44 Lifting the triangle of Problem 42 up to the plane $z = 1$ gives corners $(a_1, b_1, 1)$, $(a_2, b_2, 1)$, $(a_3, b_3, 1)$. The area of the triangle times $\frac{1}{3}$ is the volume of the upside-down pyramid from $(0, 0, 0)$ to these corners. This pyramid volume is $\frac{1}{6}$ the box volume, so $\frac{1}{3}$ (area of triangle) $= \frac{1}{6}$ (volume of box):

$$\textit{area of triangle} = \frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}.$$

Find the area A in Problem 43 from this determinant.

45 (1) The projections of $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ onto the xy plane are ________.
(2) The parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$ projects to a parallelogram with area ________.
(3) General fact: The projection onto the plane normal to the unit vector $\mathbf{n}$ has area $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n}$. Verify for $\mathbf{n} = \mathbf{k}$.

46 (a) For $\mathbf{A} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $\mathbf{B} = -\mathbf{i} + \mathbf{j}$, compute $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{i}$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{j}$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{k}$. By Problem 45 those are the areas of projections onto the yz and xz and xy planes.
(b) Square and add those areas to find $|\mathbf{A} \times \mathbf{B}|^2$. This is the Pythagoras formula in space (Remark 2).

47 (a) The triple cross product $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ is in the plane of $\mathbf{A}$ and $\mathbf{B}$, because it is perpendicular to the cross product ________.
(b) Compute $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ when $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{C} = \mathbf{i}$.
(c) Compute $(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$ when $\mathbf{C} = \mathbf{i}$. The answers in (b) and (c) should agree. This is also true if $\mathbf{C} = \mathbf{j}$ or $\mathbf{C} = \mathbf{k}$ or $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. That proves the tricky formula

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}. \qquad (*)$$

48 Take the dot product of equation (*) with $\mathbf{D}$ to prove

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}).$$

49 The plane containing $P = (0, 1, 1)$ and $Q = (1, 0, 1)$ and $R = (1, 1, 0)$ is perpendicular to the cross product $\mathbf{N} =$ ________. Find the equation of the plane and the area of triangle PQR.

50 Let $P = (1, 0, -1)$, $Q = (1, 1, 1)$, $R = (2, 2, 1)$. Choose S so that $PQRS$ is a parallelogram and compute its area. Choose T, U, V so that $OPQRSTUV$ is a box (parallelepiped) and compute its volume.

11.4 Matrices and Linear Equations

We are moving from geometry to algebra. Eventually we get back to calculus, where functions are nonlinear—but linear equations come first. In Chapter 1, $y = mx + b$ produced a line. Two equations produce two lines. If they cross, the intersection point solves both equations—and we want to find it.

Three equations in three variables x, y, z produce three planes. Again they go through one point (*usually*). Again the problem is to find that intersection point —which solves the three equations.

The ultimate problem is to solve n equations in n unknowns. There are n hyperplanes in n-dimensional space, which meet at the solution. We need a test to be sure they meet. We also want the solution. These are the objectives of ***linear algebra***, which joins with calculus at the center of pure and applied mathematics.†

Like every subject, linear algebra requires a good notation. To state the equations and solve them, we introduce a "*matrix*." ***The problem will be*** $A\mathbf{u} = \mathbf{d}$. ***The solution will be*** $\mathbf{u} = A^{-1}\mathbf{d}$. It remains to understand where the equations come from, where the answer comes from, and what the matrices A and A^{-1} stand for.

TWO EQUATIONS IN TWO UNKNOWNS

Linear algebra has no reason to choose one variable as special. The equation $y - y_0 = m(x - x_0)$ separates y from x. A better equation for a line is $ax + by = d$. (A vertical

†Linear algebra dominates some applications while calculus governs others. Both are essential. A fuller treatment is presented in the author's book *Linear Algebra and Its Applications* (Harcourt Brace Jovanovich, 3rd edition 1988), and in many other texts.

line like $x = 5$ appears when $b = 0$. The first form did not allow slope $m = \infty$.) This section studies two lines:

$$\begin{aligned} a_1 x + b_1 y &= d_1 \\ a_2 x + b_2 y &= d_2. \end{aligned} \tag{1}$$

By solving both equations at once, we are asking (x, y) to lie on both lines. The practical question is: Where do the lines cross? The mathematician's question is: Does a solution exist and is it unique?

To understand everything is not possible. There are parts of life where you never know what is going on (until too late). But two equations in two unknowns can have no mysteries. There are three ways to write the system—by ***rows***, by ***columns***, and by ***matrices***. Please look at all three, since setting up a problem is generally harder and more important than solving it. After that comes the concession to the real world: we compute x and y.

EXAMPLE 1 How do you invest \$5000 to earn \$400 a year interest, if a money market account pays 5% and a deposit account pays 10%?

Set up equations by rows: With x dollars at 5% the interest is $.05x$. With y dollars at 10% the interest is $.10y$. One row for principal, another row for interest:

$$\begin{aligned} x + \quad y &= 5000 \\ .05x + .10y &= \ \ 400. \end{aligned} \tag{2}$$

Same equations by columns: The left side of (2) contains x times one vector plus y times another vector. The right side is a third vector. The equation by columns is

$$x\begin{bmatrix} 1 \\ .05 \end{bmatrix} + y\begin{bmatrix} 1 \\ .10 \end{bmatrix} = \begin{bmatrix} 5000 \\ 400 \end{bmatrix}. \tag{3}$$

Same equations by matrices: Look again at the left side. There are two unknowns x and y, which go into a vector $\mathbf{u}$. They are multiplied by the four numbers 1, .05, 1, and .10, which go into a *two by two matrix* A. The left side becomes ***a matrix times a vector***:

$$A\mathbf{u} = \begin{bmatrix} 1 & 1 \\ .05 & .10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5000 \\ 400 \end{bmatrix}. \tag{4}$$

Now you see where the "rows" and "columns" came from. They are the rows and columns of a matrix. The rows entered the separate equations (2). The columns entered the vector equation (3). The matrix-vector multiplication $A\mathbf{u}$ is defined so that all these equations are the same:

$$A\mathbf{u} \text{ by rows: } \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 x + b_1 y \\ a_2 x + b_2 y \end{bmatrix} \quad \text{(each row is a dot product)}$$

$$A\mathbf{u} \text{ by columns: } \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + y\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{(combination of column vectors)}$$

A is the ***coefficient matrix***. The unknown vector is $\mathbf{u}$. The known vector on the right side, with components 5000 and 400, is $\mathbf{d}$. The matrix equation is $A\mathbf{u} = \mathbf{d}$.

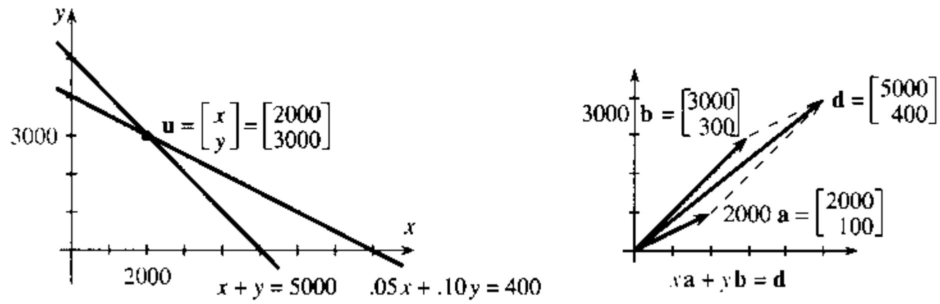


Fig. 11.16 Each row of $A\mathbf{u} = \mathbf{d}$ gives a line. Each column gives a vector.

This notation $A\mathbf{u} = \mathbf{d}$ continues to apply when there are more equations and more unknowns. The matrix A has a ***row for each equation*** (usually m rows). It has a ***column for each unknown*** (usually n columns). For 2 equations in 3 unknowns it is a 2 by 3 matrix (therefore rectangular). For 6 equations in 6 unknowns the matrix is 6 by 6 (therefore square). The best way to get familiar with matrices is to work with them. Note also the pronunciation: "matri*sees*" and never "matrixes."

Answer to the practical question The solution is $x = 2000$, $y = 3000$. That is the intersection point in the row picture (Figure 11.16). It is also the correct combination in the column picture. The matrix equation checks both at once, because matrices are multiplied by rows *or* by columns. The product either way is $\mathbf{d}$:

$$\begin{bmatrix} 1 & 1 \\ .05 & .10 \end{bmatrix} \begin{bmatrix} 2000 \\ 3000 \end{bmatrix} = \begin{bmatrix} 2000 + 3000 \\ (.05)2000 + (.10)3000 \end{bmatrix} = \begin{bmatrix} 5000 \\ 400 \end{bmatrix} = \mathbf{d}.$$

Singular case In the row picture, the lines cross at the solution. But there is a case that gives trouble. ***When the lines are parallel***, they never cross and there is *no* solution. When the lines are the same, there is an *infinity* of solutions:

$$\textit{parallel lines} \quad \begin{matrix} 2x + y = 0 \\ 2x + y = 1 \end{matrix} \qquad \textit{same line} \quad \begin{matrix} 2x + y = 0 \\ 4x + 2y = 0 \end{matrix} \tag{5}$$

This trouble also appears in the column picture. The columns are vectors $\mathbf{a}$ and $\mathbf{b}$. The equation $A\mathbf{u} = \mathbf{d}$ is the same as $x\mathbf{a} + y\mathbf{b} = \mathbf{d}$. We are asked to find the combination of $\mathbf{a}$ and $\mathbf{b}$ (with coefficients x and y) that produces $\mathbf{d}$. In the singular case $\mathbf{a}$ and $\mathbf{b}$ lie along the same line (Figure 11.17). No combination can produce $\mathbf{d}$, unless it happens to lie on this line.

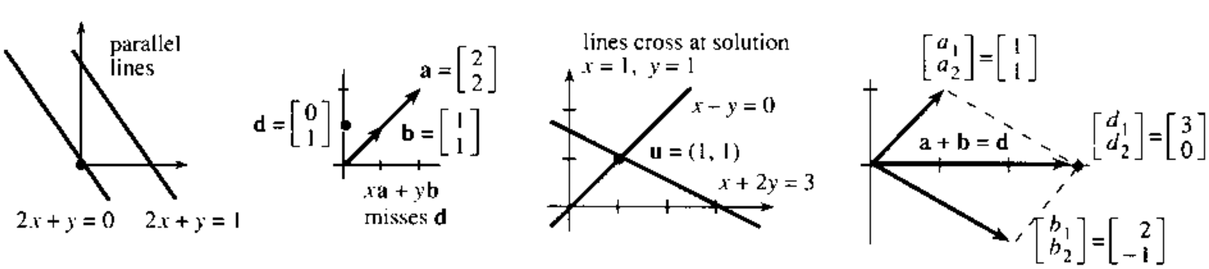


Fig. 11.17 Row and column pictures: *singular* (no solution) and *nonsingular* ($x = y = 1$).

The investment problem is *non*singular, and 2000 **a** + 3000 **b** equals **d**. We also drew **Example 2**: The matrix A multiplies $\mathbf{u} = (1, 1)$ to solve $x + 2y = 3$ and $x - y = 0$:

$$A\mathbf{u} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \qquad \text{By columns } \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

The crossing point is (1, 1) in the row picture. The solution is $x = 1$, $y = 1$ in the column picture (Figure 11.17b). Then 1 times **a** plus 1 times **b** equals the right side **d**.

SOLUTION BY DETERMINANTS

Up to now we just wrote down the answer. The real problem is to find x and y when they are unknown. We solve two equations with letters not numbers:

$$a_1 x + b_1 y = d_1$$
$$a_2 x + b_2 y = d_2.$$

The key is to eliminate x. Multiply the first equation by a_2 and the second equation by a_1. Subtract the first from the second and the x's disappear:

$$(a_1 b_2 - a_2 b_1) y = (a_1 d_2 - a_2 d_1). \tag{6}$$

To eliminate y, subtract b_1 times the second equation from b_2 times the first:

$$(b_2 a_1 - b_1 a_2) x = (b_2 d_1 - b_1 d_2). \tag{7}$$

What you see in those parentheses are 2 by 2 determinants! Remember from Section 11.3:

$$\textit{The determinant of } \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \textit{ is the number } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

This number appears on the left side of (6) and (7). The right side of (7) is also a determinant—but it has d's in place of a's. The right side of (6) has d's in place of b's. So x and y are ***ratios of determinants***, given by Cramer's Rule:

11H *Cramer's Rule* The solution is $x = \dfrac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \dfrac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$

The investment example is solved by three determinants from the three columns:

$$\begin{vmatrix} 1 & 1 \\ .05 & .10 \end{vmatrix} = .05 \qquad \begin{vmatrix} 5000 & 1 \\ 400 & .10 \end{vmatrix} = 100 \qquad \begin{vmatrix} 1 & 5000 \\ .05 & 400 \end{vmatrix} = 150.$$

Cramer's Rule has $x = 100/.05 = 2000$ and $y = 150/.05 = 3000$. This is the solution. The singular case is when *the determinant of A is zero*—and we can't divide by it.

11I Cramer's Rule breaks down when det $A = 0$—which is the singular case. Then the lines in the row picture are parallel, and one column is a multiple of the other column.

EXAMPLE 3 The lines $2x + y = 0$, $2x + y = 1$ are parallel. The determinant is zero:

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ has } \det A = \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0.$$

The lines in Figure 11.17a don't meet. Notice the columns: $\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]$ is a multiple of $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.

One final comment on 2 by 2 systems. They are small enough so that all solution methods apply. Cramer's Rule uses ***determinants.*** Larger systems use ***elimination*** (3 by 3 matrices are on the borderline). A third solution (the same solution!) comes from the ***inverse matrix*** A^{-1}, to be described next. But the inverse is more a symbol for the answer than a new way of computing it, because to find A^{-1} we still use determinants or elimination.

THE INVERSE OF A MATRIX

The symbol A^{-1} is pronounced "*A inverse.*" It stands for a matrix—the one that solves $A\mathbf{u} = \mathbf{d}$. I think of A as a matrix that takes $\mathbf{u}$ to $\mathbf{d}$. Then A^{-1} is a matrix that takes $\mathbf{d}$ back to $\mathbf{u}$. If $A\mathbf{u} = \mathbf{d}$ then $\mathbf{u} = A^{-1}\mathbf{d}$ (provided the inverse exists). This is exactly like functions and inverse functions: $g(x) = y$ and $x = g^{-1}(y)$. Our goal is to find A^{-1} when we know A.

The first approach will be very direct. Cramer's Rule gave formulas for x and y, the components of $\mathbf{u}$. From that rule we can read off A^{-1}, *assuming that* $D = a_1 b_2 - a_2 b_1$ *is not zero.* D is det A and we divide by it:

$$\textbf{\textit{Cramer}}\text{: } \mathbf{u} = \frac{1}{D}\begin{bmatrix} b_2 d_1 - b_1 d_2 \\ -a_2 d_1 + a_1 d_2 \end{bmatrix} \quad \textbf{This is } A^{-1}\mathbf{d} = \frac{1}{D}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{8}$$

The matrix on the right (including $1/D$ in all four entries) is A^{-1}. Notice the sign pattern and the subscript pattern. The inverse exists if D is not zero—this is important. Then the solution comes from a matrix-vector multiplication, A^{-1} times $\mathbf{d}$. We repeat the rules for that multiplication:

DEFINITION A matrix M times a vector $\mathbf{v}$ equals a vector of dot products:

$$M\mathbf{v} = \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \end{bmatrix}\begin{bmatrix} \mathbf{v} \end{bmatrix} = \begin{bmatrix} (\text{row } 1)\cdot\mathbf{v} \\ (\text{row } 2)\cdot\mathbf{v} \end{bmatrix}. \tag{9}$$

Equation (8) follows this rule with $M = A^{-1}$ and $\mathbf{v} = \mathbf{d}$. Look at Example 1:

$$A = \begin{bmatrix} 1 & 1 \\ .05 & .10 \end{bmatrix}, \qquad \det A = .05, \qquad A^{-1} = \frac{1}{.05}\begin{bmatrix} .10 & -1 \\ -.05 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix}.$$

There stands the inverse matrix. It multiplies $\mathbf{d}$ to give the solution $\mathbf{u}$:

$$A^{-1}\mathbf{d} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix}\begin{bmatrix} 5000 \\ 400 \end{bmatrix} = \begin{bmatrix} (2)(5000) - (20)(400) \\ (-1)(5000) + (20)(400) \end{bmatrix} = \begin{bmatrix} 2000 \\ 3000 \end{bmatrix}.$$

The formulas work perfectly, but you have to see a direct way to reach $A^{-1}\mathbf{d}$. ***Multiply both sides of*** $A\mathbf{u} = \mathbf{d}$ ***by*** A^{-1}. The multiplication "cancels" A on the left side, and leaves $\mathbf{u} = A^{-1}\mathbf{d}$. This approach comes next.

MATRIX MULTIPLICATION

To understand the power of matrices, we must multiply them. The product of A^{-1} with $A\mathbf{u}$ is a matrix times a vector. But that multiplication can be done another way. First A^{-1} multiplies A, a matrix times a matrix. The product $A^{-1}A$ is another matrix (a very special matrix). Then this new matrix multiplies $\mathbf{u}$.

The matrix-matrix rule comes directly from the matrix-vector rule. Effectively, a vector $\mathbf{v}$ is a matrix V with only one column. When there are more columns, M times V splits into separate matrix-vector multiplications, side by side:

DEFINITION A matrix M times a matrix V equals a matrix of dot products:

$$MV = \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} (\text{row } 1)\cdot\mathbf{v}_1 & (\text{row } 1)\cdot\mathbf{v}_2 \\ (\text{row } 2)\cdot\mathbf{v}_1 & (\text{row } 2)\cdot\mathbf{v}_2 \end{bmatrix}. \tag{10}$$

EXAMPLE 4 $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1\cdot 5+2\cdot 7 & 1\cdot 6+2\cdot 8 \\ 3\cdot 5+4\cdot 7 & 3\cdot 6+4\cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$$

EXAMPLE 5 Multiplying A^{-1} times A produces the "***identity matrix***" $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$$A^{-1}A = \frac{\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}}{D}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \frac{\begin{bmatrix} a_1b_2 - a_2b_1 & 0 \\ 0 & -a_2b_1 + a_1b_2 \end{bmatrix}}{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{11}$$

This identity matrix is denoted by I. It has 1's on the diagonal and 0's off the diagonal. It acts like the number 1. ***Every vector satisfies*** $I\mathbf{u} = \mathbf{u}$.

> **11J** (***Inverse matrix and identity matrix***) $AA^{-1} = I$ and $A^{-1}A = I$ and $I\mathbf{u} = \mathbf{u}$:
>
> $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{12}$$
>
> Note the placement of a, b, c, d. With these letters D is $ad - bc$.

The next section moves to three equations. The algebra gets more complicated (and 4 by 4 is worse). It is not easy to write out A^{-1}. So we stay longer with the 2 by 2 formulas, where each step can be checked. Multiplying $A\mathbf{u} = \mathbf{d}$ by the inverse matrix gives $A^{-1}A\mathbf{u} = A^{-1}\mathbf{d}$—and the left side is $I\mathbf{u} = \mathbf{u}$.

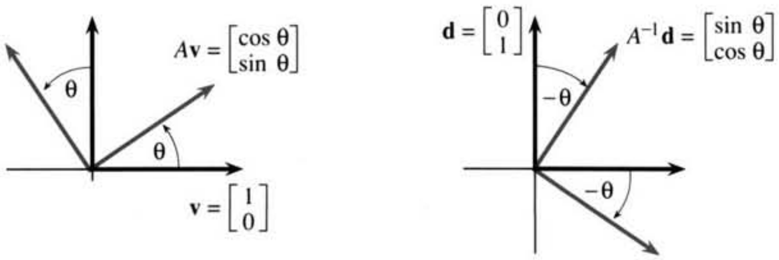


Fig. 11.18 Rotate $\mathbf{v}$ forward into $A\mathbf{v}$. Rotate $\mathbf{d}$ backward into $A^{-1}\mathbf{d}$.

EXAMPLE 6 $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates every $\mathbf{v}$ to $A\mathbf{v}$, through the angle θ.

***Question* 1** Where is the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotated to?

***Question* 2** What is A^{-1}?

***Question* 3** Which vector $\mathbf{u}$ is rotated into $\mathbf{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

***Solution* 1** $\mathbf{v}$ rotates into $A\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

***Solution* 2** $\det A = 1$ so $A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ = rotation through $-\theta$.

***Solution* 3** If $A\mathbf{u} = \mathbf{d}$ then $\mathbf{u} = A^{-1}\mathbf{d} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$.

Historical note I was amazed to learn that it was Leibniz (again!) who proposed the notation we use for matrices. ***The entry in row** i **and column** j **is** a_{ij}.* The identity matrix has $a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = 0$. This is in a linear algebra book by Charles Dodgson—better known to the world as Lewis Carroll, the author of *Alice in Wonderland*. I regret to say that he preferred his own notation $i\rceil j$ instead of a_{ij}. "I have turned the symbol toward the left, to avoid all chance of confusion with $\rceil$." It drove his typesetter mad.

PROJECTION ONTO A PLANE = LEAST SQUARES FITTING BY A LINE

We close with a genuine application. It starts with three-dimensional vectors $\mathbf{a}, \mathbf{b}, \mathbf{d}$ and leads to a 2 by 2 system. One good feature: $\mathbf{a}, \mathbf{b}, \mathbf{d}$ can be n-dimensional with no change in the algebra. In practice that happens. Second good feature: There is a calculus problem in the background. The example is ***to fit points by a straight line***.

There are three ways to state the problem, and they look different:

1. Solve $x\mathbf{a} + y\mathbf{b} = \mathbf{d}$ as well as possible (three equations, two unknowns x and y).
2. Project the vector $\mathbf{d}$ onto the plane of the vectors $\mathbf{a}$ and $\mathbf{b}$.
3. Find the closest straight line ("***least squares***") to three given points.

Figure 11.19 shows a three-dimensional vector $\mathbf{d}$ above the plane of $\mathbf{a}$ and $\mathbf{b}$. Its projection onto the plane is $\mathbf{p} = x\mathbf{a} + y\mathbf{b}$. The numbers x and y are unknown, and our goal is to find them. The calculation will use the dot product, which is always the key to right angles.

The difference $\mathbf{d} - \mathbf{p}$ is the "*error*." There has to be an error, because no combination of $\mathbf{a}$ and $\mathbf{b}$ can produce $\mathbf{d}$ exactly. (Otherwise $\mathbf{d}$ is in the plane.) The projection $\mathbf{p}$ is the closest point to $\mathbf{d}$, and it is governed by one fundamental law: ***The error is perpendicular to the plane***. That makes the error perpendicular to both vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a}\cdot(x\mathbf{a} + y\mathbf{b} - \mathbf{d}) = 0 \qquad \text{and} \qquad \mathbf{b}\cdot(x\mathbf{a} + y\mathbf{b} - \mathbf{d}) = 0. \tag{13}$$

Rewrite those as two equations for the two unknown numbers x and y:

$$\begin{aligned}(\mathbf{a}\cdot\mathbf{a})x + (\mathbf{a}\cdot\mathbf{b})y &= \mathbf{a}\cdot\mathbf{d}\\ (\mathbf{b}\cdot\mathbf{a})x + (\mathbf{b}\cdot\mathbf{b})y &= \mathbf{b}\cdot\mathbf{d}.\end{aligned} \tag{14}$$

These are the famous ***normal equations*** in statistics, to compute x and y and $\mathbf{p}$.

EXAMPLE 7 For $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (1, 2, 3)$ and $\mathbf{d} = (0, 5, 4)$, solve equation (14):

$$\begin{matrix} 3x + 6y = 9 \\ 6x + 14y = 22\end{matrix} \quad \text{gives} \quad \begin{matrix} x = -1 \\ y = 2\end{matrix} \quad \text{so} \quad \mathbf{p} = -\mathbf{a} + 2\mathbf{b} = (1, 3, 5) = \textit{projection}.$$

Notice the three equations that we are not solving (we can't): $x\mathbf{a} + y\mathbf{b} = \mathbf{d}$ is

$$\begin{matrix} x + y = 0 \\ x + 2y = 5 \\ x + 3y = 4 \end{matrix} \quad \text{with the 3 by 2 matrix} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3\end{bmatrix}. \tag{15}$$

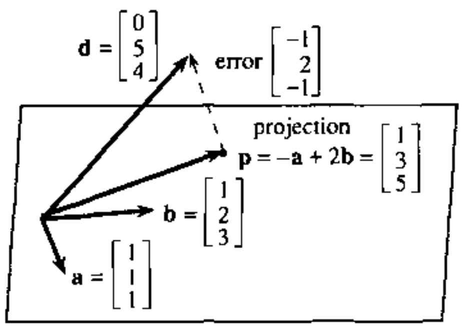
For $\mathbf{d} = (0, 5, 4)$ there is no solution; $\mathbf{d}$ is not in the plane of $\mathbf{a}$ and $\mathbf{b}$. For $\mathbf{p} = (1, 3, 5)$ there is a solution, $x = -1$ and $y = 2$. The vector $\mathbf{p}$ is in the plane. The error $\mathbf{d} - \mathbf{p}$ is $(-1, 2, -1)$. This error is perpendicular to the columns $(1, 1, 1)$ and $(1, 2, 3)$, so it is perpendicular to their plane.

SAME EXAMPLE *(written as a line-fitting problem)* Fit the points $(1, 0)$ and $(2, 5)$ and $(3, 4)$ as closely as possible ("least squares") by a straight line.

Two points determine a line. The example asks the line $f = x + yt$ to go through *three* points. That gives the three equations in (15), which can't be solved with two unknowns. We have to settle for the closest line, drawn in Figure 11.19b. This line is computed again below, by calculus.

Notice that the closest line has heights 1, 3, 5 where the data points have heights 0, 5, 4. Those are the numbers in $\mathbf{p}$ and $\mathbf{d}$! The heights 1, 3, 5 fit onto a line; the heights 0, 5, 4 do not. In the first figure, $\mathbf{p} = (1, 3, 5)$ is in the plane and $\mathbf{d} = (0, 5, 4)$ is not. Vectors in the plane lead to heights that lie on a line.

Notice another coincidence. The coefficients $x = -1$ and $y = 2$ give the projection $-\mathbf{a} + 2\mathbf{b}$. They also give the closest line $f = -1 + 2t$. All numbers appear in both figures.



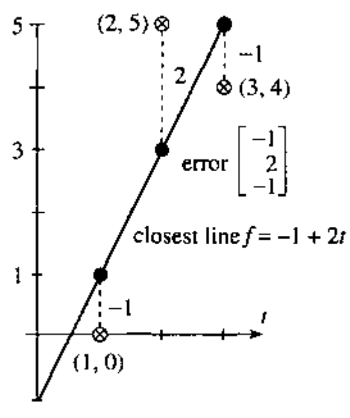


Fig. 11.19 Projection onto plane is (1, 3, 5) with coefficients −1, 2. Closest line has heights 1, 3, 5 with coefficients −1, 2. Error in both pictures is −1, 2, −1.

Remark Finding the closest line is a *calculus problem*: ***Minimize a sum of squares***. The numbers x and y that minimize E give the least squares solution:

$$E(x, y) = (x + y - 0)^2 + (x + 2y - 5)^2 + (x + 3y - 4)^2. \tag{16}$$

Those are the three errors in equation (15), squared and added. They are also the three errors in the straight line fit, between the line and the data points. The projection minimizes the error (by geometry), the normal equations (14) minimize the error (by algebra), and now calculus minimizes the error by setting the derivatives of E to zero.

The new feature is this: E depends on two variables x and y. *Therefore E has two derivatives.* They both have to be zero at the minimum. That gives two equations for x and y:

$$\begin{aligned} &x \text{ derivative of } E \text{ is zero: } 2(x + y) + 2(x + 2y - 5) \quad + 2(x + 3y - 4) \quad = 0 \\ &y \text{ derivative of } E \text{ is zero: } 2(x + y) + 2(x + 2y - 5)(2) + 2(x + 3y - 4)(3) = 0. \end{aligned}$$

When we divide by 2, those are the normal equations $3x + 6y = 9$ and $6x + 14y = 22$. The minimizing x and y from calculus are the same numbers -1 and 2.

The x derivative treats y as a constant. The y derivative treats x as a constant. These are ***partial derivatives***. This calculus approach to least squares is in Chapter 13, as an important application of partial derivatives.

We now summarize the *least squares problem*—to find the closest line to n data points. In practice n may be 1000 instead of 3. The points have horizontal coordinates $b_1, b_2, \ldots, b_n$. The vertical coordinates are $d_1, d_2, \ldots, d_n$. These vectors $\mathbf{b}$ and $\mathbf{d}$, together with $\mathbf{a} = (1, 1, \ldots, 1)$, determine a projection—the combination $\mathbf{p} = x\mathbf{a} + y\mathbf{b}$ that is closest to $\mathbf{d}$. This problem is the same in n dimensions—the error $\mathbf{d} - \mathbf{p}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$. That is still tested by dot products, $\mathbf{p} \cdot \mathbf{a} = \mathbf{d} \cdot \mathbf{a}$ and $\mathbf{p} \cdot \mathbf{b} = \mathbf{d} \cdot \mathbf{b}$, which give the normal equations for x and y:

$$\begin{matrix} (\mathbf{a} \cdot \mathbf{a})x + (\mathbf{a} \cdot \mathbf{b})y = \mathbf{a} \cdot \mathbf{d} & & (n)\ x + (\Sigma b_i)y = \Sigma d_i \\ & \text{or} & \\ (\mathbf{b} \cdot \mathbf{a})x + (\mathbf{b} \cdot \mathbf{b})y = \mathbf{b} \cdot \mathbf{d} & & (\Sigma b_i)x + (\Sigma b_i^2)y = \Sigma b_i d_i. \end{matrix} \tag{17}$$

> **11K** The least squares problem projects $\mathbf{d}$ onto the plane of $\mathbf{a}$ and $\mathbf{b}$. The projection is $\mathbf{p} = x\mathbf{a} + y\mathbf{b}$, in n dimensions. The closest line is $f = x + yt$, in two dimensions. The normal equations (17) give the best x and y.

11.4 EXERCISES

Read-through questions

The equations $3x + y = 8$ and $x + y = 6$ combine into the vector equation x ___a___ $+ y$ ___b___ = ___c___ $= \mathbf{d}$. The left side is $A\mathbf{u}$, with coefficient matrix A = ___d___ and unknown vector $\mathbf{u}$ = ___e___. The determinant of A is ___f___, so this problem is not ___g___. The row picture shows two intersecting ___h___. The column picture shows $x\mathbf{a} + y\mathbf{b} = \mathbf{d}$, where $\mathbf{a}$ = ___i___ and $\mathbf{b}$ = ___j___. The inverse matrix is A^{-1} = ___k___. The solution is $\mathbf{u} = A^{-1}\mathbf{d}$ = ___l___.

A matrix-vector multiplication produces a vector of dot ___m___ from the rows, and also a combination of the ___n___:

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}\begin{bmatrix} - \\ \mathbf{u} \\ - \end{bmatrix} = \begin{bmatrix} - \\ - \end{bmatrix}, \quad \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} - \\ - \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} - \\ - \end{bmatrix}.$$

If the entries are a, b, c, d, the determinant is D = ___o___. A^{-1} is [___p___] divided by D. Cramer's Rule shows components of $\mathbf{u} = A^{-1}\mathbf{d}$ as ratios of determinants: x = ___q___$/D$ and y = ___r___$/D$.

A matrix–matrix multiplication MV yields a matrix of dot products, from the rows of $\underline{\quad s \quad}$ and the columns of $\underline{\quad t \quad}$:

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}=\begin{bmatrix} - & - \\ - & - \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}=\begin{bmatrix} - & - \\ - & - \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}=\begin{bmatrix} - & - \\ - & - \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A \end{bmatrix}=\begin{bmatrix} - & - \\ - & - \end{bmatrix}.$$

The last line contains the $\underline{\quad u \quad}$ matrix, denoted by I. It has the property that $IA = AI = \underline{\quad v \quad}$ for every matrix A, and $I\mathbf{u} = \underline{\quad w \quad}$ for every vector $\mathbf{u}$. The inverse matrix satisfies $A^{-1}A = \underline{\quad x \quad}$. Then $A\mathbf{u} = \mathbf{d}$ is solved by multiplying both sides by $\underline{\quad y \quad}$, to give $\mathbf{u} = \underline{\quad z \quad}$. There is no inverse matrix when $\underline{\quad A \quad}$.

The combination $x\mathbf{a} + y\mathbf{b}$ is the projection of $\mathbf{d}$ when the error $\underline{\quad B \quad}$ is perpendicular to $\underline{\quad C \quad}$ and $\underline{\quad D \quad}$. If $\mathbf{a} = (1, 1, 1)$, $\mathbf{b} = (1, 2, 3)$, and $\mathbf{d} = (0, 8, 4)$, the equations for x and y are $\underline{\quad E \quad}$. Solving them also gives the closest $\underline{\quad F \quad}$ to the data points $(1, 0)$, $\underline{\quad G \quad}$, and $(3, 4)$. The solution is $x = 0$, $y = 2$, which means the best line is $\underline{\quad H \quad}$. The projection is $0\mathbf{a} + 2\mathbf{b} = \underline{\quad I \quad}$. The three error components are $\underline{\quad J \quad}$. Check perpendicularity: $\underline{\quad K \quad} = 0$ and $\underline{\quad L \quad} = 0$. Applying calculus to this problem, x and y minimize the sum of squares $E = \underline{\quad M \quad}$.

In 1–8 find the point (x, y) where the two lines intersect (if they do). Also show how the right side is a combination of the columns on the left side (if it is). Also find the determinant D.

1 $\begin{aligned} x + y &= 7 \\ x - y &= 3 \end{aligned}$

2 $\begin{aligned} 2x + y &= 11 \\ x + y &= 6 \end{aligned}$

3 $\begin{aligned} 3x - y &= 8 \\ x - 3y &= 0 \end{aligned}$

4 $\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 7 \end{aligned}$

5 $\begin{aligned} 2x - 4y &= 0 \\ x - 2y &= 0 \end{aligned}$

6 $\begin{aligned} 10x + y &= 1 \\ x + y &= 1 \end{aligned}$

7 $\begin{aligned} ax + by &= 0 \\ 2ax + 2by &= 2 \end{aligned}$

8 $\begin{aligned} ax + by &= 1 \\ cx + dy &= 1 \end{aligned}$

9 Solve Problem 3 by Cramer's Rule.

10 Try to solve Problem 4 by Cramer's Rule.

11 What are the ratios for Cramer's Rule in Problem 5?

12 If $A = I$ show how Cramer's Rule solves $A\mathbf{u} = \mathbf{d}$.

13 Draw the row picture and column picture for Problem 1.

14 Draw the row and column pictures for Problem 6.

15 Find A^{-1} in Problem 1.

16 Find A^{-1} in Problem 8 if $ad - bc = 1$.

17 A 2 by 2 system is *singular* when the two lines in the row picture ________. This system is still solvable if one equation is a ________ of the other equation. In that case the two lines are ________ and the number of solutions is ________.

18 Try Cramer's Rule when there is no solution or infinitely many:

$$\begin{aligned} 3x + y &= 0 \\ 6x + 2y &= 2 \end{aligned} \quad \text{or} \quad \begin{aligned} 3x + y &= 1 \\ 6x + 2y &= 2. \end{aligned}$$

19 $A\mathbf{u} = \mathbf{d}$ is singular when the columns of A are ________. A solution exists if the right side $\mathbf{d}$ is ________. In this solvable case the number of solutions is ________.

20 The equations $x - y = d_1$ and $9x - 9y = d_2$ can be solved if ________.

21 Suppose $x = \frac{1}{4}$ billion people live in the U.S. and $y = 5$ billion live outside. If 4 per cent of those inside move out and 2 per cent of those outside move in, find the populations d_1 inside and d_2 outside after the move. Express this as a matrix multiplication $A\mathbf{u} = \mathbf{d}$ (and find the matrix).

22 In Problem 21 what is special about $a_1 + a_2$ and $b_1 + b_2$ (the sums down the columns of A)? Explain why $d_1 + d_2$ equals $x + y$.

23 With the same percentages moving, suppose $d_1 = 0.58$ billion are inside and $d_2 = 4.92$ billion are outside *at the end*. Set up and solve two equations for the original populations x and y.

24 What is the determinant of A in Problems 21–23? What is A^{-1}? Check that $A^{-1}A = I$.

25 The equations $ax + y = 0$, $x + ay = 0$ have the solution $x = y = 0$. For which two values of a are there other solutions (and what are the other solutions)?

26 The equations $ax + by = 0$, $cx + dy = 0$ have the solution $x = y = 0$. There are other solutions if the two lines are ________. This happens if a, b, c, d satisfy ________.

27 Find the determinant and inverse of $A = \left[\begin{smallmatrix} 2 & 4 \\ 3 & 5 \end{smallmatrix}\right]$. Do the same for $2A$, A^{-1}, $-A$, and I.

28 Show that the determinant of A^{-1} is $1/\det A$:

$$A^{-1} = \begin{bmatrix} d/(ad - bc) & -b/(ad - bc) \\ -c/(ad - bc) & a/(ad - bc) \end{bmatrix}$$

29 Compute AB and BA and also BC and CB:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Verify the *associative law*: AB times C equals A times BC.

30 (a) Find the determinants of A, B, AB, and BA above.
(b) Propose a law for the determinant of BC and test it.

31 For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ write out AB and factor its determinant into $(ad - bc)(eh - fg)$. Therefore $\det(AB) = (\det A)(\det B)$.

32 Usually $\det(A+B)$ does *not* equal $\det A + \det B$. Find examples of inequality and equality.

33 Find the inverses, and check $A^{-1}A = I$ and $BB^{-1} = I$, for

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}.$$

34 In Problem 33 compute AB and the inverse of AB. Check that this inverse equals B^{-1} times A^{-1}.

35 The matrix product $ABB^{-1}A^{-1}$ equals the ______ matrix. Therefore the inverse of AB is ______. *Important*: The associative law in Problem 29 allows you to multiply BB^{-1} first.

36 The matrix multiplication $C^{-1}B^{-1}A^{-1}ABC$ yields the ______ matrix. Therefore the inverse of ABC is ______.

37 The equations $x + 2y + 3z$ and $4x + 5y + cz = 0$ always have a nonzero solution. The vector $\mathbf{u} = (x, y, z)$ is required to be ______ to $\mathbf{v} = (1, 2, 3)$ and $\mathbf{w} = (4, 5, c)$. So choose $\mathbf{u} =$ ______.

38 Find the combination $\mathbf{p} = x\mathbf{a} + y\mathbf{b}$ of the vectors $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (-1, 0, 1)$ that comes closest to $\mathbf{d} = (2, 6, 4)$. (a) Solve the normal equations (14) for x and y. (b) Check that the error $\mathbf{d} - \mathbf{p}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

39 Plot the three data points $(-1, 2)$, $(0, 6)$, $(1, 4)$ in a plane. Draw the straight line $x + yt$ with the same x and y as in Problem 38. Locate the three errors up or down from the data points and compare with Problem 38.

40 Solve equation (14) to find the combination $x\mathbf{a} + y\mathbf{b}$ of $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (-1, 1, 2)$ that is closest to $\mathbf{d} = (1, 1, 3)$. Draw the corresponding straight line for the data points $(-1, 1)$, $(1, 1)$, and $(2, 3)$. What is the vector of three errors and what is it perpendicular to?

41 Under what condition on d_1, d_2, d_3 do the three points $(0, d_1)$, $(1, d_2)$, $(2, d_3)$ lie on a line?

42 Find the matrices that reverse x and y and project:

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \quad \text{and} \quad P \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

43 Multiplying by $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ projects $\mathbf{u}$ onto the 45° line.

(a) Find the projection $P\mathbf{u}$ of $\mathbf{u} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$.

(b) Why does P times P equal P?

(c) Does P^{-1} exist? What vectors give $P\mathbf{u} = \mathbf{0}$?

44 Suppose $\mathbf{u}$ is not the zero vector but $A\mathbf{u} = \mathbf{0}$. Then A^{-1} can't exist: It would multiply ______ and produce $\mathbf{u}$.

11.5 Linear Algebra

This section moves from two to three dimensions. There are three unknowns x, y, z and also three equations. This is at the crossover point between formulas and algorithms—it is real linear algebra. The formulas give a direct solution using determinants. The algorithms use elimination and the numbers x, y, z appear at the end. In practice that end result comes quickly. *Computers solve linear equations by elimination.*

The situation for a nonlinear equation is similar. Quadratic equations $ax^2 + bx + c = 0$ are solved by a formula. Cubic equations are solved by Newton's method (even though a formula exists). For equations involving x^5 or x^{10}, algorithms take over completely.

Since we are at the crossover point, we look both ways. This section has a lot to do, in mixing geometry, determinants, and 3 by 3 matrices:

1. The row picture: three planes intersect at the solution
2. The column picture: a vector equation combines the columns
3. The formulas: determinants and Cramer's Rule
4. Matrix multiplication and A^{-1}
5. The algorithm: Gaussian elimination.

Part of our goal is three-dimensional calculus. Another part is n-dimensional algebra. And a third possibility is that you may not take mathematics next year. If that happens, I hope you will *use* mathematics. Linear equations are so basic and important, in such a variety of applications, that the effort in this section is worth making.

An example is needed. It is convenient and realistic if the matrix contains zeros. Most equations in practice are fairly simple—a thousand equations each with 990 zeros would be very reasonable. Here are three equations in three unknowns:

$$\begin{aligned} x + y \qquad &= 1 \\ x \qquad + 2z &= 0 \\ -2y + 2z &= -4. \end{aligned} \tag{1}$$

In matrix-vector form, the unknown $\mathbf{u}$ has components x, y, z. The right sides $1, 0, -4$ go into $\mathbf{d}$. The nine coefficients, including three zeros, enter the matrix A:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \quad \text{or} \quad A\mathbf{u} = \mathbf{d}. \tag{2}$$

The goal is to understand that system geometrically, and then solve it.

THE ROW PICTURE: INTERSECTING PLANES

Start with the first equation $x + y = 1$. In the xy plane that produces a line. In three dimensions it is a *plane*. It has the usual form $ax + by + cz = d$, except that c happens to be zero. The plane is easy to visualize (Figure 11.20a), because it cuts straight down through the line. The equation $x + y = 1$ allows z to have any value, so the graph includes all points above and below the line.

The second equation $x + 2z = 0$ gives a second plane, which goes through the origin. *When the right side is zero, the point* $(0, 0, 0)$ *satisfies the equation.* This time y is absent from the equation, so the plane contains the whole y axis. All points $(0, y, 0)$ meet the requirement $x + 2z = 0$. *The normal vector to the plane is* $\mathbf{N} = \mathbf{i} + 2\mathbf{k}$. The plane cuts across, rather than down, in 11.20b.

Before the third equation we combine the first two. ***The intersection of two planes is a line.*** In three-dimensional space, two equations (not one) describe a line. The points on the line have to satisfy $x + y = 1$ and also $x + 2z = 0$. A convenient point is $P = (0, 1, 0)$. Another point is $Q = (-1, 2, \frac{1}{2})$. The line through P and Q extends out in both directions.

The solution is on that line. The third plane decides where.

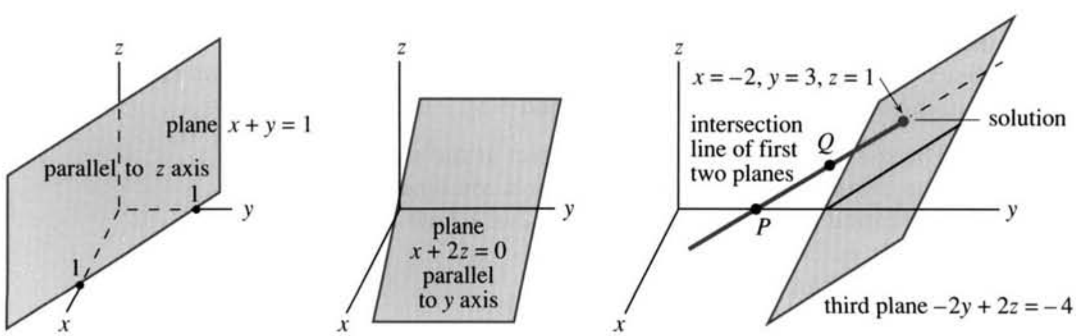


Fig. 11.20 First plane, second plane, intersection line meets third plane at solution.

The third equation $-2y + 2z = -4$ gives the third plane—which misses the origin because the right side is not zero. What is important is *the point where the three planes meet.* The intersection line of the first two planes crosses the third plane. We used determinants (but elimination is better) to find $x = -2$, $y = 3$, $z = 1$. This solution satisfies the three equations and lies on the three planes.

A brief comment on 4 by 4 systems. The first equation might be $x + y + z - t = 0$. It represents a three-dimensional "hyperplane" in four-dimensional space. (In physics this is space-time.) The second equation gives a second hyperplane, and its intersection with the first one is two-dimensional. The third equation (third hyperplane) reduces the intersection to a line. The fourth hyperplane meets that line at a point, which is the solution. It satisfies the four equations and lies on the four hyperplanes. In this course three dimensions are enough.

COLUMN PICTURE: COMBINATION OF COLUMN VECTORS

There is an extremely important way to rewrite our three equations. In (1) they were separate, in (2) they went into a matrix. Now they become a vector equation:

$$x\begin{bmatrix}1\\1\\0\end{bmatrix} + y\begin{bmatrix}1\\0\\-2\end{bmatrix} + z\begin{bmatrix}0\\2\\2\end{bmatrix} = \begin{bmatrix}1\\0\\-4\end{bmatrix}. \tag{3}$$

The columns of the matrix are multiplied by x, y, z. That is a special way to see matrix-vector multiplication: $A\mathbf{u}$ ***is a combination of the columns of*** A. We are looking for the numbers x, y, z so that the combination produces the right side $\mathbf{d}$.

The column vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are shown in Figure 11.21a. The vector equation is $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$. The combination that solves this equation must again be $x = -2$, $y = 3$, $z = 1$. That agrees with the intersection point of the three planes in the row picture.

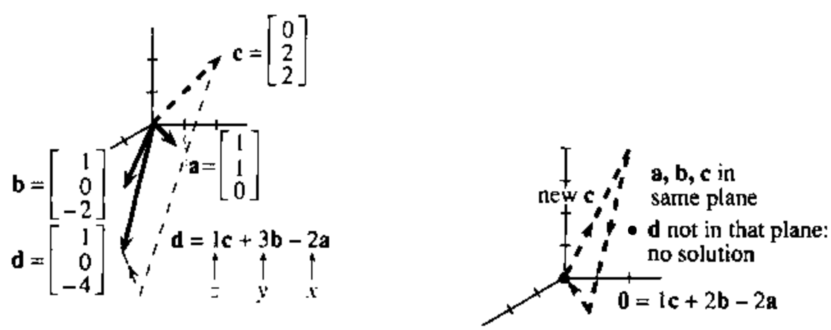


Fig. 11.21 Columns combine to give $\mathbf{d}$. Columns combine to give **zero** (singular case).

THE DETERMINANT AND THE INVERSE MATRIX

For a 3 by 3 determinant, the section on cross products gave two formulas. One was the triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. The other wrote out the six terms:

$$\det A = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1).$$

Geometrically this is *the volume of a box*. The columns **a**, **b**, **c** are the edges going out from the origin. In our example the determinant and volume are 2:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 2 \end{vmatrix} = \begin{matrix} (1)(0)(2)-(1)(-2)(2)+(1)(-2)(0) \\ -(1)(1)(2)+(0)(1)(2)-(0)(0)(0) \end{matrix} = 2.$$

A slight dishonesty is present in that calculation, and will be admitted now. In Section 11.3 the vectors **A**, **B**, **C** were *rows*. In this section **a**, **b**, **c** are *columns*. It doesn't matter, because the determinant is the same either way. Any matrix can be "transposed"—exchanging rows for columns—without altering the determinant. The six terms ($a_1b_2c_3$ is the first) may come in a different order, but they are the same six terms. Here four of those terms are zero, because of the zeros in the matrix. The sum of all six terms is $D = \det A = 2$.

Since D is not zero, the equations can be solved. The three planes meet at a point. The column vectors **a**, **b**, **c** produce a genuine box, and are not flattened into the same plane (with zero volume). The solution involves ***dividing by D***—which is only possible if $D = \det A$ is not zero.

> **11L** **When the determinant D is not zero, A has an inverse: $AA^{-1} = A^{-1}A = I$. Then the equations $A\mathbf{u} = \mathbf{d}$ have one and only one solution $\mathbf{u} = A^{-1}\mathbf{d}$.**

The 3 by 3 identity matrix I is at the end of equation (5). Always $I\mathbf{u} = \mathbf{u}$.

We now compute A^{-1}, first with letters and then with numbers. The neatest formula uses cross products of the columns of A—it is special for 3 by 3 matrices.

Every entry is divided by D: ***The inverse matrix is*** $A^{-1} = \dfrac{1}{D}\begin{bmatrix} \mathbf{b}\times\mathbf{c} \\ \mathbf{c}\times\mathbf{a} \\ \mathbf{a}\times\mathbf{b} \end{bmatrix}$. (4)

To test this formula, multiply by A. ***Matrix multiplication produces a matrix of dot products***—from the rows of the first matrix and the columns of the second, $A^{-1}A = I$:

$$\frac{1}{D}\begin{bmatrix} \mathbf{b}\times\mathbf{c} \\ \mathbf{c}\times\mathbf{a} \\ \mathbf{a}\times\mathbf{b} \end{bmatrix}\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix} = \frac{1}{D}\begin{bmatrix} \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) & \mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) & \mathbf{c}\cdot(\mathbf{b}\times\mathbf{c}) \\ \mathbf{a}\cdot(\mathbf{c}\times\mathbf{a}) & \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) & \mathbf{c}\cdot(\mathbf{c}\times\mathbf{a}) \\ \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) & \mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) & \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (5)$$

On the right side, six of the triple products are zero. They are the off-diagonals like $\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})$, which contain the same vector twice. Since $\mathbf{b}\times\mathbf{c}$ is perpendicular to **b**, this triple product is zero. The same is true of the others, like $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0$. That is the volume of a box with two identical sides. The six off-diagonal zeros are the volumes of completely flattened boxes.

On the main diagonal the triple products equal D. The order of vectors can be **abc** or **bca** or **cab**, and the volume of the box stays the same. Dividing by this number D, which is placed outside for that purpose, gives the 1's in the identity matrix I.

Now we change to numbers. The goal is to find A^{-1} and to test it.

EXAMPLE 1 *The inverse of* $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix}$ *is* $A^{-1} = \dfrac{1}{2}\begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ -2 & 2 & -1 \end{bmatrix}$.

That comes from the formula, and it absolutely has to be checked. Do not fail to multiply A^{-1} times A (or A times A^{-1}). Matrix multiplication is much easier than the formula for A^{-1}. We highlight row 3 times column 1, with dot product zero:

$$\frac{1}{2}\begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4-2 & 4-4 & -4+4 \\ -2+2 & -2+4 & 4-4 \\ -2+2 & -2+2 & 4-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark on A^{-1} Inverting a matrix requires $D \neq 0$. We divide by $D = \det A$. The cross products $\mathbf{b} \times \mathbf{c}$ and $\mathbf{c} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$ give A^{-1} in a neat form, but errors are easy. We prefer to avoid writing $\mathbf{i}, \mathbf{j}, \mathbf{k}$. There are nine 2 by 2 determinants to be calculated, and here is A^{-1} in full—containing the nine "***cofactors***" divided by D:

$$A^{-1} = \frac{1}{D}\begin{bmatrix} b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \\ c_2a_3 - c_3a_2 & c_3a_1 - c_1a_3 & c_1a_2 - c_2a_1 \\ a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \end{bmatrix}. \tag{6}$$

Important: The first row of A^{-1} does not use the first column of A, except in $1/D$. In other words, $\mathbf{b} \times \mathbf{c}$ does not involve $\mathbf{a}$. Here are the 2 by 2 determinants that produce $4, -2, 2$—which is divided by $D = 2$ in the top row of A^{-1}:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}. \tag{7}$$

The second highlighted determinant looks like $+2$ not -2. But the ***sign matrix*** on the right assigns a minus to that position in A^{-1}. We reverse the sign of $b_1c_3 - b_3c_1$, to find the cofactor $b_3c_1 - b_1c_3$ in the top row of (6).

To repeat: ***For a row of A^{-1}, cross out the corresponding column of A. Find the three 2 by 2 determinants, use the sign matrix, and divide by D.***

EXAMPLE 2 $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has $D = 1$ and $B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. (8)

The multiplication $BB^{-1} = I$ checks the arithmetic. Notice how $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$ in B leads to a zero in the top row of B^{-1}. To find row 1, column 3 of B^{-1} we ignore column 1 and row 3 of B. (Also: the inverse of a triangular matrix is triangular.) The minus signs come from the sign matrix.

THE SOLUTION $\mathbf{u} = A^{-1}\mathbf{d}$

The purpose of A^{-1} is to solve the equation $A\mathbf{u} = \mathbf{d}$. Multiplying by A^{-1} produces $I\mathbf{u} = A^{-1}\mathbf{d}$. The matrix becomes the identity, $I\mathbf{u}$ equals $\mathbf{u}$, and the solution is immediate:

$$\mathbf{u} = A^{-1}\mathbf{d} = \frac{1}{D}\begin{bmatrix} \mathbf{b} \times \mathbf{c} \\ \mathbf{c} \times \mathbf{a} \\ \mathbf{a} \times \mathbf{b} \end{bmatrix}\begin{bmatrix} \\ \mathbf{d} \\ \\ \end{bmatrix} = \frac{1}{D}\begin{bmatrix} \mathbf{d} \cdot (\mathbf{b} \times \mathbf{c}) \\ \mathbf{d} \cdot (\mathbf{c} \times \mathbf{a}) \\ \mathbf{d} \cdot (\mathbf{a} \times \mathbf{b}) \end{bmatrix}. \tag{9}$$

By writing those components x, y, z as *ratios of determinants*, we have Cramer's Rule:

11M ***(Cramer's Rule)***

$$\text{The solution is } x = \frac{|\mathbf{d}\ \mathbf{b}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|}, \quad y = \frac{|\mathbf{a}\ \mathbf{d}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|}, \quad z = \frac{|\mathbf{a}\ \mathbf{b}\ \mathbf{d}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|}. \tag{10}$$

The right side $\mathbf{d}$ replaces, in turn, columns $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{c}$. All denominators are $D = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. The numerator of x is the determinant $\mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})$ in (9). The second numerator agrees with the second component $\mathbf{d} \cdot (\mathbf{c} \times \mathbf{a})$, because the cyclic order is correct. The third determinant with columns $\mathbf{abd}$ equals the triple product $\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})$ in $A^{-1}\mathbf{u}$. Thus (10) is the same as (9).

EXAMPLE A: Multiply by A^{-1} to find the known solution $x = -2$, $y = 3$, $z = 1$:

$$\mathbf{u} = A^{-1}\mathbf{d} = \frac{1}{2}\begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4-8 \\ -2+8 \\ -2+4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}.$$

EXAMPLE B: Multiply by B^{-1} to solve $B\mathbf{u} = \mathbf{d}$ when $\mathbf{d}$ is the column (6, 5, 4):

$$\mathbf{u} = B^{-1}\mathbf{d} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}. \quad \textit{Check } B\mathbf{u} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}.$$

EXAMPLE C: Put $\mathbf{d} = (6, 5, 4)$ in each column of B. Cramer's Rule gives $\mathbf{u} = (1, 1, 4)$:

$$\begin{vmatrix} 6 & 1 & 1 \\ 5 & 1 & 1 \\ 4 & 0 & 1 \end{vmatrix} = 1 \quad \begin{vmatrix} 1 & 6 & 1 \\ 0 & 5 & 1 \\ 0 & 4 & 1 \end{vmatrix} = 1 \quad \begin{vmatrix} 1 & 1 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{vmatrix} = 4 \quad \text{all divided by } D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

This rule fills the page with determinants. Those are good ones to check by eye, without writing down the six terms (three + and three −).

The formulas for A^{-1} are honored chiefly in their absence. They are not used by the computer, even though the algebra is in some ways beautiful. In big calculations, the computer never finds A^{-1}—just the solution.

We now look at the singular case $D = 0$. Geometry–algebra–algorithm must all break down. After that is the algorithm: Gaussian elimination.

THE SINGULAR CASE

Changing one entry of a matrix can make the determinant zero. The triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, which is also the volume, becomes $D = 0$. The box is flattened and the matrix is singular. That happens in our example when the lower right entry is changed from 2 to 4:

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 4 \end{bmatrix} \text{ has determinant } D = 0.$$

This does more than change the inverse. It *destroys* the inverse. We can no longer divide by D. There is no S^{-1}.

What happens to the row picture and column picture? For 2 by 2 systems, the singular case had two parallel lines. Now the row picture has three planes, which need not be parallel. Here the planes are *not parallel*. Their normal vectors are the rows of S, which go in different directions. But somehow the planes fail to go through a common point.

What happens is more subtle. The intersection line from two planes misses the third plane. The line is parallel to the plane and stays above it (Figure 11.22a). When all three planes are drawn, they form an open tunnel. The picture tells more than the numbers, about how three planes can fail to meet. The third figure shows an end view, where the planes go directly into the page. Each pair meets in a line, but those lines don't meet in a point.

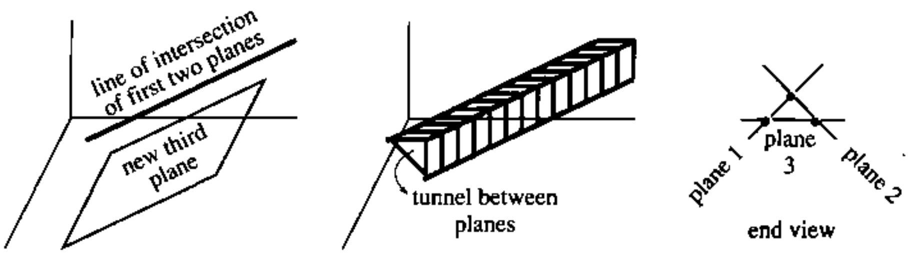


Fig. 11.22 The row picture in the singular case: no intersection point, no solution.

When two planes are parallel, the determinant is again zero. One row of the matrix is a multiple of another row. The extreme case has all three planes parallel—as in a matrix with nine 1's.

The column picture must also break down. In the 2 by 2 failure (previous section), the columns were on the same line. *Now the three columns are in the same plane.* The combinations of those columns produce $\mathbf{d}$ only if it happens to lie in that particular plane. Most vectors $\mathbf{d}$ will be outside the plane, so most singular systems have no solution.

When the determinant is zero, $A\mathbf{u} = \mathbf{d}$ ***has no solution or infinitely many.***

THE ELIMINATION ALGORITHM

Go back to the 3 by 3 example $A\mathbf{u} = \mathbf{d}$. If you were given those equations, you would never think of determinants. You would—*quite correctly*—start with the first equation. It gives $x = 1 - y$, which goes into the next equation to eliminate x:

$$\begin{aligned} x + y \quad &= 1 \\ x \quad + 2z &= 0 \\ -2y + 2z &= -4 \end{aligned} \quad \xrightarrow{x = 1 - y} \quad \begin{aligned} 1 - y + 2z &= 0 \\ -2y + 2z &= -4. \end{aligned}$$

Stop there for a minute. On the right is a 2 by 2 system for y and z. The first equation and first unknown are eliminated—exactly what we want. But that step was not organized in the best way, because a "1" ended up on the left side. Constants should stay on the right side—the pattern should be preserved. It is better to take the same step by ***subtracting the first equation from the second***:

$$\begin{aligned} x + y \quad &= 1 \\ x \quad + 2z &= 0 \\ -2y + 2z &= -4 \end{aligned} \longrightarrow \begin{aligned} -\ y + 2z &= -1 \\ -2y + 2z &= -4. \end{aligned} \tag{11}$$

Same equations, better organization. Now look at the corner term $-y$. Its coefficient -1 is the ***second pivot***. (The first pivot was $+1$, the coefficient of x in the first corner.) We are ready for the next elimination step:

Plan: Subtract a multiple of the "pivot equation" from the equation below it.
Goal: To produce a zero below the pivot, so y is eliminated.
Method: Subtract 2 times the pivot equation to cancel $-2y$.

$$\begin{aligned} -\ y + 2z &= -1 \\ -2y + 2z &= -4 \end{aligned} \rightarrow \quad -2z = -2. \tag{12}$$

The answer comes by ***back substitution***. Equation (12) gives $z = 1$. Then equation (11) gives $y = 3$. Then the first equation gives $x = -2$. This is much quicker than determinants. You may ask: *Why use Cramer's Rule*? Good question.

With numbers elimination is better. It is faster and also safer. (To check against error, substitute $-2, 3, 1$ into the original equations.) The algorithm reaches the answer *without the determinant and without the inverse*. Calculations with letters use $\det A$ and A^{-1}.

Here are the steps in a definite order (top to bottom):

Subtract a multiple of equation 1 to produce $0x$ in equation 2
Subtract a multiple of equation 1 to produce $0x$ in equation 3
Subtract a multiple of equation 2 (new) to produce $0y$ in equation 3.

EXAMPLE (notice the zeros appearing under the pivots):

$$\begin{aligned} x + y + z &= 1 \\ 2x + 5y + 3z &= 7 \\ 4x + 7y + 6z &= 11 \end{aligned} \rightarrow \begin{aligned} x + y + z &= 1 \\ 3y + z &= 5 \\ 3y + 2z &= 7 \end{aligned} \rightarrow \begin{aligned} x + y + z &= 1 \\ 3y + z &= 5 \\ z &= 2. \end{aligned}$$

Elimination leads to a ***triangular system***. The coefficients below the diagonal are zero. First $z = 2$, then $y = 1$, then $x = -2$. *Back substitution solves triangular systems* (fast).

As a final example, try the singular case $S\mathbf{u} = \mathbf{d}$ when the corner entry is changed from 2 to 4. With $D = 0$, there is no inverse matrix S^{-1}. Elimination also fails, by reaching an impossible equation $0 = -2$:

$$\begin{aligned} x + y \quad &= 1 \\ x \quad + 2z &= 0 \\ -2y + \underline{4}z &= -4 \end{aligned} \rightarrow \begin{aligned} x + y \quad &= 1 \\ -\ y + 2z &= -1 \\ -2y + \underline{4}z &= -4 \end{aligned} \rightarrow \begin{aligned} x + y \quad &= 1 \\ -y + 2z &= -1 \\ \underline{0 = -2} \end{aligned}$$

The three planes do not meet at a point—a fact that was not obvious at the start. Algebra discovers this fact from $D = 0$. Elimination discovers it from $0 = -2$. The chapter is ending at the point where my linear algebra book begins.

One final comment. In actual computing, you will use a code written by professionals. The steps will be the same as above. A multiple of equation 1 is subtracted from each equation below it, to eliminate the first unknown x. With one fewer unknown and equation, elimination starts again. (A parallel computer executes many steps at once.) Extra instructions are included to reduce roundoff error. You only see the result! But it is more satisfying to know what the computer is doing.

In the end, solving linear equations is the key step in solving nonlinear equations. The central idea of differential calculus is to *linearize* near a point.

11.5 EXERCISES

Read-through questions

Three equations in three unknowns can be written as $A\mathbf{u} = \mathbf{d}$. The ___a___ $\mathbf{u}$ has components x, y, z and A is a ___b___. The row picture has a ___c___ for each equation. The first two planes intersect in a ___d___, and all three planes intersect in a ___e___, which is ___f___. The column picture starts with vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ from the columns of ___g___ and combines them to produce ___h___. The vector equation is ___i___ $= \mathbf{d}$.

The determinant of A is the triple product ___j___. This is the volume of a box, whose edges from the origin are ___k___. If det $A =$ ___l___ then the system is ___m___. Otherwise there is an ___n___ matrix such that $A^{-1}A =$ ___o___ (the ___p___ matrix). In this case the solution to $A\mathbf{u} = \mathbf{d}$ is $\mathbf{u} =$ ___q___.

The rows of A^{-1} are the cross products $\mathbf{b} \times \mathbf{c}$, ___r___, ___s___, divided by D. The entries of A^{-1} are 2 by 2 ___t___, divided by D. The upper left entry equals ___u___. The 2 by 2 determinants needed for a row of A^{-1} do not use the corresponding ___v___ of A.

The solution is $\mathbf{u} = A^{-1}\mathbf{d}$. Its first component x is a ratio of determinants, $|\mathbf{d}\,\mathbf{b}\,\mathbf{c}|$ divided by ___w___. Cramer's Rule breaks down when det $A =$ ___x___. Then the columns $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie in the same ___y___. There is no solution to $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$, if $\mathbf{d}$ is not on that ___z___. In a singular row picture, the intersection of planes 1 and 2 is ___A___ to the third plane.

In practice $\mathbf{u}$ is computed by ___B___. The algorithm starts by subtracting a multiple of row 1 to eliminate x from ___C___. If the first two equations are $x - y = 1$ and $3x + z = 7$, this elimination step leaves ___D___. Similarly x is eliminated from the third equation, and then ___E___ is eliminated. The equations are solved by back ___F___. When the system has no solution, we reach an impossible equation like ___G___. The example $x - y = 1, 3x + z = 7$ has no solution if the third equation is ___H___.

Rewrite 1–4 as matrix equations $A\mathbf{u} = \mathbf{d}$ (do not solve).

1 $\mathbf{d} = (0, 0, 8)$ is a combination of $\mathbf{a} = (1, 2, 0)$ and $\mathbf{b} = (2, 3, 2)$ and $\mathbf{c} = (2, 5, 2)$.

2 The planes $x + y = 0$, $x + y + z = 1$, and $y + z = 0$ intersect at $\mathbf{u} = (x, y, z)$.

3 The point $\mathbf{u} = (x, y, z)$ is on the planes $x = y$, $y = z$, $x - z = 1$.

4 A combination of $\mathbf{a} = (1, 0, 0)$ and $\mathbf{b} = (0, 2, 0)$ and $\mathbf{c} = (0, 0, 3)$ equals $\mathbf{d} = (5, 2, 0)$.

5 Show that Problem 3 has no solution in two ways: find the determinant of A, and combine the equations to produce $0 = 1$.

6 Solve Problem 2 in two ways: by inspiration and Cramer's Rule.

7 Solve Problem 4 in two ways: by inspection and by computing the determinant and inverse of the *diagonal matrix*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

8 Solve the three equations of Problem 1 by elimination.

9 The vectors $\mathbf{b}$ and $\mathbf{c}$ lie in a plane which is perpendicular to the vector ________. In case the vector $\mathbf{a}$ also lies in that plane, it is also perpendicular and $\mathbf{a} \cdot$ ________ $= 0$. The ________ of the matrix with columns in a plane is ________.

10 The plane $a_1x + b_1y + c_1z = d_1$ is perpendicular to its normal vector $\mathbf{N}_1 =$ ________. The plane $a_2x + b_2y + c_2z = d_2$ is perpendicular to $\mathbf{N}_2 =$ ________. The planes meet in a line that is perpendicular to both vectors, so the line is parallel to their ________ product. If this line is also parallel to the third plane and perpendicular to $\mathbf{N}_3$, the system is ________. The matrix has no ________, which happens when $(\mathbf{N}_1 \times \mathbf{N}_2) \cdot \mathbf{N}_3 = 0$.

Problems 11–24 use the matrices A, B, C.

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 6 & 4 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & -3 \\ -1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}.$$

11 Find the determinants $|A|$, $|B|$, $|C|$. Since A is triangular, its determinant is the product ________.

12 Compute the cross products of each pair of columns in B (three cross products).

13 Compute the inverses of A and B above. Check that $A^{-1}A = I$ and $B^{-1}B = I$.

14 Solve $A\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $B\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. With this right side $\mathbf{d}$, why is $\mathbf{u}$ the first column of the inverse?

15 Suppose all three columns of a matrix add to zero, as in C above. The dot product of each column with $\mathbf{v} = (1, 1, 1)$ is ________. All three columns lie in the same ________. The determinant of C must be ________.

16 Find a nonzero solution to $C\mathbf{u} = \mathbf{0}$. Find all solutions to $C\mathbf{u} = \mathbf{0}$.

17 Choose any right side $\mathbf{d}$ that is perpendicular to $\mathbf{v} = (1, 1, 1)$ and solve $C\mathbf{u} = \mathbf{d}$. Then find a second solution.

18 Choose any right side $\mathbf{d}$ that is not perpendicular to $\mathbf{v} = (1, 1, 1)$. Show by elimination (reach an impossible equation) that $C\mathbf{u} = \mathbf{d}$ has no solution.

19 Compute the matrix product AB and then its determinant. How is det AB related to det A and det B?

20 Compute the matrix products BC and CB. All columns of CB add to ________, and its determinant is ________.

21 Add A and C by adding each entry of A to the corresponding entry of C. Check whether the determinant of $A + C$ equals det A + det C.

22 Compute $2A$ by multiplying each entry of A by 2. The determinant of $2A$ equals ________ times the determinant of A.

23 Which four entries of A give the upper left corner entry p of A^{-1}, after dividing by $D = \det A$? Which four entries of A give the entry q in row 1, column 2 of A^{-1}? Find p and q.

24 The 2 by 2 determinants from the first two rows of B are -1 (from columns 2, 3) and -2 (from columns 1, 3) and ________ (from columns 1, 2). These numbers go into the third ________ of B^{-1}, after dividing by ________ and changing the sign of ________.

25 Why does every inverse matrix A^{-1} have an inverse?

26 From the multiplication $ABB^{-1}A^{-1} = I$ it follows that the inverse of AB is ________. The separate inverses come in ________ order. If you put on socks and then shoes, the inverse begins by taking off ________.

27 Find the determinants of these four *permutation matrices*:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and $QP =$ ________. Multiply $\mathbf{u} = (x, y, z)$ by each permutation to find $P\mathbf{u}$, $Q\mathbf{u}$, $PQ\mathbf{u}$, and $QP\mathbf{u}$.

28 Find all six of the 3 by 3 permutation matrices (including I), with a single 1 in each row and column. Which of them are "even" (determinant 1) and which are "odd" (determinant -1)?

29 How many 2 by 2 permutation matrices are there, including I? How many 4 by 4?

30 Multiply any matrix A by the permutation matrix P and explain how PA is related to A. In the opposite order explain how AP is related to A.

31 Eliminate x from the last two equations by subtracting the first equation. Then eliminate y from the new third equation by using the new second equation:

$$\text{(a)} \quad \begin{aligned} x + y + z &= 2 \\ x + 3y + 3z &= 0 \\ x + 3y + 7z &= 2 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x + y \quad\;\; &= 1 \\ x + \quad\; z &= 3 \\ y + z &= 5. \end{aligned}$$

After elimination solve for z, y, x (back substitution).

32 By elimination and back substitution solve

$$\text{(a)} \quad \begin{aligned} x + 2y + 2z &= 0 \\ 2x + 3y + 5z &= 0 \\ 2y + 2z &= 8 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x - y \quad\;\; &= 1 \\ x \quad\; - z &= 4 \\ y - z &= 7. \end{aligned}$$

33 Eliminate x from equation 2 by using equation 1:

$$\begin{aligned} x + 2y + 2z &= 0 \\ 2x + 4y + 5z &= 0 \\ 2y + 2z &= 8. \end{aligned}$$

Why can't the new second equation eliminate y from the third equation? Is there a solution or is the system singular?

Note: If elimination creates a zero in the "pivot position," try to exchange that pivot equation with an equation below it. Elimination succeeds when there is a full set of pivots.

34 The pivots in Problem 32a are 1, -1, and 4. Circle those as they appear along the diagonal in elimination. Check that the product of the pivots equals the determinant. (This is how determinants are computed.)

35 Find the pivots and determinants in Problem 31.

36 Find the inverse of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and also of $B = A^2$.

37 The symbol a_{ij} stands for the entry in row i, column j. Find a_{12} and a_{21} in Problem 36. The formula $\Sigma a_{ij} b_{jk}$ gives the entry in which row and column of the matrix product AB?

38 Write down a 3 by 3 singular matrix S in which no two rows are parallel. Find a combination of rows 1 and 2 that is parallel to row 3. Find a combination of columns 1 and 2 that is parallel to column 3. Find a nonzero solution to $S\mathbf{u} = 0$.

39 Compute these determinants. The 2 by 2 matrix is invertible if ________. The 3 by 3 matrix (is)(is not) invertible.

$$D = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad D = \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \\ 7 & 8 & 9 & 7 & 8 \end{array}\right]$$

$$- \qquad + \qquad\qquad - \quad - \quad - \qquad + \quad + \quad +$$

MIT OpenCourseWare
http://ocw.mit.edu

Resource: Calculus Online Textbook
Gilbert Strang

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.