PROFESSOR: Hi. Today we will discuss the inverse hyperbolic functions and, with this lecture, finish our block on the logarithmic exponential and hyperbolic functions.

And what we're going to find is that much of what we have to say today is simply a specific application to a special function of the same theory that we were talking about in general before. Recall that we can always talk about an inverse function if the original function is a one-to-one function.

For example, to introduce today's topic, suppose we take the function 'y' equals hyperbolic sine 'x'. 'y' equals 'sinh x'. Well, as we saw last time, the graph of 'y' equals 'sinh x' is this. And we can see in a glance that this clearly is a one-to-one function. In fact, the derivative of sinh is cosh. And as we've seen also last time, the cosh can never be negative. In fact, the cosh can't be less than one. So here is a curve that's always rising-- and, in fact, in general rising quite steeply.

In any event, we can therefore talk about the inverse hyperbolic sine. All right? And how do we get the inverse function in general?

Well we've always seen that to invert a function, all we have to do is to reflect the original graph with respect to the line, 'y' equals 'x'. Or, again, in more slow motion, we rotate through 90 degrees and then flip the graph over. And, in any event, whichever way you want to look at this thing, we obtain that the graph 'y' equals inverse hyperbolic sine 'x' is this particular curve over here. This is the graph of the inverse hyperbolic sine.

Now, again, what was the main issue or the main property of inverse functions? The idea was that once we knew the original function, we could, by a change in emphasis, study the inverse function.

Well, by way of illustration, let's suppose we take the functional relationship 'y' equals inverse hyperbolic sine 'x' and we say, let's find the 'ydx'.

Recall that what we know how to do, is how to differentiate the hyperbolic sine. Consequently, given that y equals the inverse hyperbolic sine 'x', we simply switch the emphasis and say, this is the same as saying that 'x' equals 'sinh y'. And if 'x' equal 'sinh y', we already know that the derivative of 'x' with respect to 'y' is 'cosh y'. And since the derivative of 'y' with respect to 'x' is the reciprocal of the derivative of 'x' with respect to 'y', we can conclude from this that the 'ydx' is '1 over cosh y'.

In a certain manner of speaking, we're all finished now. We have found that the derivative off inverse 'sinh x' is '1 over cosh y'.

The only problem is, is as we've said many times also before that if 'y' is given as a function of 'x', we would like the 'ydx' expressed in terms of 'x'. Or, to put it in still different words, usually when you're given an expression like the 'ydx', you're asked to evaluate this when 'x' is some particular value, not when 'y' is some particular value.

At any rate, we do much the same here as we did with the inverse circular trig functions. Namely, having arrived at '1 over cosh y' and remembering that the relationship that ties in 'x' and 'y' is at 'x' equals 'sinh y', we invoke the fundamental identity that 'cosh squared y' minus 'sinh squared y' is 1. From which we can conclude that 'cosh y'-- and here we have to be a little bit careful-- if we solve this algebraically we find that 'cosh y' is plus or minus the square root of '1 plus 'sinh squared y''.

But recalling the cosh could never be negative, it means that for this particular problem, the minus sign does not apply. In other words, since cosh has to be at least as big one, you see, that 'cosh y' is plus the square root of '1 plus 'sinh squared y''.

Now keep in mind why we went to this identity in the first place. The reason we went to this identity in the first place, is if we come back to the beginning of our problem we see that we had that 'x' is equal to 'sinh y'. In other words, down here now, all we do is replace 'sinh y' by a synonym, namely 'x'. And we arrive at the fact that 'cosh y' is the square root of '1 plus 'x squared''.

Therefore, since 'dy/dx' is the reciprocal of 'cosh y', the derivative of 'y' with respect to 'x' is '1 over the 'square root of '1 plus 'x squared'''. And this is summarized in our last step over here. In other words, the derivative of the inverse hyperbolic sine of 'x' with respect to 'x' is '1 over the 'square root of '1 plus 'x squared'''.

Now, you see, the first thing I want to point out here is to observe that without worrying about what's important about this result, is to observe again that we obtain this information virtually free of charge by knowing the calculus of the regular hyperbolic functions. You see, this result was obtained without any new knowledge.

The other thing I'd like to point out here is somewhat more subtle. And we also mentioned this in the same context, but from a different point of view, when we dealt with the inverse circular functions. Many people will say things like, who needs the inverse hyperbolic functions? How many times am I going to be confronted with having to work with inverse hyperbolic functions?

And the interesting point that's typified by this result is that if we now invert this emphasis-- in other words, if we now read this equation from right to left-- observe that if you start with the function '1 over the 'square root of '1 plus 'x squared''''-- which is hardly a hyperbolic function, this is a fairly straightforward algebraic function-- notice that the inverse derivative leads to an inverse hyperbolic sine. In other words, stated from a different perspective and using our language of the indefinite integral, notice that what we have here is that the indefinite integral 'dx' over the 'square root of '1 plus 'x squared''' is inverse hyperbolic sine 'x' plus a constant.

Now what does this mean, say, geometrically?

Suppose we take the curve 'y' equals '1 over the 'square root of '1 plus 'x squared'''. Without beating this thing to death, it should be fairly straightforward at this stage of the game that the graph of this function can be obtained, and it looks something like this. In fact, intuitively notice that 'y' will be maximum when my denominator is smallest. My denominator is smallest when 'x' is zero. So the maximum value of 'y' occurs when 'x' is 0, at which case, 'y' is 1.

Also, if I replace 'x' by minus 'x', I don't change the function. And therefore the graph-- it's an even function-- the graph is symmetric with respect to the y-axis, et cetera. At any rate, I have a picture like this.

And now suppose I want to find the area of the region 'R', where 'R' is bounded above by this curve, below by the x-axis, on the left by the y-axis, and on the right by the line 'x' equals 't'. The area of the region 'R', which is a function of 't', is given by what? The definite integral from 0 to 't', 'dx' over the 'square root of '1 plus 'x squared'''.

The point is I could, as we talked about in the previous block, try to evaluate this as the limit of a sum-- in other words, an infinite sum-- and go through all sorts of work to try to do this thing.

But the first fundamental theorem tells us in this particular case that this particular area just turns out to be the inverse hyperbolic sine of 't'. Notice that a non-trigonometric region has as its answer an inverse hyperbolic trigonometric function.

Or, if you want this thing more specifically, for example, notice that if you want the area of this region from 0 to 1, the answer to this problem would have just been the inverse hyperbolic sine of 1. In other words, 'e to the 1' minus 'e to the minus 1' over 2. Notice how 'e' sneaks into a problem which basically doesn't seem to have any relationship to 'e'.

By the way-- as a very brief aside-- for what it's worth, it's rather interesting to observe that this last equation that we've written down gives a rather interesting definition of the inverse hyperbolic sine without having to refer to a hyperbola or anything trigonometric. In other words, notice that the inverse hyperbolic sine can be defined as an integral, which is what we've really done over here.

But again, that's just an aside. The main point that I wanted us to get a hold of over here was the fact that you solve non-hyperbolic functions conveniently if we have mastered the hyperbolic functions.

Well, at any rate, here's another interesting question that comes up. And I thought that we should mention this, also. Notice that we arrived at this result by doing the thing in reverse. You'll recall that we started with 'y' equals inverse hyperbolic sine 'x' and show the derivative of that function was '1 over the 'square root of '1 plus 'x squared'''. An interesting question might have been, what if we had started with the integral being given and we hadn't have differentiated the inverse hyperbolic sine? How could we have got from here to the inverse hyperbolic sine?

And I thought I would mention this because there may be some confusion, especially if you've taken to heart certain advice that I gave you earlier when we dealt with the inverse circular functions.

Remember I told you that whenever you see something like the sum of two squares, to think of a right triangle? In other words, if you have the square root of '1 plus 'x squared'', it seems to me that the triangle that suggests itself is something like this. In other words, if I call this side 'x', I call this side 1, and this the square root of '1 plus 'x squared'', it would seem to me that I could make a circular trigonometric substitution over here. In fact, what seems to dictate itself over here, is to make the substitution, let tan theta equal 'x'.

Now if I let tan theta equal 'x', watch what happens here. I get 'secant squared theta 'd theta'' equals 'dx', taking the differential of both sides. I also get, looking at my reference triangle, that the square root of '1 plus 'x squared'' is secant theta. See, this over this is secant theta. At any rate, then, making the substitution in here, replacing 'dx' by 'secant squared theta 'd theta'', and the square root of '1 plus 'x squared'' by secant theta, I wind up with integral of 'secant theta 'd theta''.

Now you see what I've done here, is I have successfully transformed an integral in terms of 'x' into one in terms of theta. But without belaboring this point, it turns out that at this stage of the game, we do not know how to exhibit a function whose derivative with respect to theta is secant theta. In other words, we've made the substitution but we wind up with an integral that's just as tough to handle as the one that we started with. You see, in this case, trying to make a circular trigonometric substitution wouldn't have helped us very much.

What I'd like to show you is, again, an interesting connection between the circular functions and the hyperbolic functions. Namely, when we did this particular thing over here using our reference triangle, what was the reference triangle really taking the place of?

Notice that if we let 'x' equal tan theta, certainly '1 plus 'x squared'' is 1 plus tan squared theta. But there is a trigonometric identity that says 1 plus tan squared theta is secant squared theta. I think the usual way that's given is that secant squared theta minus tan squared theta is 1. This is the result that we used. We didn't really use the triangle other than to get this result more visually.

The point is, is there a hyperbolic function that has the same format? Is there a hyperbolic identity which says that the difference of two squares is one?

The answer is, well remember our basic hyperbolic identity is the cosh squared theta minus sinh squared theta is 1. Structurally, notice that sinh theta does for the hyperbolic functions what tan theta does for the circular functions.

See the common structure, here? We're going to reinforce this in the next block by doing problems like this again. But for the time being, I thought I would like to point this thing out to you.

What the approach is, is that when you try a circular function substitution and it doesn't give you the answer that you want-- meaning that you wind up with an integral that's just as tough to handle as the original one-- you look for the corresponding hyperbolic function.

What hyperbolic function plays to the hyperbolic identity the same role that this trigonometric function play to the circular identity? In this case, we replace tan theta by sinh theta. Instead of making the substitution 'x' equals tangent theta, we make the substitution 'x' equals sinh theta.

And now watch what happens as we work this thing quite mechanically. The differential of sinh theta is 'cosh theta 'd theta''. And the square root of '1 plus 'x squared'' is the square root of 1 plus sin squared theta. But notice that because of the relationship between sinh and cosh-- that's how we rig this thing, that's why we chose 'x' to be sinh theta-- notice that the square root of 1 plus sinh squared theta is just cosh theta.

And therefore, when we now substitute for the 'dx' over the square root of '1 plus 'x squared'', we get what? For 'dx' we have 'cosh theta 'd theta''. For the square root of '1 plus 'x squared'' we have cosh theta. 'Cosh theta 'd theta'' over cosh theta is just 'd theta'. And now we see the answer is, quite simply, theta plus 'c'. But what was theta? Since sinh theta was 'x', theta was inverse hyperbolic sine 'x'.

And, you see, this is a technique whereby starting with the integral 'dx' over the 'square root of '1 plus 'x squared''', we can show that we must have started with inverse sinh.

At any rate, this will be reinforced in homework problems, it will be reinforced in the next block when we talk about techniques of integration. But I just wanted to again show the similarity, the things in common, between the hyperbolic functions and the circular functions and how they're intertwined.

Let's make a few more comments while we're at it.

You know, we mentioned that the hyperbolic functions were really combinations of exponential functions. Remember, 'cosh x' was ''e to the x' plus 'e to the minus x'' over 2, et cetera. So somehow or other, if the hyperbolic functions can be expressed in terms of exponentials, it would seem that the inverse hyperbolic functions should be expressible in terms of the inverse of exponentials-- namely, in terms of logarithms.

And so I thought that I would try to go through some of these finer points with you. And, for example, ask the following question.

Given that 'y' equals inverse 'sinh x', is there a way of writing this in terms of something that uses our natural logarithms?

Another reason being, what? That if we've already learned natural logs and exponentials, it would seem that whenever we can reduce unfamiliar names to more familiar ones, psychologically we feel much more at home in dealing with the concepts. In other words, one might feel strange working with inverse hyperbolic sine because he hasn't seen that very much. But if he's used to seeing logarithms, that wouldn't seem quite as strange. At any rate, let's see how one could proceed here.

Starting out with 'y' equals inverse 'sinh x', notice that by the property, the basic definition of inverse functions, I can now write that 'x' equals 'sinh y'. Now, for obvious reasons, since I want to get the inverse of exponentials in here, it would seem to me that I should express 'sinh y' in terms of exponentials. And going again back to basic definitions, 'sinh y' is ''e to the y' minus 'e to the minus y'' over 2. In other words, in terms of exponentials, 'x' is equal to ''e to the y' minus 'e to the minus y'' over 2.

If I now cross multiply, I get that '2x' is equal to 'e to the y' minus-- now notice that 'e to the minus y' is just '1 over 'e to the y'', so I wind up now with this particular equation. And multiplying through by 'e to the y', to clear fractions in my denominator, to clear my denominators, I wind up with what? 'e to the 2y' minus ''2x 'e to the y'' - 1' equals 0.

And if I now recall that 'e to the 2y' is the square of 'e to the y', observe that what I now have is a quadratic equation in 'e to the y'. I have a quadratic equation in 'e to the y'.

Now, since I have a quadratic equation in 'e to the y', I can use the quadratic formula to solve for 'e to the y'. If I do this I get what?

Remember how this thing works. I take the coefficient of this term minus that, that's '2x' plus or minus the square root of this squared minus 4 times this times the coefficient of 'e to the y' squared. Leaving the details as being fairly obvious, 'e to the y' is '2x' plus or minus the square root of ''4x' squared' plus 4 all over 2. And noticing now that the 4 can be factored out here as a 2, and that I can cancel a 2, then from both the numerator and denominator, I wind up with 'e to the y' is 'x' plus or minus the square root of 'x squared' plus 1.

The point to keep in mind, now, is remember that in terms of exponentials, 'e to the y' can never be negative. Observe that the square root of 'x squared plus 1' is bigger than 'x' in magnitude, you see. See, 'x' would be just the positive square root of 'x squared'. So the positive square root of 'x squared plus 1' is bigger than 'x' in magnitude.

Consequently, if I use the minus sign here, I'd be taking away more than what I had. That would make my answer negative, which would be a contradiction, since 'e to the y' can't be negative. Again, in terms of this particular problem, the minus root, the minus sign here is extraneous. And we therefore wind up with what? 'e to the y' is 'x' plus the square root of 'x squared plus 1'.

Therefore 'y' itself is the 'log of x' plus the square root of 'x squared plus 1' to the base 'e', which we've already seen is called the natural log.

Going back now, say, from the first step to the last, I guess we can now fill in what's really happened here. A synonym for the inverse 'sinh x' is the natural log of 'x' plus the square root of 'x squared plus 1'.

So notice that we can study the inverse sinh, for example, in terms of a suitably chosen natural log. And of course there are many more examples that we could use along these lines. But again, I think that with the previous explanation, coupled with the fact that there will be ample exercises in the like, I think the message has become clear as far as the inverse hyperbolic sine is concerned.

What I would like to do now is to turn to another facet of inverse functions, something that involves principal values the same as it did with the circular functions. We wind up with the same problem as before when we come to the idea that, technically speaking, you cannot talk about an inverse function unless the original function is one-to-one.

And so therefore, when one talks about the inverse hyperbolic cosine, one is in a way looking for trouble if one doesn't keep his eye on exactly what's going on around here.

Namely, if we look at the graph 'y' equals hyperbolic cosine 'x', observe that whereas the function is single valued, it is not one-to-one. In fact, there is a zero derivative at this point here, which leads us to believe that maybe what we should have done was to have broken this curve down into the union of two one-to-one functions.

Let me call this curve 'y' equals 'c1 of x' and let me call this branch here 'y' equals 'c2 of x'. Notice that both 'c1 of x' and 'c2 of x' are one-to-one functions.

In fact, let's write this more formally using the picture as a guide. Let's do the following analytically. Let's say this. Define 'c sub 1 of x' to be 'cosh x', provided that 'x' is at least as big as 0.

Again, I mentioned this with the circular functions, let me reinforce this again. To define a function, it's not enough to tell the rule. You must also tell the domain.

Notice that 'c1' is not the same as cosh, because the domain of cosh is all real numbers. The domain of 'c1' is just the non-negative reals.

At any rate, I define 'c1' to be 'cosh x', where 'x' is at least as big as 0. I define 'c2 of x' to be 'cosh x', where 'x' is no bigger than 0. In other words, these two functions are different, because even though the functional relations are the same, the domains are different.

The interesting point is what? That 'cosh x' is the union of 'c1' and 'c2'. But the important point is that both 'c1' and 'c2' are one-to-one. And because they are one-to-one, their inverses exist. In other words, I can talk meaningfully about 'c1 inverse' and 'c2 inverse'.

In fact, pictorially, what I have is this. See, if I take the curve 'y' equals 'cosh x' and reflect it about the 45 degree line, this is the curve that I get. You see, it's a double value curve. All I'm saying is if we look at 'y' equals 'c1x', which is a one-to-one function, its inverse is 'c1 inverse x', which is this piece over here.

And if, on the other hand, we look at 'y' equals 'c2x', that's this branch over here, its inverse is this.

You see, notice that these two pieces are symmetric with respect to the line 'y' equals 'x', and these two pieces are symmetric with respect to the line 'y' equals 'x'. As long as we break this down into the union of two pieces, we can talk about inverse functions.

Now you see, the interesting point is that what most authors traditionally refer to as the inverse hyperbolic cosine of 'x' is really what we call 'c1 inverse of x'. In other words, the definition 'y' equals inverse hyperbolic cosine 'x' is 'x' equals cosh 'y'. And this is very important, and 'y' is at least as big as 0. Notice that the domain of 'cosh inverse x' is really 'x' has to be at least as big as one.

But that's not the important point here. What I do want to see over here is that when you put this restriction on, instead if you left this restriction out, there would be no inverse function here.

I'll come back to that in a moment. Let me just reinforce what we've talked about before, and let's find the derivative of 'inverse cosh x'. In other words, let's find 'dy/dx', if 'y' equals 'inverse cosh x'.

Well again, what is the definition, 'y' equals 'inverse cosh x'? It means 'x' equals cosh y', where 'y' is positive.

OK. If 'x' equals 'cosh y', 'dx/dy' is 'sinh y'. And we'll keep track of the fact that 'y' is positive. Actually, 'y' is non-negative. Therefore the reciprocal of 'dx/dy' will be 'dy/dx'. In other words, 'dy/dx' is '1 over sinh y'. And this would be a correct answer, except, as usual, we would like to be able to express 'dy/dx' for a given value of 'x'.

What we do now is, remembering that 'x' is 'cosh y', we invoke the identity again. 'Cosh squared y' minus 'sinh squared y' is one. From which we can solve and find that 'sinh y' is plus or minus the 'square root of 'x squared minus 1'.

And by the way, I'm not going to remove the extraneous sign here. Because in a certain manner of speaking, it is only extraneous because we are imposing the condition that 'y' is positive. See, in other words, once we assume that y' is positive-- remember that 'sinh y' is positive for positive values of 'y', and negative for negative values of 'y'-- consequently, the assumption that 'y' is positive forces us to accept the fact that 'sinh y' is positive. And that's what forces us, in terms of the restriction that we imposed the fact that y has to be at least as big as 0, why we can get rid of the minus sign here.

And so we wind up with what? 'Sinh y' is positive 'square root of 'cosh squared y' minus 1''. But 'cosh y' is 'x' in this problem. In other words, 'sinh y', in this problem, is the positive square root of 'x squared minus 1'.

By the way, don't be nervous here. You might say, couldn't this be imaginary? In other words, what happens if 'x squared' is less than 1? Remember, 'x' is at least as big as 1. So this thing here in the square root sign can never be negative.

But at any rate, what we now wind up with is that the derivative of 'inverse cosh x', with respect to 'x', is '1 over the 'square root of ''x squared' minus 1'''.

Now again, there's no law that says that a person couldn't have been on the negative branch of this curve. In other words, if all you mean by inverse cosh is the inverse of cosh with no restriction to branch, what we've really proven is this.

And let me summarize on this particular point. What we've really proven is this. That the derivative of 'c1' inverse is '1 over the 'square root of ''x squared' minus 1'''. The derivative of 'c2' inverse is '1 over minus the 'square root of ''x squared' minus 1'''.

I've taken the liberty of putting the minus down with the square root sign, rather than with the fraction itself, to emphasize the fact that which of the two signs we choose depends on whether we're looking at the branch for which 'y' is above the x-axis, or the branch which 'y' is below the x-axis.

In other words, what we don't want to happen here is for people to lose track of the fact that all we have done is made a convention so we can talk about one-to-one functions and inverse functions more meaningfully. But you can be on either of these particular branches.

In any event, this does complete our discussion of the hyperbolic functions. And we will now turn our attention to utilizing these results. We will learn some techniques of integration, and the like. But at any rate, until next time, goodbye.

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