NARRATOR: The following content is provided under a Creative Commons license. Your support will help MIT OpenCourseware continue to offer high-quality educational resources for free. To make a donation or to view additional materials from hundreds of MIT courses, visit MIT OpenCourseWare at ocw.mit.edu.

PROFESSOR: Hi. The aim of our lecture today is to present what is perhaps the singly most powerful recipe for evaluating indefinite or, for that matter, definite integrals. It's called integration by parts, and therefore, today's lecture is called 'Integration by Parts'. And what I want to emphasize is essentially that this is a technique which shows us how to find indefinite integrals, but then in theory, it changes nothing over what we've had before. This is essentially, as we've mentioned, a recipe for being able to simplify or reduce indefinite integrals to things that we can handle.

Well, it centers around the rule for differentiating a product. And to show you what I mean by that, let's just barge right into this thing. In differential notation, we call the 'u' and 'v' our differentiable functions of 'x'. The differential of 'uv' is 'u' times the differential of 'v', plus 'v' times the differential of 'u'. And solving for 'udv', just transposing, we have that 'udv' is the differential of 'u' times 'v' minus 'vdu'.

Now, what we do is we integrate both sides of this equality. On the left-hand side, we have interval udv. On the right-hand side, recall that the integral of the derivative just gives us back the original function. In other words, if we integrate the differential of 'uv' up to an arbitrary constant, we get back 'u' times 'v'. The integral of 'vdu' we'll just denote by integral 'vdu'.

And observe that there's really an arbitrary constant coming from this term, an arbitrary constant coming from this term, an arbitrary constant coming from this term. But since the sum and difference of arbitrary constants is still an arbitrary constant, we can neglected all the constants here, and just lump them together, and at the end write plus 'c'. Or we can leave the 'c' off, making it implied that the arbitrary constant is contained in the symbol integral 'vdu'.

All right, now we've done this, but a much more important question would be-- or exclamation, for that matter-- would be so what? See, who cares whether this happens to be true or not? The answer is we care. That's why we're making up a lecture on it. And the reason that we care is this. That frequently, for a given 'u' and 'v', 'udv' may have a completely different nature about it than does 'vdu' insofar as being able to find the integral.

Now again, the best way to show what we mean by this is to choose a particular example. Let me start, just arbitrarily here, picking a couple of simple functions for us to differentiate or to integrate. Let's pick 'u' to equal 'x' and 'v' to equal 'sine x'. Now, it's simple to see from this example that in this case, if 'u' equals 'x' and 'v' is 'sine x', then 'dv' is ''cosine x' dx'. In fact, let's write that down so we can see it here. Notice that in this particular case, integral 'udv' is integral of 'x 'cosine x' dx'-- 'udv'. On the other hand, what is 'vdu'? 'v' is 'sine x'. Then since 'u' equals 'x', 'du' is 'dx'. Therefore, 'vdu' is just integral ''sine x' dx'.

Now, look at these two integrals. This one we can integrate just by looking at it. It's minus 'cosine x'. This one, well, offhand, we don't know a function whose derivative with respect to 'x' is 'x 'cosine x''. However, by this recipe, we now have a way of showing how to 'udv' once we know integral 'vdu'.

In other words, in this particular problem, letting 'u' equal 'x' and 'dv' being ''cosine x' dx', we can proceed as follows. Namely, by this recipe, we get what? Integral 'x 'cosine x' dx' is equal to-- this is 'udv' equals 'u' times 'v'. Recall that 'u' was 'x', 'v' is 'sine x', so 'u' times 'v' is 'x sine x', minus integral 'vdu'. But 'v' is 'sine x', 'du' is 'dx', so we wind up with this. But now we know that the integral of 'sine x' with respect to 'x' is minus the 'cosine x'. The minus and the minus cancel to give me a plus. And I wind up with that the integral of 'x 'cosine x' dx' is 'x sine x' plus 'cosine x' plus an arbitrary constant, OK.

That's all integration by parts really is. All we do here is we evaluate one integral by knowing another integral. And by the way, what I meant here when I said that one did not have to change any of the theory, what does the integral mean here anyway? What we mean by this is what? Find all functions whose derivative with respect to 'x' is 'x cosine x'.

Now, what integration by parts did for us was it showed us how to get this result. But if we didn't know how to get this result, observe that we check the answer the same way as always. Namely, one thing that hopefully we can do is differentiate this thing. And if we differentiate it, let's see. This is a product. It's the first factor times the derivative of the second. That's 'x cosine x'. Plus the second factor times the derivative of the first. That's just 'sine x'. The derivative of 'cosine x' is minus 'sine x'. The derivative of a constant is 0. And if we now sum up the pieces that we have here, we wind up with 'x cosine x'. Exactly what we were supposed to get, OK?

Now, you see again, nothing has happened in theory. All we have now is a technique that

allows us to explicitly solve certain problems that we couldn't have solved otherwise, even though we could've stated them before. For example, let's go back to a problem that technically speaking we could have presented before we got to this unit of our material.

Let's support that the region 'R' is the region bounded above by 'y' equals 'cosine x', below by the x-axis between the y-axis at 'x' equals pi/2. What we want to do is take this region 'R' and revolve it about the y-axis. And what we'd like to find is what volume is generated when the region 'R' is revolved about the y-axis.

Now, when we talked about volumes and cylindrical shells, we already had the recipe. Recall that to rotate this about the y-axis, just to refresh memories here, if we call this 'x' and we call this 'y', the element of volume was what? Well, you swung out a circumference '2 pi x' times height 'y' times thickness 'dx'. In other words, that the volume when this was rotated about the y-axis was just 2 pi integral 'x 'cosine x' dx' as 'x' goes from 0 to pi/2.

You see, I rigged this particular problem so we wouldn't have to waste the fact that we've already found out what the indefinite integral of 'x 'cosine x' dx' is. In other words, we get down to this step here. Now, by the first fundamental theorem of integral calculus, we know that whatever this integral is, it's also equal to 'G of pi/2' minus 'G of 0', where 'G' is any function of 'x' whose derivative with respect to 'x' is the integrand here, which is 'x cosine x'.

Now, you see, this part, or up to here, we could have got this result without today's lecture. The only thing that we learned to do today that we couldn't do before was the idea of how we find a function whose derivative is 'x cosine x'. In fact, what we found was what? A function whose derivative is 'x cosine x' is just 'x sine x' plus 'cosine x'.

So by the first fundamental theorem, the answer to our volume problem is that it's 2 pi times the quantity 'x sine x' plus 'cosine x' evaluated between 0 and pi/2. Putting in 'x' equals pi/2, this term is pi/2. This term is 0. Putting in the lower limit when 'x' is 0, this term is 0. This term is 1. We subtract the lower limit. That gives us a minus 1. And therefore, the volume that we get is simply 2 pi times pi/2 minus 1. Again, notice then, that the only new thing that we had to do here was devise a way for finding a function whose derivative with respect to 'x' was 'x cosine x'.

By the way, I think you'll observe over here that I started in kind of a rigged fashion, namely, I started with a specific 'u' and 'v'. It might be interesting to see how would one tackle this

problem if one had just been given the 'x 'cosine x' dx' and had to work from there. In other words, instead of working backwards now, let's present the problem as it would come up in a real problem, namely, all of a sudden, we find that we have to integrate 'x cosine x'.

And the idea go something like this. You see, we say lookit. If this 'x' weren't in here, this would be pretty easy to handle. So why don't we do this? Why don't we call this piece here 'u' and call this piece here 'dv'. Then, you see, 'du' would just be 'dx'. 'v' would just be 'sine x'. In other words, the differential of 'sine x' is 'cosine x dx'. See, if 'u' is 'x', 'du' is 'dx'. If 'dv' is 'cosine x dx', 'v' is equal to 'sine x'.

I suppose, technically speaking here, I should put in plus a constant, but I'll show later why we can disregard this constant. Roughly speaking, it just hinges on the fact that we want only one solution. But we'll talk about that later. For the time being, I'll just leave out the fact that there's a whole family of functions. All I want is one function whose differential is 'cosine x dx'.

At any rate, I now apply my recipe, namely, integral 'udv' is equal to 'u' times 'v'-- that's 'x sine x'-- minus integral 'v', which is 'sine x', times 'du', which is just 'dx'. In other words, integral 'x 'cosine x' dx' is just 'x sine x' minus 'sine x dx'. But I know how to integrate 'sine x'. It's just minus 'cosine x'. Therefore, replacing integral 'sine x dx' by minus 'cosine x' plus a constant. The minus cancels here. I wind up with the same result as before, only now hopefully in terms of a smooth flow not that didn't presuppose the answer. Namely, this integral is 'x sine x' plus 'cosine x' plus a constant.

And by the way, I should mention here that this technique does require a certain amount of ingenuity. In other words, in a given problem, where you're not told what to call 'u' and what to call 'v', where you have the choice of picking what's going to be 'u' and what's going to be 'dv', there is a certain element of insight that one needs to guess wisely. In this particular problem, it's kind of obvious which way to guess. But let me, just to give you the experience, show you what happens when you guess incorrectly.

For example, given the same problem, why couldn't I have said gee, I think I'll let 'cosine x' be 'u', and what's left over, namely, 'xdx'. That will be 'dv'. Now, notice that mathematically, no harm is done this way. Namely, if 'u' is 'cosine x', 'du' is minus 'sine x dx'. If 'dv' is 'xdx', 'v' is 1/2 'x squared'.

And now, applying the recipe, integral 'udv' is what? It's 'uv'. That's 1/2 'x squared' times 'cosine x' minus the integral 'vdu'. But 'vdu' is already minus 1/2 'x squared' 'sine x dx', so

minus that is plus 1/2 integral ''x squared' 'sine x' dx'. And I wind up with the result that the integral 'x 'cosine x' dx' is 1/2 'x squared' 'cosine x' plus 1/2 integral ''x squared' 'sine x' dx'. And there is absolutely nothing at all wrong with this, except that the chances are if I didn't know how to find the function whose derivative was 'x cosine x', what is the likelihood that I know a function whose derivative is ''x squared' 'sine x' dx'?

In other words, here's a case where replacing 'udv' by 'vdu' gave me an equivalent integral to evaluate which was more cumbersome than the first. In fact, somehow or other, it would seem that by multiplying through here by 2 and solving for ''x squared' 'sine x' dx', that it would seem that somehow or other this would be a way of simplifying the integral ''x squared' 'sine x' dx'. In other words, if you had started with this, I think you'd accept the fact, well, this still isn't as nice as I'd like it, but it does seem less complicated than this. You see, at least the power of 'x' is only to the first power here, whereas here it's the second power.

In fact, this leads to a technique called a reduction formula. Again, using hindsight rather than foresight, having motivated the integral ''x squared' 'sine x' dx', let's see how I could have solved this particular problem if I didn't know what the answer was going to be in advance. You see, I say to myself, you know, this x squared here is bothering me. If that 'x squared' weren't here, this would be pretty easy to handle. Well, the best way to knock the 'x squared' down is to let 'u' equal 'x squared', in which case, 'du' will be '2xdx'.

In other words, your second power will now be replaced by a first power. It's not perfect, but at least there's hope here. At any rate, what I'm saying is motivated into calling 'u' 'x squared' and therefore letting 'dv' be the rest of this integrand, which is 'sine x dx', we wind up with the fact that 'du' is '2xdx' and 'v' is minus 'cosine x'.

Well, you see now, remembering that 'u' is 'x squared', 'dv' is 'sine x dx', the recipe tells us that integral 'udv' is 'u' times 'v'. That's minus 'x cosine x' minus integral 'vdu'. But integral 'dvu' is integral twice 'x 'cosine x' dx' with a minus sign. The minus sign in the recipe makes it plus. And we therefore wind up with that integral ''x squared' 'sine x' dx' is equal to minus ''x squared' 'cosine x'' plus twice integral 'x 'cosine x' dx'.

By the way, just to refresh your memories on this, if we come back and compare this with what we had over here, it's exactly the same thing. The only difference now is that we arrived at this thing directly. Now you see the point is, having reduced this to this, I can now use integration by parts again to simplify this. In other words, what I would do next is let 'u' equal 'x' and 'dv'

be 'cosine x dx'. But let's face it, I've already done this problem. Time is getting short. Why do it again? The point is that integral 'x cosine x' using integration by parts again is going to come out to be 'x sine x' plus 'cosine x' plus 'c' because that's what we got before when we used integration by parts to do this.

At any rate, what I would now do is replace integral 'x cosine x' in here by this, OK? And if I do that, I have what? Integral ''x squared' 'sine x' dx' is equal to what? Minus ''x squared' 'cosine x'' plus twice times this, which is twice this. That's what? 2 'x sine x' plus 2 'cosine x' plus a constant.

And by the way, again notice, if I wanted to check whether this was right, the one thing I can do if somebody gives me this is what? I can differentiate it with respect to 'x'. When I differentiate it with respect to 'x', if I get ''x squared' 'sine x'' as my result, this is correct. Otherwise, I've made a mistake. But the idea is keep in mind that these can always be checked because they're inverse derivatives. They can always be checked by actual differentiation.

And again, notice the beauty of this technique. If somebody said construct the function whose derivative with respect to 'x' is ''x squared' 'sine x'', notice how difficult it would be to try to piece this thing together. See, notice the difference between being able to check that this is a right answer and how one would find this. This is the technique of a power of integration by parts.

In fact, in making up exam questions, this is exactly how professor sometimes do this. That what you start with is you start with this, then give a person this problem and tell them to differentiate this. And that whole mess simply comes out to be ''x squared' 'sine x''. And that's why a lot of times when you have a messy problem to differentiate and you look at the answer in the back of the book, you become amazed to see that the thing really simplified.

Again, notice the idea. The integrand here is relatively simple. What we have to differentiate to get that relatively simple integrand is not quite so simple. But enough about that. Let me just go on with a few more examples. See, I have plenty of examples in the exercises to keep you busy and to give you drill on this. All I'm trying to do now is to motivate places where one would want to use integration by parts. Well, let me show you one place it might be used.

One place that it's always worth trying is when you look at the integrand, haven't the vaguest notion as to how to integrate it, but you are able to differentiate it. You see, you've got nothing

to lose. The worst that will happen is that the resulting integral vdu will be no easier to handle than the original udv, but at least there's a chance that something will work. But let me give you a for instance.

I see I've written this rather badly. This is not 'Stan'. This is the integral of inverse 'tangent x dx', OK? Integral inverse 'tangent x dx'. What function has its derivative equal to inverse 'tangent x'? Well, you see, the thing I do know hopefully is what the derivative of inverse 'tangent x' is. The derivative of inverse 'tangent x' with respect to 'x' is '1 over '1 plus x squared''. And by the way, if you don't know this, there's no hope of even beginning the integration by parts. in other words, we have to assume that we can do something in a problem like this.

At any rate, knowing this, what do I do next? I say, well, since I know how to differentiate the inverse tangent, that's a very natural thing to call 'u' because I can certainly find 'du'. Moreover, the nice thing is that calling 'dv' 'dx', if I can integrate anything, it's certainly 'dx'. Namely, what I'm saying is if 'u' equals inverse 'tan x' and 'dv' is 'dx', then 'du' is 'dx' over '1 plus x squared', and 'v' is just equal to 'x'. Now, therefore, integral inverse 'tan x dx', in other words, integral 'udv', is just what? It's still just 'u' times 'v'. That's 'x' times inverse 'tan x' minus 'vdu'. That's minus integral 'x' times 'dx' over '1 plus x squared'.

Now, look what happens in this case. Obviously, with time being short and having other places to show you my failures, namely in the exercises, I picked one that worked nicely. Here's the case, you see, that the integral inverse 'tan x dx', by using parts, becomes replaced by integral 'xdx' over '1 plus x squared'. This is one I can handle from before. This is one where the substitution 'u' equals '1 plus x squared' works very nicely for me.

Or if you're getting glib with these things, just observe that this integrand is 1/2 natural log '1 plus x squared'. Because how would you differentiate natural log '1 plus x squared'? It would just be what? '1 over '1 plus x squared'' times the derivative of this, which is '2x'. The 2 and the 1/2 would cancel, and we just have left 'x over '1 plus x squared''. In other words, again, notice that the answer here is what? This is the function that one would have to differentiate with respect to 'x' to wind up with inverse 'tan x'.

I'll show you another example in a few minutes. What I did just remember is that I mentioned why we can drop the arbitrary constant. Let me just take a minute here and show how this problem should have really been worked. If we wanted to be precise here, we'd say OK, 'u'

equals inverse 'tan x'. Therefore, 'du' is 'dx over '1 plus x squared''. That part's fine.

Then we say, OK, 'dv' is 'dx'. Now, if 'dv' is 'dx', what must 'v' be? Well, 'v' could be 'x', but it must be something of the form 'x' plus some constant, which I'll call 'c1'.

Could I have chosen 'v' to be 'x plus c1'? Why not? In fact, what would have happened if I did that? Notice that what would have happened in this case is that 'u' times 'v' now would simply have been 'x plus c1' times inverse 'tan x', whereas integral 'vdu', instead of being 'xdx' over '1 plus x squared' would have been ''x plus c1' dx' over '1 plus x squared'. In other words, leaving the constant in, I would get this.

And now all I want you to see here, and I've done this just for this particular example, and in the exercises we'll work it out in general. The whole idea is this. This, of course, can be written as two terms, one of which is 'x inverse tan x', and the other of which is 'c1 inverse tan x'. This also can be written as two integrals. Namely, with the minus sign kept aside here, this is integral of 'xdx' over '1 plus x squared' and plus integral 'c1 dx' over '1 plus x squared', the minus making this minus. But we know that the integral 'dx' over '1 plus x squared' is inverse 'tan x'.

In other words, if we now separate these into pieces, we have what? 'x inverse tan x' minus integral 'xdx' over '1 plus x squared'. And notice that the 'c1' term drops out because we have 'c1 inverse tan x' occurring twice. It's this term here, and it's also this part of our integral. And see, with the minus sign here, they just cancel out. And that's why we really don't have to keep the arbitrary constant in there, but if we wish to, we may. At any rate, let me summarize with just one more problem to make sure that we have the technique down fairly pat.

Let's suppose we were given integral 'log x dx'. What we certainly do know I hope by this time is that the derivative of 'log x' with respect to 'x' is '1/x'. Since I can differentiate 'log x', it becomes very meaningful to try letting 'u' equal 'log x'. If I do this, 'u' is 'log x'. 'dv' is 'dx'. 'du' then becomes ''1 over x' dx', and 'v' is 'x'. And now I put in plus 'c' in parentheses here to indicate that if you wanted to use it, you could.

At any rate, what happens now? Integral udv, integral 'log x dx' in this case is 'u' times 'v', which is 'x log x', minus integral 'vdu'. Well, you see the 'x' times the '1/x' is just 1. Therefore, 'vdu' is just 'dx', so minus that is just minus 'dx'. In other words, minus the integral 'vdu' is just minus integral 'dx'. And now, you see I have what? 'x log x'. And this is very easy to integrate. It just happens to be minus 'x', then I tack on the arbitrary constant. Again, to check whether

this was the right answer, all I have to do is differentiate 'x log x' minus 'x' with respect to 'x' and see if I get 'log x', and I will, OK?

Now, how would I use a problem like this, say, in a practical situation? Well, for example, suppose I wanted to find the area of the region 'R' where 'R' was the region bounded above by the curve 'y' equals 'natural log x', below by the x-axis, on the left by 'x' equals 1, and on the right by 'x' equals 'b', where 'b' can be any number you want right now as long as it's bigger than 1 in this diagram. Notice that the definite integral says that the area of the region 'R' is the definite integral from 1 to 'b', 'log x dx'. And the first fundamental theorem of calculus says lookit. If you happen to know a function whose derivative with respect to 'x' is 'log x', you can evaluate this thing very quickly. Namely, it's just 'G of b' minus 'G of 1' where 'G prime of x' is 'log x'.

Well, you see, we do know a function now whose derivative with respect to 'x' is 'log x', namely, 'x log x' minus 'x'. Consequently, to solve this problem, all I have to do is evaluate this between the limits of 1 and 'b', which I won't bother doing right now, OK?

Now, here's the idea. What we've really learned here is one more powerful technique for evaluating integrals. How we arrive at integrals in no way changes in this particular lecture. All we have now is what? A new technique whereby we evaluate udv by being able to evaluate equivalently vdu. At any rate, that finishes up what we had in mind for today, and so until next time, goodbye.

NARRATOR: