PROFESSOR: Hi. Our lecture today concerns mathematical induction, which, roughly speaking, is a technique that one uses to prove something when one already has a pretty good suspicion as to what the right answer is. Now, rather than to philosophize about this too long, let's tear right into a problem and see, in action, just what the concept means.

Recall that, in an earlier lecture, we have proven that the limit of a sum is equal to the sum of the limits provided, of course, that there are only two terms in the sum. That is, if we have two functions, 'f 1' and 'f 2', the limit of 'f1 of x' plus 'f2 of x' as 'x' approaches 'a' is the limit of 'f1 of x' as 'x' approaches 'a' plus the limit of 'f2 of x' as 'x' approaches 'a'.

The question, now, is how about the limit of 'x' approaches 'a' of 'f1 of x', plus 'f2 of x', plus 'f3 of x'? Now, to tackle a problem like this, we do what is so often done in any mathematical logical procedure. We try to reduce an unfamiliar problem to a familiar problem which has already been solved. Let's see what I mean by that. If this had been only two terms in here that we're adding, we would have known how to handle this problem. So what we observe is that since the sum of two functions is, again, a function, we can assume that our expression is written this way.

Now, you see, we've reduced our problem to the sum of two functions, namely, 'f1 of x' plus 'f2 of x' being one of our functions, 'f3 of x' being another of our functions. We know, now, what? That the limit of a sum is the sum of the limits. If we have two functions, you see this is, what? The limit 'f1 of x' plus 'f2 of x', you see, plus the limit as 'x' approaches 'a', 'f3 of x'.

But now, look what's in this expression over here. This is now, what? The limit of the sum of two functions. And, you see, we know that the limit of a sum is the sum of the limits is true if we have only two functions, so, from here, we get to here. And, from here, we can say, what? This is the limit 'x' approaches 'a', 'f1 of x', plus the limit as 'x' approaches 'a', 'f2 of x', plus the limit as 'x' approaches 'a', 'f3 of x'.

In other words, what we have shown is that the limit of a sum equals the sum of the limits is true not just the sum of two functions, but for the sum of 3 as well. And, more importantly, the truth for 3 hinged directly on the truth for 2. In other words, it wasn't just that we proved that the formula was true for the sum of three functions, we proved it on the assumption that it was already true for the sum of two functions.

And, by the way, notice how we may now begin to suspect that this idea generalizes. For example, let's take a look over here. Suppose I now said how about the limit as 'x' approaches 'a'? And we'll now take the sum of four functions: 'f1 of x', plus 'f2 of x', plus 'f3 of x', plus 'f4 of x'. See, what about something like that?

And, again, we argue the same way as we did before. We say, you know, if we had only had two functions in here-- and this gives us the hint to do this-- see, we do know that the limit of the sum is the sum of the limits if we have only two functions. You see, now, I could write this as, what? It's the limit of the first one, as 'x' approaches 'a', but what is the first one? The first one is the function 'f1 of x', plus 'f2 of x', plus 'f3 of x', plus the limit of the second. The second now you see, is 'f sub four of x' as 'x' approaches 'a'.

Now, you see our previous case told us, what? That the limit of a sum is equal to the sum of the limits if you have the sum of three functions. That's exactly what we have over here, and now you see we can say, what? Ah, this is the limit of the first as 'x' approaches 'a', plus the limit of the second as 'x' approaches 'a', plus the limit of the third as 'x' approaches 'a', plus the limit the fourth as 'x' approaches 'a'. In other words, what have we done now? We've shown that knowing that the limit of a sum is the sum of the limits was true for a sum of two functions and of three functions. We've shown, inescapably, that the same result holds for the sum of four functions.

Let's take a little breather here and make a few asides. I think, sometimes, when one starts to work too much with mathematical symbolism, we lose track of the fact that things are not quite as difficult as they might otherwise have seemed. You see, for one thing, my claim is that we have already tackled this problem as recently as first grade arithmetic. Namely, we learned, what? We learned tables.

Remember the addition tables? You learned how to add two numbers. All of a sudden somebody says what is the sum of 1, 2, 3, and 4? What is this number? And what we said was, look at, we'll just add these two at a time. 1 plus 2 is a number, which is 3. 3 plus 3 is a number, namely 6. And 6 plus 4 is a number: 10. In other words, we essentially did, what? We added the first to the second. Then we added the sum of the first two to the third, the sum of the first three to the fourth, and that was how we did this. Of course, what we assumed in doing this was that the sum of two numbers was, again, a number.

Up above, we assumed that the sum of two functions was a function. And this is not quite as trivial as it might otherwise have seemed. Namely, look at, if you add two odd numbers, do you get like things when you combine like things? The sum of two odd numbers is always even. You see, you can't say let's replace the sum of two odd numbers by another odd number. You can say it, but it would be wrong.

On the other hand, another example. How about subtraction? That's a nice operation. If you subtract a positive number from a positive number, are you guaranteed that the result will be positive? Well, for example, what about 3 less 5? The answer would be negative 2. Positive minus positive can very well be negative, so we must be sure, what? That, when we combine like objects, we get like objects. And another thing that we assumed was that our answer did not depend upon voice inflection.

Now, what does that mean? Let me show you something over here. Look at the expression 12 divided by 6 divided by 2. If you read this as if it said 12 divided by 6 divided by 2, the answer is 1. On the other hand, if you read the same thing as if it said 12 divided by 6 divided by 2, the answer is, what? 12 divided by 3, which is 4. In other words, division depends on voice inflection, whereas, addition doesn't.

And I simply point out these asides to show you that, as we go through advanced mathematical analysis, we are always making use of the same assumptions that we were making when we dealt with more simple things. And I think this is the healthiest way of seeing how our subject develops. We will leap from things that we already know into generalizations that are less familiar to us.

At any rate, let's now return to our main theme of mathematical induction and try to summarize what's happened so far. We are working with a certain conjecture, let's call it. The conjecture is that the limit of the sum is the sum of the limits. We know that the conjecture was true for two terms. We proved that, once it was true for two terms, it was true for three terms. We then prove that, if it was true for three terms, it was true for four terms. And now, if we have any imagination at all, we might become suspicious and say, you know, I think, if it's true, in general, for 'n' terms, it's going to be true for 'n + 1' terms.

And that brings us to our next stage in our mathematical induction, namely, suppose the limit of the sum is equal to the sum of the limits in the case that there are 'n' terms in our sum. What can we conclude about the limit of a sum in the case of 'n + 1' terms? And, without going through the proof here, we'll do these things in our supplementary notes in our learning exercises, but, here, I just want to focus our attention on what the main theme is.

What we say is, since we can add without worrying about voice inflection, why don't we throw in a pair of braces here, thus reducing our problem to the limit of a sum when we're adding but two functions, use the theorem there. And, now, given the limit of the sum of 'n' functions, we know that that's the limit of a sum. And now, it appears that the truth for 'n' is going to imply the truth for 'n + 1'.

Now, of course, there may be other problems that work structurally this way other than the limit of a sum equals the sum of the limits. So let's generalize that result. And the generalization is what is known as mathematical induction. What mathematical induction says is this, let's suppose we have a conjecture. Now, how we get the conjecture is something we'll talk about in a while, but let's suppose we have the conjecture.

Well, to try to show that the conjecture is true all the time, we had better be sure it's true at least sometimes. So we say, OK, let's show that the conjecture is true for 'n' equals 1. That's just a simple verification. We show that it's true for 'n' equals 1. Then we say OK, next, prove that the truth for 'n' equals 'k' implies the truth for 'n' equals 'k + 1'. In other words, we're not saying that it's true for 'k', all we're saying is that, if it's true for 'k', if it's true for 'k', the truth for 'n' equals 'k' implies the truth for 'n' equals 'k + 1'. Then, if that's true, our conjecture is true for all whole numbers. Why is that? Well, let's take a look, informally, here.

Let's suppose that both of these conditions are obeyed. We know the conjecture is true when 'n' equals 1. Now, take 'k' to be 1. Since it's true for 1, this part tells us it's going to be true for one more than 1, which is 2. Now that the conjecture is true for 2, this says, what? It's going to be true for 3. And knowing that it's true for 3, this will say it's true for 4. And now I'll loosely use the word et cetera, and come back and reinforce that as we go along.

I think now, perhaps, the best thing to do is to look at a second example. You see, what we did first of all was we used an example to lead in to what the definition would be. Now that we have the definition, let's proceed directly to use it. Let me give you a conjecture. The conjecture which I have in mind is that the sum of the first n positive numbers-- and this is an interesting formula-- it's the last number multiplied by one more than the last number divided by 2.

Well, let's just see if that's true at all for a while. Look at, if 'n' is 1, the left-hand side here is 1. And 1 times 2 divided by 2 is also 1. If 'n' is 2, the sum of the first two numbers here is, what? One plus 2 is three. On the other hand, 2 times 3 divided by 2 is also 3. So, at least, our conjecture is true for 'n' equals 1 and 'n' equals 2.

What does mathematical induction say? Let's take a look again, now. You see, we showed that the conjecture was true for 'n' equals 1. And, for good measure, we also showed that it was true for 'n' equals 2. So, we can check this off. We've done that. Now, what do we do? We assume the conjecture is true for 'n' equals 'k'. That means, what? Well, just replace 'n' by 'k'.

We're assuming that this is true. From the truth of this, what must we do next? Well, what we must do next is investigate what happens if you add, what? Not 'k' numbers, but 'k + 1'. So, in other words, what happens when you replace 'n' by 'k + 1'? Now, watch how we do this. The same thing that we did in theory before, we say, look at, we already know how to handle this amount. We're told that that's going to be ''k' times 'k + 1' over 2', so let's rewrite this in this way. Now, we can replace the bracketed expression by ''k' times 'k + 1' over 2'. We add on, of course, 'k + 1' because that's the last term that's over here. Now, we factor out 'k + 1' from this factor here. That leaves us with, what? 'k/2 + 1'. And this, in turn, says, what? That the sum of the first 'k + 1' numbers is ''k + 1' times 'k + 2' over 2'.

And notice that that's exactly what the conjecture should say when 'n' equals 'k + 1', namely, what? The sum of the first 'n' numbers, no matter how many you have, is, what? The last number times one more than the last number divided by 2. The sum of the first 'k + 1' numbers is, what? It's 'k + 1', the last one, times 'k + 2', which is one more than the last one, divided by 2. And now, what we've done is we have verified the second part of our mathematical induction setup, namely, if we go back to our basic definition over here, we have to show, what? Prove that the truth for 'n' equals 'k' implies the truth for 'n' equals 'k + 1', which is exactly what we did.

And, while you're thinking about that, let's take a break for a few more asides which, I think, may cement down this idea a little bit more strongly. I mentioned before that induction is something that one uses when one already has a suspicion as to what the right answer is. I don't know how this grabs you, buy my own particular feeling is that you do not look at the sum of the first 'n' numbers and say aha, it's the last one times one more than the last one divided by 2. You see, that's the nice thing about textbook problems. When they give you a problem on induction and they say prove this conjecture, notice that they've already given you a tremendous hint, namely, they've told you what the conjecture is.

You see, in the textbook of real life, one usually has to find out what the conjecture is for oneself. In fact, in the form of a rather interesting aside, there is a very interesting mathematical anecdote connected with this particular problem. It's an anecdote attributed to the mathematician, Gauss, who, when he was a young chap, was a discipline problem in school. And the story is that his teacher, as a punishment, asked him to add the first 100 numbers. And Gauss wrote down the answer very, very rapidly.

And what he did was, he didn't add these all up. What he observed was, what? The first one plus the last one added up to 101. The second plus the next to the last added up to 101, you see? And each pair going in this way added up to 101. And how many pairs were there, all together? Well, there were 100 numbers, so there were 50 pairs. In other words, there were, what? 100 divided by 2 pairs, each pair adding up to 101. And now, notice the recipe: 100, namely the last number, times one more than the last number, 101, divided by 2.

By the way, in the exercises on this assignment we have another problem, and that is, if you think this one was already cumbersome, try guessing what the recipe for this one is: what is the sum of the first n squares? In other words, not 1, plus 2, plus 3, et cetera, but 1 squared, plus 2 squared, plus 3 squared, et cetera. I give you the answer, but try to think for a while as to how likely it is that you would have conjectured this in the first place. It turns out to be ''n' times 'n + 1' times '2 n + 1' over 6'.

You see, the reason I bring this out is that I call this a contrived example. It is not the case where the mathematician would most likely have invented mathematical induction. The case where he would have invented mathematical induction is the case that we did earlier, for example, where we talk about the limit of a sum being the sum of the limits. You can actually see what's happening, how the truth that 'k' implies the truth for 'k + 1'.

And by the way, let me make one more aside here that I forgot to mention earlier. In our definition of mathematical induction, we said show the conjecture is true for 'n' equals 1. This was quite hypocritical because the very first example that I picked didn't even start until 'n' was 2. We talked about the limit of a sum being the sum of the limits, and the smallest sum we talked about was the sum of two terms.

The point that I wanted to mention is, for example, suppose that the first number you can prove the conjecture for is, for example, 'n' equals 7. I don't know why I picked 7. I had to pick something. Let's just call it 'n' equals 7. Suppose I can also show that, if the conjecture is true for 'k', it's true for 'k + 1'. Then, you see, what I can conclude is I can conclude that it's true for 'n' greater than or equal to 7. Namely, if it's true for 7, this says it will be true for one more than 7, which is 8. If it's true for 8, this says it will be true for 9. If it's true for 9, this says it will be true for 10, et cetera. And we go on this way.

Now, again, this may be a very naive way of looking at it, but I always look at mathematical induction as a bunch of toy soldiers stacked up in a line in such a way that, if any one of the toy soldiers falls down, he knocks down the one that's immediately behind him, OK? You see what I'm driving at here? If the first falls-- and, by the way, that's a big if-- if the first falls, he knocks down the second. The second falling knocks down the third. The third falling knocks down the fourth. The fourth falling down knocks down the fifth, et cetera. Notice that, if the first one doesn't fall, none of them fall. Or, for that matter, going back to my previous analogy, if the seventh one is the first one that falls, all the ones behind him fall down.

Now, be very careful. Mathematical induction doesn't say the first 50 fall down, or the first 100 fall down, it says they all have to fall down. For example, here's a case where several fall down, but, all of a sudden, one isn't knocked down by the one in front of him. In other words, what mathematical induction really involves is the idea not just that something is true, but that it's true because the previous one was true.

You see, for example, is it possible that all of the soldiers fall down even though the one in front didn't knock the other guy down? I mean, it's possible all of these go down, but for different reasons. Mathematical induction says much more than that. Mathematical induction says yes, they all go down, but each goes down because of the one before.

And, by the way, one more little aside. Notice that, in our analog, we assumed that there was a next one. You say, well, what do you mean you assume there was a next one? Obviously, there has to be a next one. But, the concept of next depends on whole numbers. For example, when you deal with fractions, there is no next one.

Let me show you what I mean by that. Let's talk about the real numbers in general. On the number line, here's 0. What is the first fraction, the first real number? I don't care what you call it. What is the first number which is greater than 0? What is the first number which is greater than 0? And the answer is, there is none because whichever one you pick-- call it 'r'. Let 'r' stand for the first number which you think is bigger than 0, OK? How about 'r/2'? 'r/2' is still bigger than 0, but 'r/2' is less than 'r'. In other words, given any number you pick that's bigger than 0, you can fit in another one. So, there is no number which is immediately next to 0. In other words, as a caution, notice that mathematical induction is used when we're dealing with a whole number of objects.

Now, let's emphasize some of these little asides from a more specific point of view. Let me, first of all, give you an example in which something is true for a whole bunch of numbers, but, all of a sudden, isn't true in general. Now, how anybody ever stumbled across the example I'm going to give you next, I have no idea, but I find it's a very interesting concept. Let's look at this.

Let's write down the following function: 'p' of 'n', where 'n' is any positive whole number, will be defined to be 'n' squared, minus 'n', plus 41. For example, 'p' of 1 will be 1 squared, minus 1, plus 41, which happens to be 41. 41 happens to be a prime number: a number which has no factors other than itself and 1. Let's try 2 in here. 2 squared, minus 2, plus 41, is 43: also a prime number. Let's try 3 in here. 3 squared is 9, minus 3 is 6, plus 41 is 47: also a prime number.

The amazing thing is that, as you go all the way through to 40, you get nothing but prime numbers. And you say, ah, it's right so far, it must be right all the time. And this is wishful thinking. This is like the man who wants to count the deck of cards to see if the cards are all there, and he says one, two, three, four, five. Well, so far, so good. They must all be here. It's like falling off the Empire State building, and halfway down, somebody says, "How you doing?" You say, "So far, so good." No, we're in a little bit of trouble here because, as soon as we pick 'n' to be 41, watch what happens here. This becomes 41 squared, minus 41, plus 41. And this is 41 squared, which, obviously, is not a prime. It's 41 times 41.

And here is an interesting example where a certain formula generates nothing but primes for the first 40 integers, but fails on the 41st integer. Why did this happen? Because, evidently, the fact that this was a prime in no way depended structurally on the fact that this one was a prime. There is our mathematical induction again, that not only must the conjecture be true, but it must follow inescapably from the case before.

If you'd like a more fascinating, realistic example, it's something that we call the unique factorization theorem of elementary number theory. This says that every positive whole number greater than one can be factored uniquely into a product of primes, unique up to the order in which you write them. For example, 2 is already a prime, it's 2. 3 is already a prime: 3. 4 can be factored as 2 times 2. 5 can be factored as-- well, it's already a prime, it's 5. 6 can be written as, what? 2 times 3. 7 can be written, of course, as 7. It's already a prime. 8 is 2 times 2 times 2. 9 is 3 times 3. 10 is 2 times 5, and 11 is already a prime.

Well, this doesn't prove anything. I'm just trying to demonstrate what the theorem says. The interesting thing is do you notice how 'n + 1' factors-- I don't know what phrase to use here, but let's call it considerably different than 'n'. In other words, look at what happened to 10 when I added 1 onto it. It factored into 2 times 5. I add 1 onto it, and, all of a sudden, the factorization properties change. Let me give you an example. Look at 59. 59 happens to be a prime. Look at 61. 61 also happens to be a prime.

By the way, for those of you are number theory buffs, these are called twin primes. Consecutive odd numbers, which are both prime, are called twin primes. By the way, 2 is the only even prime, of course, because, if a number is greater than 2, and it's even, it's divisible by 2. So, 2 is the only even prime. Twin primes are, what? Consecutive odd numbers both of which are primes. So, 59 and 61 are a pair of twin primes. One might intuitively suspect, therefore, that the number between them must be sort of a prime too, or whatever that means. Obviously, it can't be a prime because in between them comes 60, which is, in particular, an even number.

But look at all the nice factors that sixty has. 60 is, what? It's one more than 59, one less than 61, but look at how different it factors. In fact, a pseudo induction-type thing is look at the factors of 60: 1, 2, 3, 4, 5, 6. We say, what? 7, experimental error? No, no. But, this is not induction, by the way. The fact that 1, 2, 3, 4, 5, and 6 are all factors of 60 does not mean that 7 is going to be a factor of 60. We do not say just because it works so far, it's going to keep working.

And I'm not going to belabor this point anymore. Suffice it to say that look at how differently 60 factors compare to the number the came just before it and the number that came just after it. In other words, assuming that the unique factorization theorem is true, the truth for 'n + 1', somehow or other, has nothing to do with the truth for 'n'. And you see this is, again, another weakness of, what's called, induction.

Well, so, that's a weakness of induction. The point that I'm making is why shouldn't induction have some weaknesses? After all, if it could solve every problem, what we would do is have a calculus book that was three pages long. It would be called, *The Principle of Mathematical Induction.* When we solved that problem by induction, everything else would be done. No, there are problems that do not lend themselves to induction.

In summary, induction is a particularly effective technique which one uses to prove that something is true for all whole numbers provided that one has a suspicion that this thing is true in the first place. And secondly that, even if the suspicion is true, the truth for the 'n' plus first case follows inescapably from the truth for the n-th case. At any rate, this completes our lecture for today. And, until next time, good bye.

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