 that $\frac{d\vec{R}}{dq}$ is tangent to the curve and its magnitude $\left|\frac{d\vec{R}}{dq}\right|$ is equal to $\left|\frac{ds}{dq}\right|$.

In other words, then, if $\vec{R}$ is expressed as a function of q, we may find $\vec{T}$ quite conveniently simply by letting

$$\vec{T} = \frac{d\vec{R}/dq}{|d\vec{R}/dq|}. \tag{2}$$

Equation (2) supplies us with an excellent hint as to why equation (1) is correct. Namely, in the special case that we choose q to equal s, we have that $\frac{ds}{dq} = 1$, whereupon equation (2) becomes equation (1) by the chain rule (proven in Exercise 2.1.3). That is,

$$\frac{d\vec{R}/dq}{|d\vec{R}/dq|} = \frac{d\vec{R}/dq}{ds/dq} = \frac{d\vec{R}}{ds}.$$

Yet, there is an even better philosophical reason for defining $\vec{T}$ by (1) rather than by (2), even though, in a given case (2) may be more direct to compute than is (1). The thought is that a unit tangent vector to a curve is determined by the shape of the curve itself, not by either the particular coordinate system in

2.2.1(L) continued

which we elect to write the equation of the curve nor the variable used as the parameter. For example, there are many different parameters that might have been chosen to represent the vector equation for the curve. That is, in terms of our previous notation, q could have been chosen in many ways, among which are time and arclength. However, arclength (s) is an invariant of the curve. In other words, if we measure the length of a given curve between two points P and Q on the curve, the length depends only on the curve and the two points, not on any particular coordinate system: (although the degree of complexity of the actual computations involved may well depend on the coordinate system).

In summary, then, the definition that $\vec{T} = \frac{d\vec{R}}{ds}$ gives us a definition of the unit tangent vector which is independent of any particular coordinate system under consideration. In this exercise, however, it is our aim to emphasize that the actual computations are best done in terms of the particular variable with respect to which $\vec{R}$ is expressed.

With this discussion in mind we have

a. $\vec{R} = t\,\vec{i} + \frac{1}{3}(t^2 + 2)^{3/2}\,\vec{j}$.

Therefore,

$$\vec{v} = \frac{d\vec{R}}{dt} = \vec{i} + t\sqrt{t^2 + 2}\;\vec{j}. \qquad (3)$$

From (3)

$$|\vec{v}| = \sqrt{1 + (t\sqrt{t^2 + 2})^2}$$

$$= \sqrt{1 + t^2(t^2 + 2)}$$

$$= \sqrt{t^4 + 2t^2 + 1}$$

$$= \sqrt{(t^2 + 1)^2}$$

Therefore, $|\vec{v}| = t^2 + 1$. (4)

2.2.1(L) continued

Since $\vec{v}$ is a tangent vector, $\frac{\vec{v}}{|\vec{v}|}$ will be a unit tangent vector; hence, (3) and (4) yield

$$\vec{T} = \frac{\vec{i} + t\sqrt{t^2 + 2}\ \vec{j}}{t^2 + 1}$$

or

$$\vec{T} = \left(\frac{1}{t^2 + 1}\right)\vec{i} + \left(\frac{t\sqrt{t^2 + 2}}{t^2 + 1}\right)\vec{j}. \qquad (5)$$

b. It should seem clear that no matter how cleverly we disguise it, equation (5) somehow must contain the arclength s.

The easiest way to see that it does is to recall from the previous unit the fact that

$$|\vec{v}| = \frac{ds}{dt}\ . \qquad (6)$$

Quite simply, then, when we formed $\frac{\vec{v}}{|\vec{v}|}$ we were in fact computing

$$\frac{d\vec{R}/dt}{ds/dt} = \frac{d\vec{R}}{ds}\ .$$

In any event, equation (6) combined with (4) tells us that

$$\frac{ds}{dt} = t^2 + 1$$

whereupon

$$s = \int_0^6 (t^2 + 1)dt$$

$$= \frac{1}{3}t^3 + t\Big|_0^6$$

2.2.1(L) continued

$$= \frac{6^3}{3} + 6$$

$$= 78.$$

2.2.2

We have

$$\vec{R} = \frac{t^2}{2}\vec{i} + \frac{1}{3}(2t + 1)^{3/2}\vec{j}. \qquad (1)$$

Therefore $\vec{v} = t\,\vec{i} + (2t + 1)^{1/2}\vec{j}$. (2)

$$\vec{a} = \vec{i} + \frac{1}{\sqrt{2t + 1}}\vec{j}. \qquad (3)$$

a. At $t = 4$ the particle is at $\left(\frac{4^2}{2}, \frac{1}{3}[2(4) + 1]^{3/2}\right) = (8,9)$, since from (1), $x = \frac{t^2}{2}$ and $y = \frac{1}{3}(2t + 1)^{3/2}$.

From (2),

$\vec{v} = 4\vec{i} + 3\vec{j}$ therefore, $|\vec{v}| = \sqrt{4^2 + 3^2} = 5$,

and, from (3),

$$\vec{a} = \vec{i} + \frac{1}{3}\vec{j}.$$

b. $|\vec{v}| = \frac{ds}{dt}$ and by (2)

$|\vec{v}| = \sqrt{t^2 + [(2t + 1)^{1/2}]^2} = \sqrt{t^2 + 2t + 1} = |t + 1|$, but $0 \leqslant t \leqslant 4$ implies that $|t + 1| = t + 1$.

Therefore $\frac{ds}{dt} = t + 1$

Therefore, $s = \frac{1}{2}t^2 + t\Big|_0^4$

$$= 12.$$

2.2.2 continued

c. At $t = 4$, $\vec{v} = 4\vec{i} + 3\vec{j}$ and $|\vec{v}| = 5$.

Therefore, $\vec{T} = \frac{d\vec{R}}{ds} = \frac{d\vec{R}/dt}{ds/dt} = \frac{\vec{v}}{|\vec{v}|} = \frac{4\vec{i} + 3\vec{j}}{5} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$.

2.2.3(L)

a. The result asked for here has been derived twice before--once in the lecture and once in the text (in fact, it is essentially done twice in the text). Certainly, we could justify this part of the exercise on the grounds that the result is important enough for you to make sure you can derive it. Yet, there is still a better reason for assigning this exercise. It allows us to introduce a rather interesting subtlety concerning the difference between a curve and the path traced out by a particular particle.

The idea is this: if we are given a particular particle and we study its motion, it is very natural that at any point on its path we will pick as the unit tangent vector the one whose sense is the same as the velocity vector. That is, we visualize the curve as being generated by the moving particle. Of course, the curve, itself, is inanimate. It does not know how the particle is traversing it, and what could be even more complicated is the fact that different particles may traverse this same curve in different ways.

The key thought is that if we are mainly interested in the curve, we are free to assign it a sense in any way we please, and, once this is done, the particle may traverse the curve in any way it wishes. Its speed, however, will be negative if the particle has the opposite sense as the curve at that point, and positive if it has the same sense as the curve at that point. (This is much like the convention we have adopted with respect to the x-axis. If a particle is moving from left-to-right at a given point, we call its speed positive, while if it moves from right-to-left, we call its speed negative.)

The beauty of the definition $\vec{T} = \frac{d\vec{R}}{ds}$ is that no matter how we elect to define the sense of the curve, $\vec{T}$ will always have the

2.2.3(L) continued

same sense as the curve. (This is explained in the text as well as in the lecture.) If we now think of a particle traversing the curve according to some rule $\vec{R} = \vec{R}(t)$, we have by the chain rule,

$$\vec{T} = \left(\frac{d\vec{R}}{dt}\right)\left(\frac{dt}{ds}\right)$$

from which we may solve for $\vec{v}$ $\left(= \frac{d\vec{R}}{dt}\right)$, to obtain

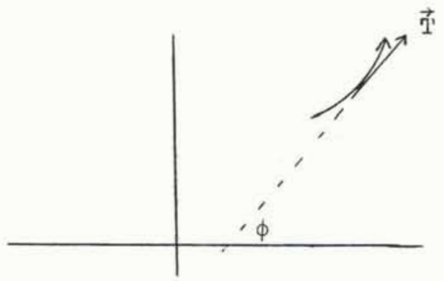
$$\vec{v} = \frac{ds}{dt}\vec{T} \qquad (1)$$

where all we can be sure of is that $\vec{T}$ and $\vec{v}$ have the same direction. We cannot be sure that they have the same sense. That is, once the sense of the curve is assigned, a particle may traverse it so that its sense is opposite to that of the curve, in which case $\frac{ds}{dt}$ is negative.

If we differentiate (1) with respect to t (remembering to use the product rule since both $\frac{ds}{dt}$ and $\vec{T}$ are functions of time [i.e., $\vec{T}$ has constant magnitude, but except for straight lines, its direction varies with time]) we obtain

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\frac{d\vec{T}}{dt}. \qquad (2)$$

To replace $\frac{d\vec{T}}{dt}$ by more "familiar" terms, recall that in the lecture we saw



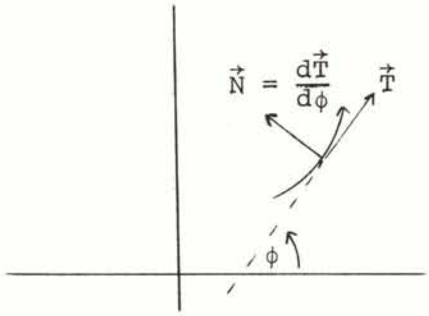
$$\vec{T} = \cos\phi\,\vec{i} + \sin\phi\,\vec{j},$$

whence,

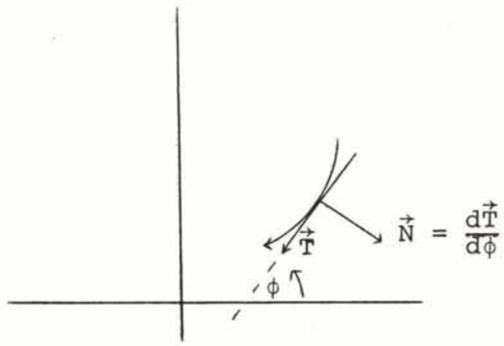
2.2.3(L) continued

$$\frac{d\vec{T}}{d\phi} = -\sin\phi\ \vec{i} + \cos\phi\ \vec{j} = \cos\ (\phi + 90°)\vec{i} + \sin\ (\phi + 90°)\vec{j}.$$

This led us to the conclusion that $\frac{d\vec{T}}{d\phi}$ was <u>the</u> unit vector obtained by rotating $\vec{T}$ 90° in the positive (counterclockwise) direction and this unit vector is defined to be $\vec{N}$, i.e., $\vec{N} \equiv \frac{d\vec{T}}{d\phi}$. That is,



(Note: had we assigned the sense of our curve in the opposite direction, we would have



In other words, $\vec{N}$ may point "in" or "out" depending on the sense of the curve.)

To get the right side of (2) into the desired form (i.e., to replace $\frac{d\vec{T}}{dt}$ by $\vec{N}$), we again use the chain rule to obtain

$$\vec{a} = \frac{d^2s}{dt^2}\ \vec{T} + \frac{ds}{dt}\ \frac{d\vec{T}}{d\phi}\ \frac{d\phi}{dt},$$

and since $\frac{d\vec{T}}{d\phi} = \vec{N}$,

2.2.3(L) continued

$$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\frac{d\phi}{dt}\vec{N}\ . \qquad (3)$$

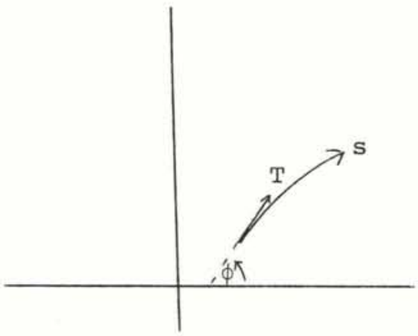
With respect to (3), we observe that $\frac{d\phi}{dt} = \frac{d\phi}{ds}\frac{ds}{dt}$ and that $\frac{d\phi}{ds}$, being the measure of how the direction of the tangent to the curve is changing at any instant, is an excellent parameter by which to measure curvature. We, therefore, let $\kappa = \frac{d\phi}{ds}$, and (3) becomes

$$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\left(\frac{d\phi}{ds}\right)\left(\frac{ds}{dt}\right)\vec{N}$$

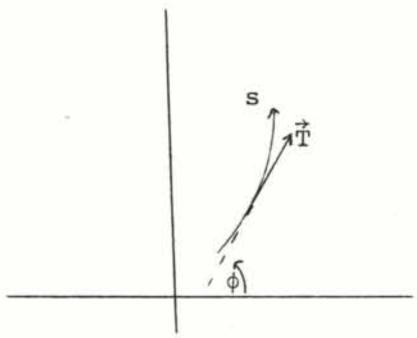
$$= \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2\kappa\,\vec{N}\ . \qquad (4)$$

Equation (4) is the desired result, but notice that our proof offers an alternative to the text in the sense that we used the derivation of (4) to motivate the "invention" of κ (while the text derived κ in its own right first and then applied the result to the derivation of (4)).

Notice also that (4) is true no matter how the sense of the curve is initially chosen. Once the sense is chosen, observe that $\frac{d\phi}{ds}(= \kappa)$ may be either positive or negative. That is,



$\frac{d\phi}{ds} < 0$

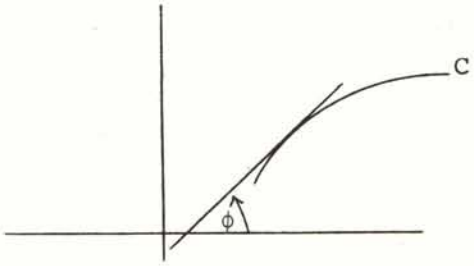


$\frac{d\phi}{ds} > 0$

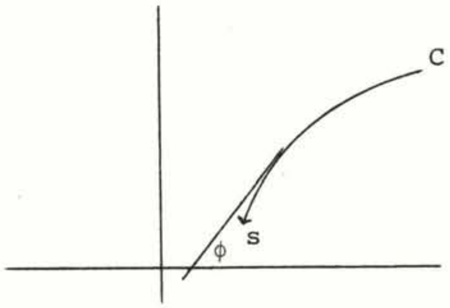
With this in mind, we complete our problem by reserving our choice of sense of the curve until this moment (since no harm is done by waiting because (4) is true for any sense that we choose) and we then assign the sense to our curve that makes

2.2.3(L) continued

$\frac{d\phi}{ds}$ non-negative. For example, given the curve C without an indicated sense, where C is



We choose the sense of C as shown below in order to make $\frac{d\phi}{ds} > 0$.



As the particle traverses C in the indicated direction (sense) ϕ increases.

In cases where no sense is assigned to the curve, $\frac{d\phi}{ds}$ has an ambiguous sign, and in this event it is customary to define κ to be $\left|\frac{d\phi}{ds}\right|$ which agrees with our definition of sense and also with the text's definition of κ.

There now remains only one major issue that must be resolved between our approach and that of the text. At the moment $\vec{N}$ has been defined in two different ways. On the one hand, using our initiative visual model, $\vec{N}$ was a positive 90° rotation of $\vec{T}$. In the text, $\vec{N}$ is the unit vector in the direction <u>and</u> <u>sense</u> of $\frac{d\vec{T}}{ds}$. That is, $\frac{d\vec{T}}{ds} = \left|\frac{d\vec{T}}{ds}\right|\vec{N}$. Now $\left|\frac{d\vec{T}}{ds}\right| = \left|\left(\frac{d\vec{T}}{d\phi}\right)\left(\frac{d\phi}{ds}\right)\right| = \left|\frac{d\vec{T}}{d\phi}\right|\left|\frac{d\phi}{ds}\right| = (1)(\kappa) = \kappa$. Thus, if we rewrite (2) from this point of view, we obtain

2.2.3(L) continued

$$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \frac{d\vec{T}}{ds} = \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa \vec{N}$$

and comparing this with (4) we see that $\vec{N}$ has the same meaning in either case. The beauty of the text's approach is that we need no knowledge of geometry to see that $\vec{T}$ and $\frac{d\vec{T}}{ds}$ are perpendicular. The drawback is that the approach is sufficiently abstract (if you are not used to thinking in terms of vectors) that you may get no feeling for the result. The advantage to our approach is that it is sufficiently visual that you can see what is happening. On the other hand, our approach depends on being able to discover a few geometric results that might not have seemed that obvious. In particular, when we want to investigate 3-dimensional space curves, the geometry may be enough more complex so that this approach will be even more difficult to apply, in which case we may prefer to use the more analytic approach of the text.

Notice that the validity of (4) does not require that we be able to interpret the result physically, but (4) does have a rather easy interpretation. Namely, the tangential component of the acceleration is the acceleration of the particle along the curve (since this is precisely what $\frac{d^2s}{dt^2}$ measures). The normal component of the acceleration is the product of the curvature and the square of the speed of the particle along the curve. While this may not seem familiar, the student of elementary physics might remember the formula for centripetal acceleration in circular motion as being $\frac{v^2}{r}$ where v is the speed of the particle and r is the radius of the circle. If we define the radius of curvature ρ to be the reciprocal of the curvature, i.e., $\rho = \frac{1}{\kappa}$, equation (4) yields that the normal component of acceleration is $\frac{v^2}{\rho}$.

b. From (1) and (4), we have that

$$\vec{v} \times \vec{a} = \frac{ds}{dt}\vec{T} \times \left(\frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2 \vec{N}\right)$$

$$= \frac{ds}{dt}\frac{d^2s}{dt^2}(\vec{T} \times \vec{T}) + \kappa\left(\frac{ds}{dt}\right)^3(\vec{T} \times \vec{N}). \qquad (5)$$

2.2.3(L) continued

But, since $\vec{T} \perp \vec{N}$ and $|\vec{T}| = |\vec{N}| = 1$, $|\vec{T} \times \vec{N}| = 1$. Hence, (5) becomes

$$|\vec{v} \times \vec{a}| = |\kappa| \left| \left(\frac{ds}{dt}\right) \right|^3 = \kappa |\vec{v}|^3.$$

Therefore, $\kappa = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$. (6)

Equation (10) yields κ in terms of the motion ($\vec{v}$ and $\vec{a}$) of the path, a very practical form in many applications, since $\vec{v}$ and $\vec{a}$ may well be the only available parameters.

c. We had

$$\vec{v} = 4\vec{i} + 3\vec{j}$$

$$\vec{a} = \vec{i} + \frac{1}{3}\vec{j}.$$

Therefore, $|\vec{v}| = 5$

and

$$\vec{v} \times \vec{a} = \frac{4}{3}(\vec{i} \times \vec{j}) + 3(\vec{j} \times \vec{i})$$

$$= \frac{4}{3}\vec{k} - 3\vec{k}$$

$$= -\frac{5}{3}\vec{k}.$$

Therefore, $|\vec{v} \times \vec{a}| = \frac{5}{3}$.

Therefore, from b.

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{\frac{5}{3}}{5^3} = \frac{1}{75}.$$

2.2.4

Starting with the "if" part, we assume the speed is constant. That is, $\frac{ds}{dt}$ = constant. Therefore $\frac{d^2s}{dt^2} = 0$. In this case,

$$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N}$$

implies

$$\vec{a} = 0\,\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N}. \qquad (1)$$

From (1) $\vec{a}$ is a scalar multiple of $\vec{N}$ so that the acceleration is normal to the path in this case.

Conversely, if we now assume that the acceleration is normal to the path then the $\vec{T}$-component of acceleration must be 0. But the $\vec{T}$-component is $\frac{d^2s}{dt^2}$. Since $\frac{d^2s}{dt^2} = 0$, $\frac{ds}{dt}$ is constant, and the assertion is proved.

(From a rigorous grammatical point of view "only if" should be translated that if $\frac{ds}{dt}$ is not constant then the acceleration is not normal to the path. In this case the argument would be that since $\frac{ds}{dt}$ is not constant, $\frac{d^2s}{dt^2}$ cannot be identically 0. Hence the $\vec{T}$-component of the acceleration (i.e. $\frac{d^2s}{dt^2}$) is not always 0. Hence the direction of the acceleration is not always in the direction of $\vec{N}$.)

2.2.5(L)

a. We have $\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\phi}{ds}\right|$, but when the curve is given in the form $y = f(x)$, neither ϕ nor s may be convenient parameters.

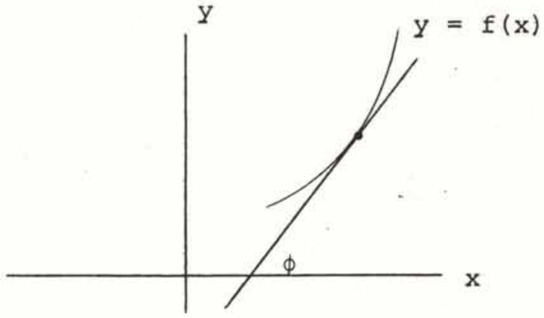
In still other words, when the equation is in the form $y = f(x)$ it is usually convenient to compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Now, what we do know is that

$$\tan\phi = \frac{dy}{dx}. \qquad (1)$$

2.2.5(L) continued

Pictorially,



$$m = \tan\phi = \frac{dy}{dx}$$

Differentiating (1) with respect to x, yields

$$\sec^2\phi \, \frac{d\phi}{dx} = \frac{d^2y}{dx^2}.$$

Therefore, $\frac{d\phi}{dx} = \cos^2\phi \, \frac{d^2y}{dx^2}$. (2)

Now , $\tan\phi = \frac{dy}{dx}$ implies that

$$\cos^2\phi = \frac{1}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} \qquad (\text{i.e., } \sec^2\phi - \tan^2\phi = 1).$$

Hence, (2) becomes

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}. \qquad (3)$$

By the chain rule

$$\frac{d\phi}{ds} = \frac{d\phi}{dx}\frac{dx}{ds} = \frac{d\phi}{dx} \Big/ \frac{ds}{dx} \, .$$

2.2.5(L) continued

Therefore $\kappa = \left|\frac{d\phi}{ds}\right| = \left|\frac{d\phi}{dx} \Big/ \pm\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ ^{*}\right|$. (4)

Substituting (3) into (4) yields

$$\kappa = \left| \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \right|.$$

Had we wished, we could vectorize y = f(x) into a motion problem by writing

$$\vec{R} = t\,\vec{i} + f(t)\vec{j}$$

(i.e., x = t and y = f(t) → y = f(x)).

Then

$$\vec{v} = \vec{i} + f'(t)\vec{j}$$

and $\vec{a} = f''(t)\vec{j}$.

By our previous result,

$$|\kappa| = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{|f''(t)\vec{k}|}{\left(\sqrt{1 + [f'(t)]^2}\right)^3}$$

$$= \frac{|f''(t)|}{\left(1 + [f'(t)]^2\right)^{\frac{3}{2}}}.$$

*The sign, even though it will make no difference when we compute κ, depends upon whether $\frac{ds}{dx}$ is positive or negative, that is, whether s increases or decreases in the direction of the positive x-axis. Our definition that κ is positive avoids this problem.

2.2.5(L) continued

Note

Unfortunately there are several mathematical concepts that lend themselves to more than one convention. Curvature is one of these. We have elected to use the approach that $\kappa = \left|\frac{d\vec{T}}{ds}\right|$, whence $\kappa \geq 0$.

Some people define κ to be positive or negative depending upon whether the curve is "convex" or "concave" with respect to the position vector.

In this case, one writes that

$$\kappa = \pm\left[\frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}\right].$$

2.2.6

a. $y = e^{2x}$

Therefore, $\frac{dy}{dx} = 2e^{2x}$

$$\frac{d^2y}{dx^2} = 4e^{2x}.$$

Therefore, $|\kappa| = \frac{4e^{2x}}{(1 + 4e^{4x})^{3/2}}$ since $|\kappa| = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$

$$= \frac{4}{2\sqrt{2}} = \sqrt{2}.$$

b. $\vec{R} = t\,\vec{i} + e^{2t}\,\vec{j}$

Therefore, $\vec{v} = \vec{i} + 2e^{2t}\,\vec{j}$

$$\vec{a} = 4e^{2t}\,\vec{j}.$$

Therefore, $\frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{|4e^{2t}\,\vec{k}|}{(1 + 4e^{4t})^{3/2}}$, and letting $\kappa = t$, we obtain

2.2.6 continued

$$\kappa = \frac{4e^{2x}}{(1 + 4e^{4x})^{3/2}} .$$

2.2.7

a. From Exercise 2.2.5, $y = ax + b \to \frac{d^2y}{dx^2} = 0$. Therefore,

$\kappa = 0.$

(Note: this is what we would expect intuitively for straight line. Had we wished to see the $\frac{d\phi}{ds}$ for curvature, notice that for a straight line ϕ is constant. Hence $\frac{d\phi}{ds} = 0$.)

b. $y = (a^2 - x^2)^{1/2}$

Therefore, $\frac{dy}{dx} = \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x) = \frac{-x}{(a^2 - x^2)^{1/2}}$; $\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{a^2 - x^2}$;

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{a^2}{a^2 - x^2}$$

$$\frac{d^2y}{dx^2} = \frac{(a^2 - x^2)^{1/2}(-1) - (-x)\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)}{\left[(a^2 - x^2)^{1/2}\right]^2}$$

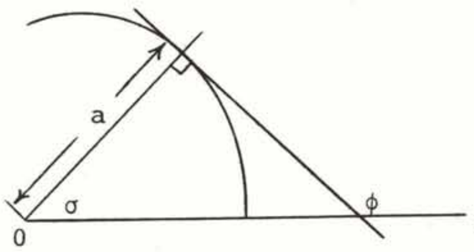
$$= \frac{-\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}}}{a^2 - x^2}$$

$$= \frac{-a^2}{(a^2 - x^2)^{3/2}} .$$

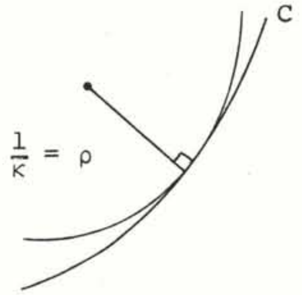
2.2.7 continued

Therefore, $\kappa = \left| \dfrac{-a^2 \Big/ (a^2 - x^2)^{3/2}}{(a^2/a^2 - x^2)^{3/2}} \right| = \left| \dfrac{-a^2}{a^3} \right| = \dfrac{1}{a}$.

(Note: $y = \sqrt{a^2 - x^2}$ is the equation of a semicircle of radius a, centered at (0,0) with $y \geqslant 0$.)



For this circle, of course, a is constant and $\phi = \sigma + 90°$ or $d\phi = d\sigma$. Also $s = a\sigma$ hence $ds = a d\sigma$. Therefore, $\frac{d\phi}{ds} = \frac{d\sigma}{a d\sigma} = \frac{1}{a}$ as before. In this case, the curvature is the reciprocal of the radius. With this in mind, given κ for any curve, we define $\frac{1}{\kappa} = \rho$ to be the radius of curvature at that point. Pictorially, ρ is the radius of an "instantaneous" circle called the osculatory circle. I.e.,



In many kinematics problems, we study the motion of a particle by assuming the particle is on the osculating circle at the given instant.

2.2.8(L)

We have

a. $\vec{R} = t\vec{i} + (t^2 + 1)\vec{j}$ (1)

$\vec{v} = \vec{i} + 2t\,\vec{j}$ (2)

$\vec{a} = 2\vec{j}$. (3)

2.2.8 continued

We already know that $\vec{v}$ is a tangent vector but not necessarily a unit tangent. This is easily remedied by dividing $\vec{v}$ by its magnitude. From (2) we then obtain

$$\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i} + 2t\,\vec{j}}{\sqrt{1 + 4t^2}}\,.$$

Letting $\vec{T} = \dfrac{\vec{v}}{|\vec{v}|}$ we have our desired unit tangent vector. Namely,

$$\vec{T} = \frac{1}{\sqrt{1 + 4t^2}}(\vec{i} + 2t\,\vec{j}). \tag{4}$$

We saw in the previous section that $|\vec{v}| = \frac{ds}{dt}$ and we know that $|\vec{v}| = \sqrt{1 + 4t^2}$ in this exercise. See note at end of this exercise. Hence,

$$\frac{ds}{dt} = \sqrt{1 + 4t^2}\,. \tag{5}$$

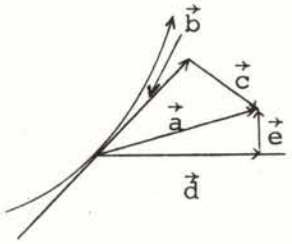
But, we know that $a_T = \dfrac{d^2s}{dt^2}$.

Therefore, $a_T = \dfrac{d^2s}{dt^2} = \dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) = \dfrac{d\left(\sqrt{1 + 4t^2}\right)}{dt}$.

Therefore, $$a_T = \frac{8t}{2\sqrt{1 + 4t^2}} = \frac{4t}{\sqrt{1 + 4t^2}}\,. \tag{6}$$

From (3) we know that $|\vec{a}| = |2\vec{j}| = 2$, and we also know that $|\vec{a}| = \sqrt{{a_T}^2 + {a_N}^2}$. (Here it is most important to observe that $\vec{R}$, $\vec{v}$, and $\vec{a}$ are properties of the motion of the particle, not of the coordinate system. Thus, while the $\vec{i}$ and $\vec{j}$ components of $\vec{a}$ are different from the $\vec{T}$ and $\vec{N}$ components of $\vec{a}$, $\vec{a}$ is the same vector in either case. Pictorially,

2.2.8 continued



$\vec{a} = \vec{b} + \vec{c} = a_T\,\vec{T} + a_N\,\vec{N}$, but $\vec{a}$ is the same

$= \vec{d} + \vec{e} = a_i\,\vec{i} + a_j\,\vec{j}$, in either case.)

In any event,

$a_N{}^2 = |\vec{a}|^{\,2} - a_T{}^2$ (Recall a_T and a_N are scalars since they are the components of acceleration in the $\vec{T}$ and $\vec{N}$ directions.)

So from (3) and (6)

$$a_N{}^2 = (2)^2 - \left(\frac{4t}{\sqrt{1 + 4t^2}}\right)^2$$

$$= 4 - \frac{16t^2}{1 + 4t^2}$$

$$= \frac{4}{1 + 4t^2}.$$

Therefore, $a_N = \dfrac{2}{\sqrt{1 + 4t^2}}$. (7)

b. We know that $a_N = \kappa\left(\frac{ds}{dt}\right)^2$. Hence $\kappa = \dfrac{a_N}{\left(\frac{ds}{dt}\right)^2}$.

From (7) $a_N = \dfrac{2}{\sqrt{1 + 4t^2}}$ while from (5) $\dfrac{ds}{dt} = \sqrt{1 + 4t^2}$.

Therefore, $\kappa = \left(\dfrac{2}{\sqrt{1 + 4t^2}}\right)\Big/(1 + 4t^2) = \dfrac{2}{(1 + 4t^2)^{3/2}}$. (8)

2.2.8 continued

Note: the procedure used here outlines a rather common approach for finding the tangential and normal components of acceleration when the equation of motion is given in Cartesian coordinates. From the given equation we compute $\vec{v}$ and $\vec{a}$. The magnitude of $\vec{v}$ is the speed of the particle along the curve, which is also denoted by ds/dt. From ds/dt we can find the tangential component of acceleration, since this is simply d^2s/dt^2. Since we then know both the total acceleration $\vec{a}$ (which is the same whether we are in Cartesian coordinates or in $\vec{T}$ and $\vec{N}$ coordinates), we use the Pythagorian Theorem to determine the normal component of acceleration. That is,

$$a_N{}^2 = |\vec{a}|^2 - a_T{}^2.$$

Earlier we showed how to find the curvature as a "bonus" if we already know the speed of the particle and the normal component of acceleration. In part b. we want to emphasize that we do not have to resort to any special set of coordinates to compute the curvature. As we mentioned in Exercise 2.2.3 b., curvature is a property of the path and not the coordinate system. This was why we developed a formula for curvature completely in terms of the velocity and the acceleration of the particle at a particular point. Recall that this formula was given by

$$|\kappa| = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} . \qquad (9)$$

From (2) and (3) we see that

$$\begin{aligned}\vec{v} \times \vec{a} &= (\vec{i} + 2t\vec{j}) \times 2\vec{j} \\ &= 2(\vec{i} \times \vec{j}) + 4t(\vec{j} \times \vec{j}) \\ &= 2\vec{k}.\end{aligned}$$

Therefore, $|\vec{v} \times \vec{a}| = 2.$ (10)

2.2.8 continued

Moreover, $|\vec{v}| = \frac{ds}{dt} = \sqrt{1 + 4t^2}$ implies that

$$|\vec{v}|^3 = (1 + 4t^2)^{3/2}. \tag{11}$$

Putting (10) and (11) into (9) yields

$$|\kappa| = \frac{2}{(1 + 4t^2)^{3/2}}$$

which checks with our answer in step 5 in Equation (8).

In closing this exercise we would like to take a few moments to show the advantage, computationally,of the technique used in this problem. We obtained, for example, κ and $\vec{N}$ as natural by-products of relatively simple computations. We could have obtained these quantities directly from their definitions, and it is probably worthwhile to do this at least once.

We have that

$$\frac{d\vec{T}}{ds} = \left|\frac{d\vec{T}}{ds}\right| \left(\frac{d\vec{T}}{ds} \Big/ \left|\frac{d\vec{T}}{ds}\right| \right) = \kappa\, \vec{N}. \tag{12}$$

From (4), we can find $\frac{d\vec{T}}{dt}$, and by the chain rule (proven in the exercises of the previous unit) we have

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\left(\frac{dt}{ds}\right) = \frac{d\vec{T}}{dt} \Big/ \frac{ds}{dt}. \tag{13}$$

From (5), $\frac{ds}{dt} = \sqrt{1 + 4t^2}$

and by one of our product rules of vector calculus we have from (4)

$$\frac{d\vec{T}}{dt} = \frac{d}{dt}\left[(1 + 4t^2)^{-\frac{1}{2}}(\vec{i} + 2t\,\vec{j})\right]$$

$$= (1 + 4t))^{-\frac{1}{2}} \frac{d}{dt}(\vec{i} + 2t\vec{j}) + \left[\frac{d}{dt}(1 + 4t^2)^{-\frac{1}{2}}\right](\vec{i} + 2t\vec{j})$$

2.2.8 continued

$$= (1 + 4t^2)^{-\frac{1}{2}}\, 2\vec{j} - \frac{1}{2}(1 + 4t^2)^{-\frac{3}{2}}(8t)(\vec{i} + 2t\vec{j})$$

$$= \frac{2\vec{j}}{\sqrt{1 + 4t^2}} - \frac{4t\ (\vec{i} + 2t\vec{j})}{(1 + 4t^2)^{3/2}}. \qquad (14)$$

$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\Big/\frac{ds}{dt} = \frac{1}{\sqrt{1 + 4t^2}} \frac{d\vec{T}}{dt}$ from which (14) yields

$$\frac{d\vec{T}}{ds} = \frac{2\vec{j}}{1 + 4t^2} - \frac{4t}{(1 + 4t^2)^2} (\vec{i} + 2t\vec{j})$$

$$= \left[\frac{-4t}{(1 + 4t^2)^2}\right]\vec{i} + \left[\frac{2}{1 + 4t^2} - \frac{8t^2}{(1 + 4t^2)^2}\right]\vec{j}$$

$$= \left[\frac{-4t}{(1 + 4t^2)^2}\right]\vec{i} + \left[\frac{2}{(1 + 4t^2)^2}\right]\vec{j}$$

$$= \frac{2}{(1 + 4t^2)^2}\left[-2t\ \vec{i} + \vec{j}\right]$$

$$= \frac{2}{(1 + 4t^2)^2}\left[\frac{(-2t\vec{i} + \vec{j})}{\sqrt{1 + 4t^2}}\ \sqrt{1 + 4t^2}\right]$$

$$= \underbrace{\frac{2}{(1 + 4t^2)^{3/2}}}_{\kappa} \underbrace{\left[\frac{-2t\vec{i} + \vec{j}}{\sqrt{1 + 4t^2}}\right]}_{\left[\vec{N} = \frac{d\vec{T}}{ds}/\left|\frac{d\vec{T}}{ds}\right|\right]}. \qquad (15)$$

2.2.9

$$\vec{R} = t^3\ \vec{i} + \sin t\ \vec{j} \qquad (1)$$

Therefore, $\vec{v} = 3t^2\vec{i} + \cos t\ \vec{j}$. (2)

2.2.9 continued

$$\vec{a} = 6t\,\vec{i} - \sin t\,\vec{j}. \tag{3}$$

Therefore, $|\vec{v}| = \sqrt{9t^4 + \cos^2 t} = \frac{ds}{dt}$. (4)

From (4),

$$\frac{d^2s}{dt^2} = \frac{1}{2}(9t^4 + \cos^2 t)^{-\frac{1}{2}}(36t^3 - 2\cos t \sin t)$$

$$= \frac{36t^3 - \sin 2t}{2\sqrt{9t^4 + \cos^2 t}}.$$

Since the tangential component of acceleration a_T, is equal to $\frac{d^2s}{dt^2}$, we have

$$a_T = \frac{36t^3 - \sin 2t}{2\sqrt{9t^4 + \cos^2 t}}. \tag{5}$$

As for the normal component of acceleration, a_N, we have

$$|\vec{a}|^2 = a_T^2 + a_N^2 \tag{6}$$

and from (3)

$$|\vec{a}| = \sqrt{36t^2 + \sin^2 t}. \tag{7}$$

Putting the results of (5) and (7) into (6), we obtain

$$36t^2 + \sin^2 t = \frac{(36t^3 - \sin 2t)^2}{4(9t^4 + \cos^2 t)} + a_N^2.$$

2.2.9 continued

Therefore,

$$a_N{}^2 = \frac{(36)^2t^6 + 36t^2 4\cos^2 t + 36t^4\sin^2 t + 4\cos^2 t\sin^2 t - (36t^3 - \sin^2 t)^2}{4(9t^4 + \cos^2 t)}$$

$$a_N = \frac{\sqrt{36t^2(4\cos^2 t + t^2\sin^2 t + 2t\sin 2t}}{\sqrt{4(9t^4 + \cos^2 t)}} \quad \text{or}$$

$$a_N = \frac{6t(2\cos t + t\sin t)}{2\sqrt{9t^4 + \cos^2 t}} .$$