PROFESSOR: OK, today's about inverse functions, which is a new way to create one function from another one. And the reason it's so important is that we want to-- this, the logarithm, is going to be the inverse function for e to the x. We can't live without e to the x on one side and the logarithm on the other side.

So here's the idea of inverse functions. Well, here are the letters we use. Usually y is f of x. That's the standard letters. Then, for the inverse, I'm going to use f with a minus one above the line. And notice though, it will be x is f inverse of y. And let me show you why that is.

Let's just remember what a function is. So function like f is, I take an x-- that's my input-- then the function acts on that input and produces an output. At some level, this is what a function is, a bunch of inputs and the corresponding outputs.

OK, what's the inverse function? You can guess what's coming. I'll reverse those. For the inverse function, y will be the input. So y is now the input. What used to be the output is now the input-- just turning them around. Then the question is, what x did it come from? That x that y came from up here is the x that it goes to with the inverse function.

So you see the point? X and y are just getting reversed. Let me do an example, because that's letters, and we need a first example. y is x squared. So that's my function f of x. f is this squaring function. If you give me x equals three, the output is y equals nine.

Now, what's the inverse function? The inverse function, I want to find x from y. How do I do that? I take the square root. So the inverse function will be x is-- do you like to write square root of y or do you like to write y to the one half power? Both good--

That's the inverse function of the other, and, of course, before we said, if x was three y was nine. And, now, if y is nine, then x will come out to be the square root of nine, three.

Oh, one small point-- well, not so small. I was really staying there in this example with x greater or equal zero. I don't know want to allow x equal minus three. Well, why not? Because if I allowed x equal minus three as one of the inputs, if I extended the function x squared to go for all x's. So if x equal minus three was allowed input, then y would be the same answer, nine. So I would be getting nine from both three and minus three. And then, in the inverse, I wouldn't know which one to go back to. In the inverse, the input would be nine, but should the output be three or minus three. So the point is, our functions have to be-- one-to-one is a kind of nice expression that gives you the idea, one x for one y, one y for one x. And that means that they're graphs have to go steadily upwards or steadily downwards, but not down and up the way y equal x squared would if I went over all x's.

Let me do some more examples before I come to the reason for this lecture, which is the exponential and the logarithm. And let's just look ahead to what will come near the end of the lecture. We know facts about the exponential, e to the x, and those facts, when I look at the inverse function, give me some different facts, important facts still, about the logarithm.

And here is the most important one. That the log of a product of two numbers is the sum of the two logarithms, very important fact. That simple but important fact is what made logarithms famous. And it was the whole basis for the slide rule. Well, do you know what a slide rule is? Maybe you haven't ever seen one? Probably not. Everybody had them, and then suddenly nobody has any. But the point was, on a slide rule you had a little stick, another little stick-- I used to drop the thing-- And you push out log y, and then the second stick measures out log capital Y, and then you read off the answer of a multiplication. So you were able to multiply, but kind of inaccurately. So-- But that doesn't mean that this logarithm isn't still important. It is, just not for slide rules.

I promised two more examples. Also this is a chance to think about functions. So what about the radius of a circle r and the area a? There is a function there. The input is r, and the area is pi r squared. So that's some function of r, input r, output a.

Now, tell me the inverse function. The inverse function, I'm going to input a, and I want to get r. So the input for the inverse function, it's going to be like this. I have to solve that equation for r. How do I do that? I divide by pi. That gives me r squared. And then, just as there, I take the square root, and I have r. So that's the inverse function of-- is that a function of r? No way. The input is now a. This is a function of a. Divide by pi, take the square root, and you're back to r.

Let me draw the pictures that go with that. Because the graph of a function and its inverse function are really quite neat. So do you know what the graph of a equals pi r squared would look like? Again, r is only going to be positive, and, now, area's only going to be positive. And the graph of pi r squared is a parabola. Say at r equal one, I reach area equal-- What would be the area if the radius is one? Plug it in the formula, the area is pi. That's that point. And those are all the other points. OK. So that's the graph, which was nothing new.

The new graph is the graph of the inverse function. OK. What's up? This time the input is now a, and the output is r. If the area is 0, the radius is 0. If the area is pi-- Oh, look, I'm just going to take this, and it's going to go here. If the area is pi, what's the radius? Well, put it in the formula. If the area is pi, I have pi over pi, one, square root's one. The radius is one, of course it's one. One there, so that's a point on the graph of the inverse function, of this square root of a over pi.

And what's the rest of the graph. Does it look like that? No way. Everything is being flipped, you could say, Or you could say, a mirror image just turned over here. This thing, which started out like this, is now a square root function. The square root function climbs and comes around like that like it's a parabola this way, because that one was a parabola that way.

All right, let me go on to a second example. But that point that these graphs just flip over the 45 degree line, it's because y and x are getting switched. Let me do the second example. What about temperature? We could measure temperature in Fahrenheit, say f. Or we can measure it in centigrade or Celsius, say c. And what's the function? If I take f, I want to know c. The centigrade temperature is some function of f.

Let me at the same time draw the picture so we can remember. So this is now the forward function. I'm creating f first, and then I'm going to create f inverse. OK. Do you remember the point how they're connected? Here is f. And we'll start with the freezing point of water. The freezing point of water is 32 Fahrenheit but is zero centigrade. That's why that system got created. So f equals 32, c equals zero. That's on the graph.

And then what's the other key point? The boiling point of water, so say, that's one f is 212. 212 is the boiling point of water in Fahrenheit. And what's the boiling point in Celsius, centigrade? 100-- I mean that system was-- I don't know where 32 and 212 came from, but 0 to 100 is pretty sensible, 0 and then 100. So that's the other point and then, actually, the graph is just straight line.

In fact, let's find the formula. What's the equation for that line? So I take f and I subtract 32, so that gets me at the right start, the right freezing point. And now I want to multiply by the right slope to get up to the right boiling point. So when I go over 180, I want to go up 100. So it's 100 over 180. That 180 was the 32 to 212. So the ratio of 100 to 180 that's, well, five to nine would be easier to write, so let me write five to nine. Is that OK for the graph of the original function? This is my function of f giving me c.

Ready for the inverse function? Can you do this with me? c is now the input. f is now the output. What was c equals 0 and a 100, those were the key points for c. f equals 32 and 212 were they key points for f. This was on the graph, right? 0 centigrade gives 32 Fahrenheit. 100 centigrade matches 212. That's on the graph. And again, it's a line in between. Hoo! My picture isn't so fantastic. That 212 really should be higher, and that line should be steeper. Let's see that from the formula for f inverse.

What is going to be the steepness of the second line? OK. Here I've done graphs with some numbers. Here I'm going to do algebra. I mean, the point of algebra is-- you may have wondered, what was the point of algebra-- the point is to deal with all numbers at once. I could write down some other numbers like, some in between number like 122 or something, probably corresponds to a centigrade of 50. But I can't live forever with numbers. I need symbols. That's where letters, algebra, comes in.

So now, I'm going to do algebra. I want to get Fahrenheit out of centigrade. I want to solve this equation for f. How do you solve for f? Well, first thing is, get rid of that 5/9, multiply by 9/5. So now I have 9/5 of c. So that 5/9 is now over here. Now, I have an f minus 32. I want to bring the 32 over on to the c side. It'll come over as a plus. So I've solved this equation for f, and that's told me, what's the inverse function.

And you notice, it is a straight line. And what's it's slope by the way? Its slope is 9/5, where this had a slope of 5/9. That's going to happen, if you multiply. And sooner or later, in the inverse, you have to divide. So one slope is the reciprocal of the other slope. Well, it's especially easy when we see straight lines. OK.

Now, are we ready for the real thing, meaning exponentials? OK. So I come back to this board, which tells me what I'm after. And raise that a little and go for it. So, what am I saying here? I'm saying that the logarithm is going to be the inverse function of e to the x. And it's called the natural logarithm, and we use this letter n for natural.

Although, the truth is, that it's the only logarithm I ever think of. I would freely write L-O-G, because I would always mean this natural logarithm. So I'm defining it as the inverse and probably a graph is the way to see what it looks like. So I need to graph of e to the x, and then, a graph of its inverse. And then, by the way, since we're doing calculus, our next lecture is going to find derivatives. We know the derivative of e to the x. It's e to the x. That's the remarkable property that we started with. Then we'll find the derivative of the log, the inverse function. And it will come out to be remarkable too, amazing, amazing, just what we needed, in fact.

All right, but let's get an idea what that log looks like. I know you've seen logs before, but now we have this base e, e to the x that only comes in calculus. And let's graph it. So now my function of x is e to the x, and I want to graph it. This is, of course, y. Actually, I realize, x can be negative or positive, no problem. But y-- e to the x, always comes out positive. The graph is going to be above-- Here's x.

Let me draw the graph from 0 to one and say, back to minus one. Then the graph is going to be above the axis here. Let's see, where is it? When x is 0, what's y? y is e to the 0th power, which is one. e to the 0 is one.

The exponential function starts at one, right there. That height is one. Now when x is one, y is e to the first power, which is e, about 2.78. So maybe up there, somewhere about here. So that height is e, corresponding to one.

And what about when x is minus one? Then y is e to the minus one. e to the minus one is-- that minus says, divide. It's one over e to the first power, one over 2.78, something like 1/3 or so, something about there.

And now, if I put in the other points here, the graph looks like that. And, actually, the reason I didn't go beyond x equal one is that it climbs so fast. e to the x takes off. It grows exponentially, if you can allow me to say that.

Which reminds me, we don't often say, grows logarithmically. Well, let's see what grows logarithmically means. It means creeping along. If e to the x is zipping up real fast, then the log is going to go up only slowly. So I want the inverse function. Of course, this graph continues, gets very, very small. It continues up here. It gets very, very big but keeps going.

Now, ready for x equals log y. And remember, I'm going to draw its picture, and I'm defining that function as the inverse function of the one we have. So I'm not going to give a new definition, a new function. It's defined by being the inverse function. That's what it is. But now we know, from experience with two graphs, we know what its graph is going to look like.

So x is now going to be this graph, and y is going to be that one. So y only is positive. We can only take the log of positive numbers. The log of a negative number, that's something imaginary, we're not touching that. The log can come out positive, zero, negative. x could be anything. Here is x.

Let's put in the points we know. They'll be the same three points as here, but you see that x axis is now vertical, the y axis is now horizontal. I put in these points, now let me put in-- So what's the thing? When y is 0, what's x? Yeah, can you get that one? What's the log--oh, no. y doesn't make it to 0.

When y is one, that's what I meant to say. When y is one, what's the log of one? What is the log of one? That's a key point here, and we see it. We got y equal one when x is 0. The logarithm of one is-- so when y is one-- is that right? Logarithm of one, let's put in that one. The logarithm of one is 0. That's a point on our curve. That's a point on our curve. This point flips down to this point. Can I just remind myself, because you saw me hesitating, that the log of one is 0. It's nice to have a couple of numbers.

Then what are the other ones I want? I want to know the log of e, and I want to know the log of 1/e. And what are those logarithms? I could look over here. They're going to be one and minus one. But let's just begin to get the idea of the log.

The log is the exponent. That's what you should say to yourself all the time. What is the logarithm? The logarithm is the exponent in the original. So here the exponent is one, so the log is one. What's the exponent there? One over e, that's e to the minus one power. That's log of e to the minus one power. And what is that logarithm? It's the exponent minus one.

Let me plot those points. Here is e. So there is one, here is e, here is 1/e. The logarithm of one was 0. That point's on my graph. The logarithm of e is one. This point's on my graph. The logarithm of 1/e is minus one. The curve is coming up like that but bending down just the way this curve was bending up. And if I continue the curve, the logarithm would get more and more negative. It's headed down there, but y is never allowed to be 0. Headed up here, what happens? The log of a million, the log of a trillion-- I mean, we can deal with the national debt, just take its log, because it climbs so slowly. Notice it kept climbing. It doesn't peak off here.

That's a little farther than I intended to draw it. That's pretty far out on the y axis, but not very high on the x axis. Logarithms of big numbers are quite small numbers. And that's actually why, as we'll see, people use log paper. They draw a log-log graphs. That's to get big numbers on to the graph by dealing with logs.

So that's what I want to say about the logarithm. Except, to come back to these two key facts, especially the first one. Can I find space for that first one? So y is e to the x, as always. Capital Y would be e to the capital X. And now, the interesting property is what happens if I multiply.

What happens if I multiply y times Y? I have, that's e to the x times e to the X. That's what the little y and big y were. Now we're ready to use the crucial property of the exponential curve. I'm asking you, because you have to know this. What is e to the x times e to the capital X?

Suppose x was two and capital X was three? Then I have e times e, e squared, multiplying e times e times e, three e's. What do I have? I've got e times e, time e times e time e. All together five e's are getting multiplied. I just add the exponents. That's the big rule for the exponential. If I multiply exponentials, I add the exponents.

Now, I just want to convert that to a rule for logarithms. I'm going to do the inverse function. I'm going to take the log of both sides, and, I hope, we're going to get the right thing. The logarithm of this is-- Well, what's the logarithm of this result? It's the exponent. The logarithm of this number is that. Just the way the logarithm of e to that number was the one. The logarithm of e to that number was the minus one. The logarithm of this number is the exponent x plus capital X. And finally what is little x? Well, don't forget where it came from. Little x is the exponent for y, so little x is log y. And capital X is log capital Y.

Bunch of symbols on that board. And the last line is the one that we were shooting for. The logarithm of y times Y is the sum of the logs. Because this guy is also important-- Maybe I don't even give a proof. Because it's intimately related to this one, why don't I just see it. What would be the log of y squared?

Actually, we already-- here. If I wanted little y squared, what should I do? I can get that answer from what I've done. The log of little y squared, I just take big Y to be the same as little y, I take big X to be the same as little x, and I've got the log of y squared. Then is x plus x, two x's-- but x is the log of y. If you square a number, you only double its log. You're again seeing why these numbers can grow very quickly by squaring and squaring and squaring, but the logarithms only grow by multiplying by two, only going up slowly. And then the general result would be for any power, not just n equals two, not just n equals a whole number, not just n equals positive numbers, but all n, will be-- I'll have n of these-- so I'll have n logarithm of y. So that's a closely related property that takes the same y to different powers. OK.

Lots of symbols today, but you had to get that logarithm function straight before we can take its derivative. Can I tell you what its derivative is? Would you like to know in advance? The derivative-- I don't know if I should tell you. The derivative of log y, the derivative of this log function, turns out to be 1/y. Isn't that nice. A really good answer coming from this function that we created as an inverse function.

And I'll just say here that now we've created the function. We've got it. Then I don't mind if you give it a different letter, give it another name. Well, I hope you keep its name log. Most people use that name. But you could use a different letter. I'm perfectly happy for you to write this as the derivative of log x is 1/x. Between that and that, I've just changed letters. That was like after the real thinking of this lecture, which was the when x was an input and y was an output, and I really needed two different letters. OK, good, that's inverse functions. Thank you.

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