PROFESSOR: Hi, welcome to Calculus Part 2, where our theme for the entire course will essentially be functions of several variables. But a more underlying theme, a theme which will not only permeate this course, but virtually every course in the mathematics curriculum, and perhaps other curricula as well, is the idea of what we mean by a mathematical structure. And if this sounds a little bit ominous and frightening, the idea of a mathematical structure can be compared very nicely in terms of a game, and for this reason, I have chosen to entitle our first lesson "The 'Game' of Mathematics".

"The 'Game' of Mathematics." And let's emphasize the word "game" here. We do not mean game in a trivial sense, where an elementary school, the first row races the second row to see who finishes the addition problem first. The idea that we want to talk about is what is a game.

And I remember an old riddle when I was in about the fourth or fifth grade, when somebody said to me what is it that looks like a box, smells like cheese, and flies, and the answer was a flying cheese box. And the interesting point is that this is how, in the scientific world, we often make up definitions. Namely, to define a game, we tried to think of every single ingredient that is common to every game, and then we roll all of these ingredients into one long definition, and that becomes our final definition. And with that in mind, let me take the following tack.

And again, let me point out that I am going through this rather hurriedly because the main aim of the lesson is to give an overview with the idea that the supplementary notes and the exercises in the unit will give you the computational drill that you need. But for a first approximation, let us say that, in any game, you must have definitions, so to know what terminology you have. Then you must have rules of the game, and notice that the rules of the game are relationships between the terms. Oh, as a trivial example in playing cards, there are many different card games that can be played with the same deck of cards. What makes the game different are not the definitions involved, but the rules of the game.

And finally, there are objectives. Well, obviously, the objective of any game is to win. What we really mean by an objective is the art of carrying out a winning situation by successfully employing the definitions and the rules. And the way we do that is usually called strategy.

Strategy is the art of using the definitions and rules to carry out the objective in an inescapable manner. And the reason I like this particular little setup is it gives me a way to show you, in juxtaposition, the role of rote verses reason, logic verses memory, in any mathematical situation or real life situation. Namely, the strategy part of a game is logic, and things like the definitions and the rules are things that we memorize.

OK, now, the thing that we're going to do in our particular course is always come back to this particular structure. In other words, our definition of a game is any system which consists of definitions, rules, and objectives, where the objective is carried out as an inescapable consequence of the definitions and the rules by means of strategy. And by the way, don't take this lightly. This is a very serious topic. Later in the course, the computation will become sufficiently difficult that we may lose sight of the forest because of the trees.

Right now, what I want to do is emphasize this to you in terms of topics that you're already familiar with so that you can see what the overall structure of mathematics is so that you will not be preoccupied with this when we're learning more computational things. Let me just take a look at, well, a relatively trivial example. I call this a new look at counting.

We all know how to count. We start with a number called 1, and we have various definitions, 1 plus 1 is 2, 2 plus 1 is 3, 3 plus 1 is 4, 4 plus 1 is 5, 5 plus 1 is 6, 6 plus 1 is 7, et cetera. OK, so far, so good. Now, we ask the question how much is 4 plus

3?

Now, obviously, we all know that 4 plus 3 is 7. We're not saying what are we looking at this problem for. What we're trying to show now is a very important aspect of the game of mathematics. You see, notice that in our list of definitions, no place do we have the sum 4 plus 3 defined. What we are interested in now is not so much the truthful statement that 4 plus 3 is 7, but whether the result 4 plus 3 equals 7 follows inescapably from the definitions that we've listed.

Now, you see, the point is all we've listed are definitions. We haven't told any particular rules of the game yet. Well, let's make up some rules as we go along, and as I say, we'll discuss these in more details in our notes and in the exercises. We essentially do something like this.

We write down 4 plus 3, then we say, OK, we do know how to add by ones. That's what our definition says, so we rewrite 3 as 2 plus 1. In other words, we substitute 2 plus 1, which is equal to 3 by definition. Then we say, OK, 2 plus 1 is the same as 1 plus 2.

Now, by the way, notice the tacit assumption that we're making. We're assuming that the order in which you add two numbers makes no difference. Obviously, this is not a rule in every game of life. In most things in life, order does make a difference.

Consider, for example, the statements first I undress, and then I take a shower; or first I shower, and then I undress. Without meaning to pass judgment as to which is proper, at least notice there is a difference between the two. What we're saying is that, somehow or other in the game of arithmetic, we assume that addition has the property that the order in which you add makes no difference. So we say, OK, let's accept that as a rule of the game.

If we accept that as a rule the game, 2 plus 1 could then be substituted, 4 plus 1 plus 2. We make the additional assumption that, when you add three numbers, the answer does not depend on voice inflection. In other words that 4 plus 1 plus 2 is equal to 4 plus 1 plus 2, and the strategy behind doing that is that we know that another name for 4 plus 1 is 5. In other words, we now arrive at the fact that 4 plus 3 is equal to 5 plus 2.

We now rewrite 2 as 1 plus 1, and we now have the 5 plus 2 is the same as 5 plus 1 plus 1. We now again use the fact that voice inflection makes no difference, and we rewrite this as 5 plus 1 plus 1. 5 plus 1, we know by definition, is 6, and 6 plus 1, we know by definition, is seven.

In other words, subject to the rules that we've talked about implicitly here but have not stated in our game format explicitly, what we have shown is that if we accept certain rules, it follows from our definitions that 4 plus 3 equals 7 is an inescapable conclusion. Notice that the inescapability of the conclusion hinges on the fact that we've accepted certain rules. If we change the rules, we can change the conclusions. In other words, this thing called drawing inescapable conclusions is something called validity, and validity involves the art of drawing inescapable conclusions using given rules and given definitions. We'll talk about that in more detail, but for now, notice the difference here.

Before we did this, we knew as a conjecture, a past experience thing, that 4 plus 3 equals 7 is a true statement. What we now know is in terms of certain rules of the game coupled with the definitions that we've accepted, 4 plus 3 equals 7 is an inescapable conclusion, henceforth to be called a "theorem" in our game. Let's see if we can't get away from this rather simple example, and again, let me emphasize that as simple as this example is, throughout our course, we will be using the same technique, only at a more sophisticated level of computational skill. But let's take a look here and see what we're really saying.

What mathematical structure really involves is a logic machine type of thing. We have a logic machine, a machine that, being fed any kinds of definitions, rules, assumptions, et cetera, grinds these things through and has, as its output, inescapable conclusions. In other words, you feed in definitions, rules, et cetera, which by the way, don't really have to be true in the real life sense of being true. I mean, for example, in the baseball game, to say the rule is three strikes is an out, there was certainly no basic truth the said that had to be the case.

What is true is that, in the world of science, the scientist is the interpreter of nature. What happens in real life happens whether the scientist predicts it or not. What he tries to do is to make up definitions and rules, which are compatible with his experience, things which he calls truth. He feeds those through his logic machine, draws inescapable conclusions, and if the conclusions follow inescapably from the definitions and the rules, we call the resulting argument valid.

In other words, truth is a value judgment that we make about a particular statement. Validity is a more objective thing that we attribute to an argument. In other words, a statement is either true or false. An argument is either valid or invalid, meaning that we only judge whether the conclusion of the argument follows inescapably from the given assumptions independently of whether those assumptions happen to be true or not.

By the way, as an aside, one of the reasons that the scientist prefers to dodge issues such as what is truth and leaves that to the philosopher, is that truth, in many cases, is a relative thing. It's based on the available knowledge. It's also based on the situation that we want to handle.

In other words, what is truth? And my claim is that the answer depends on the situation. Well, let me give you again a trivial arithmetic situation. Does 1/2 plus 1/3 equal 5/6, or does 1/2 plus 1/3 equal 2/5? And again, the answer is it depends on what real life problem you're dealing with.

For example, if a person is working for you by the hour, and he works for you for a half hour one day and a third of an hour the next day, the total time he's worked is 5/6 of an hour. Why? Because this agrees with our real life experience that 30 minutes plus 20 minutes is 50 minutes. On the other hand, if a baseball player goes one for two in the first game of a double header and one for three in the second game of a double header, he has batted two hits in five times at bat. In fact, if you could convince the public that he had five hits and six times at bat, you could become a director of any economy program in the nation, I guess.

The point, however, is this. In most arithmetic questions that we deal with, we are used to the physical interpretation in which 1/2 plus 1/3 equals 5/6 reflects the real life situation. Whereas this example here may seem trivial, namely, how often are you going to be involved with batting averages unless you're doing sixth grade arithmetic. The point remains, ironically or whatever you want to call it, that this little batting average problem is not trivial. In the world of engineering, we know this example as the weighted average problem.

In other words, you'll notice that 2 over 5, as a fraction, is more nearly equal to one third that it is equal to 1/2, and the reason for that is that the player batted at the low average, 1 over 3, one hit in three times at bat, for more times at bat. In other words, the three has a heavier weighting factor than the two, and every time that you use a weighted average in a scientific engineering oriented investigation, this is the truth, not this. In other words, it is neither true or false that 1/2 plus 1/3 equals 2/5 or that 1/2 plus 1/3 equals 5/6. Which is true depends on the particular physical situation. You see a rather interesting point here, that we sometimes allow truth to be based on what particular problem we're trying to solve.

Another way that we manipulate truth is that we sometimes have a rule that we like to be true. In fact, I guess this is probably what happens with most political theories that one starts with the objectives, knows what it is that he wants to be true, and then invents the definitions and the rules to conform with this. In much the same way as around the fourth grade again, we learn such adages as, "Look before you leap," and two minutes later you learn, "He who hesitates is lost," and suddenly you come to the conclusion that you can make your assumptions validify any conclusion you want just by choosing your assumptions appropriately. Now, if that sounds degrading, let me show you how it's used effectively in mathematics. In other words, let me just say that predetermined rules may control truth.

Let me give you an example. Going back to exponents, why does b to the 0 equal 1? The answer is very simple, that when we use positive exponents, positive whole number exponents, we talked about that many factors of b. And one of the interesting rules that we saw that was obeyed by whole number exponents was that if you multiply b to the mth power by b to the nth power, you got as an answer b to the m plus n power. b to the m plus n.

Now, the interesting thing is that after a while you never even paid attention to why this rule worked. What you did know was that this was a pretty darn convenient rule to use. Computationally, this rule simplified many particular computations that you were doing. In particular, then, as soon as n is 0, you would still like to be able to use this rule. In other words, we want this nice rule to apply even when n equals 0.

Now, let's look at this from a computational point of view. If we want this rule to apply when n equals 0, let's simply rewrite this exactly as it appears here with n equal to 0. We then have, what? We have b, we'll just repeat everything here. b to the m times b. Now, we're replacing n by 0, so that's b to the 0, and that must equal b to the m plus n, that's m plus 0.

But the interesting point is that we know how to add numbers, and for numbers, m plus zero is m. In other words, this is still b to the m. Now, we look at this, and what we're saying is if we want the rule b to the m times b to the n to equal b to the m plus n to be true even when n is 0, this says that b to the 0 must be that number such that when we multiply it by b to the m, we get b to the m. Now, what number has the property that when you multiply it by b to the m you get b to the m? And if you're real quick, you say 1, and if you're algebra oriented, you say b to the 0 is therefore equal b to the m divided by b to the m, and again you say 1, except of course that b must be unequal to zero because, hopefully, by this time we understand why we must never divide by 0.

In other words, what we now have is the old high school rule that b to the 0 equals 1 provided b is unequal to zero. But here's the important point. If I have never defined b to the 0, and somebody says to me make up a definition for b to the 0, and I say, OK, b to the 0 is going to be 36, I have every right in the world to make up that definition. What I don't have a right to do is, when I'm ever using an expression like b to the 0, is to assume that I have the right to use this particular recipe.

In other words, if I want to be able to use the nice recipe, even when n is 0, I have no choice but to define b to the 0 to equal 1. Therefore, we make up that definition because we want that recipe to apply, and that's exactly what we're going to be doing through most of this course. We are going to look at certain real life situations, we are going to look at certain recipes that we want to apply, and we are going to make up definitions this way, and then see how our inescapable conclusions follow from these definitions. More about that will be said in the remainder of this unit, and our next lecture will pick up the game of mathematics in a different context. But until next time, good bye.

Funding for the publication of this video was provided by the Gabriella and Paul Rosenbaum Foundation. Help OCW continue to provide free and open access to MIT courses by making a donation at ocw.mit.edu/donate.