CHAPTER 3   APPLICATIONS OF DERIVATIVES

3.1   Linear Approximation   (page 95)

This section is built on one idea and one formula. The idea is to use the tangent line as an approximation to the curve. The formula is written in several ways, depending which letters are convenient.

$$f(x) \approx f(a) + f'(a)(x - a) \qquad \text{or} \qquad f(x + \Delta x) \approx f(x) + f'(x)\Delta x.$$

In the first formula, a is the "basepoint." We use the function value $f(a)$ and the slope $f'(a)$ at that point. The step is $x - a$. The tangent line $Y = f(a) + f'(a)(x - a)$ approximates the curve $y = f(x)$.

In the other form the basepoint is x and the step is Δx. Remember that the average $\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$ is close to $f'(x)$. That is all the second form says: Δf *is close to* $f'(x)\Delta x$. Here are the steps to approximate $\frac{1}{4.1}$:

Step 1: What is the function being used?   ● It is $f(x) = \frac{1}{x}$.

Step 2: At what point "a" near $x = 4.1$ is $f(x)$ exactly known?   We know that $\frac{1}{4} = 0.25$. Let $a = 4$.

Step 3: What is the slope of $f(x)$ at $x = a$?   The derivative is $f'(x) = -\frac{1}{x^2}$ so $f'(4) = -\frac{1}{4^2} = -\frac{1}{16}$.

Step 4: What is the formula for the approximation? ● $f(x) \approx \frac{1}{4} - \frac{1}{16}(x - 4)$.

Step 5: Substitute $x = 4.1$ to find $f(4.1) \approx \frac{1}{4} - \frac{1}{16}(4.1 - 4) = 0.24375$. The calculator gives $\frac{1}{4.1} = .24390\cdots$.

1. Find the linear approximation to $f(x) = \sec x$ near $x = \frac{\pi}{3}$

   - ● You will need the derivative of $\sec x$. It's good to memorize all six trigonometric derivatives at the bottom of page 76. Since $\sec x = \frac{1}{\cos x}$, you can also use the reciprocal rule. Either way, $f(x) = \sec x$ leads to $f'(x) = \sec x \tan x$ and $f'(\frac{\pi}{3}) = \sec \frac{\pi}{3} \tan \frac{\pi}{3} = (2)(\frac{1}{\sqrt{3}})$. This is the slope of the tangent line.

     The approximation is $f(x) \approx \sec \frac{\pi}{3} + \frac{2}{\sqrt{3}}(x - \frac{\pi}{3})$ or $f(x) \approx 2 + \frac{2}{\sqrt{3}}(x - \frac{\pi}{3})$. The right side is the tangent line $Y(x)$. The other form is $f(\frac{\pi}{3} + \Delta x) \approx 2 + \frac{2}{\sqrt{3}}\Delta x$. The step $x - \frac{\pi}{3}$ is Δx.

2. (This is Problem 3.1.15) Calculate the numerical error in the linear approximation to $(\sin 0.01)^2$ from the base point $a = 0$. Compare the error with the quadratic correction $\frac{1}{2}(\Delta x)^2 f''(x)$.

   - ● The function being approximated is $f(x) = (\sin x)^2$. We chose $a = 0$ since 0.01 is near that basepoint and $f(0) = 0$ is known. Then $f'(x) = 2 \sin x \cos x$ and $f'(0) = 2 \sin 0 \cos 0 = 0$. The linear approximation is $f(0.01) \approx 0 + 0(.01)$. The tangent line is the x axis! It is like the tangent to $y = x^2$ at the bottom of the parabola, where $x = 0$ and $y = 0$ and slope $= 0$.

   The linear approximation is $(\sin 0.01)^2 \approx 0$. A calculator gives the value $(\sin 0.01)^2 = 9.9997 \times 10^{-5}$ which is close to $10 \times 10^{-5} = 10^{-4}$. To compare with the quadratic correction we need $f''(x)$. From $f'(x) = 2 \sin x \cos x$ comes $f''(x) = 2 \cos^2 x - 2 \sin^2 x$. Then $\frac{1}{2}(\Delta x)^2 f''(0) = \frac{1}{2}(0.01)^2(2) = 10^{-4}$ exactly. (Note: The calculator is approximating too, but it is using much more accurate methods.)

   To summarize: Linear approximation 0, quadratic approximation .0001, calculator approximation .000099997.

3. A melting snowball of diameter six inches loses a half inch in diameter. Estimate its loss in surface area and volume.

   - ● The area and volume formulas on the inside back cover are $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$. Since $\frac{dA}{dr} = 8\pi r$ and $\frac{dV}{dr} = 4\pi r^2$, the linear corrections (the "differentials") are $dA = 8\pi r dr$ and $dV = 4\pi r^2 dr$. Note that $dr = -\frac{1}{4}$ inches since the *radius* decreased from 3 inches to $2\frac{3}{4}$ inches. Then $dA = 8\pi(3)(-\frac{1}{4}) = -6\pi$ square inches and $dV = 4\pi(3)^2(\frac{-1}{4}) = -9\pi$ cubic inches.

Note on differentials: *df* is exactly the linear correction $f'(x)dx$. When $f(x)$ is the area $A(r)$, this problem had $dA = 8\pi r dr$. The point about differentials is that they allow us to use an equal sign $(=)$ instead of an approximation sign $(\approx)$. It is a good notation for the *linear correction term*.

4. Imagine a steel band that fits snugly around the Earth's equator. More steel will be added to the band to make it lie one foot above the equator all the way around. How much more steel will have to be added? Take a guess before you turn to differentials.

   - Did you guess about a couple of yards? Here's the mathematics: $C = 2\pi r$. We want to estimate the change in circumference C when the radius increases by one foot. The differential is $dC = 2\pi dr$. Substitute $dr = 1$ foot to find $dC = 2\pi$ feet ≈ 6.3 feet. Note that the actual radius of the earth doesn't enter into the calculations, so this 2-yard answer is the same on any planet or any sphere.

Read-throughs and selected even-numbered solutions :

On the graph, a linear approximation is given by the **tangent line**. At $x = a$, the equation for that line is $Y = f(a) + \mathbf{f'(a)(x - a)}$. Near $x = a = 10$, the linear approximation to $y = x^3$ is $Y = 1000 + \mathbf{300(x - 10)}$. At $x = 11$ the exact value is $(11)^3 = \mathbf{1331}$. The approximation is $Y = \mathbf{1300}$. In this case $\Delta y = \mathbf{331}$ and $dy = \mathbf{300}$. If we know $\sin x$, then to estimate $\sin(x + \Delta x)$ we add $\mathbf{(\cos x)\Delta x}$.

In terms of x and Δx, linear approximation is $f(x + \Delta x) \approx f(x) + \mathbf{f'(x)\Delta x}$. The error is of order $(\Delta x)^p$ or $(x - a)^p$ with $p = \mathbf{2}$. The differential dy equals $\mathbf{dy/dx}$ times the differential $\mathbf{dx}$. Those movements are along the **tangent line**, where Δy is along the **curve**.

2 $f(x) = \frac{1}{x}$ and $a = 2 : Y = f(a) + f'(a)(x - a) = \frac{1}{2} - \frac{1}{4}(x - 2)$. Tangent line is $Y = 1 - \frac{1}{4}x$.

6 $f(x) = \sin^2 x$ and $a = 0 : Y = \sin^2 0 + 2\sin 0 \cos 0(x - 0) = 0$. Tangent line is x axis.

10 $f(x) = x^{1/4}, Y = 16^{1/4} + \frac{1}{4}16^{-3/4}(15.99 - 16) = 2 + \frac{1}{48}(-.01) = 1.9996875$. Compare $15.99^{1/4} = 1.9996874$.

18 Actual error: $\sqrt{8.99} - (3 + \frac{1}{6}(-.01)) = -4.6322 \ 10^{-7}$; predicted error for $f = \sqrt{x}, f'' = \frac{-1}{4x^{3/2}}$ near

   $x = 9 : \frac{1}{2}(.01)^2(\frac{-1}{4(9)^{3/2}}) = -4.6296 \ 10^{-7}$.

26 $V = \pi r^2 h$ so $dV = \pi r^2 dh = \pi(2)^2(0.5) = \pi(.2)$.

3.2   Maximum and Minimum Problems   (page 103)

Here is the outstanding application of differential calculus. There are three steps: *Find the function, find its derivative, and solve* $f'(x) = 0$. The first step might come from a word problem – you have to choose a good variable x and find a formula for $f(x)$. The second step is calculus – to produce the formula for $f'(x)$. This may be the easiest step. The third step is fast or slow, according as $f'(x) = 0$ can be solved by a little algebra or a lot of computation. Every textbook gives problems where algebra gets the answer – so you see how the whole method works. We start with those.

A maximum can also occur at a rough point (where f' is not defined) or at an endpoint. Usually those are easier to locate than the stationary points that solve $f'(x) = 0$. They are not only easier to find, they are easier to forget. In Problems 1 – 4 find stationary points, rough points, and endpoints. Decide whether each of these is a local or absolute minimum or maximum.

1. (This is Problem 3.2.5) The function is $f(x) = (x - x^2)^2$ for $-1 \le x \le 1$. The end values are $f(-1) = 4$ and $f(1) = 0$. There are no rough points.

- The derivative by the power rule is $f' = 2(x - x^2)(1 - 2x) = 2x(1 - x)(1 - 2x)$. Then $f'(x) = 0$ at $x = 0$, $x = 1$, and $x = \frac{1}{2}$. Substitute into $f(x)$ to find $f(0) = 0$, $f(1) = 0$ and $f(\frac{1}{2}) = \frac{1}{16}$. Plot all these *critical points* to see that $(-1, 4)$ is the absolute maximum and $(\frac{1}{2}, \frac{1}{16})$ is a relative maximum. There is a tie $(0,0)$ and $(1,0)$ for absolute minimum.

The next function has two parts with two separate formulas. In such a case watch for a rough point at the "breakpoint" between the parts – where the formula changes.

2. (This is Problem 3.2.8) The two-part function is $f(x) = x^2 - 4x$ for $0 \le x \le 1$ and $f(x) = x^2 - 4$ for $1 \le x \le 2$. The breakpoint is at $x = 1$, where $f(x) = -3$. The endpoints are $(0,0)$ and $(2,0)$.

   - The slope for $0 < x < 1$ is $f'(x) = 2x - 4$. Although $f'(2) = 0$, this point $x = 2$ is not in the interval $0 \le x \le 1$. For $1 < x < 2$ the slope $f'(x) = 2x$ is never zero.

This function has no stationary points. Its derivative is never zero. Still it has a maximum and a minimum! There is a rough point at $x = 1$ because the slope $2x - 4$ from the left does not equal $2x$ from the right. This is the minimum point. The endpoints where $f = 0$ are absolute maxima.

3. $f(x) = 2 + x^{2/3}$ for $-1 \le x \le 8$.

   - The endpoints have $f(-1) = 3$ and $f(8) = 6$. The slope is $f'(x) = \frac{2}{3}x^{1/3}$. Note that $f'(0)$ is not defined, so $x = 0$, $f(x) = 2$ is a rough point. Since $f'(x)$ never equals zero, there are no stationary points. The endpoints are maxima, the rough point is a minimum. All powers x^p with $0 < p < 1$ are zero at $x = 0$ but with infinite slope; x^{p-1} blows up; rough point.

4. $f(x) = 3x^4 - 40x^3 + 150x^2 - 600$ all for x. "For all x" is often written "$-\infty < x < \infty$". No endpoints.

   - The slope is $f'(x) = 12x^3 - 120x^2 + 300x = 12x(x^2 - 10x + 25) = 12x(x - 5)^2$. Then $f'(x) = 0$ at $x = 0$ and $x = 5$. Substituting 0 and 5 into $f(x)$ gives -600 and $+25$. That means $(0, -600)$ and $(5, 25)$ are stationary points. Since $f'(x)$ is defined for all x, there are no rough points.

Discussion: Look more closely at $f'(x)$. The double factor $(x - 5)^2$ is suspicious. The slope $f'(x) = 12x(x - 5)^2$ is positive on both sides of $x = 5$. The stationary point at $x = 5$ is *not* a maximum or minimum, just a "pause point" before continuing upward.

The graph is rising for $x > 0$ and falling for $x < 0$. The bottom is $(0, -600)$ – an absolute minimum. There is no maximum. Or you could say that the maximum value is *infinite*, as $x \to \infty$ and $x \to -\infty$. The maximum is at the "end" even if there are no endpoints.

5. (This is Problem 3.2.26) A limousine gets $(120-2v)/5$ miles per gallon. The chauffeur costs \$10/hour. Gas is \$1/gallon. Find the cheapest driving speed. What is to be minimized?

   - Minimize the *cost per mile*. This is (cost of driver per mile) + (cost of gas per mile). The driver costs $\frac{10 \text{ dollars/hour}}{v \text{ miles/hour}} = \frac{10}{v}$ dollars/mile. The gas costs $\frac{1 \text{ dollar/gallon}}{\frac{120-2v}{5} \text{ miles/gallon}} = \frac{5}{120-2v}$ dollars/mile. Total cost per mile is $C(v) = \frac{10}{v} + \frac{5}{120-2v}$. Note that $0 < v < 60$. The cost blows up at the speed limit $v = 60$. Now that the hard part is over, we do the calculus:

$$\frac{dC}{dv} = \frac{-10}{v^2} + \frac{-5(-2)}{(120 - 2v)^2} = 0 \quad \text{if} \quad \frac{10}{v^2} = \frac{10}{(120 - 2v)^2}. \quad \text{This gives } v^2 = (120 - 2v)^2.$$

Then $v = 40$ or $v = 120$. We reject $v = 120$. The speed with lowest cost is $v = 40$ mph.

6. Form a box with no top by cutting four squares of sides x from the corners of a $12'' \times 18''$ rectangle. What x gives a box with maximum volume?
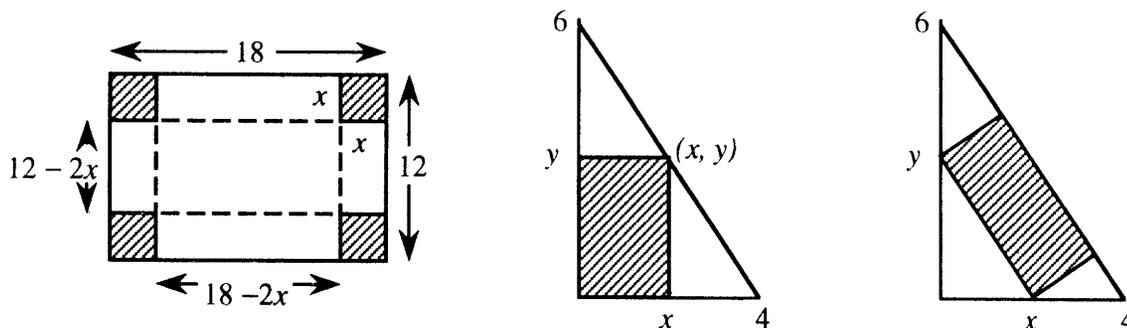
● Cutting out the squares and folding on the dotted lines gives a box x inches high and $12 - 2x$ inches wide and $18 - 2x$ inches long. We want to maximize volume = (height)(width)(length):

$$V(x) = x(12 - 2x)(18 - 2x) = 4x^3 - 60x^2 + 216x.$$

The endpoints are $x = 0$ (no height) and $x = 6$ (no width). Set $\frac{dV}{dx}$ to zero:

$$\frac{dV}{dx} = 12x^2 - 120x + 216 = 12(x^2 - 10x + 18) = 0$$

The quadratic formula gives $x = \frac{10 \pm \sqrt{100 - 4(18)}}{2} = 5 \pm \sqrt{7}$, so that $x \approx 7.6$ or $x \approx 2.35$. Since $x < 6$, the maximum volume is obtained when $x = 5 - \sqrt{7} \approx 2.35$ inches.

7. (This is Problem 3.3.43) A rectangle fits into a triangle with sides $x = 0$, $y = 0$ and $\frac{x}{4} + \frac{y}{6} = 1$. Find the point on the third side which maximizes the area xy. We want to maximize $A = xy$ with $0 < x < 4$ and $0 < y < 6$. Before we can take a derivative, we need to write y in terms of x (or vice versa). The equation $\frac{x}{4} + \frac{y}{6} = 1$ links x and y, and gives $y = 6 - \frac{3}{2}x$. This makes $a = x(6 - \frac{3}{2}x) = 6x - \frac{3}{2}x^2$. Now take the derivative: $\frac{dA}{dx} = 6 - 3x = 0$ when $x = 2$. Then $y = 6 - \frac{3}{2}x = 3$. The point $(2,3)$ gives maximum area 6.

I believe that a tilted rectangle of the same area also fits in the triangle. Correct?

8. Find the minimum distance from the point $(2,4)$ to the parabola $y = \frac{x^2}{8}$.

● Of all segments from $(2,4)$ to the parabola, we would like to find the shortest. The distance is $D = \sqrt{(x - 2)^2 + (\frac{x^2}{8} - 4)^2}$. Because of the square root, $D'(x)$ is algebraically complicated. We can make the problem easier by working with $f(x) = (x - 2)^2 + (\frac{x^2}{8} - 4)^2 = distance\ squared$. If we can minimize $f = D^2$, we have also minimized D. Using the square rule,

$$\frac{df}{dx} = 2(x - 2) + 2(\frac{x^2}{8} - 4)(\frac{x}{4}) = 2x - 4 + \frac{x^3}{16} - 2x = \frac{x^3}{16} - 4.$$

Then $\frac{df}{dx} = 0$ if $x^3 = 64$ or $x = 4$. This gives $y = \frac{x^2}{8} = 2$. The minimum distance is from $(2,4)$ to the point $(4,2)$ on the parabola. The distance is $D = \sqrt{(4 - 2)^2 + (2 - 4)^2} = \sqrt{4 + 4} = 2\sqrt{2}$.

Read-throughs and selected even-numbered solutions :

If $df/dx > 0$ in an interval then $f(x)$ is **increasing**. If a maximum or minimum occurs at x then $f'(x) = $ **0**. Points where $f'(x) = 0$ are called **stationary** points. The function $f(x) = 3x^2 - x$ has a (**minimum**) at $x = \frac{1}{6}$. A stationary point that is not a maximum or minimum occurs for $f(x) = \mathbf{x^3}$.

Extreme values can also occur when $\mathbf{f'(x)}$ is not defined or at the **endpoints** of the domain. The minima of $|x|$ and $5x$ for $-2 \le x \le 2$ are at $x = \mathbf{0}$ and $x = \mathbf{-2}$, even though df/dx is not zero. x^* is an absolute **maximum** when $f(x^*) \ge f(x)$ for all x. A **relative** minimum occurs when $f(x^*) \le f(x)$ for all x near x^*.

The minimum of $\frac{1}{2}ax^2 - bx$ is $\mathbf{-b^2/2a}$ at $x = \mathbf{b/a}$.

18 $f'(x) = \cos x - \sin x = 0$ at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. At those points $f(\frac{\pi}{4}) = \sqrt{2}$, the maximum, and $f(\frac{5\pi}{4}) = -\sqrt{2}$, the minimum. The endpoints give $f(0) = f(2\pi) = 1$.

28 When the length of day has its maximum and minimum, its derivative is zero (no change in the length of day). In reality the time unit of days is discrete not continuous; then Δf is small instead of $df = 0$.

30 $f'(t) = \frac{(1+3t^2)3 - (1+3t)(6t)}{(1+3t^2)^2} = \frac{-9t^2 - 6t + 3}{(1+3t^2)^2}$. Factoring out -3, the equation $3t^2 + 2t - 1 = 0$ gives $t = \frac{-2 + \sqrt{16}}{6} = \frac{1}{3}$. At that point $f_{\max} = \frac{2}{4/3} = \frac{3}{2}$. The endpoints $f(0) = 1$ and $f(\infty) = 0$ are minima.

36 Volume of popcorn box $= x(6-x)(12-x) = 72x - 18x^2 + x^3$. Then $\frac{dV}{dx} = 72 - 36x + 3x^2$. Dividing by 3 gives $x^2 - 12x + 24 = 0$ or $x = 6 \pm \sqrt{36 - 24} = 6 \pm \sqrt{12}$ at stationary points. **Maximum volume is at** $\mathbf{x = 6 - \sqrt{12}}$. ($V$ has a minimum at $x = 6 + \sqrt{12}$, when the box has negative width.)

46 The cylinder has radius r and height h. Going out r and up $\frac{1}{2}h$ brings us to the sphere: $r^2 + (\frac{1}{2}h)^2 = 1$. The volume of the cylinder is $V = \pi r^2 h = \pi[1 - (\frac{1}{2}h)^2]h$. Then $\frac{dV}{dh} = \pi[1 - (\frac{1}{2}h)^2] + \pi(-\frac{1}{2}h)h = 0$ gives $1 = \frac{3}{4}h^2$. The best h is $\frac{\mathbf{2}}{\sqrt{\mathbf{3}}}$, so $V = \pi[1 - \frac{1}{3}]\frac{2}{\sqrt{3}} = \frac{\mathbf{4\pi}}{\mathbf{3\sqrt{3}}}$. Note: $r^2 + \frac{1}{3} = 1$ gives $r = \sqrt{\frac{2}{3}}$.

56 *First method:* Use the identity $\sin x \sin(10 - x) = \frac{1}{2}\cos(2x - 10) - \frac{1}{2}\cos 10$. The maximum when $2x = 10$ is $\frac{1}{2} - \frac{1}{2}\cos 10 = .92$. The minimum when $2x - 10 = \pi$ is $-\frac{1}{2} - \frac{1}{2}\cos 10 = -.08$. *Second method:* $\sin x \sin(10 - x)$ has derivative $\cos x \sin(10 - x) - \sin x \cos(10 - x)$ which is $\sin(10 - x - x)$. This is zero when $10 - 2x$ equals 0 or π. Then $\sin x \sin(10 - x)$ is $(\sin 5)(\sin 5) = .92$ or $\sin(5 + \frac{\pi}{2})\sin(5 - \frac{\pi}{2}) = -.08$.

62 The squared distance $x^2 + (y - \frac{1}{3})^2 = x^2 + (x^2 - \frac{1}{3})^2$ has derivative $2x + 4x(x^2 - \frac{1}{3}) = 0$ at $\mathbf{x = 0}$. Don't just cancel the factor x! The nearest point is $(0,0)$. Writing the squared distance as $x^2 + (y - \frac{1}{3})^2 = y + (y - \frac{1}{3})^2$ we forget that $y = x^2 \ge 0$. Zero is an **endpoint** and it gives the minimum.

3.3   Second Derivatives: Bending and Acceleration   (page 110)

The first derivative gives the slope. The second derivative gives the *change of slope*. When the slope changes, the graph *bends*. When the slope $f'(x)$ does not change, then $f''(x) = 0$. This happens only for straight lines $f(x) = mx + b$.

At a minimum point, the slope is going from $-$ to $+$. Since f' is increasing, f'' must be positive.

At a maximum point, f' goes from $+$ to $-$ so f'' is negative.

At an inflection point, $f'' = 0$ and the graph is momentarily straight.

It makes sense that the approximation to $f(x + \Delta x)$ is better if we include bending. The linear part $f(x) + f'(x)\Delta x$ follows the tangent line. *The term to add is $\frac{1}{2}f''(x)(\Delta x)^2$*. With "$a$" as basepoint this is $\frac{1}{2}f''(a)(x - a)^2$.

The $\frac{1}{2}$ makes this exactly correct when $f = (x-a)^2$, because then $f'' = 2$ cancels the $\frac{1}{2}$.

1. Use $f''(x)$ to decide between maxima and minima of $f(x) = \frac{1}{3}x^3 + x^2 - 3x$.

   - $f'(x) = x^2 + 2x - 3 = (x+3)(x-1)$, so the stationary points are $x = 1$ and $x = -3$. Now compute $f'' = 2x + 2$. At $x = 1$, $f''(x)$ is positive. So this point is a local minimum. At $x = -3$, $f''(x)$ is negative. This point is a local maximum.

2. Find the minimum of $f(x) = x^2 + \frac{54}{x}, x \neq 0$.

   - $f'(x) = 2x - \frac{54}{x^2} = 0$ if $2x^3 = 54$ or $x^3 = 27$. There is a stationary point at $x = 3$. Taking the second derivative we get $f''(x) = 2 + \frac{108}{x^3}$. Since $f''(3) > 0$ the point is a minimum. This test on f'' does not say whether the minimum is relative or absolute.

3. Locate inflection points, if any, of $f(x) = \frac{1}{20}x^5 - \frac{1}{6}x^4 + \frac{1}{6}x^3 - 10x$. Here $f' = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 - 10$.

   - This example has $f''(x) = x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2$. Inflection points require $f''(x) = 0$, which means that $x = 0$ or $x = 1$. But $x = 1$ is not a true inflection point. The double root from $(x-1)^2$ is again suspicious. The sign of f'' does not change as x passes 1. The bending stays positive and the tangent line stays under the curve. At a true inflection point the curve and line cross.

   The only inflection point is at $x = 0$, where f'' goes from negative to positive: bend down then bend up.

4. Write down the quadratic approximation for the function $y = x^4$ near $x = 1$. Use the formula boxed on page 109 with $a = \underline{\hspace{1cm}}, f(a) = \underline{\hspace{1cm}}, f'(x) = \underline{\hspace{1cm}}, f'(a) = \underline{\hspace{1cm}}, f''(x) = \underline{\hspace{1cm}}, f''(a) = 12$.

   - The quadratic approximation is $x^4 \approx 1 + 4(x-1) + 6(x-1)^2$. Note $f(a) = 1, f'(a) = 4, \frac{1}{2}f''(a) = 6$.

Read-throughs and selected even-numbered solutions :

The direction of bending is given by the sign of $f''(x)$. If the second derivative is **positive** in an interval, the function is concave up (or convex). The graph bends **upward**. The tangent lines are **below** the graph. If $f''(x) < 0$ then the graph is concave **down**, and the slope is **decreasing**.

At a point where $f'(x) = 0$ and $f''(x) > 0$, the function has a **minimum**. At a point where $f'(x) = 0$ and $f''(x) < 0$, the function has a maximum. A point where $f''(x) = 0$ is an **inflection** point, provided f'' changes sign. The tangent line **crosses** the graph.

The centered approximation to $f'(x)$ is $[f(x + \Delta x) - f(x - \Delta x)]/2\Delta x$. The 3-point approximation to $f''(x)$ is $[f(x + \Delta x) - 2f(x) + f(x - \Delta x)]/(\Delta x)^2$. The second-order approximation to $f(x + \Delta x)$ is $f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2$. Without that extra term this is just the **linear** (or tangent) approximation. With that term the error is $O((\Delta x)^3)$.

2 We want inflection points $\frac{d^2y}{dx^2} = 0$ at $x = 0$ and $x = 1$. Take $\frac{d^2y}{dx^2} = x - x^2$. This is positive ($y$ is concave up) between 0 and 1. Then $\mathbf{y = \frac{1}{6}x^3 - \frac{1}{12}x^4}$. (Intermediate step: the first derivative is $\frac{1}{2}x^2 - \frac{1}{3}x^3$.)
Alternative: $y = -x^2$ for $x < 0$, then $y = +x^2$ up to $x = 1$, then $y = 2 - x^2$ for $x > 1$.

20 $f'(x) = \cos x + 3(\sin x)^2 \cos x$ gives $f'''(x) = -\sin x - 3(\sin x)^2 \sin x + 6 \sin x(\cos x)^2 = \mathbf{5 \sin x - 9 \sin^3 x}$.
Inflection points where $\sin x = 0$ (at $0, \pi, \cdots$) and also where $(\sin x)^2 = \frac{5}{9}$ (an angle x in each quadrant). Concavity is up-down-up from 0 to π. Then down-up-down from π to 2π.

34 $f(x) = \frac{1}{1-x}$ $\frac{f(\Delta x)-f(0)}{\Delta x} - f'(0)$ $\frac{f(\Delta x)-f(-\Delta x)}{2\Delta x} - f'(0)$ $\frac{f(\Delta x)-2f(0)+f(-\Delta x)}{(\Delta x)^2} - f''(0)$ $1 + x + x^2 - \frac{1}{1-x}$

$\Delta x = 1/4$ $1/3 = .333$ $1/15 \quad = .067$ $2/15 = .133$ $-1/48 = -.021$

$\Delta x = 1/8$ $1/7 = .142$ $1/63 \quad = .016$ $2/63 = .032$ $-1/448 = -.002$

36 At $x = 0.1$ the difference $\frac{1}{.9} - (1.11) = .00111 \cdots$ comes from the omitted terms $x^3 + x^4 + x^5 + \cdots$.

At $x = 2$ the difference is $\frac{1}{1-2} - (1 + 2 + 4) = -8$. This is large because $x = 2$ is far from the basepoint.

42 $f(1) = 3, f(2) = 2 + 4 + 8 = 14, f(3) = 3 + 9 + 27 = 39$. The second difference is $\frac{39-28+3}{1^2} = 14$. The true $f'' = 2 + 6x$ is also 14. The error involves f'''' which in this example is zero.

3.4   Graphs   (page 119)

1. The intercepts or axis crossings of this graph are (_____ , _____) and (_____ , _____).   • (0,0) and (2,0).

2. $f'(x)$ is positive for which x?   • $x < -1$ and $x > \frac{1}{2}$.

3. The stationary point with horizontal tangent is (_____, _____).   • $\left(\frac{1}{2}, -\frac{4}{3}\right)$.
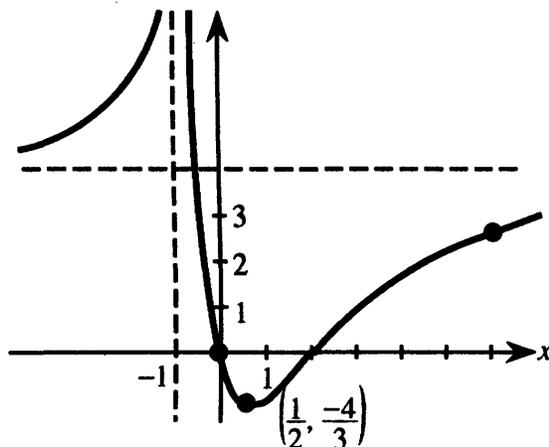
4. $f''(x) > 0$ and the curve bends up when _____.   • $x < 2$ omitting $x = -1$.

5. The inflection point where $f''(x) = 0$ is (_____ , _____).   • $(2, 0)$.

6. As $x \to \infty$, the function $f(x)$ approaches _____ .   • The limit is 4 (from below).

7. As $x \to -\infty$, the function $f(x)$ approaches _____ .   • The limit is 4 (from above).

8. $f(x) \to \infty$ as x approaches _____ .   • $x \to -1$ (from the left and from the right).

9. If this $f(x)$ is a ratio of polynomials, the denominator must have what factor?   • $(x + 1)^2$ or $(x + 1)^4 \ldots$

   Note $\frac{1}{x+1}$ causes blowup at $x = -1$. When it is squared the function goes to $+\infty$ on both sides of $x = -1$.

10. The degree of the numerator is $(<, =, >)$ the degree of the denominator?   • *Equal* degrees since $y \to 4$.

   Questions 11 – 13 are based on the graph below, which is drawn for $0 < x < 6$. In each question you complete the graph for $-6 < x < 0$.

11. Make $f(x)$ periodic with period 6.   12. *Make $f(x)$ even.*   13. *Make $f(x)$ odd.*
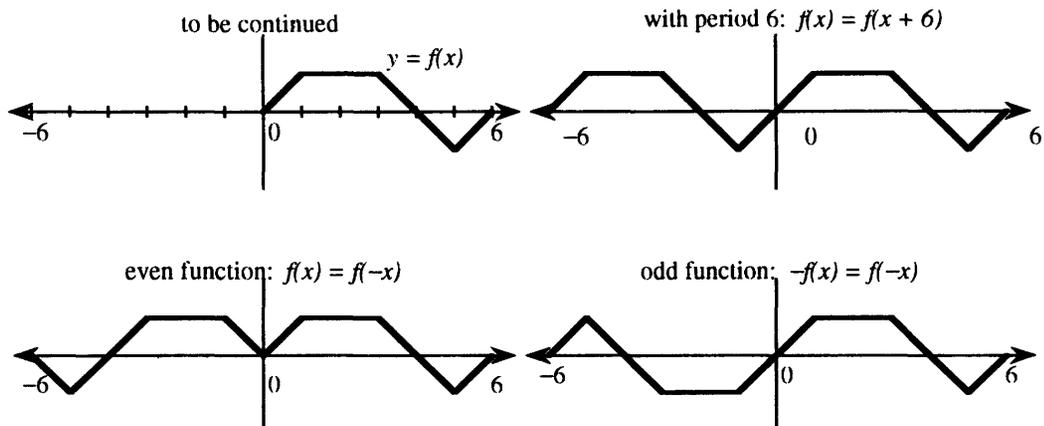
Problems 14 - 17 are designed to be done by hand. To graph a function from its equation, you need to think about x and y intercepts, asymptotes, and behavior as $x \to \infty$ and $x \to -\infty$. Consider also whether the function is even or odd. Use the information given by derivatives. (And it helps to plot a few points!)

14. Sketch $y = \frac{x}{x+1}$. Especially find the asymptotes.

   - At $x = 0$ we find the only intercept $y = 0$. Since $x = -1$ makes the denominator zero, the line $x = -1$ is a *vertical asymptote*. To find horizontal asymptotes, let x get very large. Then $\frac{x}{x+1}$ gets very near to $y = 1$. Thus $y = 1$ is a *horizontal asymptote* (also as $x \to -\infty$). The function is neither even nor odd since $f(-x) = \frac{-x}{-x+1}$ is neither $f(x)$ nor $-f(x)$. The slope is $f'(x) = \frac{1}{(x+1)^2}$, so the graph is always rising. Since $f''(x) = \frac{-2}{(x+1)^3}$ the graph is concave up when $x < -1$ and concave down when $x > -1$.
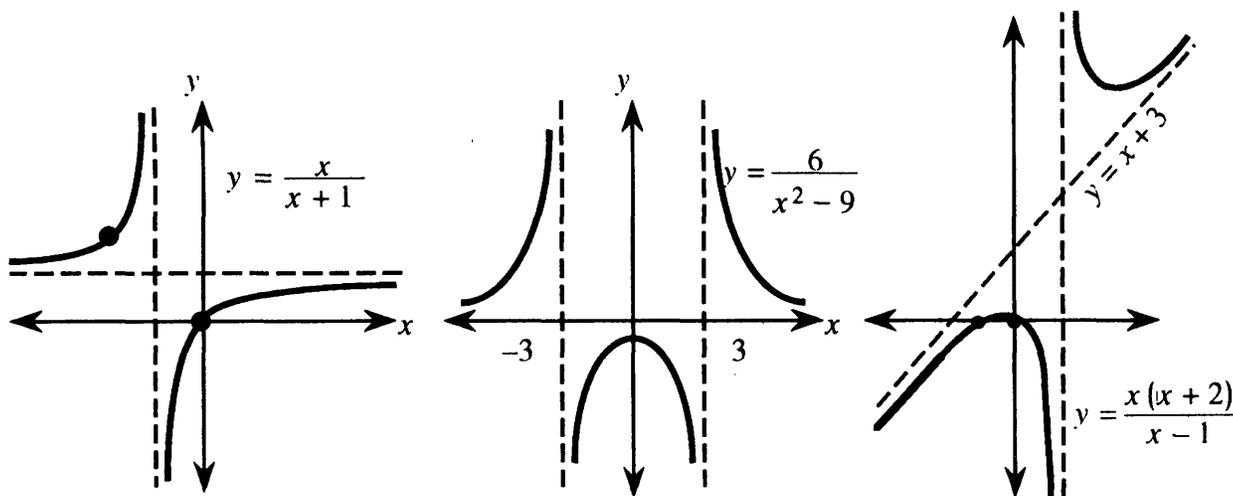
15. Sketch $y = \frac{6}{x^2-9}$. Explain why this function is even.

- Vertical asymptotes are $x = 3$ and $x = -3$, when the denominator $x^2 - 9$ is zero. The graph passes through $x = 0$, $y = -\frac{6}{9}$. There is no solution $y = 0$. In fact $y = 0$ is a horizontal asymptote (because $x^2 - 9 \to +\infty$). Note $f(-x) = \frac{6}{(-x)^2-9} = \frac{6}{x^2-9} = f(x)$ so this function is *even*. Its graph is symmetric about the y axis. The derivative is $f'(x) = -12x(x^2 - 9)^{-2}$, so there is a stationary point at $x = 0$. That point is a local maximum. *Even functions have odd derivatives. Always max or min at $x = 0$.*

16. Sketch $y = \frac{x^2-4}{x+2}$. It has a straight graph!

- $x = -2$ is *not* an asymptote even though the denominator is zero! We can factor $x^2 - 4$ into $(x-2)(x+2)$ and cancel $x + 2$ from the top and bottom. (This is illegal at the point where $x + 2 = 0$ and we have $\frac{0}{0}$.) The graph is the *straight* line $y = x - 2$ with a hole at $x = -2, y = -4$.

17. Sketch $y = \frac{x(x+2)}{(x-1)}$. Why will it have a sloping asymptote?

- There are intercepts at $(0,0)$ and $(-2,0)$. The vertical asymptote is $x = 1$ where the denominator is zero. The numerator has greater degree (2 versus 1) so we divide:

$$y = \frac{x^2 + 2x}{x-1} = x + 3 + \frac{3}{x-1}. \text{ (You should redo that division.)}$$

As x gets large, the last fraction is small. The line $y = x + 3$ is a *sloping asymptote*. (Sloping asymptotes come when "top degree = 1 + bottom degree.") The derivative is $\frac{dy}{dx} = 1 - \frac{3}{(x-1)^2}$, so there are stationary points at $(x - 1)^2 = 3$ or $x = 1 \pm \sqrt{3}$. Since $\frac{d^2y}{dx^2} = 6(x - 1)^{-3}$, the graph is concave up if $x > 1$ and concave down if $x < -1$.

Read-throughs and selected even-numbered solutions :

The position, slope, and bending of $y = f(x)$ are decided by $\mathbf{f(x), f'(x)}$, and $\mathbf{f''(x)}$. If $|f(x)| \to \infty$ as $x \to a$, the line $x = a$ is a vertical **asymptote**. If $f(x) \to b$ for large x, then $x = b$ is a **horizontal asymptote**. If $f(x) - mx \to b$ for large x, then $y = mx + b$ is a **sloping asymptote**. The asymptotes of $y = x^2/(x^2 - 4)$ are $\mathbf{x = 2, x = -2, y = 1}$. This function is even because $y(-x) = \mathbf{y(x)}$. The function $\sin kx$ has period $\mathbf{2\pi/k}$.

Near a point where $dy/dx = 0$, the graph is extremely flat. For the model $y = Cx^2, x = .1$ gives $y = .01C$. A box around the graph looks long and thin. We **zoom** in to that box for another digit of x^*. But solving $dy/dx = 0$ is more accurate, because its graph **crosses** the x axis. The slope of dy/dx is $\mathbf{d^2y/dx^2}$. Each derivative is like an infinite **zoom**.

To move (a, b) to $(0,0)$, shift the variables to $X = \mathbf{x - a}$ and $Y = \mathbf{y - b}$. This **centering** transform changes $y = f(x)$ to $Y = -b + f(X + a)$. The original slope at (a, b) equals the new slope at $(0, 0)$. To stretch the axes by c and d, set $\mathbf{x} = cX$ and $\mathbf{y} = dY$. The **zoom** transform changes $Y = F(X)$ to $\mathbf{y = dF(x/c)}$. Slopes are multiplied by $\mathbf{d/c}$. Second derivatives are multiplied by $\mathbf{d/c^2}$.

12 $\frac{x}{\sin x}$ is **even**. Vertical asymptotes at all multiples $\mathbf{x = n\pi}$, except at $x = 0$ where $f(0) = 1$.

16 $\frac{\sin x + \cos x}{\sin x - \cos x}$ is periodic, not odd or even, vertical asymptotes when $\sin x = \cos x$ at $x = \frac{\pi}{4} + n\pi$.

22 $f(x) = \frac{1}{x} + 2x + 3$

30 (a) False: $\frac{x^4}{x^2+1}$ has no asymptotes (b) True: the second difference on page 108 is even (c) False: $f(x) = 1 + x$ is not even but $f'' = 0$ is even (d) False: $\tan x$ has vertical asymptotes but $\sec^2 x$ is never zero.

38 This is the second difference $\frac{\Delta^2 f}{\Delta x^2} \approx (\sin x)'' = -\sin x$.

40 (a) The asymptotes are $y = 0$ and $x = -3.48$. (b) The asymptotes are $y = 1$ and $x = 1$ (double root).

42 The exact solution is $x^* = \sqrt{3} = 1.73205$. The zoom should find those digits.

48 The exact $x = \sqrt{5}$ solves $\frac{x}{30\sqrt{15+x^2}} = \frac{1}{60}$ or $(2x)^2 = 15 + x^2$ or $x^2 = 5$.

52 $\sqrt{3x+1}$ is defined only for $x \geq -\frac{1}{3}$ (local maximum); minimum near $x = .95$; no inflection point.

58 Inflection points and second derivatives are **harder** to compute than maximum points and first derivatives. (In examples, derivatives seem easier than integrals. For numerical computation **it is the other way:** derivatives are very sensitive, integrals are smooth.)

3.5   Parabolas, Ellipses, and Hyperbolas   (page 128)

This section is about "second degree equations," with x^2 and xy and y^2. In geometry, a plane cuts through a cone – therefore *conic* section. In algebra, we try to write the equation so the special points of the curve can be recognized. Also the curve itself has to be recognized – as circle, ellipse, parabola, or hyperbola.

The special point of a circle is its *center*. The special points for an ellipse are the two *foci*. A parabola has only one focus (the other one is at infinity) and we emphasize the *vertex*. A hyperbola has two foci and two vertices and two branches and frequently a *minus sign* in a critical place. But $B^2 - 4AC$ is positive.

1. Write the equation of a parabola with vertex at (1,3) and focus at (1,1).

   • The focus is below the vertex, so the parabola opens downward. The distance from the vertex to the focus is $a = -2$, so $\frac{1}{4a} = -\frac{1}{8}$. The equation is $(y - 3) = -\frac{1}{8}(x - 1)^2$. Notice how the vertex is at (1,3) when the equation has $x - 1$ and $y - 3$. The vertex is at (0,0) when the equation is $y = \frac{1}{4a}x^2$.

2. $4(x + 3)^2 + 9(y - 2)^2 = 36$ is the equation for an ellipse. Find the center and the two foci.

   • Divide by 36 to make the right side equal 1: $\frac{(x+3)^2}{9} + \frac{(y-2)^2}{4} = 1$. We see an ellipse centered at $(-3, 2)$. It has $a = \sqrt{9}$ and $b = \sqrt{4}$. From $c^2 = a^2 - b^2$ we calculate $c = \sqrt{9 - 4} = \sqrt{5}$. Then the foci are left and right of the center at $(-3 - \sqrt{5}, 2)$ and $(-3 + \sqrt{5}, 2)$.

3. $9x^2 - 16y^2 + 36x + 32y - 124 = 0$ is the equation of a hyperbola. Find the center, vertices, and foci.

   • The x^2 and y^2 terms have opposite signs, which indicates a hyperbola. Regroup the terms:

   $$9x^2 + 36x - 16y^2 + 32y = 124 \quad \text{or} \quad 9(x^2 + 4x + \underline{\quad}) - 16(y^2 - 2y + \underline{\quad}) = 124.$$

   The blanks are for completing the squares. They should be filled by 4 and 1 to give $(x + 2)^2$ and $(y - 1)^2$. We must also add 9×4 and -16×1 to the right side: $9(x + 2)^2 - 16(y - 1)^2 = 124 + 36 - 16 = 144$. Dividing by 144 gives the standard form $\frac{(x+2)^2}{16} - \frac{(y-1)^2}{9} = 1$.

   Now read off the main points. Center at $(-2, 1)$. Lengths $a^2 = 16$, $b^2 = 9$, $c = \sqrt{16 + 9} = 5$. Vertices are $a = 4$ from the center, at $(-6, 1)$ and (2,1). Foci are $c = 5$ from the center, at $(-7,1)$ and (3,1).

4. (This is Problem 3.5.33) Rotate the axes of $x^2 + xy + y^2 = 1$. Use equation (7) with $\sin \alpha = \cos \alpha = \frac{1}{\sqrt{2}}$.

   • The rotation is $x = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'$ and $y = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$. The original equation involves x^2 and y^2 and xy so we compute

$$x^2 = \frac{1}{2}(x')^2 - x'y' + \frac{1}{2}(y')^2 \text{ and } y^2 = \frac{1}{2}(x')^2 + x'y' + \frac{1}{2}(y')^2 \text{ and } xy = \frac{1}{2}(x')^2 - \frac{1}{2}(y')^2.$$

Substitution changes $x^2 + xy + y^2 = 1$ into $\frac{3}{2}(x')^2 + \frac{1}{2}(y')^2 = 1$. This is $\frac{(x')^2}{\frac{2}{3}} + \frac{(y')^2}{2} = 1$, an ellipse.

5. $x - 2y^2 + 12y - 14 = 0$ is the equation for a parabola. Find the vertex and focus.

 - There is no xy term and no x^2 term, so we have a sideways parabola with a horizontal axis. Group the y terms: $2y^2 - 12y = x - 14$. Divide by 2 to get $y^2 - 6y = \frac{1}{2}x - 7$. Add 9 to both sides so that $y^2 - 6y + 9 = (y-3)^2$ is a perfect square. Then $(y-3)^2 = \frac{1}{2}x + 2 = \frac{1}{2}(x+4)$. The vertex of this parabola is at $x = -4$, $y = 3$. At the vertex the equation becomes $0 = 0$.

 The form $X = \frac{1}{4a}Y^2$ is $x + 4 = 2(y-3)^2$. Thus $\frac{1}{4a} = 2$. The focus is $a = \frac{1}{8}$ from the vertex.

6. What is the equation of a hyperbola having a vertex at $(7,1)$ and foci at $(-2,1)$ and $(8,1)$?

 - The center must be at $(3, 1)$, halfway between the foci. The distance from the center to each focus is $c = 5$. The distance to $(7, 1)$ is $a = 4$, and so the other vertex must be at $(-1,1)$. Finally $b = \sqrt{c^2 - a^2} = 3$. Therefore the equation is $\frac{(x-3)^2}{4^2} - \frac{(y-1)^2}{3^2} = 1$.

 The negative sign is with y^2 because the hyperbola opens horizontally. The foci are on the line $y = 1$.

7. Find the ellipse with vertices at $(0, 0)$ and $(10, 0)$, if the foci are at $(1,0)$ and $(9,0)$.

 - The center must be at $(5, 0)$. The distances from it are $a = 5$ and $c = 4$. Since an ellipse has $a^2 = b^2 + c^2$ we find $b = 3$. The equation is $\frac{(x-5)^2}{5^2} + \frac{y^2}{3^2} = 1$. Notice $x - 5$ and $y - 0$.

Read-throughs and selected even-numbered solutions :

The graph of $y = x^2 + 2x + 5$ is a **parabola**. Its lowest point (the vertex) is $(x,y) = (-1,4)$. Centering by $X = x + 1$ and $Y = y - 4$ moves the vertex to $(0,0)$. The equation becomes $Y = \mathbf{X^2}$. The focus of this centered parabola is $(0, \frac{1}{4})$. All rays coming straight down are **reflected** to the focus.

The graph of $x^2 + 4y^2 = 16$ is an **ellipse**. Dividing by **16** leaves $x^2/a^2 + y^2/b^2 = 1$ with $a = \mathbf{4}$ and $b = \mathbf{2}$. The graph lies in the rectangle whose sides are $\mathbf{x = \pm 4}, \mathbf{y = \pm 2}$. The area is $\pi ab = \mathbf{8\pi}$. The foci are at $x = \pm c = \pm\sqrt{\mathbf{12}}$. The sum of distances from the foci to a point on the ellipse is always **8**. If we rescale to $X = x/4$ and $Y = y/2$ the equation becomes $\mathbf{X^2 + Y^2 = 1}$ and the graph becomes a **circle**.

The graph of $y^2 - x^2 = 9$ is a **hyperbola**. Dividing by 9 leaves $y^2/a^2 - x^2/b^2 = 1$ with $a = \mathbf{3}$ and $b = \mathbf{3}$. On the upper branch $y \geq 0$. The asymptotes are the lines $\mathbf{y = \pm x}$. The foci are at $y = \pm c = \pm\sqrt{\mathbf{18}}$. The **difference** of distances from the foci to a point on this hyperbola is **6**.

All these curves are conic sections – the intersection of a **plane** and a **cone**. A steep cutting angle yields a **hyperbola**. At the borderline angle we get a **parabola**. The general equation is $Ax^2 + \mathbf{B}xy + \mathbf{C}y^2 + \mathbf{D}x + \mathbf{E}y + \mathbf{F} = 0$. If $D = E = 0$ the center of the graph is at $(0,0)$. The equation $Ax^2 + Bxy + Cy^2 = 1$ gives an ellipse when $\mathbf{4AC > B^2}$. The graph of $4x^2 + 5xy + 6y^2 = 1$ is an **ellipse**.

14 $xy = 0$ gives the two lines $x = 0$ and $y = 0$, a degenerate hyperbola with vertices and foci all at $(0,0)$.

16 $y = x^2 - x$ has vertex at $(\frac{1}{2}, -\frac{1}{4})$. To move the vertex to $(0,0)$ set $X = x - \frac{1}{2}$ and $Y = y + \frac{1}{4}$. Then $Y = X^2$.

20 The path $x = t$, $y = t - t^2$ starts with $\frac{dx}{dt} = \frac{dy}{dt} = 1$ at $t = 0$ (45° angle). Then $y_{max} = \frac{1}{4}$ at $t = \frac{1}{2}$. The path is the parabola $y = x - x^2$.

32 The square has side s if the point $(\frac{s}{2}, \frac{s}{2})$ is on the ellipse. This requires $\frac{1}{a^2}(\frac{s}{2})^2 + \frac{1}{b^2}(\frac{s}{2})^2 = 1$ or $s^2 = 4(\frac{1}{a^2} + \frac{1}{b^2})^{-1} =$ area of square.

34 The Earth has $a = 149,597,870$ kilometers (Problem 19 on page 469 says $1.5 \cdot 10^8$ km). The eccentricity $e = \frac{c}{a}$ is 0.167 (or .02 on page 356). Then $c = 2.5 \cdot 10^6$ and $b = \sqrt{a^2 - c^2}$. This b is very near a; our orbit is nearly a circle. Use $\sqrt{a^2 - c^2} \approx a - \frac{c^2}{2a} \approx a - 2 \cdot 10^4$ km.

40 Complete squares: $y^2 + 2y = (y + 1)^2 - 1$ and $x^2 + 10x = (x + 5)^2 - 25$. Then $Y = y + 1$ and $X = x + 5$ satisfy $Y^2 - 1 = X^2 - 25$: the hyperbola is $\mathbf{X^2 - Y^2 = 24}$.

46 The quadratic $ax^2 + bx + c$ has two real roots if $b^2 - 4ac$ is positive and no real roots if $b^2 - 4ac$ is negative. Equal roots if $b^2 = 4ac$.

3.6   Iterations $x_{n+1} = F(x_n)$ (page 136)

This is not yet a standard topic in calculus. Or rather, the title "Iterations" is not widely used. Certainly Newton's method with $x_1 = F(x_0)$ and $x_2 = F(x_1)$ is always taught (as it should be). Other iterations should be presented too!

Main ideas: The *fixed points* satisfy $x^* = F(x^*)$. The iteration is fixed there if it starts there. This fixed point is *attracting* if $|F'(x^*)| < 1$. It is *repelling* if $|F'(x^*)| > 1$. The derivative F' at the fixed point decides whether $x_{n+1} - x^*$ is smaller than $x_n - x^*$, which means attraction. The "cobweb" spirals inward.

We gave iteration a full section because it is a very important application of calculus.

1. Problem 3.6.5 studies $x_{n+1} = 3x_n(1 - x_n)$. Start from $x_0 = .6$ and $x_0 = 2$. Compute $x_1, x_2, x_3, \cdots$ to test convergence. Then check dF/dx at the fixed points. Are they attracting or repelling?

   - This is one of the many times when a calculator comes in handy. From $x_0 = 0.6$ it gives $x_1 = 3(0.6)(1 - 0.6) = 0.72$ and $x_2 = 3(0.72)(1 - 0.72) = 0.6048$. On the TI-81 I use 3 ANS (1-ANS) and just push ENTER. Then $x_3 = 0.71705088$ and eventually $x_6 = 0.6118732$. Many iterations lead to $x = 0.6666\cdots$ (this looks like $x^* = \frac{2}{3}$). On the other hand $x_0 = 2$ leads to $x_1 = 3(2)(1 - 2) = -6$ and $x_2 = 3(6)(1 - (-6)) = 126$ and $x_3 = 3(126)(1 - 126) = -47250$. The sequence is diverging.

     To find the fixed points algebraically, solve $x^* = 3x^*(1 - x^*)$. This gives $x^* = 3x^* - 3(x^*)^2$, or $3(x^*)^2 = 2x^*$. The solutions $x^* = 0$ and $x^* = \frac{2}{3}$ are fixed points. If we start at $x_0 = 0$, then $x_1 = 0$ and we stay at 0. If we start at $x_0 = \frac{2}{3}$, then $x_1 = \frac{2}{3}$ and we stay at that fixed point.

     To decide whether the fixed points attract or repel, take the derivative of $F(x) = 3x(1 - x)$. This is $3 - 6x$. At $x^* = 0$ the derivative is 3, so $x^* = 0$ is repelling. At $x^* = \frac{2}{3}$ the derivative is -1 so we are on the borderline. That value $|F'| = 1$ is why the calculator showed very slow attraction to $x^* = \frac{2}{3}$.

2. (This is Problem 3.6.15) Solve $x = \cos\sqrt{x}$ by iteration. Get into radian mode!

   - By sketching $y = x$ and $y = \cos\sqrt{x}$ together you can see that there is just one solution, somewhere in the interval $\frac{1}{2} < x^* < \frac{3}{4}$. Pick a starting point like $x_0 = 0.6$. Press $\sqrt{}$, cos, = and repeat. On the TI-81 press cos, $\sqrt{}$, ANS. Then ENTER many times to reach three-place accuracy $x^* = 0.679$.

3. Write out the first few steps of Newton's method for solving $x^3 - 7 = 0$.

   - Note $x^* = \sqrt[3]{7}$ so $x_0 = 2$ is a good place to start. Follow the steps in this table:

	x_n	$f(x_n) = x_n^3 - 7$	$c = \frac{1}{f'(x_n)} = \frac{1}{3(x_n)^2}$	$x_{n+1} = x_n - cf(x_n)$
$n = 0$	2	1	$\frac{1}{12}$	$2 - 1(\frac{1}{12}) = \frac{23}{12}$
$n = 1$	$\frac{23}{12}$	0.041088	0.0907372	1.9129385
$n = 2$	1.9129385	0.00007987	0.0910912	1.9129312

Since x_2 and x_3 begin with 1.9129, those digits are almost certainly correct in $x^* = \sqrt[3]{7} = 1.912931\cdots$.

4. (This is Problem 3.6.26) Show that both fixed points of $x_{n+1} = x_n^2 + x_n - 3$ are repelling.

- The fixed points are solutions of $x^* = (x^*)^2 + x^* - 3$. This gives $(x^*)^2 - 3 = 0$ and $x^* = \sqrt{3}$ or $-\sqrt{3}$. Let $F(x) = x^2 + x - 3$. Then $F'(x) = 2x + 1$. Since $F'(\sqrt{3}) = 2\sqrt{3} + 1 > 1$ and $F'(-\sqrt{3}) = -2\sqrt{3} + 1 < -1$, both fixed points are repelling.

The iteration has nowhere to settle down. It probably blows up but it might cycle and it might be chaotic.

Read-throughs and selected even-numbered solutions :

$x_{n+1} = x_n^3$ describes an **iteration**. After one step $x_1 = \mathbf{x_0^3}$. After two steps $x_2 = F(x_1) = \mathbf{x_1^3} = \mathbf{x_0^9}$. If it happens that input = output, or $x^* = \mathbf{F(x^*)}$, then x^* is a **fixed** point. $F = x^3$ has **three** fixed points, at $x^* = \mathbf{0, 1}$ and $- \mathbf{1}$. Starting near a fixed point, the x_n will converge to it if $|\mathbf{F'(x^*)}| < 1$. That is because $x_{n+1} - x^* = F(x_n) - F(x^*) \approx \mathbf{F'(x^*)(x_n - x^*)}$. The point is called **attracting**. The x_n are repelled if $|\mathbf{F'(x^*)}| > \mathbf{1}$. For $F = x^3$ the fixed points have $F' = \mathbf{0}$ **or** $\mathbf{3}$. The cobweb goes from (x_0, x_0) to (x_0, x_1) to $(\mathbf{x_1}, \mathbf{x_1})$ and converges to $(x^*, x^*) = (\mathbf{0, 0})$. This is an intersection of $y = x^3$ and $y = \mathbf{x}$, and it is super-attracting because $\mathbf{F'} = \mathbf{0}$.

$f(x) = 0$ can be solved iteratively by $x_{n+1} = x_n - cf(x_n)$, in which case $F'(x^*) = \mathbf{1} - \mathbf{cf'(x^*)}$. Subtracting $x^* = x^* - cf(x^*)$, the error equation is $x_{n+1} - x^* \approx m(\mathbf{x_n - x^*})$. The multiplier is $m = \mathbf{1 - cf'(x^*)}$. The errors approach zero if $-1 < m < 1$. The choice $c_n = 1/f'(\mathbf{x^*})$ produces **Newton's** method. The choice $c = 1$ is "**successive substitution**" and $c = 1/f'(\mathbf{x_0})$ is **modified** Newton. Convergence to x^* is **not** certain.

We have three ways to study iterations $x_{n+1} = F(x_n)$: (1) compute $x_1, x_2, \cdots$ from different x_0 (2) find the fixed points x^* and test $|dF/dx| < 1$ (3) draw cobwebs.

10 $x_0 = -1, x_1 = 1, x_2 = -1, x_3 = 1, \cdots$ The double step $x_{n+2} = x_n^9$ has fixed points $x^* = (x^*)^9$, which allows $x^* = 1$ and $x^* = -1$.

18 At $x^* = (a - 1)/a$ the derivative $f' = a - 2ax$ equals $f'(x^*) = a - 2(a - 1) = 2 - a$. Convergence if $|F'(x^*)| < 1$ or $1 < a < 3$. (For completeness check $a = 1$: convergence to zero. Also check $a = 3$: with $x_0 = .66666$ my calculator gives back $x_2 = .66666$. Apparently period 2.)

20 $x^* = (x^*)^2 - \frac{1}{2}$ gives $x_+^* = \frac{1 + \sqrt{3}}{2}$ and $x_-^* = \frac{1 - \sqrt{3}}{2}$. At these fixed points $F' = 2x^*$ equals $1 + \sqrt{3}$ (greater than 1 so x_+^* repels) and $F'(x_-^*) = 1 - \sqrt{3}(x_-^*$ attracts). Cobwebs show convergence to x_-^* if $|x_0| < |x_+^*|$, convergence to x_+^* if $|x_0| = |x_+^*|$, divergence to ∞ if $|x_0| > |x_+^*|$.

26 The fixed points satisfy $x^* = (x^*)^2 + x^* - 3$ or $(x^*)^2 = 3$; thus $x^* = \sqrt{3}$ or $x^* = -\sqrt{3}$. The derivative $2x^* + 1$ equals $2\sqrt{3} + 1$ or $-2\sqrt{3} + 1$; both have $|F'| > 1$. The iterations blow up.

28 (a) Start with $x_0 > 0$. Then $x_1 = \sin x_0$ is less than x_0. The sequence $x_0, \sin x_0, \sin(\sin x_0) \cdots$ decreases to zero (**convergence**: also if $x_0 < 0$.) On the other hand $x_1 = \tan x_0$ is larger than x_0. The sequence $x_0, \tan x_0, \tan(\tan x_0), \cdots$ is increasing (slowly repelled from 0). Since $(\tan x)' = \sec^2 x \geq 1$ there

is no attractor (**divergence**). (b) F'' is $(\sin x)'' = -\sin x$ and $(\tan x)'' = 2\sec^2 x \tan x$.

Theory: When F'' changes from $+$ to $-$ as x passes x_0, the curve stays closer to the axis than the 45° line (convergence). Otherwise divergence. See Problem 22 for $F'' = 0$.

42 The graphs of $\cos x, \cos(\cos x), \cos(\cos(\cos x))$ are approaching the **horizontal line** $y = .7391\cdots$ (where $x^* = \cos x^*$). For every x this number is the limit.

3.7   Newton's Method and Chaos   (page 145)

1. Use Newton's method to approximate $\sqrt[4]{20} = (20)^{1/4}$. Choose $f(x) = x^4 - 20$ with $f'(x) = 4x^3$.

$$\textit{Newton's method is} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 20}{4x_n^3}.$$

It helps to simplify the expression on the right:

$$x_{n+1} = \frac{4x_n^4 - (x_n^4 - 20)}{4x_n^3} = \frac{3x_n^4 + 20}{4x_n^3} = \frac{3}{4}x_n + \frac{5}{x_n^3}.$$

Choose a starting point, say $x_0 = 2$. Then $x_1 = \frac{3}{4}(2) + \frac{5}{2^3} = \frac{3}{2} + \frac{5}{8} = 2.125$. The next point is $x_2 = \frac{3}{4}(2.125) + \frac{5}{(2.125)^3} = 2.1148$. Then $x_3 = 2.11474$. Comparing x_2 and x_3 indicates that $\sqrt[4]{20} = 2.115$ is correct to three places.

2. (This is Problem 3.7.4) Show that Newton's method for $f(x) = x^{1/3} = 0$ gives $x_{n+1} = 2x_n$. Draw a graph to show the iterations. They don't converge to $x^* = 0$.

   ● Newton's method for the equation $x^{1/3} = 0$ gives $x_{n+1} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}}$. The fraction is made simpler by multiplying top and bottom by $3x_n^{2/3}$ so $x_{n+1} = x_n - \frac{3x_n}{1} = -2x_n$. Every step just multiplies by -2. The tangent line at each x hits the axis at the next point $-2x$.
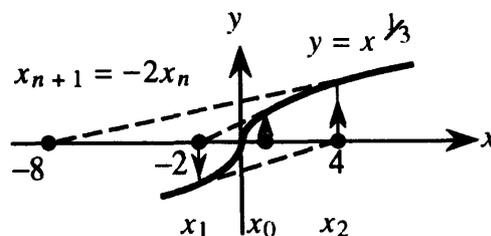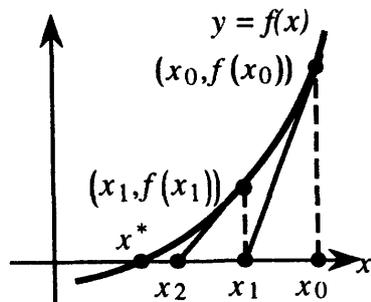
3. The *secant method* is good for those occasions when you don't know $f'(x)$. Use this method to solve $\sin x = 0.5$. (Of course we do know $(\sin x)' = \cos x!$). Start with $x_0 = 0.5$ and $x_1 = 0.6$ (*two* start points).

   ● With $f(x) = \sin x - 0.5$, this table gives x_{n+1} by the secant method (formula at end of the line)

n	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	$(\frac{\Delta f}{\Delta x})_n = \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}$	$x_{n+1} = x_n - \frac{f(x_n)}{(\frac{\Delta f}{\Delta x})_n}$
1	0.5	−0.0206	0.6	0.0646	0.8522	0.5241
2	0.6	0.0646	0.5241	0.00047	0.8449	0.5236
3	0.5241	0.00047	0.5236	−.000001	0.8659	0.5236

Since $x_3 = x_4 = 0.5236$, this should solve $\sin x = 0.5$ to four places. Actually $\sin^{-1} .5 = \frac{\pi}{6} \approx 0.523599$.

***Read-throughs and selected even-numbered solutions* :**

When $f(x) = 0$ is linearized to $f(x_n) + f'(x_n)(x - x_n) = 0$, the solution $x = \mathbf{x_n - f(x_n)/f'(x_n)}$ is Newton's x_{n+1}. The tangent line to the curve crosses the axis at x_{n+1}, while the curve crosses at x^*. The errors at x_n and x_{n+1} are normally related by $(\text{error})_{n+1} \approx M(\text{error})_{\mathbf{n}}^{\mathbf{2}}$. This is **quadratic** convergence. The number of correct decimals **doubles** at every step.

For $f(x) = x^2 - b$, Newton's iteration is $x_{n+1} = \frac{1}{2}(\mathbf{b} + \frac{\mathbf{x_n}}{\mathbf{b}})$. The x_n converge to $\sqrt{b}$ if $x_0 > 0$ and to $-\sqrt{b}$ if $x_0 < 0$. For $f(x) = x^2 + 1$, the iteration becomes $x_{n+1} = \frac{1}{2}(\mathbf{x_n - x_n^{-1}})$. This cannot converge to $i = \sqrt{-1}$. Instead it leads to chaos. Changing to $z = 1/(x^2 + 1)$ yields the parabolic iteration $z_{n+1} = \mathbf{4z_n - 4z_n^2}$.

For $a \le 3, z_{n+1} = az_n - az_n^2$ converges to a single **fixed point**. After $a = 3$ the limit is a 2-cycle, which means that the z's **alternate** between two values. Later the limit is a **Cantor set**, which is a one-dimensional example of a **fractal**. The Cantor set is **self-similar**.

2 $f(x) = \frac{x-1}{x+1}$ has $f'(x) = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$ so Newton's formula is $x_{n+1} = x_n - \frac{(x_n+1)^2}{2}\frac{x_n-1}{x_n+1} =$
$x_n - \frac{x_n^2-1}{2}$. The fixed points of this F satisfy $x^* = x^* - \frac{(x^*)^2-1}{2}$ which gives $x^* = 1$ and $x^* = -1$.
The derivatives $F' = 1 - x^*$ are 0 and -2. So the sequence approaches $x^* = 1$, the correct zero of $f(x)$.

6 $f(x) = x^3 - 3x - 1 = 0$: roots near 1.9, $-.5, -1.6$

10 Newton's method for $f(x) = x^4 - 100$ approaches $x^* = \sqrt{10}$ if $x_0 > 0$ and $x^* = -\sqrt{10}$ if $x_0 < 0$.
In this case the error at step $n + 1$ equals $\frac{3}{2x^*}$ times $(\text{error at step } n)^2$. In Problem 9 the multiplier is $\frac{1}{2x^*}$ and convergence is quicker. Note to instructors: The multiplier is $\frac{f''(x^*)}{2f'(x^*)}$ (this is $\frac{1}{2}F''(x^*)$: see Problem 31 of Section 3.8).

24 $\theta = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{16\pi}{9}, \frac{32\pi}{9}, \frac{64\pi}{9} = 7\pi + \frac{\pi}{9}$; this happened at step 6 so $x_6 = x_0$.

26 If $z_0 = \sin^2 \theta$ then $z_1 = 4z_0 - 4z_0^2 = 4\sin^2 \theta - 4\sin^4 \theta = 4\sin^2 \theta (1 - \sin^2 \theta) = 4\sin^2 \theta \cos^2 \theta = (2\sin \theta \cos \theta)^2 = \sin^2 2\theta$.

44 A Newton step goes from $x_0 = .308$ to $x_1 = x_0 + \frac{\cos x_0}{\sin x_0} = 3.45143$. Then $\frac{\Delta f}{\Delta x} = \frac{\cos x_1 - \cos x_0}{x_1 - x_0} = -.606129$
and a secant step leads to $\quad x_2 = x_1 + \frac{\cos x_1}{.606129} = 1.88$.

3.8   The Mean Value Theorem and l'Hôpital's Rule   (page 152)

Be careful to apply l'Hôpital's Rule only when the limit of $\frac{f(x)}{g(x)}$ leads to $\frac{0}{0}$ or $\frac{\infty}{\infty}$. (The limit can be as $x \to a$ or $x \to \infty$ or $x \to -\infty$. If you get a form "$\infty - \infty$" or "$0 \cdot \infty$" use algebra to transform it into $\frac{0}{0}$.

1. Find $\lim_{x \to 0} \frac{\sin^2 x}{x}$.

   - This approaches $\frac{0}{0}$. The rule takes the derivative of $\sin^2 x$ and x to find $2 \sin x \cos x$ and 1. Then $\lim_{x \to 0} \frac{2 \sin x \cos x}{1} = \frac{0}{1} = 0$. The limiting answer is 0.

   Direct answer without l'Hôpital: $\sin x \to 0$ times $\frac{\sin x}{x} \to 1$ gives $\frac{\sin^2 x}{x} \to (0)(1) = 0$.

2. Find the limit of $\cot x - \csc x$ as $x \to \pi$.

   - This has the form "$\infty - \infty$". Rewrite $\cot x - \csc x$ as $\frac{\cos x}{\sin x} - \frac{1}{\sin x} = \frac{\cos x - 1}{\sin x}$. Now $\lim_{x \to \pi} \frac{\cos x - 1}{\sin x}$ is of the form $\frac{0}{0}$. We can apply l'Hôpital's rule to get $\lim_{x \to \pi} \frac{-\sin x}{\cos x} = 0$. The answer is 0.

3. Find the limit of $x - \sqrt{x^2 + 4}$ as $x \to \infty$.

   - This is also of the form $\infty - \infty$. An algebraic trick works (no l'Hôpital):

   $$x - \sqrt{x^2 - 4} = \frac{(x - \sqrt{x^2 + 4})(x + \sqrt{x^2 + 4})}{(x + \sqrt{x^2 + 4})} = \frac{x^2 - (x^2 + 4)}{x + \sqrt{x^2 + 4}} = \frac{-4}{x + \sqrt{x^2 + 4}}.$$

   As $x \to \infty$, the numerator stays at -4 while the denominator goes to ∞. The limit is 0.

4. Find $\lim_{x \to \frac{\pi}{4}} \sec 2x (1 - \cot x)$. Since $\sec 2(\frac{\pi}{4})$ is infinite and $1 - \cot \frac{\pi}{4} = 0$, we have $\infty \cdot 0$.

   - Rewrite the problem to make it $\frac{0}{0}$, by changing $\sec 2x$ to $\frac{1}{\cos 2x}$. Then use l'Hôpital's rule:

   $$\lim_{x \to \frac{\pi}{4}} \frac{1 - \cot x}{\cos 2x} = \lim_{x \to \frac{\pi}{4}} \frac{\csc^2 x}{-2 \sin 2x} = \frac{2}{-2} = -1$$

5. $\lim_{x \to 1} \frac{x^3 - x^2 - x + 1}{x^3 - 1}$ has the form $\frac{0}{0}$. One application of l'Hôpital's rule gives $\lim_{x \to 1} \frac{3x^2 - 2x - 1}{3x^2}$. Don't be tempted to apply the rule again! The limit is $\frac{0}{3} = 0$.

   Questions 6 – 8 are about the Mean Value Theorem.

6. What does the Mean Value Theorem say about the function $f(x) = x^3$ on the interval $[-1, 2]$?

   - The graph tells the story. The secant line connecting $(-1, 1)$ to $(2, 8)$ has slope $\frac{8-1}{2-(-1)} = \frac{7}{3}$. The MVT says that there is some c between -1 and 2 where the slope of the tangent line at $x = c$ is $\frac{7}{3}$.

7. What is the number c in question 6?

   - The slope of $y = x^3$ is $\frac{dy}{dx} = 3x^2$. The slope of the tangent at $x = c$ is $3c^2 = \frac{7}{3}$. So $c^2 = \frac{7}{9}$ and $c \approx \pm 0.882$. There are actually *two* values of c where the tangent line is parallel to the secant. (The Mean Value Theorem states there is at least one c, not "exactly" one.)

8. Use the Mean Value Theorem to show that $\tan a > a$ when $0 < a < \frac{1}{2}\pi$. Choose $f(x) = \tan x$.

   - On the interval $[0, a]$, the Theorem says that there exists a point c where $f'(c) = \sec^2 c = \frac{\tan a - \tan 0}{a - 0} = \frac{\tan a}{a}$. Since $\sec^2 c$ always exceeds 1 inside the interval we have $\frac{\tan a}{a} > 1$ and $\tan a > a$.

Read-throughs and selected even-numbered solutions :

The Mean Value Theorem equates the average slope $\Delta f/\Delta x$ over an **interval** $[a, b]$ to the slope df/dx at an unknown **point**. The statement is $\Delta f/\Delta x = f'(x)$**for some point** $a < c < b$. It requires $f(x)$ to be **continuous** on the **closed** interval $[a, b]$, with a **derivative** on the open interval (a, b). Rolle's theorem is the special case when $f(a) = f(b) = 0$, and the point c satisfies $f'(c) = 0$. The proof chooses c as the point where f reaches its **maximum or minimum**.

Consequences of the Mean Value Theorem include: If $f'(x) = 0$ everywhere in an interval then $f(x) =$ **constant**. The prediction $f(x) = f(a) + f'(c)(x - a)$ is exact for some c between a and x. The quadratic prediction $f(x) = f(a) + f'(x)(x - a) + \frac{1}{2}f''(c)(x - a)^2$ is exact for another c. The error in $f(a) + f'(a)(x - a)$ is less than $\frac{1}{2}M(x - a)^2$ where M is the maximum of $|f''|$.

A chief consequence is l'Hôpital's Rule, which applies when $f(x)$ and $g(x) \to 0$ as $x \to a$. In that case the limit of $f(x)/g(x)$ equals the limit of $f'(x)/g'(x)$, provided this limit exists. Normally this limit is $f'(a)/g'(a)$. If this is also $0/0$, go on to the limit of $f''(x)/g''(x)$.

2 $\sin 2\pi - \sin 0 = (\pi \cos \pi c)(2 - 0)$ when $\cos \pi c = 0$: then $\mathbf{c} = \frac{1}{2}$ or $\mathbf{c} = \frac{3}{2}$.

10 $f(x) = \frac{1}{x^2}$ has $f(1) = 1$ and $f(-1) = 1$, but no point c has $f'(c) = 0$. MVT does not apply because $f(x)$ is **not continuous** in this interval.

12 $\frac{d}{dx}\csc^2 x = 2\csc x(-\csc x \cot x)$ is equal to $\frac{d}{dx}\cot^2 x = 2\cot x(-\csc^2 x)$. Then $f(x) = \csc^2 x - \cot^2 x$ has $f' = 0$ at every point c. By the MVT $f(x)$ must have the same value at every pair of points a and b. By trigonometry $\csc^2 x - \cos^2 x = \frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x}{\sin^2 x} = 1$ at all points.

16 l'Hôpital's Rule does not apply because $\sqrt{1 - \cos x}$ has no derivative at $x = 0$. (There is a corner.) The knowledge that $1 - \cos x \approx \frac{x^2}{2}$ gives $\sqrt{1 - \cos x} \approx \frac{|x|}{\sqrt{2}}$. Then $\frac{|x|}{\sqrt{2}}x$ has one-sided limits. The limit from the right (where $x > 0$) is $\mathbf{L} = \dfrac{1}{\sqrt{2}}$. The limit from the left (where $x < 0$) is $\mathbf{L} = -\dfrac{1}{\sqrt{2}}$. These limits also come from "one-sided l'Hôpital rules."

18 $\lim_{x \to 1} \frac{x-1}{\sin x} = \frac{0}{\sin 1} = 0$ (not an application of l'Hôpital's Rule).

20 $\lim_{x \to 0} \frac{(1+x)^n - 1 - nx}{x^2} = \lim_{x \to 0} \frac{n(1+x)^{n-1} - n}{2x} = $ (l'Hôpital again) $\lim_{x \to 0} \frac{n(n-1)(1+x)^{n-2}}{2} = \mathbf{\dfrac{n(n-1)}{2}}$.

30 Mean Value Theorem: $f(x) - f(y) = f'(c)(x - y)$. Therefore $|f(x) - f(y)| = |f'(c)||x - y| \le |x - y|$ since we are given that $|f'| \le 1$ at all points. Geometric interpretation: If the tangent slope stays between -1 and 1, so does the slope of any secant line.

32 No: The converse of Rolle's theorem is false. The function $f(x) = x^3$ has $f' = 0$ at $x = 0$ (horizontal tangent). But there are no two points where $f(a) = f(b)$ (no horizontal secant line).

3   Chapter Review Problems

Computing Problems

C1   Find the fixed points of $f(x) = 1.2 \sin x$. Which are attracting and which are repelling?

C2   Find the real roots of $f(x) = x^3 + 2x + 1$ using Newton's method, correct to four places.

C3   Solve $f(x) = 2\sin x - 1 = 0$ correct to four places.

C4   Minimize $y = \frac{2x^{1/3}}{2-x}$ (you will find a *relative* minimum: why?).

Review Problems

R1   Linear Approximation: Sketch a function $f(x)$ and a tangent line through $(a, f(a))$. Label $dy, dx, \Delta y, \Delta x$. Based on Figure 3.2 indicate the error between the curve and its tangent line.

R2   To find the maximum (or minimum) of a function, what are the three types of critical point to consider? Sketch a function having maxima or minima at each type of critical point.

R3   Explain the second derivative test for local maxima and minima of $f(x)$.

R4   Are stationary points always local maxima or minima? Give an example or a counterexample.

R5   Show by a picture why a fixed point of $y = F(x)$ is attracting if $|\frac{dF}{dx}| < 1$ and repelling if $|\frac{dF}{dx}| > 1$.

R6   For $f(x) = x^3 - 3x + 1 = 0$, Newton's method is $x_{n+1} = $ _____.

R7   Draw a picture to explain the Mean Value Theorem. With another sketch, show why this theorem may not hold if $f(x)$ is not differentiable on (a, b). With a third sketch, show why the theorem may not hold if $f(x)$ is defined on (a, b) but not at the endpoints.

R8   State l'Hôpital's Rule. How do you know when to use it ? When do you use it twice? Illustrate with $\lim_{x \to 0} \frac{\tan x - x}{x^3}$.

Drill Problems

D1   Find a linear approximation for $y = 1 + \frac{1}{x}$ near $a = 2$. Then find a quadratic approximation.

D2   Approximate $\sqrt[3]{500}$ using a linear approximation.

D3   The volume of a cylinder is $V = \pi r^2 h$. What is the percent change in volume if the radius decreases by 3% and the height remains the same? Is this exact or approximate?

In **4 – 8** find stationary points, rough points, and endpoints. Also relative and absolute maxima and minima.
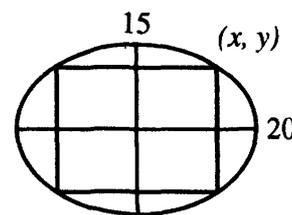
D4 $y = 3x^2 - 2x^3$ 　　　　　**D5** $y = x^4 - 8x^2 + 16$ 　　　　　**D6** $y = \frac{1}{x+1}, 0 < x < 99$

D7 $y = \sin x + |\sin x|, -2\pi \le x \le 2\pi$ 　　　　**D8** $f(x) = \frac{1}{x} - \frac{x^2}{16}$ on $[-4, -1]$

D9   Where are these functions concave up and concave down? Sketch their graphs.

(a) $(x^2 - 1)^3$ 　　(b) $12x^{2/3} - 4x$ 　　(c) $x^2 + \frac{2}{x}$ 　　(d) $\frac{2x^2}{x^2+3}$

D10   A 20 meter wire is formed into a rectangle. Maximize the area.

D11   A level rectangle is to be inscribed in the ellipse $\frac{x^2}{400} + \frac{y^2}{225} = 1$. Find the rectangle of maximum area. The ellipse is drawn above.

D12   Write these equations in standard form and identify parabola, ellipse, or hyperbola. Draw rough graphs.

(a) $4x^2 + 9y^2 = 36$ 　(b) $16x^2 - 7y^2 + 112 = 0$ 　(c) $x^2 - 2x + y^2 + 6y + 2 = 0$ 　(d) $x^2 = 2y + 2x$

D13   Compute $\lim_{x \to 0} \frac{x + \sin 2x}{x - \sin 2x}$ and $\lim_{x \to \pi} \frac{\sin 2x}{x - \pi}$.

D14   Find $\lim_{x \to \infty} \frac{\sqrt{2 + x^2}}{x}$ and $\lim_{x \to 0^+} x^2 \cot x$ and $\lim_{x \to 0^+} (\frac{1}{x} - \frac{1}{\sqrt{x}})$.

D15   Find the "c" guaranteed by the Mean Value Theorem for $y = x^3 + 2x$ on the interval $[-1, 2]$.