FEMALE ANNOUNCER:

HERBERT GROSS: Hi. In our last few lectures we were trying to establish the identity of integral calculus and differential calculus in their own right, independently of one another, and then by the fundamental theorems of integral calculus to show the amazing relationship between these two subjects. Now what we would like to do today is to emphasize this topic in terms of an application unlike what we've been doing before.

In particular, what we will do today is discuss the question of finding volumes. And in doing this, several interesting things should happen, not the least of which is that we will rederive certain results that we've been taking for granted about solid regions-- volumes of regions-- for quite awhile. And also, it will give us an excellent chance to understand what we really mean by a mathematical structure.

Well, at any rate, to emphasize the structure part, I've called today's lesson, '3-dimensional Area'. See, instead of calling it volume, I call it 3-dimensional area, and the reason for this is I'd like to show you how one can study volumes in a completely analogous way to how we studied areas.

Remember, we had three basic assumptions for area. I'm now going to assume three similar assumptions for volume except where I have to amend them by necessity. And the only place this amendment has to take place is whereas the rectangle was the basic building block of areas, the so-called cylinder will be the basic building block of volumes.

Let me take a moment to digress here and explain to you mathematically what the mathematician calls a cylinder. We start with any closed curve, say, in a plane, and we then take a line perpendicular to that plane. And with that line we trace along the curve. And we then take another plane parallel to the plane that the curve is in and slice this thing off someplace.

In other words, by definition, a cylinder has congruent cross sections all the way through. And what we're saying is that-- in fact, the familiar form of a cylinder is the one where the cross section is a circle. That's called the right circular cylinder.

Remember, the volume of a right circular cylinder is 'pi r squared h', the area of the cross section times the height. Well, that's the generalization that we make. In other words, our first assumption is that for any cylinder the volume of the cylinder is the cross-sectional area times the height. Or you could call it the area of the base times the height, since the cross-sectional area is the same for all slices. OK?

The next assumption says, if we think of volume as meaning the amount of space only in three dimensions, whereas area means the amount of space in two dimensions, our next assumption is that if the three dimensional region 'R' is contained in the three dimensional region 'S', then the volume of 'R' is less than or equal to the volume of 'S'.

And finally, we assume an analogous result about the area of the whole equals the sum of the areas of the parts. We assume that if a region is made up of the union of 'n' regions which do not overlap-- notice the union notation here. The union of 'R sub 1' up to 'R sub n' and if the 'R's do not overlap, then the volume of the region 'R' is the sum of the volumes of the constituent parts.

In other words, notice that except for the fact that cylinder replaces rectangle, the basic axioms for studying volume are precisely the same as the axioms for studying area. In particular, then, what this means is that structurally the same results that we were able to show for area should follow word for word, essentially, for volumes.

And I thought what we would do is start with a rather familiar example. You may recall-- of course, we've used this result many times-- that the volume of a cylinder is given by 1/3-- that the volume of a cone is '1/3 pi r squared h' where 'r' is the radius of the base and 'h' is the height. And you may remember that in solid geometry as a traditional high school curriculum went, these results were given but seldom if ever proved.

So what I thought we would do now is see how we can use these axioms to arrive at these results, and to get the spirit of what we've been trying to do, I will do this by integral calculus at least first and then by differential calculus later. But the idea is something like this. To visualize the cone, the radius of whose base is 'r' and whose height is 'h', we can think of the straight line that joins the origin to the point (r, h), this region here, this right triangle. And we can think of that as being revolved about the x-axis to give the cone.

Now when we were dealing with areas, you may recall that we broke things down into rectangles that were too big, rectangles that were too small, and we computed, say, 'U sub n' and 'L sub n', et cetera. We can do the same thing now. What we do is we again circumscribe rectangles.

Now the idea is whatever volume is traced out by this piece here, whatever volume is traced out when we rotate this triangle about the x-axis, that volume will be less than the volume generated by this particular rectangle, because, you see, the volume that we're looking for is contained inside the rectangle when we revolve this particular thing. Now notice that this particular rectangle when revolved gives me a right circular cylinder, and we're assuming that we know how to find the volume of a cylinder. It's the cross sectional area times the height.

Let's focus our attention on what I call the k-th region here and see what this thing looks like. First of all, notice that if we've divided this length, which is 'h' units long into 'n' equal parts, each of these pieces is 'h' over 'n'.

Secondly, notice that the radius of the base of the cylinder that we're going to get-- well, let's see. It's going to be this y-coordinate. Given the x-coordinate, 'y' is determined by multiplying the x-coordinate by 'r/h'. The x-coordinate is 'kh' over 'n'. I multiply that by 'r/h'. That gives me 'kr' over 'n'. That's the height of this-- the radius of the base of the cylinder that we're going to revolve.

Now what is the volume of this cylinder? The area of the base is 'pi y squared', and I'm now going to multiply that by the height, which is 'h/n'. And if I do that, I obtain what? The volume of this particular cylinder is 'pi r squared h' times 'k squared' over 'n cubed'.

And now, if I add up all of these volumes as 'k' goes from 1 to 'n', that will give me a bunch of stacked cylinders which enclose my cone. In other words, an answer that will be too large will be what? This sum as 'k' goes from 1 to 'n', notice that this is the only portion that depends on 'k', so the upper approximation-- in other words, the volume that's too large to be the right answer-- is 'pi r squared h' over 'n cubed' times the sum as 'k' goes from 1 to 'n', 'k squared''.

Now you notice I always stick to problems where we have something fairly simple like this, because this limit process, as we've mentioned in the previous lectures, becomes very, very difficult to do in general, the beauty or one of the beauties of our fundamental theorem.

But the idea is I do know that this sum is 'n' times 'n + 1' times '2n + 1' over 6. Now distributing the 'n cubed' one factor at a time, the way we have before, I can now write that this is '1/6 pi r squared h'. 'n + 1' over 'n' is '1 + '1/n''. '2n + 1' over 'n' is '2 + '1/n''. And I find that my upper approximation is given by this expression.

And if I now take the limit of 'U sub n' as 'n' goes to infinity, this factor approaches 1. This factor approaches 2. Therefore, our entire product approaches in the limit '1/3 pi r squared h', which is the familiar result of solid geometry.

Of course, we've taken a lot for granted over here. What we've really proven here is not that the volume of a cone is '1/3 pi r squared h'. What we have proven is at the limit of 'U sub n' as 'n' approaches infinity, it's '1/3 pi r squared h'. The question that comes up is how do you know that as these divisions get small that the volume-- the upper approximation gets arbitrarily close to the correct answer.

And again, notice how we can reason analogously to what we did in the case of area. Namely, what we could do next, you see, is take the smallest cylinder that can be inscribed here. In other words, this would give us an approximation which is too small. The total error is no more than the solid generated by this hatch region revolving about the x-axis. Notice, however, that each of these pieces fits very nicely in here, so that the total error between an approximation which is too big and an approximation which is too small is this height, which is 'r'. OK?

Let's see, the cross-sectional area-- this is 'r'. So 'pi r squared' is the area of the base. The height is 'h/n'. Notice that 'r' and 'h' are given constants, therefore, as 'n' goes to infinity, the numerator stays constant. The denominator goes to infinity. The difference goes to 0. In other words, again, we can show that 'U sub n' and 'L sub n' have a common limit.

In fact, we can generalize this result rather nicely. Take this drawing to be whatever you'd like it to be. I've simply tried to visualize here a solid region. This is a 3-dimensional region. It has various cross sections. And I know that as I look at it in the 'x' direction, the region begins at 'x' equals 'a' and terminates at 'x' equals 'b'.

And the question is how can I find the volume of this particular region, assuming I know the cross-sectional area for any slice? And the idea, again, is what? We can slice this solid up into 'n' parts, which I call 'delta V1' up to 'delta Vn'. The sum of these would be the true volume that we're looking for. Now, I focus my attention on the k-th piece here, and again, what I do is I inscribe and circumscribe cylinders, one of which is contained entirely within 'delta V sub k', and the other of which surrounds 'delta V sub k'.

In other words, what I do is is I find the biggest possible cross-sectional area I have in this interval, and I denote the 'x' value at which that occurs by 'M sub k'. I find the value of 'x', which I call 'm sub k', at which I get the smallest cross-sectional area. And therefore, the inscribed cylinder has volume given by this. The circumscribed cylinder has volume given by this, and the piece that I'm looking for, the true volume, is caught between these two.

Therefore, if I add these up as 'k' goes from 1 to 'n', I've caught 'V' between 'U sub n' and 'L sub n'. Assuming only that the area is a continuous function, the difference between the largest cross section and the smallest cross section approaches 0 as 'delta x sub k' approaches 0. In essence then, the same as we did before, what we can show is that as 'n' goes to infinity, both 'U sub n' and 'L sub n' approach a common limit, and therefore, the 'V' being caught between these two is equal to the common limit.

Now here's the interesting point. When we talked about the definite integral, no one asked physically what the function 'f' was or what the 'c sub k's that we're using here were, only that they be in the proper interval and 'f' be a continuous function. Notice that we're assuming that 'A' is continuous. In essence then, by the definition of the definite integral, that volume is just what? The definite integral from 'a' to 'b', 'f of x' 'dx'. In other words, it's this sum taken in the limit as 'n' approaches infinity, or another way of saying that, as the maximum 'delta X sub k' approaches 0. That, by the way, is the integral calculus approach.

If we want the differential calculus approach, remember what we do, we say look it. The change in volume is less than or equal to the maximum cross-sectional area times 'delta X' and greater than or equal to the minimum cross-sectional area times 'delta X'. Same as we did for area, you see, we divide through by 'delta X'. We have that 'delta V' divided by 'delta X' is caught between 'A of M', 'A of m', where 'm' and 'M' are in this interval.

And now you see as 'delta X' approaches 0, 'm' and 'M' both approach 'x'. You see the same procedure as we had before. You see what we're saying going back to this diagram is, for example, here is a 'delta X', and 'm' and 'M' are points in here, and as 'delta X' goes to 0, 'm' and 'M' both approach the end point 'x'. And since 'A' is assumed to be continuous, if 'M' and 'm' approach 'x', 'A of M', 'A of m' approach 'A of x', and we arrive at, by differential calculus, that 'dV dx' is 'A of x', and therefore, 'V' is again equal to integral ''A of x' 'dx'' as 'x' goes from 'a' to 'b'. Where now by differential calculus, this means what? It simply means 'G of b' minus 'G of a', where 'G' is any function whose derivative is 'f'.

Now again, I have to go through this thing rather hurriedly because I want to get some examples done. But what I hope is that we went slowly enough so that you can again sense how we're using integral calculus, differential calculus, and the relationship between them.

Let's, at any rate, illustrate some of these results more concretely in terms of-- well, first of all, let's talk about one particular type of solid, what's called a solid of revolution. That's the particular type of solid where you have a region in the 'xy' plane, and you take that region and rotate it either around the x-axis or the y-axis, thus generating a 3-dimensional region. See, in other words, a plane area is rotated through 360 degrees either with respect to the x-axis or the y-axis.

I'll consider the x-axis here. Notice that this is a special case of what we've just studied, namely, in this particular case, if 'y' equals 'f of x' is a continuous function, notice that every cross section here, every cross section will be a circle. The area of the circle is 'pi y squared'. That's 'pi 'f of x' squared'.

And therefore, according to our fundamental theorem, since the area is continuous-- and why is the area continuous? Well, if 'f' is continuous, remember the product of continuous functions is continuous. If 'f' is continuous, 'f squared' is continuous. So according to our result, the volume of the region 'R' is just the integral from 'a' to 'b', 'pi 'f of x' squared' 'dx'.

And you see, using differential calculus, all we need now is a function whose derivative is 'pi 'f of x' squared', and we call that function 'G'. We compute 'G of b' minus 'G of a' and that gives us the volume that we're looking for.

Remember, when we talked about areas, we mentioned this was highly specialized. What if you had a region like this? And again, sparing the details, observe that if we have a region like this, we can draw in the lines where the curve doubles back. We can now visualize this as the volume-- the difference of two volumes. Namely, we can find this volume and subtract from that volume this volume. See, in other words, both of these have the right form. And by this difference, what's left? The difference of the big volume minus the small volume is the volume generated by 'R'.

And as I say, these are rather simple details that we can check out computationally in terms of exercises, but the reason I wanted to mention the solid of revolution is that not only is this a rather common and important category, but it also happens to be the type of solid that we opened our program with. Remember, the cone may be viewed as what? The solid generated by a particular right triangle being revolved about the x-axis.

In fact, I thought we could try that same problem now doing it by the antiderivative method. Namely, we take this particular region 'R' and notice now, if we revolve this about the x-axis--let's see, the cross-sectional area will be what? Well, it's a circle of radius 'y'. For a given value of 'x', 'y' is equal to 'r' times 'x' over 'h'. See the slope of this line is 'r/h'. It passes through the origin here. So the cross-sectional area is 'y squared' times pi. That's 'pi 'r squared h' squared' over 'h squared' times 'x squared'.

And to find that volume, I simply integrate this between 0 and 'h'. Recalling that pi, 'r', and 'h' are constants, I can take the constants outside of the integral sign. The integral of 'x squared', meaning what? The inverse derivative is '1/3 x cubed'. If I evaluate that between 0 and 'h', I get '1/3 h cubed'. The 'h cubed' in the numerator cancels the 'h squared' in the denominator, leaving a factor of 'h' in the numerator, and I wind up, as I saw before, that the volume of this cone is '1/3 pi r squared h'.

And this is nice that I get the same answer as by the limit method, because according to the fundamental theorem, the first fundamental theorem, this is precisely what was supposed to happen. In other words, I can do these either by limits or by derivatives. I want you to see these things side by side, because in certain cases, as I've emphasized in the previous lectures, there will be times when we cannot, by differential calculus, find a function 'G' whose derivative is equal to a given function 'f of x'. But enough about that for the time being.

The next question that comes up gives us a review of what happens with inverse functions. It's a rather interesting type of situation. It's called the method of cylindrical shells and it's motivated by the following. Let's suppose again we're given a very nice region 'R'. What do I mean by very nice? Well, to simplify the computation, even though it doesn't change the theory at all, I'm assuming that 'y' equals 'f of x' is an increasing curve. In other words, I'm even assuming that we have a one-to-one function here.

Now the idea is here's this nice region and instead of revolving this about the x-axis, I would like to revolve it about the y-axis. Now you see, to use the method of revolution here, to revolve this about the y-axis, essentially what I do is I pick a washer-shaped region, you see? I have to compute the volume generated by the 'y' part.

See, in other words, I do this as two separate parts. I find the volume of the big piece minus the volume of the small piece, and what's left is the volume generated by 'R'.

But notice a rather difficult computational thing occurs here. Namely, notice that this length here has to be expressed as 'x' goes from one value to another value. Now, you see, if this is 'b', and this is 'a', you see, notice what's happening here, how our strips are being chosen. You see, for a given strip, the final 'x' value is 'b', but what is the initial 'x' value? See, down here the 'x' value is 'a', but what happens up here? In other words, how do you find what the 'x' value is for a given value of 'y' here?

Well, you see what you must do is invert the relationship. Now even though I've picked a case where the inverse exists-- see, this is a one-to-one function-- we've already had ample examples in which we've shown that computationally it's extremely difficult if it's even possible to explicitly perform the inversion.

And this is where the method of cylindrical shells comes from. Essentially what the method of cylindrical shells says is wouldn't it have been nice if we chose our generating element to be this way? In other words, what we say is look at this piece of area over here. One way of visualizing this solid being rotated is to think of this particular region being rotated about the x-axis, and it generates a certain volume.

By the way, what volume will it generate? The volume that it will generate will be less than the volume that this rectangle generates but greater than the volume that this rectangle here generates. Now what is the volume generated by the large rectangle?

And, by the way, notice that I mean by the volume generated by the rectangle think of this as being a slab of a certain amount of material and I rotate that slab around through 360 degrees. The volume I'm thinking of is the volume of the material in that slab, not the material that's enclosed. It would be like, if you're thinking in terms of a tin can. I'm not thinking of the volume enclosed by the tin can. I'm thinking of the volume of the tin itself that makes up the can.

Well, you see again, to go through this thing as rapidly as possible but still hitting the main points, you see, notice that the volume that I'm looking for, what is the volume that's cut out by this big rectangle? Well, notice that the area of the base from-- if I look at this as being this cylinder minus this cylinder, the volume of the big cylinder is pi times ''x + delta X' squared' times the height here, which is 'f of 'x + delta X'.

And the volume of the hollow part from here to here is what? It's 'pi x squared'-- that's the radius of the base-- times the height, which is still 'f of 'x + delta X''. In other words, 'delta V' is bounded above by this volume. In other words, as messy as this looks, that's only what? That's the volume of the region generated by this big rectangle.

If we take the smallest rectangle, namely the one that's inscribed inside this region, we get the same results, except that the height is now replaced by 'f of x' rather than by 'f of 'x + delta X''. In other words, we catch 'delta V' between two expressions involving 'x'. By the way, notice how the bracketed expression simplifies the 'pi x squared' here cancels with the 'pi x squared' here leaving inside the parentheses just '2x delta X' plus 'delta X squared'. In other words, simplifying this thing, I can now show that 'delta V' is caught between these two expressions now, this expression here, which is too big, and this expression here, which is too small.

Now I divide by 'delta X'. The usual procedure to find 'dV dx', it's 'delta V' divided by 'delta X'. Then I will take the limit as 'delta X' approaches 0. You see, so I divide through by 'delta X'. We're assuming, of course, that 'delta X' is not 0. That's what the limit means as 'delta X' approaches 0. You see, it's not zero, but it gets arbitrarily close to 0.

Notice then that my 'delta V' divided by 'delta X' is caught between pi times '2x + delta X' times 'f of 'x + delta X'', and pi times '2x + delta X' times 'f of x'. And I now let 'delta X' approach 0. And here's the key point. As 'delta X' approaches 0, notice that the left hand side becomes '2 pi x' times 'f of x'.

Notice also what happens to the right hand side. This factor, as 'delta X' approaches 0, becomes '2x'. And because 'f' is continuous, as 'delta X' approaches 0, 'f of 'x + delta X'' approaches 'f of x'. In other words, then, in the limit, as 'delta X' approaches 0, I have that 'dV dx' on the one hand can't be any greater than '2 pi x' times 'f of x'. On the other hand, it can't be any less than '2 pi x' times 'f of x'. Consequently, it must equal '2 pi x' times 'f of x'.

Therefore, if this is 'dV dx', then 'V' itself is the integral of this thing evaluated between 'a' and 'b', because that's where we're adding these things up from. In other words, if we're using differential calculus, this is 'G of b' minus 'G of a' where 'G prime' equals 'f'. If we're using integral calculus, we've found the 'U sub n' and 'L sub n' and we've caught 'V' between 'U sub n' and 'L sub n'.

But, in any event, what we've shown rigorously now is that by the cylindrical shell method-- and we'll illustrate these with examples to finish off today's lesson-- that the volume is given by integral from 'a' to 'b' '2 pi x'-- and let me just replace 'f of x' by 'y' to make my diagram simpler-- times 'dx'.

And if you want to think of this in what I call the traditional engineering point of view where you think of a thin rectangle generating a volume, what we're saying is if you think of a little thin piece like this being revolved to generate, say, some material in a tin can, notice that the amount of material in here will be what? Well, when you unroll this thing-- see, this thing sort of like this. When you unroll this thing, the radius is 'x', so the circumference when you unroll it will be '2 pi x'. The height is 'y'.

So the cross-sectional area is '2 pi x y' and the thickness is 'dx'. So if I multiply that by dx, that gives me the volume generated by this piece. And then in the proud tradition of the sigma notation, which I'll come back to in the next lecture to show how dangerous this really is, but the shortcut method is what? Add up all of these contributions as 'x' goes from 'a' to 'b'.

At any rate, that's called the method of cylindrical shells. Essentially, one uses cylindrical shells when we think of a generating element being parallel to the axis of rotation. We use revolution, when it's perpendicular to the axis of rotation. Which of the two is easier depends on the particular computational technique necessitated by the relationship between the variables in the problem.

Well, at any rate, let's do a couple of examples. The first example I'd like to do is to take that same region 'R', namely, the right triangle whose legs are 'r' and 'h'. We've already solved this problem of finding the volume when we rotate this about the x-axis. What I'd now like to do is see what volume is generated by this as I rotate it about the y-axis.

And again, I find that I can do this problem in several ways, but I thought it was an easy enough problem to do by cylindrical shells, because, as we so often do, I thought the first problem that we do by cylindrical shells should be one that we can check by another method. But at any rate, using cylindrical shells, let's see what happens over here.

The volume is what? It's the integral from 0 to 'h', '2 pi x' times this height, which is 'rx' over 'h' integrated with respect to 'x'. That's just '2 pi r' over 'h'-- we can take that outside, because that's a constant factor-- integral 'x squared dx'. The integral of 'x squared' is '1/3 x cubed'. We evaluate that between 0 and 'h'. That gives us '1/3 h cubed'. We cancel the 'h' in the denominator with one of the 'h's in the numerator, and we find that the volume that's generated is '2/3 pi r h squared', not 'r squared h', 'r h squared', not 1/3, 2/3.

Remember, by the way, what this thing looks like. I think you can visualize this. This is a cylinder with a cone cut out of it. See, in other words, if this thing had been solid, we'd have called it a right circular cylinder, and then what's missing is the cone shaped region over here. In fact, that's how we can check this.

See, what would the volume be that's generated by this rectangle? This would be a cylinder the radius of whose base is 'h', whose height is 'r', and the volume of that cylinder is 'pi h squared r'. The cone that's missing, the cone that was cut out of this thing, has the radius of a space equal to 'h' and its height equal to 'r', so its volume is '1/3 pi h squared r'. And, therefore, the volume that's left when we subtract this off is '2/3 pi h squared r', which does check with this.

By the way, just as an aside, notice that the region 'R' generates a different volume if we rotate it about the y-axis than it did if we rotate it about the x-axis. Numerically, what we're saying is that we just found that the volume when rotated around the y-axis is '2/3 pi h squared r'. We know that when we revolve that about the x-axis, it's '1/3 pi r squared h', and these two expressions are not identical.

In fact, if we divide both sides by 'pi r h' over 3, we find that equality holds only if we have that highly specialized case that '2h' equals 'r', which is not really important. I just threw that in as an aside. But I do want you to notice that the same area, of course, generates different volumes depending on what you rotate it with respect to.

Well, at any rate, at least this was a problem that we could check by another method. Let me just use cylindrical shells for a problem which is slightly tougher but one that can still be checked by another method. Let's take the following region. Let's take the curve 'y' equals '2x - x squared' between 'x' equals 0 and 'x' equals 2. Leaving the details as a rather trivial exercise, it is not difficult to see that this is the parabola that peaks at 1, 1 and crosses the x-axis at 'x' equals 0 and 'x' equals 2.

If I now want to compute this volume as I rotate the region 'R' about the y-axis-- see, I'm going to rotate this about the y-axis. I want to find out what volume is generated by this region 'R' in this case. Remember I can use either cylindrical shells or I can use revolution here.

Notice the problem I'm going to be in. Notice that this particular function is single valued but not one-to-one. When I try to find these two 'x' values I'm going to run into multi-values. I'm going to have to invert. All sorts of computational skills are going to come into play here.

On the other hand, if I take my generating element parallel to the y-axis, I have a very simple expression for this, and now that indicates what? By the method of cylindrical shells, this will be the integral from 0 to 2. The generating arm is 'x', so that cuts out '2 pi x'. The height is 'y', which is '2x - x squared'. The thickness is 'dx'. In other words, mechanically I must evaluate this particular integral. OK?

At any rate, it's factoring out the 2 pi and just observing that the integral of '2 x squared' is '2/3 x cubed'. The integral of 'x' to the fourth is '1/4 'x to the 4th''. Evaluating that between 0 and 2, I get 2 pi times 16/3 minus 4. The lower limit is 0 here. This just comes out to be 16 minus 12, 4/3 times 2 pi. That's 8 pi over 3.

By the way, before I go any further with this, let's make the interesting observation. See, 8 pi over 3 is the volume generated by this whole thing being revolved about the y-axis. If I'd drawn in this line, which is a line of symmetry, notice that these two areas are congruent. These two regions are congruent. However, it's also interesting to observe that the volume generated by this piece as you revolve it about the y-axis is not twice the volume generated by this piece.

See, notice that the integral from 0 to 1-- in other words, if we just took this region here, integrated this from 0 to 1, we would get 5/6, 2/3 minus 1/4 times 2 pi, 5/6. And if we double that, we would get 5/3. In other words, this area is 5/6. Double it would be 5 pi over 6. Double it would be 5 pi over 3, and 5 pi over 3 is not the same as 8 pi over 3.

The thing to keep in mind here is notice how the distance comes in again. You see, for example, these two lines here are symmetric with respect to the line 'x' equals 1. But notice that this generates a much larger volume than this because its generating arm is longer. It's further away. But at any rate, that's just an aside. Notice how by the method of cylindrical shells, we determine the volume is 8 pi over 3.

Suppose we'd wanted to do this by the solid method-- the solid revolution method. Notice that we would first have to invert this relationship. We would first have to solve for 'x' in terms of 'y'. Notice that 'y' equals '2x - x squared' is the same as saying that ''x squared' - 2x + y' is 0. Using the quadratic formula, we can solve for 'x', and we now find that 'x' is 1 plus the square root of '1 - y' or 1 minus the square root of '1 - y'.

What that means, by the way, geometrically, is simply this. For a given value of 'y', there are two values of 'x' located symmetrically with respect to 'x' equals 1. See, they're on symmetrical portions. Well, this doesn't make that much difference. The thing now is what? What is my cross-sectional area? My cross-sectional area is pi times this length squared minus pi times this length squared. That's pi times 1 plus the square root of ''1 - y' squared' minus pi times the square root of 1 minus the square root of ''1 - y' squared'.

When I square this and subtract, all but the middle term drops out. In other words, I have twice the square root of '1 - y' here minus twice the square root of '1 - y' here. When I subtract, I get 4 times the square root of '1 - y'. I multiply that by pi. That's my cross-sectional area.

And now to find the volume, I just integrate that as 'y' goes from 0 to 1. You see, and if I carry out this integration, noticing that the integral of ''1 - y' to the 1/2' is minus 2/3. Remember, the derivative of '1 - y' with respect to 'y' is minus 1. ''1 - y' to the 3/2'. Evaluate that between 0 and 1. The upper limit gives me 0. The lower limit is minus 2/3. I subtract the lower limit. It gives me 2/3. 4 pi times 2/3 is 8 pi over 3, the same answer as I got before.

Notice, by the way, that this was messy, but we could handle it. If this had been much tougher-- say a 6 over here or something like that instead of a 2-- how would we have solved for 'x' in terms of 'y'? You see, in other words, this would have been a case where the shell method would have been necessitated because of the impossibility of the algebra.

But at any rate, we have plenty of opportunity to illustrate that in terms of exercises and supplementary notes and reading material and what have you. That is actually the easiest part.

The hard part is to understand the significance of what's going on, so I thought that to summarize today's lecture, let's keep in mind that whether you call it area or whether you call it volume or whether you call it distance traveled in velocity, the fact remains that if 'f' is a function continuous on the closed interval from 'a' to 'b', and we partition that interval into 'n' parts, and we form the sum as 'k' goes from 1 to 'n', 'f of 'c sub k'' times 'delta x sub k', where 'c sub k' is in the k-th interval, and 'delta x sub k' is just 'x sub k' minus 'x sub 'k - 1''. If we take that limit as the largest 'delta x' approaches 0 and call that 'Q', that limit 'Q' exists.

Symbolically, it's written by the definite integral from 'a' to 'b', ''f of x' dx', and, more to the point, if you happen to know differential calculus, you can compute 'Q' just by computing 'G of b' minus 'G of a', where 'G prime' is any function whose derivative is 'f'. OK?

Now again, this is why I'm calling it a summary. If you separate this out from all of the computational stuff that we did in today's lecture, this is the part that's left. OK? And what I want to do next time is to show you that things are not quite this straightforward all the time, that certain nice things have been happening here that allow us, essentially, to get away with murder. And what I mean by that will become clearer next time, but until next time then goodbye.

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