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The Amplitude of Convection

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13.1 Introduction

As a ubiquitous source of motion, both astrophysical and geophysical, convection has attracted theoretical attention since the last century. In the ocean, many different scales are called convection; from the deep circulation due to the seasonal production of Arctic bottom water (see chapter 1) to micromixing of salt fingers (see chapter 8). In the atmosphere, convection dominates the flow from subcloud layers to Hadley "cells." It is proposed that convection in the earth's core powers the geomagnetic field. The nonperiodic reversals of that field, captured in the rock, define the evolution of the ocean basin. Recent recognition that this latter process is caused by convection in the mantle has produced a new geophysics.

In the past, understanding the central features of convection has come from the isolation of "simplest" mechanistic examples. Although large-scale geophysical convection never coincides with the idealized simplest problem, these examples (e.g., Lord Rayleigh's study of the Bénard cells) have generated much of the formal language of inquiry used in the field. Students of dynamic oceanography have favored this formal language mixed in equal parts with more pragmatic engineering tongues when interpreting oceanic convective processes.

Speculations beyond these mathematically accessible problems take the form of hypotheses, experiments, and numerical experiments in which one seeks to isolate the central processes responsible for the qualitative and quantitative features of fully evolved flow fields. The many facets of turbulent convection represent the frontier. This chapter reviews only a narrow path toward that frontier. This path is aimed at an understanding of the elementary processes responsible for the amplitude of convection, in the belief that quantitative theories permit the theorist the least self-deception.

Of course the heat flux due to a prescribed thermal contrast, like the flow due to a given stress, has been observed for a century. The relation between force and flux has been rationalized with models emerging largely from linear theory and kinetic theory—in particular, with the use of observationally determined "eddy conductivities" (estimated for the oceans in Sverdrup, Johnson, and Fleming, 1942). Early theoretical interpretations of oceanic transport processes that go beyond these simple beginnings were explored by Stommel (1949), while current usage and extensions of "mixing" theories are discussed in chapter 8.

Central to the most recent of such proposals is the idea that some large scale of the motion or density field is steady or statistically stable, while turbulent transport due to smaller scales can be parameterized. Changing the amplitude of the small-scale transports is pre-

sumed to lead to a new equilibrium for the large scale, so that the statistical equilibrium is marginally stable. This view lurks behind most traditional oceanic model building and its quasi-linear form is used on small-scale phenomena as well—from inviscid marginal stability for the purpose of quantifying aspects of the wind mixed layer (Pollard, Rhines, and Thompson 1973) to viscous marginal stability for the purpose of quantifying double diffusion (Linden and Shirtcliffe, 1978).

It has not yet been possible to establish either the limits of validity or generalizability of this quasi-linear use of marginal stability in the geophysical setting. There can be little doubt that it is “incorrect”—that fluids typically are destabilized by the extreme fluctuations—yet it appears to be the only quantifying concept of sufficient generality to have been used in oceanic phenomena from the largest to the smallest scales. Of course, our idealizations in the realm of geophysics are all “incorrect.” We turn to observation to establish in what sense and in what degree these idealizations are good “first-order” descriptions of reality.

This chapter explores the hierarchy of quantifying idealizations in convection theory. The quasi-linear marginal-stability problem is drawn from the full formal statement for stability of the flow. A theory of turbulent convection based on marginal stability is presented, incorporating both the qualitative features determined by inviscid processes and the quantitative aspects determined by dissipative processes.

Observations provide better support for both the quantitative and qualitative results from quasi-linear marginal-stability theory than might have been anticipated, encouraging its continued application in the oceanic setting.

13.2 Basic Boussinesq Description

The primary simplification that permitted mathematical progress in the study of motion driven by buoyancy was the Boussinesq statement of the equations of motion. In retrospect, the central problem was to translate the correct energetic statement

$$\langle \mathbf{u} \cdot \nabla P \rangle = \Phi,$$

into the approximate form

$$\langle \gamma \overline{WT} \rangle = \Phi,$$

where \mathbf{u} is the vector velocity of the fluid, P the pressure, γ the coefficient of thermal expansion times the acceleration of gravity, W the vertical component of velocity, T the temperature field, Φ the total dissipation by viscous processes in the fluid, and the brackets a spatial average over the entire fluid. This has been achieved (e.g., Spiegel and Veronis, 1960; Malkus 1964) by recognizing that the Boussinesq equations are the

leading terms in an a sonic asymptotic expansion away from a basic adiabatic hydrostatic temperature distribution. This expansion is usually made in two small parameters; one is the ratio of the height of the convecting region to the total “adiabatic depth” of the fluid, while the second is the ratio of the superadiabatic temperature contrast across the convecting region to the mean temperature.

In suitably scaled variables, the leading equations of the expansion are

$$\nabla \cdot \mathbf{u} = 0, \tag{13.1}$$

$$\frac{1}{\sigma} \frac{D\mathbf{u}}{Dt} = -\nabla P + \nabla^2 \mathbf{u} + Ra T \mathbf{k}, \tag{13.2}$$

$$\frac{DT}{Dt} = \nabla^2 T, \tag{13.3}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \quad \sigma = \frac{\nu}{\kappa}, \quad Ra = \frac{\gamma \Delta T d^3}{\kappa \nu},$$

\mathbf{k} is the unit vector in the antirection of gravitational acceleration, d the depth of the convecting region, ΔT the superadiabatic temperature contrast, κ the thermometric conductivity of the fluid and ν is its kinematic viscosity, Ra the Rayleigh number, and σ the *Prandtl number*. Other symbols are defined above. The Boussinesq equations retain the principal advective nonlinearity, but have no sonic solutions. Higher-order equations are linear and inhomogeneous, forced by the lower-order solutions.

The most accessible problem in free convection has been the study of motion in a horizontal layer of fluid bounded by good thermal conductors at prescribed temperatures. Such a layer is the thermal equivalent of the constant-stress layer in shear flow. This is seen by taking an average over the horizontal plane of each term in the heat equation. One writes from 13.3

$$-\frac{\partial \overline{T}}{\partial t} = \frac{\partial}{\partial z} \left(-\frac{\partial \overline{T}}{\partial z} + \overline{WT} \right), \tag{13.4}$$

where the overbar indicates the horizontal average. For steady or statistically steady convection, $\partial \overline{T} / \partial t$ vanishes, and one may integrate (13.4) twice to obtain

$$Nu = -\frac{\partial \overline{T}}{\partial z} + \overline{WT} = 1 + \langle WT \rangle, \tag{13.5}$$

where the constant of integration Nu is called the Nusselt number and is the ratio of the total heat flux to that due to conduction alone.

Two other integrals of considerable interest can be constructed from the Boussinesq equations. The first of these is the power integral found by taking the scalar product of (13.2) with \mathbf{u} and integrating over the entire fluid. One obtains

$$\langle -\mathbf{u} \cdot \nabla^2 \mathbf{u} \rangle = \langle |\nabla \mathbf{u}|^2 \rangle = Ra \langle WT \rangle, \quad (13.6)$$

due to the vanishing of the conservative advective terms. Multiplying (13.6) by the fluctuation temperature,

$$T = T - \bar{T}, \quad (13.7)$$

one obtains an integral similar to (13.6), which may be written by means of (13.5) as

$$\begin{aligned} \langle -T \nabla^2 T \rangle &= \langle |\nabla T|^2 \rangle = \langle -WT \frac{\partial \bar{T}}{\partial z} \rangle \\ &= \langle WT \rangle + \langle WT \rangle^2 - \langle \bar{W} \bar{T}^2 \rangle. \end{aligned} \quad (13.8)$$

The integrals (13.5), (13.6), and (13.8) are the principal constraints used in the "upper bound" theories of convection (see section 13.3).

This section would not be complete without a statement of those equations that determine the stability of any solution, say \mathbf{u}_0, P_0, T_0 , of the basic equations (13.1)-(13.3). Consider a general disturbance \mathbf{v}, p, θ to the solution \mathbf{u}_0, P_0, T_0 . Then

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \quad P = P_0 + p, \quad T = T_0 + \theta. \quad (13.9)$$

One concludes from (13.1)-(13.3) that the time evolution of \mathbf{v}, p, θ is determined by

$$\nabla \cdot \mathbf{v} = 0, \quad (13.10)$$

$$\begin{aligned} \frac{1}{\sigma} \left(\frac{D\mathbf{v}}{Dt} + \mathbf{v} \cdot \nabla \mathbf{u}_0 + \mathbf{v} \cdot \nabla \mathbf{v} \right) \\ = -\nabla p + \nabla^2 \mathbf{v} + Ra \theta \mathbf{k}, \end{aligned} \quad (13.11)$$

$$\left(\frac{D\theta}{Dt} + \mathbf{v} \cdot \nabla T_0 + \mathbf{u}_0 \cdot \nabla \theta \right) = \nabla^2 \theta, \quad (13.12)$$

where, as before, $D/Dt = \partial/\partial t + \mathbf{u}_0 \cdot \nabla$.

For a solution \mathbf{u}_0, P_0, T_0 to be stable (hence realizable), arbitrary infinitesimal disturbances, \mathbf{v}, p, θ must eventually decay. For a solution, \mathbf{u}_0, P_0, T_0 to be absolutely stable, arbitrary disturbances of any amplitude must eventually decay. This latter problem is tractable in some instances and has been addressed using the "power" integrals for \mathbf{v} and θ . These are written

$$\begin{aligned} \frac{1}{\sigma} \frac{D}{Dt} \langle \frac{1}{2} \mathbf{v}^2 \rangle + \langle \mathbf{v} \cdot \nabla \cdot \nabla \mathbf{u}_0 \rangle \\ = - \langle |\nabla \mathbf{v}|^2 \rangle + Ra \langle \mathbf{v} \cdot \mathbf{k} \theta \rangle, \end{aligned} \quad (13.13)$$

$$\frac{D}{Dt} \langle \frac{1}{2} \theta^2 \rangle + \langle \theta \mathbf{v} \cdot \nabla T_0 \rangle = - \langle |\nabla \theta|^2 \rangle. \quad (13.14)$$

Equations (13.10), (13.13), and (13.14) constitute a linear problem for \mathbf{v} and θ whose solution can determine a minimum $Ra = Ra(\sigma)$ for which \mathbf{u}_0, P_0, T_0 is absolutely stable. In contrast, the solutions to the linear

form of (13.10)-(13.12) can determine a maximum $Ra = Ra(\sigma)$ beyond which \mathbf{u}_0, P_0, T_0 is assuredly unstable. In principal these linear problems can be solved when \mathbf{u}_0, P_0, T_0 is time independent, and solved at least approximately when \mathbf{u}_0, P_0, T_0 is periodic in time. However, analytic techniques for determining the conditions leading to eventual decay of \mathbf{v}, p, θ on a nonperiodic solution \mathbf{u}_0, P_0, T_0 have not been developed.

In concluding this description of the "simple" Boussinesq fluid one should note how many interesting problems lie outside the formalism. Just outside the framework, but capable of incorporation, are porous boundaries, variable viscosity, and nonlinear equations of state. Much farther outside the framework are convection through several scale heights and velocities comparable to the sound speed.

13.3 Initial Motions

The state of pure conduction without motion, $\mathbf{u}_0 = 0, T_0 = -z$, where z is the vertical coordinate measured from the lower surface, is a solution to the Boussinesq equations at all Ra . The determination of that critical value of Ra at which this conduction solution is unstable is the classical Rayleigh convection problem (e.g., Chandrasekhar, 1961). This problem is the linearized form of (13.10)-(13.12) for $\mathbf{u}_0 = 0, T_0 = -z$, and is written

$$\nabla \cdot \mathbf{u} = 0, \quad (13.15)$$

$$\frac{1}{\sigma} \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \nabla^2 \mathbf{v} + Ra \theta \mathbf{k}, \quad (13.16)$$

$$\frac{\partial \theta}{\partial t} = w + \nabla^2 \theta, \quad w = \mathbf{v} \cdot \mathbf{k}. \quad (13.17)$$

If one takes the \mathbf{k} component of the curl of the curl of (13.16) and, using (13.17), eliminates θ , one may write the Rayleigh problem as a constant coefficient, sixth-order partial differential equation in the single variable w :

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial t} - \nabla^2 \right) \left(\frac{\partial}{\partial t} - \nabla^2 \right) \nabla^2 w = -Ra \nabla_{\parallel}^2 w, \quad (13.18)$$

where ∇_{\parallel}^2 is the horizontal Laplacian. The problem is separable so that

$$\nabla_{\parallel}^2 w = -\alpha^2 w \quad (13.19)$$

defines the horizontal wavenumber α^2 . Also, it is not difficult to establish that the disturbance w first starts to grow without temporal oscillations; the instability is "marginal." Hence, subject to appropriate boundary conditions one is to find the eigenstructure of the ordinary differential operator

$$\left[\left(\frac{\partial^2}{\partial z^2} - \alpha^2 \right)^3 - \alpha^2 Ra \right] w = 0. \quad (13.20)$$

The richness of solutions to this simple problem and its "adjacent" modifications have been explored for more than two generations and now enters a third. In the present generation, formal techniques that had been developed to find the initial postcritical amplitude of convection are used to clarify the finite-amplitude stability problems. These finite-amplitude studies include the resolution of the infinite plan form degeneracy of the classical linear problem and the determination of conditions causing subcritical instabilities (or "snap-through" instabilities). The formal finite-amplitude technique, often called modified perturbation theory, is the subject of a recent lengthy review [Busse, 1978]. In brief, one expands \mathbf{v} , p , θ in terms of an amplitude ϵ (typically the amplitude of w), and in addition employs a parametric expansion of the controlling parameters, also in terms of ϵ , to permit solvability of the sequence of equations generated by the nonlinear terms. One writes

$$\mathbf{v} = \sum_{n=1}^{\infty} \epsilon^n \mathbf{v}_n, \quad Ra = \sum_{n=0}^{\infty} Ra_n \epsilon^n, \quad (13.21)$$

with similar expansions for p and θ , and where Ra_0 is the critical Rayleigh number determined from the linear problem (13.20). Here, each of the Ra_n is determined to permit a steady solution to the ϵ^n set of equations generated by inserting (13.20) into (13.10)–(13.12). Paralleling (13.10)–(13.12) and (13.20), an expansion can be constructed for potential disturbances \mathbf{v}' , p' , θ' in order to determine their stability at each order ϵ^n .

Among the many interesting conclusions reached in these studies is that the amplitude of the stable convection forms are determined principally by a modification of \bar{T} due to the convective flux $\bar{w}\bar{\theta}$. The finite-amplitude distortions of w and θ from their infinitesimal form also affects their equilibrium amplitude, and although a smaller effect than the modification of the mean, it is always present. Conditions for subcritical instability, from (13.21), are seen to be that either $Ra_1 < 0$, $Ra_2 > 0$ (as is the case when hexagonal cellular convection is observed), or $Ra_2 < 0$, $Ra_4 > 0$ (as occurs in penetrative convection in water cooled below 4°C), or some mixture of these two conditions. It is found to order ϵ^2 in each case that the preferred (stable) convection is that which transports the most heat. However, there is now an example of a convective process in which the maximum heat-flux solution at large amplitude is not the most stable form. No simple integral criterion has been found that assures the stability of a Boussinesq solution at large Ra .

Many papers have been written using modified perturbation theory, and more appear each year. Unusual current studies include nonperiodic behavior of initial convection in rotating systems and the onset of magnetic instabilities due to finite-amplitude convection in electrically conducting fluids. Busse's review ex-

hibits the significant enrichment of our knowledge and language of inquiry of fluid dynamics by these finite-amplitude studies. This same review, however, clarifies the intractable character of convection mathematics beyond ϵ^2 .

This section on initial motions will be concluded by noting E. Lorenz's (1963b) minimal nonlinear convection model, first explored by Saltzman (1962), is an abruptly truncated ϵ^2 Rayleigh convection process. This third-order autonomous system exhibits a transition to "convection," then at higher "Rayleigh" number a transition to exact solutions with nonperiodic behavior. A group of mathematicians and physicists has emerged to explore these "strange attractors" and similar models, hoping among other things to find new access to the turbulent process in fluids. One can be confident that the study of these autonomous models will lead eventually to elementary insights of value to the geophysical dynamicist, yet such study is only one facet of the third generation beyond linear convective instability mentioned earlier. The path to be followed here through the ever-growing convection literature must include the theory of upper bounds on turbulent heat flux, for this theory deductively addresses convection amplitude. The blaze mark along this path, however, continues to be stability theory, and the final section explores its relation to the observed fully evolved flow.

13.4 Quantitative Theories for High Rayleigh Number

Beyond modified perturbation theory one finds hypotheses, speculation, and assorted ad hoc models related to aspects of turbulent convection or designed to rationalize a body of data. In this section three unique theories that predict an amplitude for convection are discussed. The first of these is the mean-field theory of Herring (1963). The second is the theory of the upper bound on heat flux, first formulated by Howard (1963), and its "multi- α " extension by Busse (1969). The third theory, by Chan (1971), is also a heat-flux upper-bound theory, but for the idealized case of infinite Prandtl number. The link between the first and last theories proves to be most informative.

The mean-field theory is equivalent to the first approximation of many formal statistical closure proposals, and is the only contact this study will make with such proposals. As implemented by Herring (1963), the equation describing the motion and temperature field are obtained by retaining only those nonlinear terms in the basic equations that contain a nonzero horizontally averaged part. From (13.1)–(13.4) and (13.7) one writes the mean-field equations thusly:

$$\nabla \cdot \mathbf{u} = 0, \quad (13.22)$$