Chapter 27

The Real Options Model of Land Value and Development Project Valuation

Major references include*:

•J.Cox & M.Rubinstein, *"Options Markets"*, Prentice-Hall, 1985

•L.Trigeorgis, *"Real Options"*, MIT Press, 1996

•T.Arnold & T.Crack, *"Option Pricing in the Real World: A Generalized Binomial Model with Applications to Real Options"*, Dept of Finance, University of Richmond, Working Paper, April 15, 2003 (available on the Financial Economics Network (FEN) on the Social Science Research Network at www.ssrn.com).

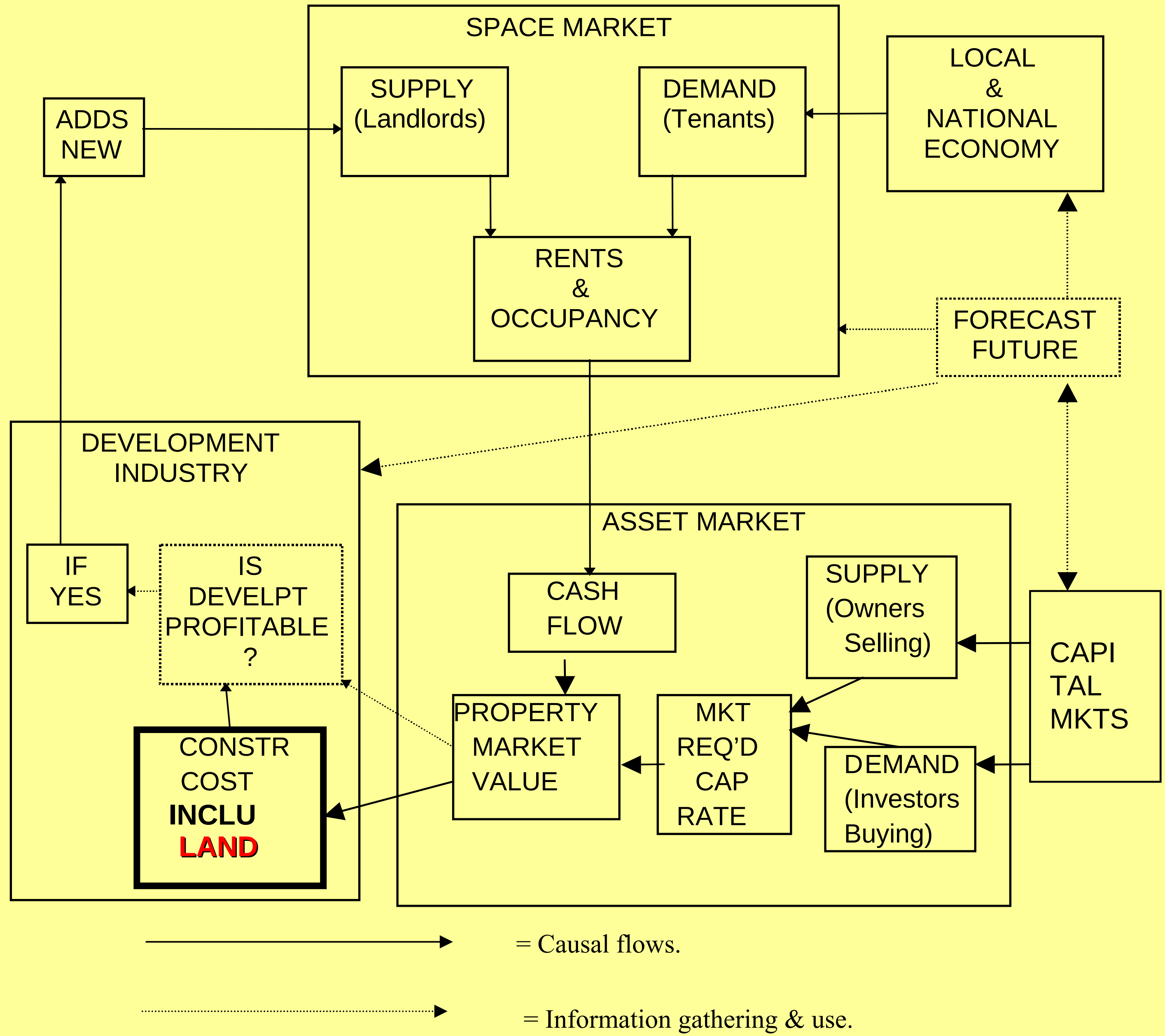
Chapter 27 in perspective . . .

In the typical development project (or parcel of developable land), there are three major types of options that may present themselves:*

- ***"Wait Option"*: The option to *delay* start of the project construction (Ch.27);**
- ***"Phasing Option"*: The breaking of the project into sequential *phases* rather than building it all at once (Ch.29);**
- ***"Switch Option"*: The option to choose among *alternative types* of buildings to construct on the given land parcel.**

All three of these types of options can affect optimal investment decision-making, add significantly to the value of the project (and of the land), affect the risk and return characteristics of the investment, and they are difficult to accurately account for in traditional DCF investment analysis.

Exhibit 2-2: The "Real Estate System": Interaction of the Space Market, Asset Market, & Development Industry



Land value plays a pivotal role in determining whether, when, and what type of development will (and should) occur.

Relationship is two-way:

Land Value ⟷ **Optimal Devlpt**

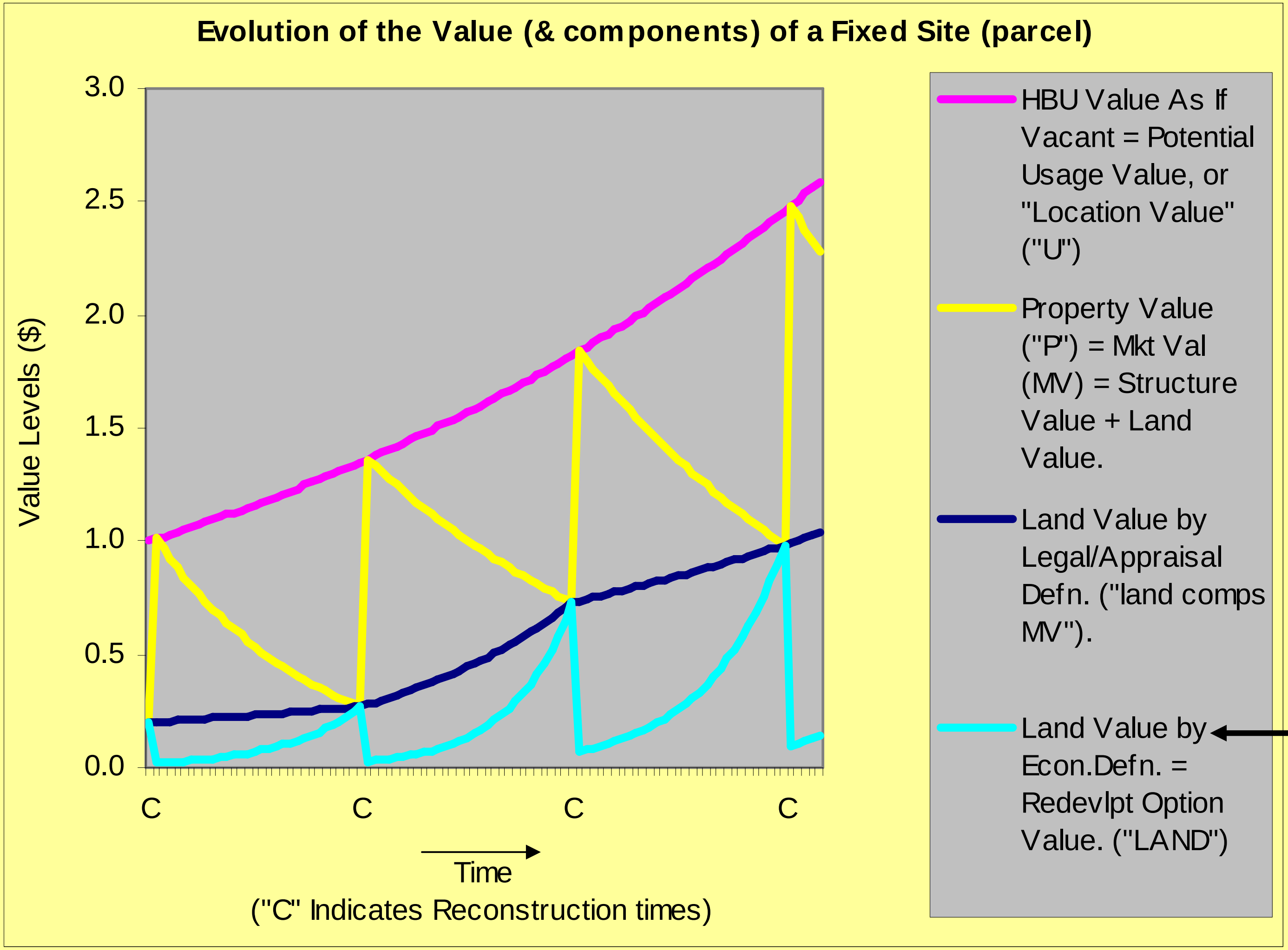
- **From a finance/investments perspective:**
  - ***Development activity links the asset & space markets;***
  - ***Determines L.R. supply of space, ➔ L.R. rents.***
  - ***Greatly affects profitability, returns in the asset market.***
- **From an urban planning perspective:**
  - ***Development activity determines urban form;***
  - ***Affects physical, economic, social character of city.***

Recall relation of land value to land use boundaries noted in Ch.5…

Different conceptions of "land value" (Recall Property Life Cycle theory from Ch.5) . . .

Property Value, Location Value, & Land Value

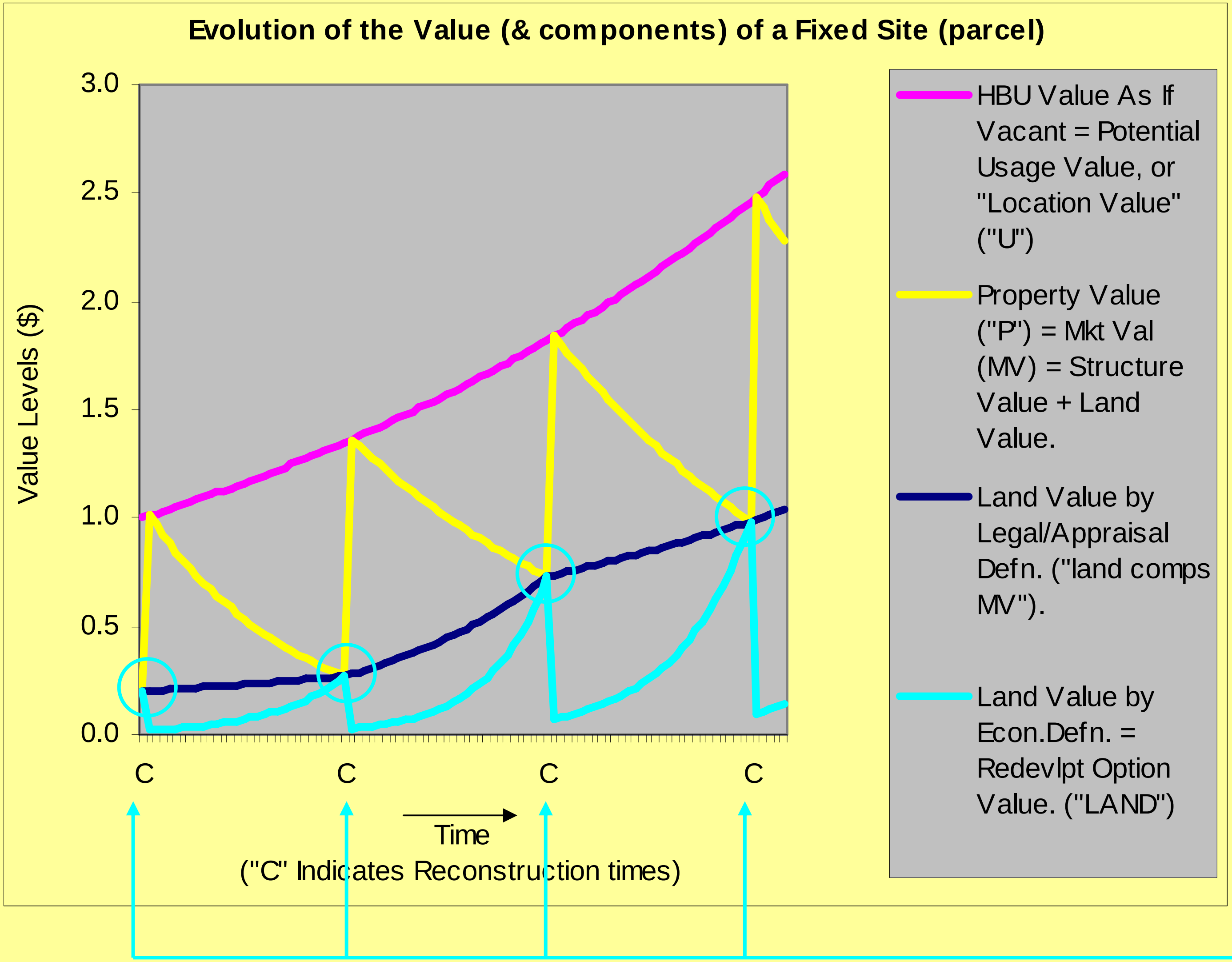
Exh.5-10, Sect.5.4



In Ch.27we focus on the Econ.Defn.: "LAND"

Different conceptions of "land value" . . .

Property Value, Location Value, & Land Value

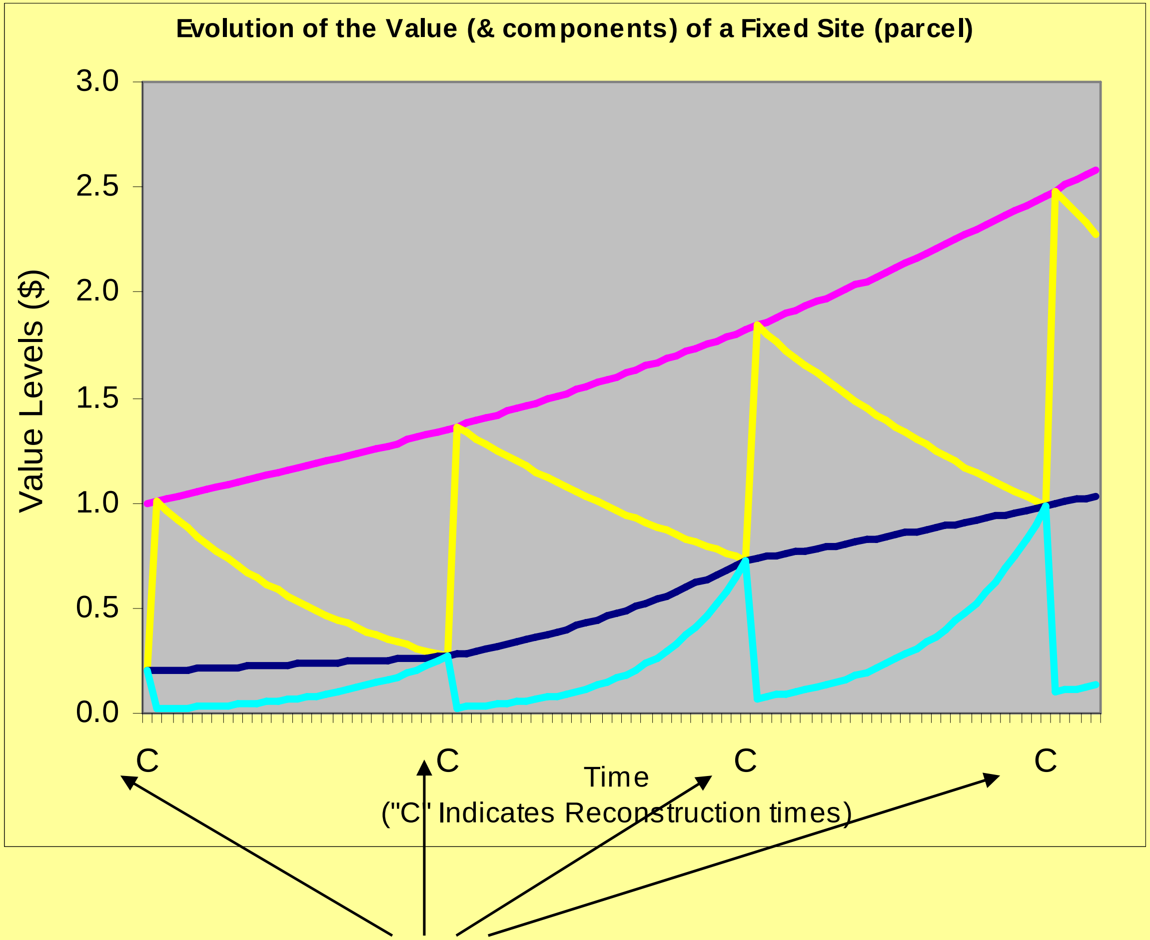


Note that there are points in time when three of the four definitions all give the same value, namely, property value = land value defined by either defn at the times of optimal redevelopment (construction) on the site.

The economic definition of land value ("LAND") is based on nothing more or less than the fundamental capability that land ownership gives to the landowner (unencumbered):

The right without obligation to develop (or redevelop) the property.

This definition of land value is most relevant . . .



Just prior to the times when development or redevelopment occurs on the site.

To understand the economic conception of land value, a famous theoretical development from financial economics is most useful: *"Option Valuation Theory" (OVT)* :

In particular, a branch of that theory known a *"Real Options"*.

Some history:

Call option model of land arose from two strands of theory:

- ***Financial economics*** **study of corporate capital budgeting,**
- ***Urban economics*** **study of urban spatial form.**

Capital Budgeting:

- **How corporations should make capital investment decisions (constructing physical plant, long-lived productive assets).**
- **Includes question of optimal timing of investment.**
- **e.g., McDonald, Siegel, Myers, (others), 1970s-80s.**

Urban Economics:

- **What determines density and rate of urban development.**
- **Titman, Williams, Capozza, (others), 1980s.**

It turned out the 1965 Samuelson-McKean Model of a perpetual American warrant was the essence of what they were all using.

27.1 *Real Options*: The Call Option Model of Land Value

Real Options:

Options whose underlying assets (either what is obtained or what is given up on the exercise of the option) are real assets (i.e., physical capital).

The call option model of land value (introduced in Chapter 5) is a real option model:

Land ownership gives the owner the *right without obligation* to develop (or redevelop) the property upon payment of the construction cost. Built property is underlying asset, construction cost is exercise price (including the opportunity cost of the loss of any pre-existing structure that must be torn down).

In essence, all real estate development projects are real options, though in some simple cases the optionality may be fairly trivial and can be safely ignored.

27.2 A Simple Numerical Example of OVT Applied to Land Valuation and the Development Timing Decision

	Today	Next Year	
Probability	100%	30%	70%
Value of Developed Property	$100.00	$78.62	$113.21
Development Cost (exclu land)	$88.24	$90.00	$90.00
NPV of exercise	$11.76	-$11.38	$23.21
(Action)		(Don't build)	(Build)
Future Values		0	$23.21
Expected Values	$11.76	$16.25	
= Sum[Probability X Outcome]	(1.0)11.76	(0.3)0 + (0.7)23.21	
PV(today) of Alternatives @20%	$11.76	16.25 / 1.2 = $13.54	

Note: In this example the *expected growth* in the HBU value of the built property is *2.83%*: as *(.3)78.62 + (.7)113.21 = $102.83*.

What is the value of this land today? **Answer: = *MAX[11.76, 13.54] = $13.54***

Should owner build now or wait? **Answer: = *Wait. (100.00 – 88.24 – 13.54< 0.)***

The $13.54 – $11.76 = $1.78 option premium is due to *uncertainty or volatility*.

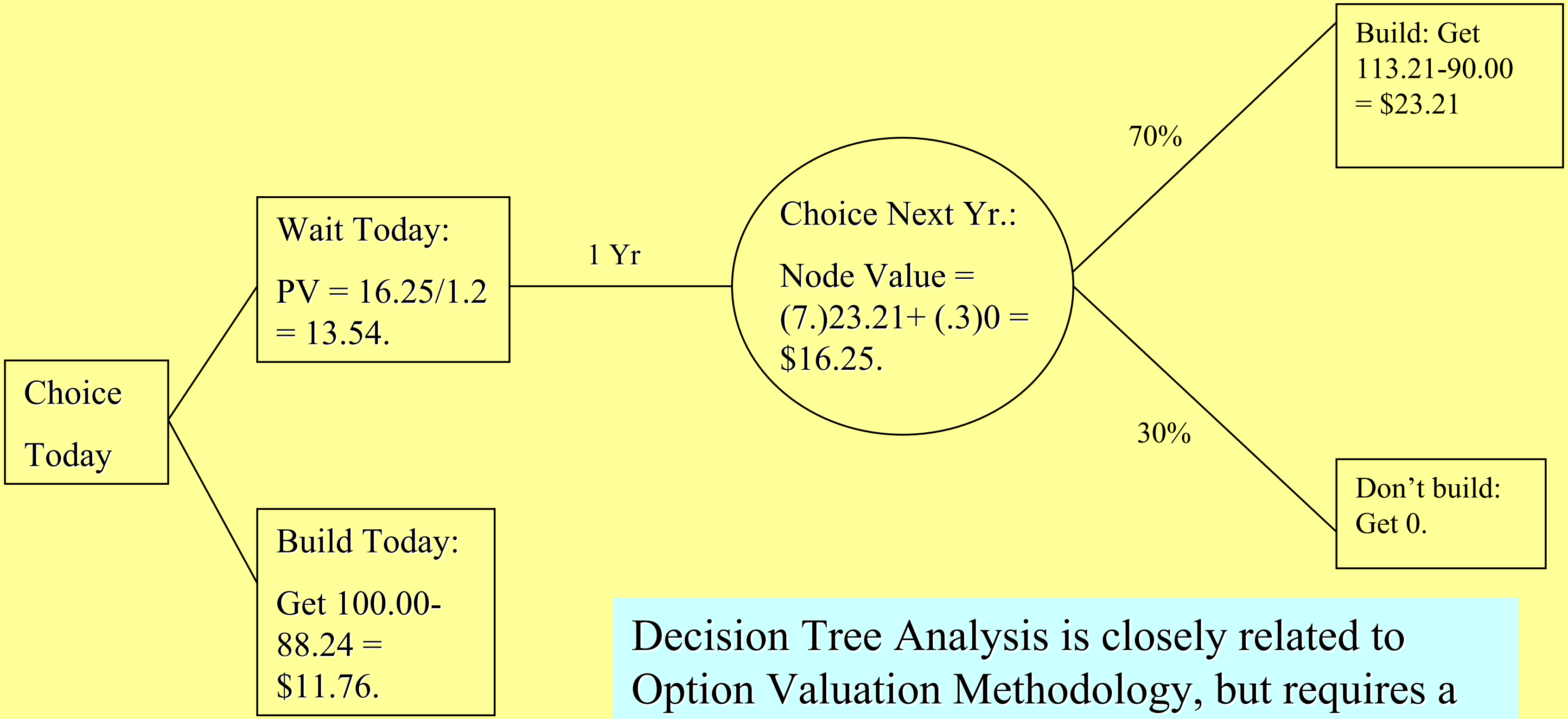
Consider the effect of *uncertainty* (or volatility) in the evolution of the built property value (for whatever building would be built on the site), and the fact that development at any given time is *mutually exclusive* with development at any other time on the same site *("irreversibility")*. e.g.:

	Today	Next Year	
Probability	100%	30%	70%
Value of Developed Property	$100.00	$78.62	$113.21
Development Cost (exclu land)	$88.24	$90.00	$90.00
NPV of exercise	$11.76	-$11.38	$23.21
(Action)		(Don't build)	(Build)
Future Values		0	$23.21
Expected Values	$11.76	$16.25	
= Sum[Probability X Outcome]	(1.0)11.76	(0.3)0 + (0.7)23.21	
PV(today) of Alternatives @20%	$11.76	16.25 / 1.2 = $13.54	

Note the importance of *flexibility* inherent in the option ("right *without obligation*"), which allows the negative downside outcome to be avoided. This gives the option a positive value and results in the *"irreversibility premium"* in the land value (noted in Geltner-Miller Ch.5).

Representation of the preceding problem as a *"decision tree"*:

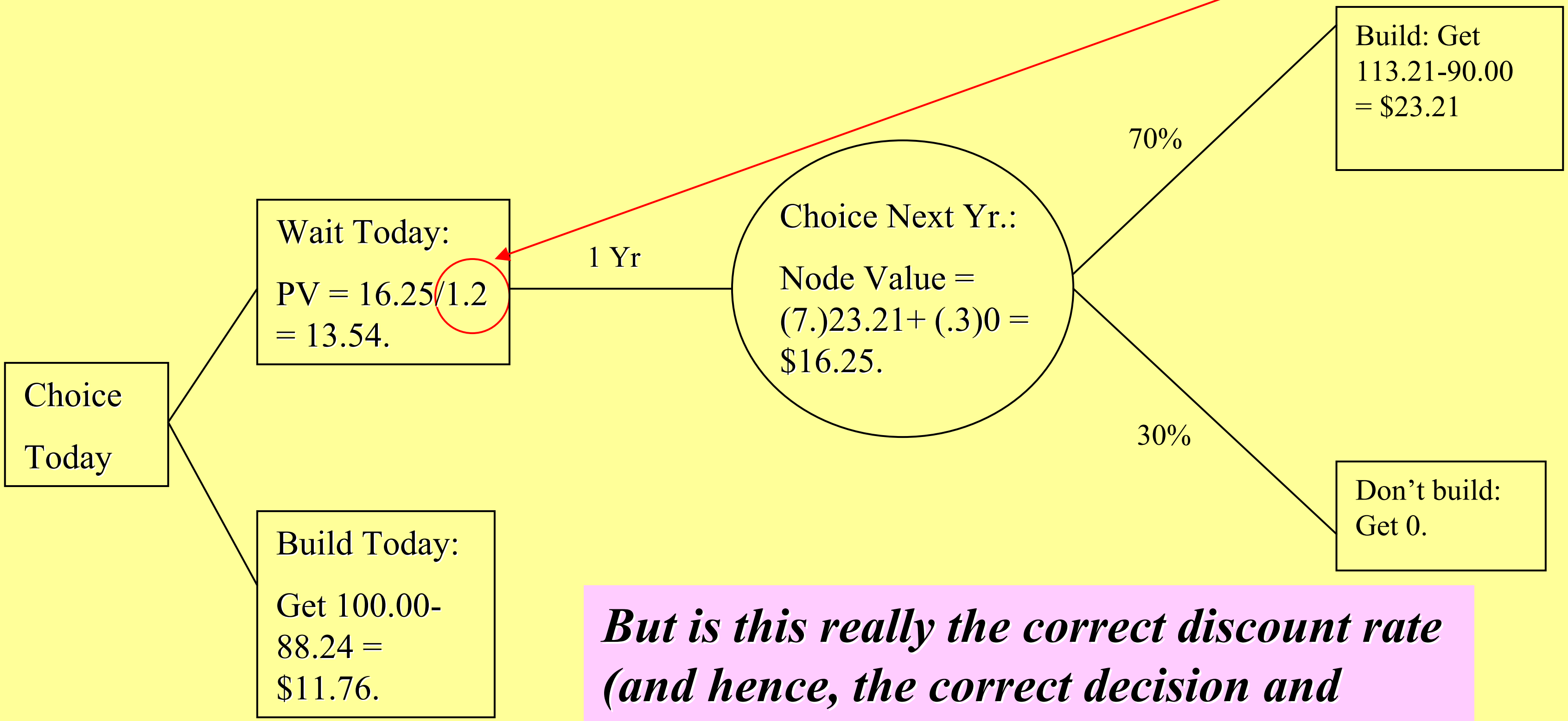
- **Identify decisions and alternatives (nodes & branches).**
- **Assign probabilities (sum across all branches @ ea. node = 100%).**
- **Locate nodes in time.**
- **Assume "rational" (highest value) decision will be made at each node.**
- **Discount node expected values (means) across time reflecting risk.**



Decision Tree Analysis is closely related to Option Valuation Methodology, but requires a different type of simplification (finite number of discrete alternatives).

A problem with traditional decision tree analysis…

We were only able to completely evaluate this decision because we somehow knew what we thought to be the appropriate risk-adjusted discount rate to apply to it (here assumed to be 20%).

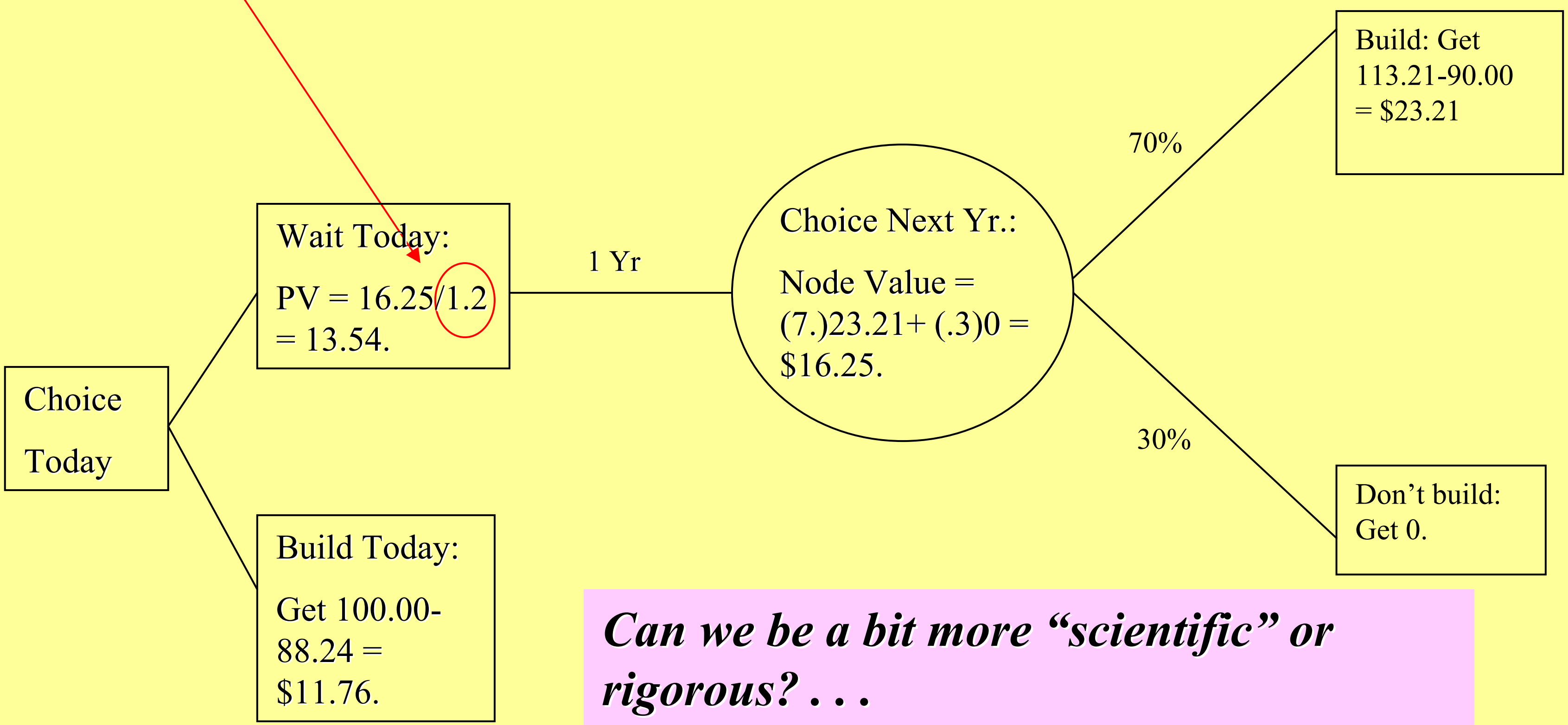


But is this really the correct discount rate (and hence, the correct decision and valuation of the project)?...

Where *did* the 20% discount rate (OCC) come from anyway?...

To be honest…

It was a nice round number that seemed "in the ballpark" for required returns on development investment projects.



Can we be a bit more "scientific" or rigorous? . . .

27.3.1 *An Arbitrage Analysis…*

Suppose there were "complete markets" in land, and buildings, and bonds, such that we could buy or sell (short if necessary) infinitely divisible quantities of each, including land and buildings like our subject development project…

Thus, we could buy today:

• 0.67 units of a building just like the one our subject development would produce next year that will either be worth $113.21 or $78.62 then.

And we could partially finance this purchase by issuing:

• $51.21 worth of riskless bonds (with a 3% interest rate).

Then this *"replicating portfolio"* (long in the bldg, short in the bond) next year will be worth:

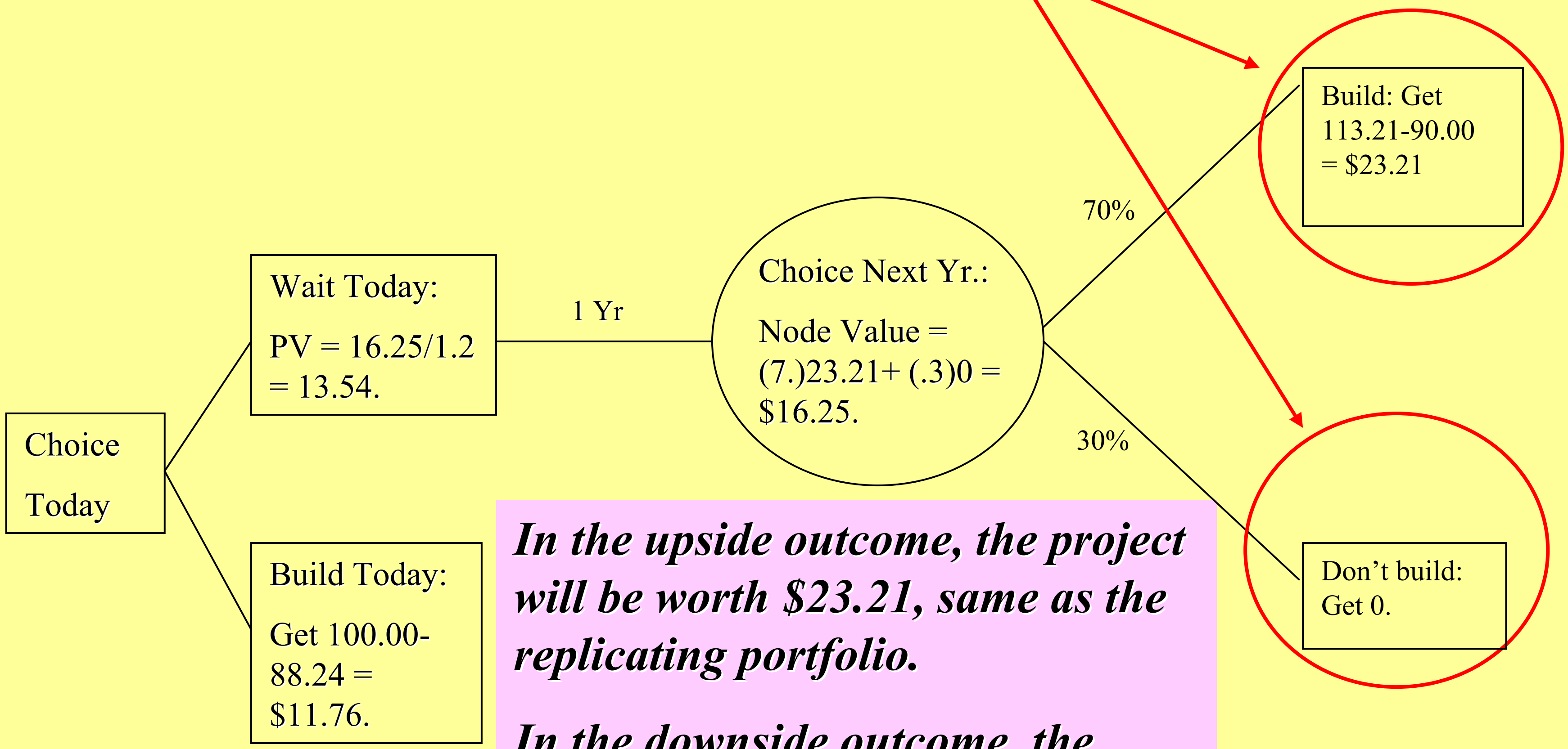
- **In the "up" scenario: (0.67)$113.21 - $51.21(1.03) = $75.95 – $52.74 = $23.21, or:**
- **In the "down" scenario: (0.67)$78.62 - $ 51.21(1.03) = $52.74 – $52.74 = 0.**

Exactly Equal to the Development Project in <u>All Future Scenarios</u>!

27.3.1

Recall:

These are the future scenarios, describing all possible future outcomes.



In the upside outcome, the project will be worth $23.21, same as the replicating portfolio.

In the downside outcome, the project will be worth 0, same as the replicating portfolio.

Thus, this "replicating portfolio" *must* be worth the same as the land (the development option) today.

Suppose not:

• If the land can be bought for less than the replicating portfolio, then I can sell the replicating portfolio short, buy the land, pocket the difference as profit today, and have zero net value impact next year (as the land and replicating portfolio will in all cases be worth the same next year, so my long position offsets my short position exactly).

• If the land costs more than the replicating portfolio, then I can sell the land short, buy the replicating portfolio, pocket the difference as profit today, and once again have zero net impact next year.

This is what is known as an "arbitrage" – riskless profit!

In equilibrium (within and across markets), arbitrage opportunities cannot exist, for they would be bid away by competing market participants seeking to earn super-normal profits.

In real estate, markets are not so perfect and complete to enable actual construction of technical arbitrage. But nevertheless competition tends to eliminate super-normal profit, so we can use this kind of analysis to model prices and values.

Fundamentally, this approach will always equalize the expected return risk premium per unit of risk, across the asset markets.

So, how much *is* the land worth in our example . . .

The replicating portfolio is:

$$\mathbf{(0.67)V(0) - \$51.21}$$

And thus must have this value.

The only question is, what is the value of V(0), the value of the underlying asset (the project to be developed) *today* (time-0)?...

27.3.1

We know that a similar asset already completed today is worth \$100.00.

However, this value includes the value of the net cash flow (dividends, rents) that asset will pay between today and next year.

Our development project won't produce those dividends, because it won't produce a building until next year.

So, we need a little more analysis…

Suppose that the underlying asset (the built property) has an expected total return of 9%.

If a similar building has a value today of \$100.00, and an (ex dividend) value next year of either \$113.21 (70% chance) or \$78.62 (30% chance), then the expected value next year is (0.7)113.21+(0.3)78.62 = \$102.83 (i.e., expected growth is $E[g_V]=2.83\%$).

Thus, the PV today of a building that would not exist until next year (i.e., PV of similar pre-existing building net of its cash flow between now and next year) is:

$$\mathbf{PV[V_1] = V(0) = \$102.83 / 1.09 = \$94.34.}$$

(versus $V_0 = \$100.00$ for pre-existing bldg.)

Now we can value the option by valuing the replicating portfolio:

$$
\begin{aligned}
C_0 &= (0.67)V(0) - \$51.21 \\
&= (0.67)\$94.34 - \$51.21 \\
&= \$63.29 - \$51.21 \\
&= \$12.09.
\end{aligned}
$$

Thus, our previous estimate of \$13.54 (based on the 20% OCC) was apparently not correct. The option is actually worth \$1.45 less.

The general formula for the Replicating Portfolio in a Binomial World is:

Replicating Portfolio = NV-B, where:

"N" is "shares" (proportional value) of the underlying asset (built property) to purchase,

"B" is current (time 0) dollar value of bond to issue (borrow), and:

N=(Cu-Cd)/(Vu-Vd); and

B=(NVd-Cd)/(1+r_f).

With: Cu = MAX[Vu-K, 0]; Cd = MAX[Vd-K, 0];

Vu, Vd, = "up" & "down" values of property to be built; K = constr cost.

In the preceding example:
N = (23.21-0)/(113.21-78.62) = 23.21/34.59 = 0.67; and
B = (0.67(78.62)-0)/1.03 = $52.74/1.03 = $51.21.

Suppose we could sell the option for $13.54…

Then we could (with complete markets):

- Sell the option (short) for $13.54, take in $13.54 cash.
- Borrow $51.21 at 3% interest (with no possibility of default), thereby take in another $51.21 cash.
- Use part of the resulting $64.75 proceeds to buy 0.67 units of a building just like the one to be built (minus its net rent for this coming year), for a price of (0.67)$94.34 = $63.29.
- Our **net cash flow at time 0 is: +$64.74 – $63.29 = + $1.45.**
- A year from now, we face:
  - In the **"up"** outcome:
    - We must pay to the owner of the option we sold $23.21 = $113.21 - $90, the value of the development option.
    - We must pay off our loan for (1.03)$51.21 = $52.74.
    - We will sell our .67 share of the building for (.67)$113.21 = $75.95 cash proceeds.
    - Giving us a total **net cash flow next year of $75.95 – ($23.21 + $52.74) = $75.95 - $75.95 = 0.**
  - In the **"down"** outcome:
    - We owe the owner of the option nothing, but we still owe the bank $52.74.
    - We sell our .67 share of the building for (.67)$78.62 = $52.74 cash proceeds.
    - Giving us a total **net cash flow next year of 0.**
- Thus, we make a **riskless profit** at time 0 of **+$1.45.** (= $13.54 - $12.09.)
- **We could perform arbitrage for any option price other than $12.09.**

27.3.1

Here is another way of depicting what we have just suggested (Exh.27-3):

	Today	Next Year	
Development Option Value $C = Max[0,V-K]$	$PV[C_1] = x$ "***x***" = *unkown value,* $\boldsymbol{x} = P_0$ *, otherwise arbitrage..*	$C_1^{up} = 113.21-90$ **= \$23.21**	C_1^{down} **= 0** *(Don't build)*
Built Property Value	$PV[V_1] = E[V_1] / (1+OCC)$ = [(.7)113.21+(.3)78.62]/1.09 = *\$102.83/1.09 =\$94.34*	V_1^{up} = *\$113.21*	V_1^{down} = *\$78.62*
Bond Value	B = *\$51.21*	$B_1 = (1+r_f)B =$ *(1.03)51.21* = *\$52.74*	$B_1 = (1+r_f)B =$ *(1.03)51.21* = *\$52.74*
Replicating Portfolio: $P = (N)V - B$	$P_0 = (N)\ PV[V_1] - B$ = *(0.67)\$94.34 - \$51.21* = *\$63.29 - \$51.21* = ***\$12.09***	P_1^{up} =*(0.67)113.21 - \$52.74* = *\$75.95 - \$\$52.74* **= \$23.21**	P_1^{down} =*(0.67)78.62 - \$52.74* = *\$52.74 - \$52.74* **= \$0**

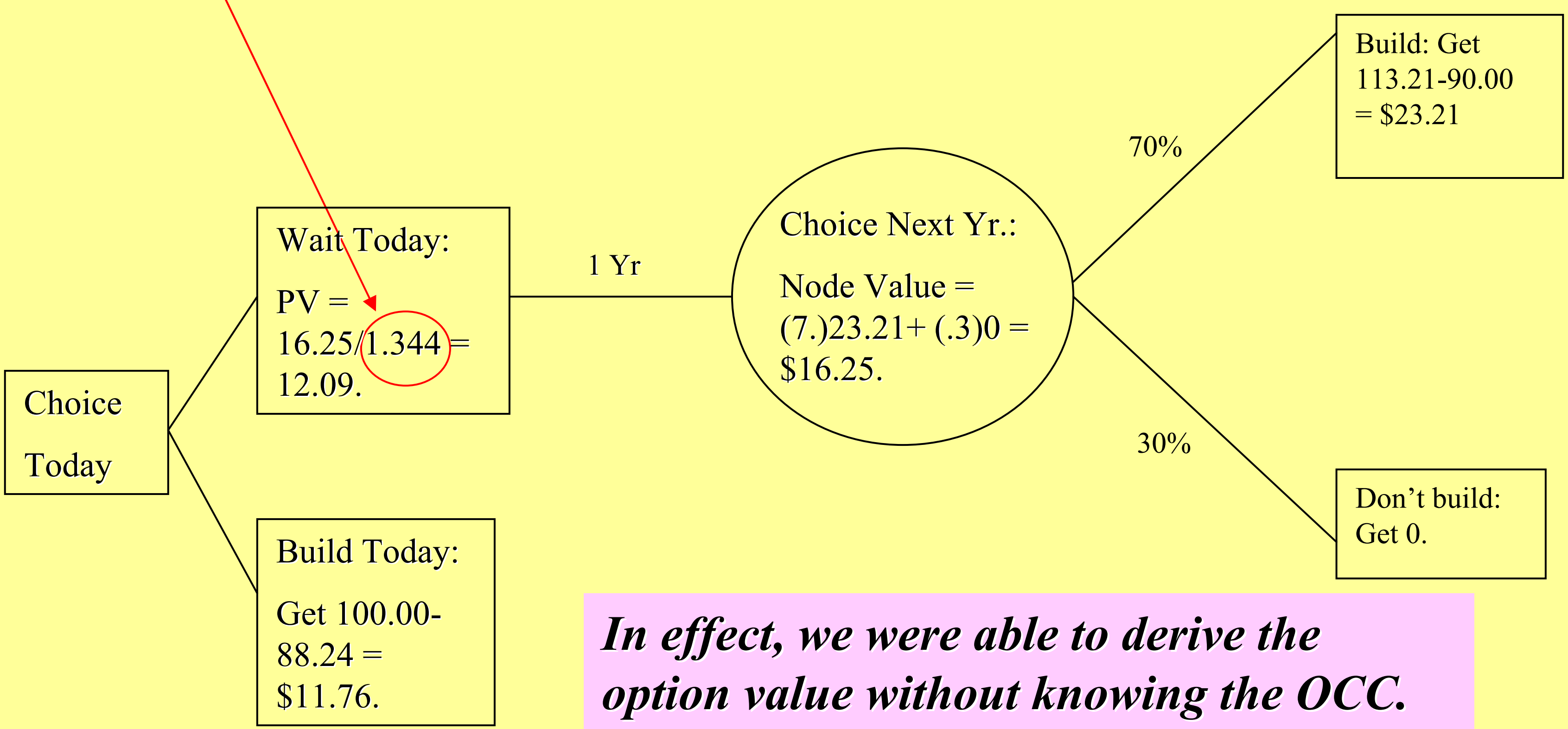
The replicating portfolio duplicates the option value in all future scenarios, hence its present value must be the same as the option's present value: C_0.

Thus, the option is worth \$12.09.

We can now correct our decision tree:

The correct OCC was not 20%, but rather 34.4%.

We know this because this is the rate that gives the correct PV of the option: $12.09 = E[C] / (1+E[r_c]) = \$16.25 / 1.344$.



In effect, we were able to derive the option value without knowing the OCC. If we want to know the OCC we can "back it out" from the option value.

Note:

This options-based derivation of the OCC of developable land is completely consistent with Chapter 29's formula for development project OCC:

$$PV[C_T] = \frac{V_T - L_T}{(1+E[r_C])^T} = \frac{V_T}{(1+E[r_V])^T} - \frac{L_T}{(1+E[r_D])^T}$$

Only in the circumstance where the option will definitely be developed next period (e.g., in the previous example, if the construction cost were \$78.62 million instead of \$90 million, the option would be worth \$18.01 million and it would be "ripe" for immediate development):

$$\$18.01 = \frac{\$102.83 - 78.62}{1.344} = \frac{\$102.83}{1.09} - \frac{\$78.62}{1.03}$$

In all cases, the result is to provide the same expected return risk premium *per unit of risk* across all the asset markets (land, buildings, bonds): the *equilibrium* condition within and across the relevant markets.

Here is the corrected summary of the analysis of the development project:

The land is worth: ***MAX[\$100.00 – \$88.24, C_0] = \$12.09:***

➔ 34.4% OCC for the option

	Today	Next Year	
Probability	100%	30%	70%
Value of Developed Property	\$100.00	\$78.62	\$113.21
Development Cost (exclu land)	\$88.24	\$90.00	\$90.00
NPV of exercise	\$11.76	-\$11.38	\$23.21
(Action)		(Don't build)	(Build)
Future Values		0	\$23.21
Expected Values	\$11.76	\$16.25	
= Sum[Probability X Outcome]	(1.0)11.76	(0.3)0 + (0.7)23.21	
PV(today) of Alternatives @ 34%	\$11.76	16.25 / 1.344 = \$12.09	

What is the value of this land today? **Answer: =** ***MAX[11.76, 12.09] = \$12.09***

Should owner build now or wait? **Answer: =** ***Wait. (100.00 – 88.24 – 12.09 < 0.)***

27.3.2

The previously described option valuation of a development project is completely consistent with the *"Certainty Equivalent Valuation"* form of the DCF valuation model presented earlier in the Chapter 10 lecture in this course.

The general 1-period Certainty Equivalent Valuation Formula is:

$$C_0 = \frac{CEQ_0[C_1]}{1+r_f} = \frac{E_0[C_1] - \left(C_{up} - C_{down}\right)\left(\dfrac{E[r_V] - r_f}{V_{up}\% - V_{down}\%}\right)}{1+r_f}$$

e.g., in our example:

$$C_0 = \frac{\left((.7)\$23.21 + (.3)\$0\right) - \left(\$23.21 - \$0\right)\left(\dfrac{9\% - 3\%}{(113.21/94.34)\% - (78.62/94.34)\%}\right)}{1+3\%}$$

$$= \frac{(\$16.25) - (\$23.21)\left(\dfrac{6\%}{120\% - 83.33\%}\right)}{1.05} = \frac{(\$16.25) - (\$23.21)\left(\frac{6\%}{36.67\%}\right)}{1.03} = \frac{(\$16.25) - (\$23.21)(0.1636)}{1.03}$$

$$= \frac{\$16.25 - \$3.80}{1.03} = \frac{\$12.45}{1.03} = \$12.09$$

I'm hoping you developed some intuition for the certainty equivalence valuation model back in Chapter 10. But in case not, let's try this . . .

The certainty equivalent value next year is the downward adjusted value of the risky expected value for which the investment market would be indifferent between that value and a riskfree bond value of the same amount…

$$C_0 = \frac{CEQ_0[C_1]}{1+r_f} = \frac{\boxed{E_0[C_1]} - \boxed{(C_{up} - C_{down})\left(\frac{E[r_V] - r_f}{V_{up}\% - V_{down}\%}\right)}}{1+r_f}$$

The certainty equivalent value next year is the expected value minus a risk discount.

$$\boxed{(C_{up} - C_{down})}\boxed{\left(\frac{E[r_V] - r_f}{V_{up}\% - V_{down}\%}\right)}$$

The risk discount consists of the amount of risk in the next year's value as indicated by the range in the possible outcomes times… ➔

times the market price of risk.

The market price of risk is the market expected return risk premium per unit of return risk, the ratio of…

the market expected return risk premium divided by the range in the corresponding return possible outcomes.

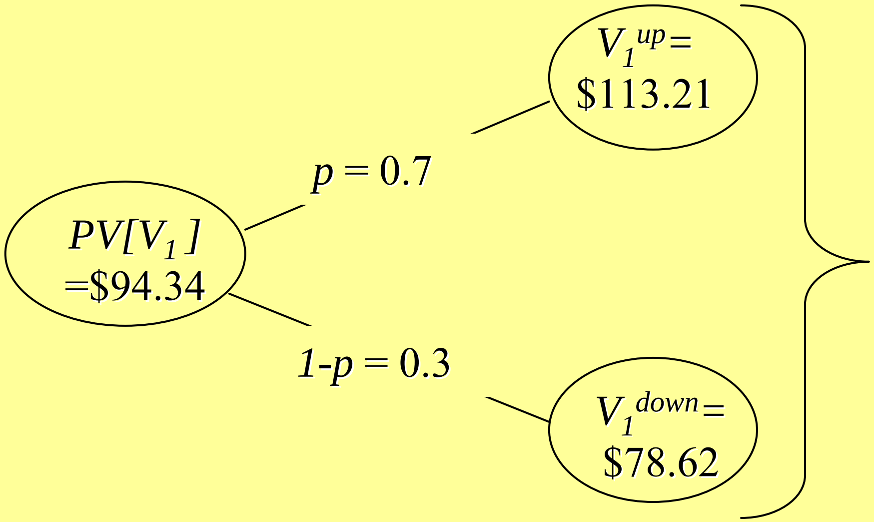
$$\frac{\boxed{E[r_V] - r_f}}{\boxed{V_{up}\% - V_{down}\%}}$$

Thus, we can derive the same present value of the option through two completely consistent (indeed, mathematically equivalent) approaches:

- **The "arbitrage analysis" based on the *replicating portfolio*, or;**
- **The certainty equivalent valuation model.**

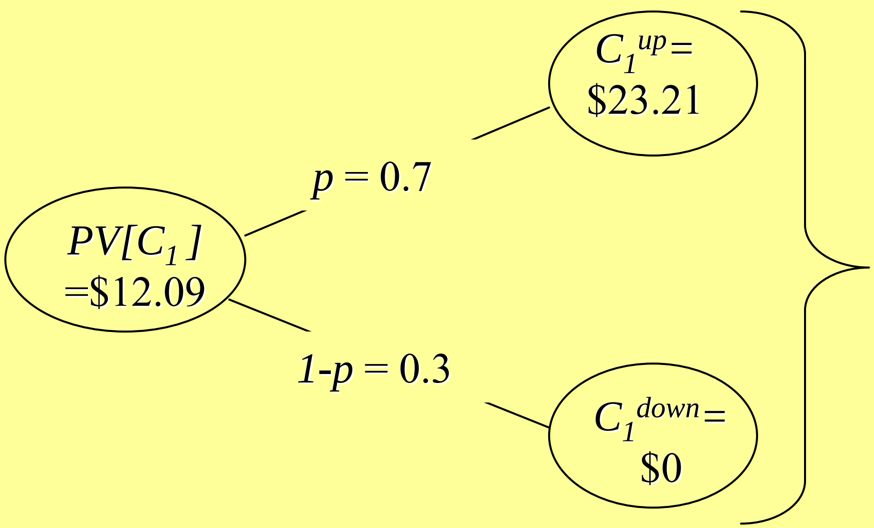
The latter is more convenient for computations.

27.3.3 *What is fundamentally going on with this framework:*



Underlying asset (built property) outcome % range:

$$\frac{\$113.21-\$78.62}{\$94.34}=37\%$$



Option (land) outcome % range:

$$\frac{\$23.21-\$0}{\$12.09}=192\%$$

With *perfect* correlation between the two (land is *derivative*).

27.3.3 *What is fundamentally going on with this framework:*

With *perfect* correlation between the two (land is *derivative*).

Hence, relative risk exactly equals ratio of outcome ranges:

$$\frac{192\%}{37\%} = 5.24$$

Option (land) is 5.24 times more risky than investment in the underlying asset (built property).

Thus, option value must be such that $E[r_C]$ risk premium in land is 5.24 times greater than that in built property.

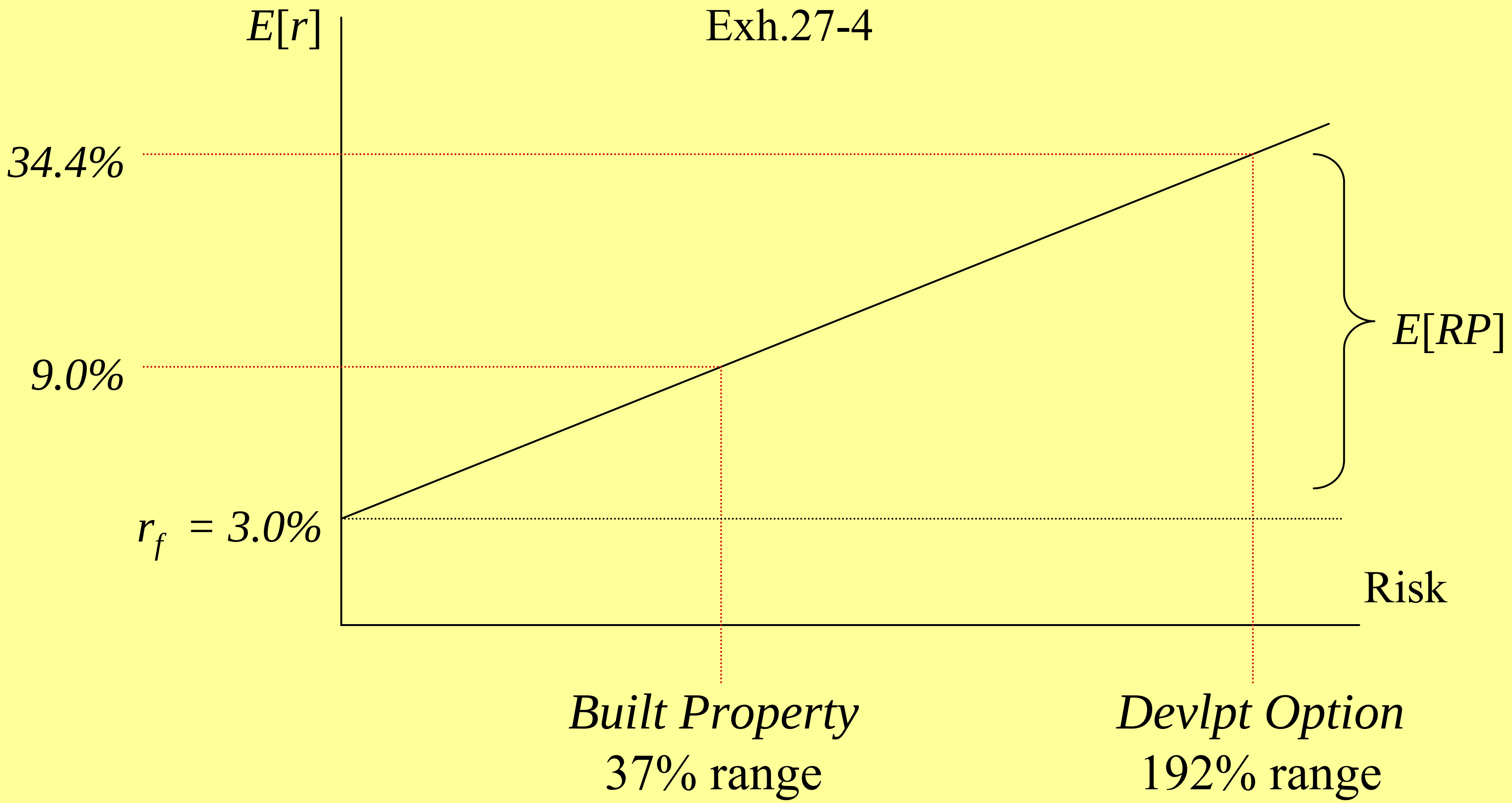
Built property risk premium is: $RP_V = 9\% - 3\% = 6\%$.

Thus, land risk premium must be: $RP_C = 6\% * 5.24 = 31.4\%$.

Thus land's $E[r_C] = r_f + RP_C = 3\% + 31.4\% = 34.4\%$.

27.3.3 *What is fundamentally going on with this framework:*

The "price of risk" (the ex ante investment return risk premium per unit of risk) is being equated across the markets for land and built property:



If this relationship does not hold, then there are "super-normal" (disequilibrium) profits (expected returns) to be made somewhere, and correspondingly "sub-normal" profits elsewhere, across the markets for: *Land*, *Stabilized Property*, and *Bonds ("riskless" CFs)*.

“Real” vs “Risk-neutral” dynamics . . .

Note that the probabilities and expected future values used in this model are “real”, not “risk-neutral dynamics” values. i.e., (0.7)113.21 + (0.3)78.62 = $102.83 is the real or true expected value of the to-be-built building next year, and 70% and 30% are the true probabilities of the “up” and “down” outcomes.

It is necessary in this formulation to know:

- The expected total return (OCC) to the underlying asset (the $E[r_V]$ = 9% in our example), and
- The underlying asset’s cash payout rate (the $E[y_V]$ = 6% in our example).

It is also possible to obtain an exactly equivalent solution using so-called “risk-neutral dynamics”, in which case it is not necessary to know the OCC of the underlying asset. However, this poses little additional advantage in the case of real estate, and it results in a less intuitive formulation.

27.4 The Binomial Option Value Model

Think of an individual binomial element (1 period, either "up" or "down") as like a financial economic "molecule": the smallest, simplest representation of the essential characteristics dealt with by financial economics: value over *time* with *risk*.

We can have as many periods of time as we want (individual "molecules" stitched together as in a "crystal": as layers or rows & columns in a table, or nodes & branches in a "tree).

Each period can represent as short a span of calendar time as we want. We can have as many periods as we want.

Result:

Binomial "tree" can very realistically model actual evolution of values over time.

CD27.4 The Binomial Option Value Model

Here are the rules for constructing the underlying asset value tree.

Let:

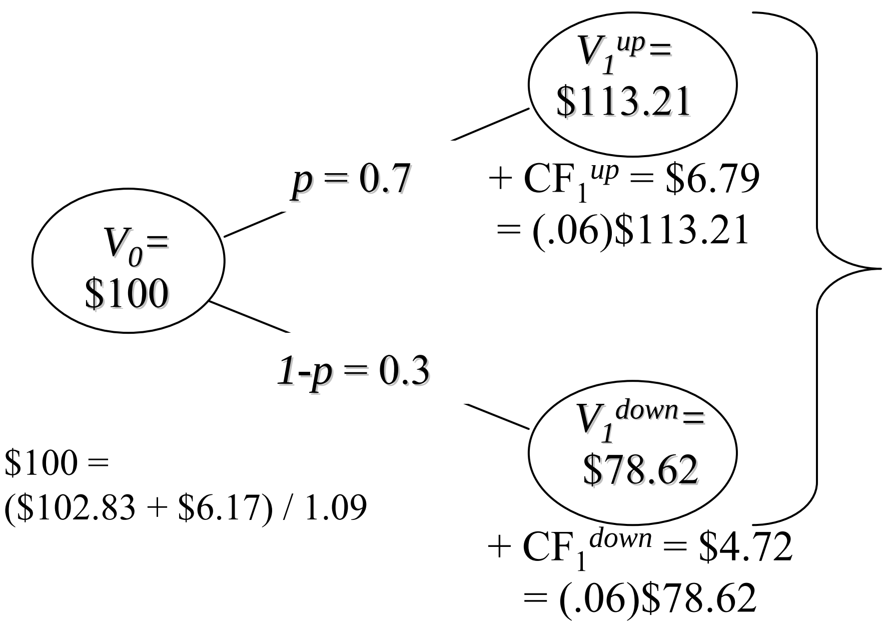
- **r_V = Expected total return rate on the underlying asset (built property).**
- **y_V = Payout rate (dividend yield or net rent yield).**
- **r_f = Riskfree interest rate**
- **σ = Annual volatility of underlying asset (instantaneous rate)*.**
- **V_t = Value of the underlying asset at time (end of period) t, *ex dividend* (i.e., net of current cash payout, i.e., the value of the asset itself based only on forward-looking cash flows beyond time t). The asset is assumed to pay out cash at a rate of y_V every period: $y_V = CF_{t+1} / V_{t+1}$.****

All rates are simple periodic rates: $r = i/m$, where r is the simple periodic rate, i is the nominal annual rate, and m is the number of periods per year. The implied effective annual rate (EAR) is thus given by: $1+EAR = (1+r)^m$

CD27.4 The Binomial Option Value Model

For example, in our previous illustration…

- r_V **= 9%**
- y_V **= 6%**
- r_f **= 3%**
- **σ = Let's say this is 20%.**
- V_t **= \$100 at time 0, E[**V_{t+1}**] = \$102.83 at time 1.**



$E[V_1] = (1.09)\$100/(1.06) = \$102.83.$

$E[V_1] = (0.7)\$113.62 + (0.3)\$78.62 = \$102.83.$

$E[CF_1] = (0.06)\$102.83 = \$6.17.$

$E[CF_1] = (0.7)\$6.79 + (0.3)\$4.72 = \$6.17.$

"going-in cap rate" = \$6.17 / \$100 = 6.17%.

$E[g_V] = (1+r_V)/(1+y_V) = 1.09/1.06 - 1 = 2.83\%$
$= r_V - (\textit{going-in cap rate}) = 9\% - 6.17\%.$

CD27.4 The Binomial Option Value Model

Now define the 1-period "up" movement ratio as:

$u = V_{up} / V(0)$. e.g., in our last example: u = \$113.21 / \$94.34 = 1.20.

For the binomial model to work, the "down" movement ratio must be the *inverse* of the "up" movement ratio:

$d = V_{down} / V(0) = 1 / u$. e.g., in our last example: d = \$78.62 / \$94.34 = 0.833 = 1/1.20.

The magnitude of the "up" movement is determined so that the binomial tree will converge to a "normal" (Gaussian) distribution of periodic returns with annual volatility σ as the period lengths approach zero ($m \rightarrow \infty$, or $T/n \rightarrow 0$). This requires:

$$u = 1 + \sigma\sqrt{T / n}$$

where T is the total calendar time in the tree (in years) and n is the total number of periods (hence, T/n is the fraction of a year in any one period, and $m = n/T$ is the number of periods per year).

In our previous example: $T = 1$, $n = 1$, and $\sigma = 20\%$.

CD27.4 The Binomial Option Value Model

The probability of the "up" move, *p*, is determined so that the binomial tree will converge to a normal (Gaussian) distribution of periodic returns with a mean annual total return based on r_V as the period lengths approach zero ($m \rightarrow \infty$, or $T/n \rightarrow 0$). [Or equivalently, an appreciation return of approximately: $g_V = (1+r_V)/(1+y_V)-1$.] This requires:

$$p = \frac{(1+r_V)-d}{u-d} = \frac{(1+r_V)-1/\left(1+\sigma\sqrt{T/n}\right)}{\left(1+\sigma\sqrt{T/n}\right)-1/\left(1+\sigma\sqrt{T/n}\right)}$$

The probability of the "down" movement is of course just $1-p$.

Note: These are <u>*actual*</u> *probabilities, not "risk-neutral" pseudo-probabilities. They produce a tree that reflects actual real underlying value distributions.*

CD27.4 The Binomial Option Value Model

Example based on our previous illustration…

$$p = \frac{(1+r_V)-d}{u-d} = \frac{(1+r_V)-1/\left(1+\sigma\sqrt{T/n}\right)}{\left(1+\sigma\sqrt{T/n}\right)-1/\left(1+\sigma\sqrt{T/n}\right)}$$

$$= \frac{(1.09)-1/1.20}{1.20-1/1.20} = \frac{1.09-0.833}{1.20-0.833} = \frac{0.2567}{0.3667} = 0.70$$

The probability of the "down" movement is of course just:

1 – p = 1 – 0.7 = 0.3 .

CD27.4 The Binomial Option Value Model

The binomial tree for the underlying asset ex-dividend values is then constructed as follows.

For any given value node with current (observable) ex-dividend value V_t, the subsequent "up" and "down" values in the two possible subsequent value nodes are:

$$V_{t+1}^{up} = uV_t / (1 + y_V) = \left(1 + \sigma\sqrt{T/n}\right) V_t / (1 + y_V)$$

$$V_{t+1}^{down} = dV_t / (1 + y_V) = V_t / \left(\left(1 + \sigma\sqrt{T/n}\right)(1 + y_V)\right)$$

CD27.4 The Binomial Option Value Model

The binomial tree for the underlying asset ex-dividend values is then constructed as follows.

For example in our previous illustration…

$$V_{t+1}^{up} = uV_t / (1 + y_V) = \left(1 + \sigma\sqrt{T/n}\right)V_t / (1 + y_V)$$

$$= (1.20)\$100 / (1.06) = \$113.21$$

$$V_{t+1}^{down} = dV_t / (1 + y_V) = V_t / \left(\left(1 + \sigma\sqrt{T/n}\right)(1 + y_V)\right)$$

$$= (0.833)\$100 / (1.06) = \$78.62$$

CD27.4 The Binomial Option Value Model
Numerical Example:

Suppose:

Expected total return on the underlying asset is 10%, with a cash payout rate of 6%, and a riskfree interest rate of 3% (all nominal annual rates).

Option expires in T = 1 year. n = 12: There are 12 periods (beyond 0), of one month each (1 year total). Hence:

$r_V = 10\%/12 = .833\%$, $y_V = 6\%/12 = 0.5\%$, $r_f = 3\%/12 = 0.25\%$;

➔ $g_V = (1+r_V)/(1+y_V)-1 = 1.00833/1.005 - 1 = 0.0033 = 0.33\%$.

Suppose the volatility of the underlying asset (built property) is σ = 15%.

V_0 = \$100, the time 0 value of the underlying asset (as if pre-existing).

Thus: u = [1+.15*SQRT(1/12)] = 1.0433; $d = 1 / u = 1 / 1.0433 = 0.9585$.

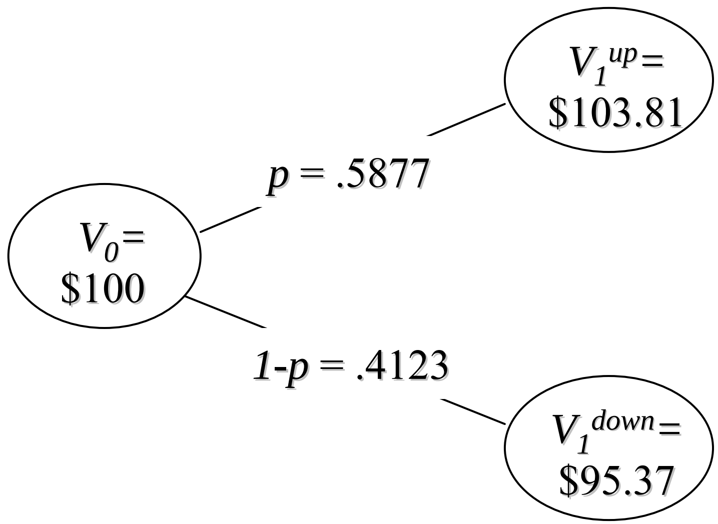
$V^{up} = u(\$100)/(1+.005) = \$104.33/(1.005) = \$103.81$.

$V^{down} = d(\$100)/(1+.005) = \$95.85/(1.005) = \$95.37$.

p = ((1+.10/12)-1/(1+.15*SQRT(1/12)))/((1+.15*SQRT(1/12))-1/(1+.15*SQRT(1/12)))
= (1.00833 – .9585)/(1.0433 – .9585) = 0.5877; *hence:* $1 - p = 0.4123$.

CD27.4 The Binomial Option Value Model

Numerical Example (cont.)…



Note that: $E[V_1] = .5877(103.81) + .4123(95.37) = \$100.33 = (1.0033)\$100 = (1+g_V)V_0$

Note that: $V(0) = \$100.33 / (1+(.10/12)) = \$100.33/1.00833 = \$99.50 \neq \$100 = V_0$

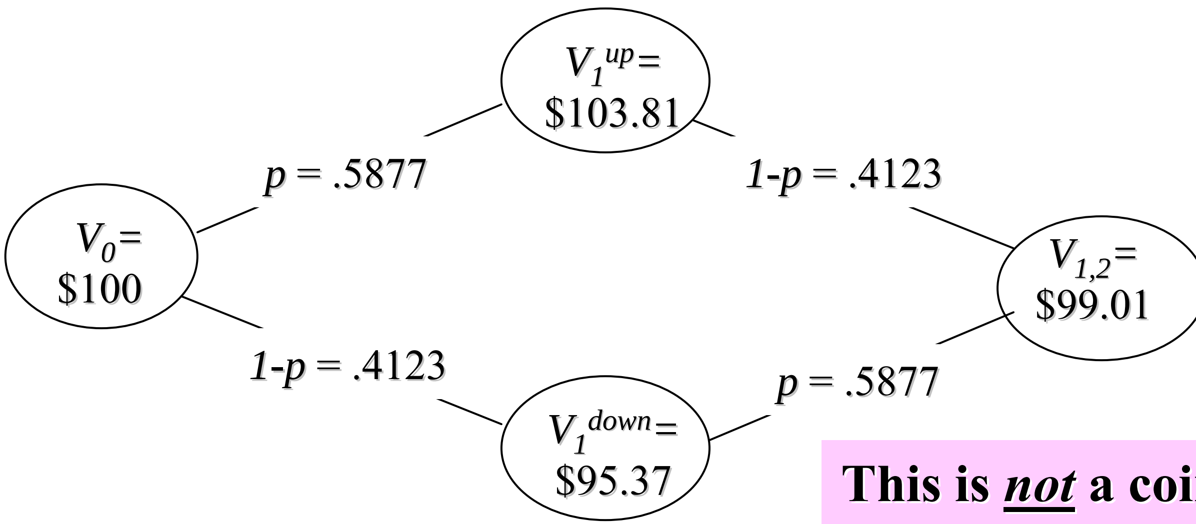
Equivalently: $V(0) = V_0 / (1+y_V) = \$100 / (1+(.06/12)) = \$100/1.005 = \$99.50.$

CD27.4 The Binomial Option Value Model

Now consider the "down" jump from V_1^{up} , and the "up" jump from V_1^{down} (call this value "$V_{1,2}$", because it is in the 1st row down from the top, 2nd column over from the left in the overall binomial tree) . . .

$V_{1,2}$ = *up* from $V_1^{down} = u(\$95.37)/(1+.06/12) = 1.0433(95.37)/1.005 = \$99.01$.
$V_{1,2}$ = *down* from $V_1^{up} = d(\$103.81)/(1+.06/12) = .9585(103.81)/1.005 = \$99.01$.

It's the same value!



This is *not* a coincidence.

It is a general property of the way we have constructed the binomial tree. (constant u, $d = 1/u$.)

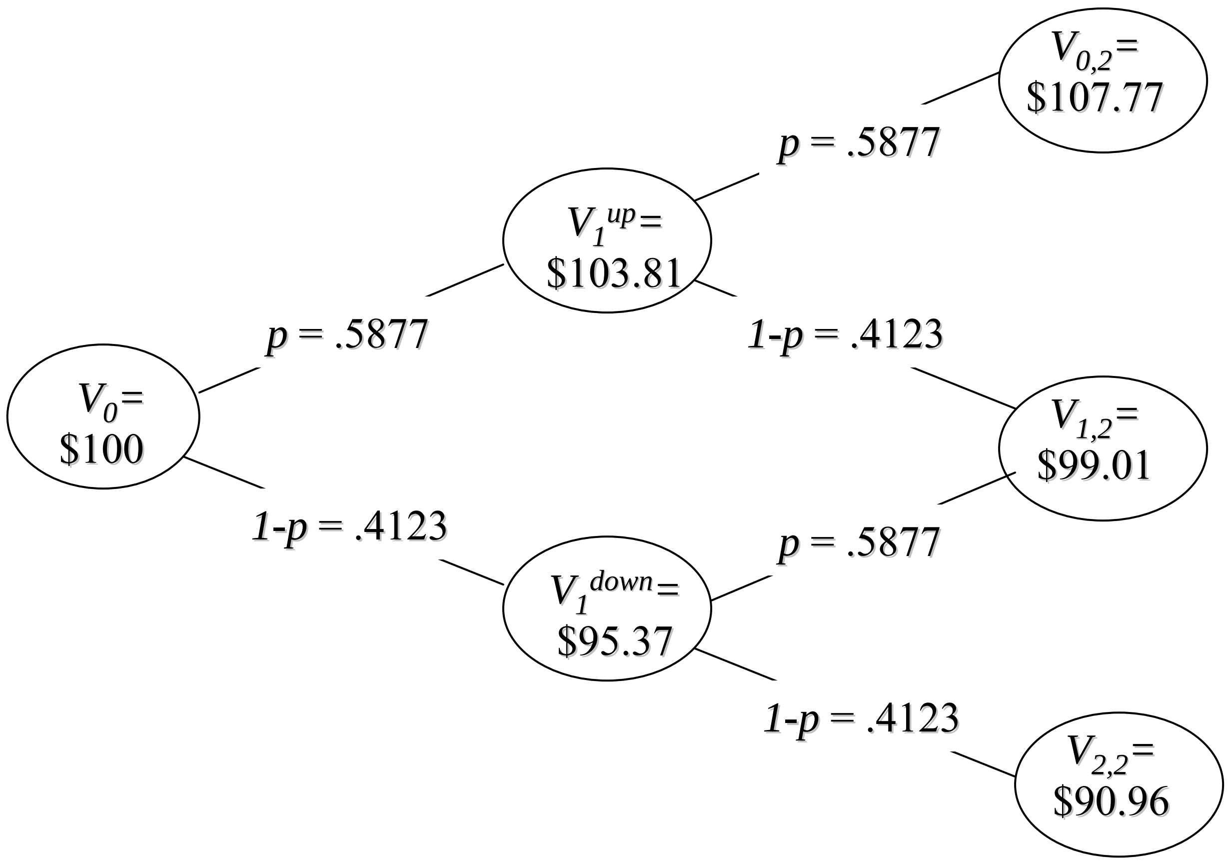
CD2.4 The Binomial Option Value Model

We build the underlying asset value tree forward in this manner…

Here is the tree up through the $V_{0,2}$, $V_{1,2}$, and $V_{2,2}$ value nodes . . .

$V_{0,2}$ = *up* from $V_1^{up} = u(\$\ 103.81)/(1+.06/12) = 1.0433(103.81)/1.005 = \$107.77$.

$V_{2,2}$ = *down* from $V_1^{up} = d(\$95.37)/(1+.06/12) = .9585(95.37)/1.005 = \$90.96$.



27.4 The Binomial Option Value Model

We build the underlying asset value tree forward in this manner…

			$116.58
		$110.77	
	$105.25		$104.20
$100.00		$99.01	
	$94.07		$93.14
		$88.49	
			$83.25

27.4 The Binomial Option Value Model

We build the underlying asset value tree forward in this manner…

												$184.73
											$175.52	
										$166.77		$165.11
									$158.45		$156.88	
								$150.55		$149.06		$147.58
							$143.05		$141.63		$140.22	
						$135.91		$134.57		$133.23		$131.91
					$129.14		$127.86		$126.59		$125.33	
				$122.70		$121.48		$120.28		$119.08		$117.90
			$116.58		$115.43		$114.28		$113.15		$112.02	
		$110.77		$109.67		$108.58		$107.50		$106.44		$105.38
	$105.25		$104.20		$103.17		$102.14		$101.13		$100.13	
$100.00		$99.01		$98.02		$97.05		$96.09		$95.13		$94.19
	$94.07		$93.14		$92.21		$91.30		$90.39		$89.49	
		$88.49		$87.62		$86.75		$85.89		$85.03		$84.19
			$83.25		$82.42		$81.60		$80.79		$79.99	
				$78.31		$77.53		$76.77		$76.00		$75.25
					$73.67		$72.94		$72.21		$71.50	
						$69.30		$68.61		$67.93		$67.26
							$65.19		$64.55		$63.91	
								$61.33		$60.72		$60.12
									$57.69		$57.12	
										$54.27		$53.73
											$51.05	
												$48.03

27.4 The Binomial Option Value Model

Here is the 12-period, monthly periods (1 year) numerical example tree we have been working on (from Excel)…

V tree (net of payout, "ex dividend" values):

Period ("*j* "): "*n* " =

0	1	2	3	4	5	6	7	8	9	10	11	12

V tree (net of payout, "ex dividend" values):

100.00	103.81	107.77	111.87	116.14	120.56	125.16	129.93	134.88	140.02	145.36	150.90	156.65
	95.37	99.01	102.78	106.70	110.76	114.99	119.37	123.92	128.64	133.54	138.63	143.91
		90.96	94.43	98.02	101.76	105.64	109.66	113.84	118.18	122.69	127.36	132.22
			86.75	90.06	93.49	97.05	100.75	104.59	108.58	112.71	117.01	121.47
				82.74	85.89	89.16	92.56	96.09	99.75	103.55	107.50	111.60
					78.91	81.92	85.04	88.28	91.64	95.13	98.76	102.52
						75.26	78.12	81.10	84.19	87.40	90.73	94.19
							71.77	74.51	77.35	80.30	83.36	86.53
								68.45	71.06	73.77	76.58	79.50
									65.29	67.77	70.36	73.04
										62.26	64.64	67.10
											59.38	61.65
												56.64

Notice the first element here, as we previously calculated it.

Each node in the tree (each row, column cell in the table) represents a possible future "state of the world", as indicated by a possible value of the underlying asset as of the given future time period (month in this case).

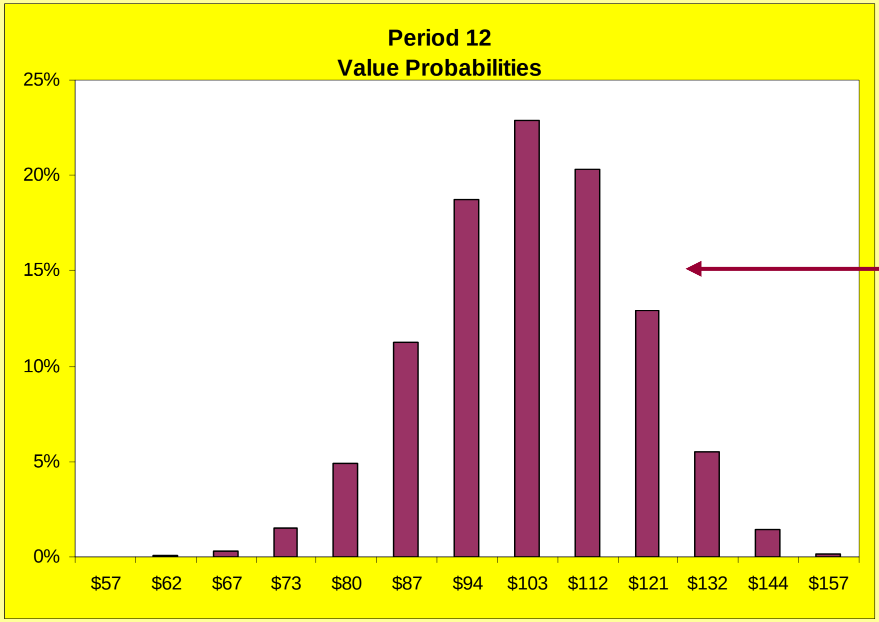
27.4 The Binomial Option Value Model

Although the conditional probabilities are *p* = .5877 and *1-p* = .4123 going forward one period (*up* and *down*) from any given node, over multiple periods the unconditional probabilities become bell-shaped over all the possible outcomes…

Tree real probabilities (p based):

Period ("*j* "): "*n* " =

0	1	2	3	4	5	6	7	8	9	10	11	12
1.0000	0.5877	0.3454	0.2030	0.1193	0.0701	0.0412	0.0242	0.0142	0.0084	0.0049	0.0029	0.0017
	0.4123	0.4846	0.4272	0.3347	0.2459	0.1734	0.1189	0.0798	0.0528	0.0345	0.0223	0.0143
		0.1700	0.2997	0.3523	0.3451	0.3042	0.2503	0.1961	0.1482	0.1088	0.0782	0.0551
			0.0701	0.1648	0.2421	0.2846	0.2926	0.2752	0.2426	0.2036	0.1645	0.1289
				0.0289	0.0849	0.1497	0.2053	0.2413	0.2553	0.2500	0.2309	0.2035
					0.0119	0.0420	0.0864	0.1355	0.1791	0.2105	0.2268	0.2285
						0.0049	0.0202	0.0475	0.0838	0.1231	0.1591	0.1870
							0.0020	0.0095	0.0252	0.0494	0.0798	0.1125
								0.0008	0.0044	0.0130	0.0280	0.0493
									0.0003	0.0020	0.0065	0.0154
										0.0001	0.0009	0.0032
											0.0001	0.0004
												0.0000



Actually, although the model converges toward continuously-compounded return probabilities that are normally distributed, the asset value level probabilities are log-normally distributed (skewed bell shape).

CD27.4 The Binomial Option Value Model

Define the tree as a table of rows and columns. The j^{th} column is the number of periods after the present (time 0), where $j = 0, 1, 2, \ldots, n$ (where n is the total number of periods). The i^{th} row is the number of "down" moves in the asset price since time 0, where $i = 0, 1, 2, \ldots, j$. Each row, column cell (i, j) defines a "state of the world" j periods in the future. $V_{i,j}$ is the value of the underlying asset in that state.

V tree (net of payout, "ex dividend" values):

Period ("*j*"):

"down" moves ("*i*"):	0	1	2	3	4	5	6	7	8	9	10	11	"*n*" = 12
	V tree (net of payout, "ex dividend" values):												
0	100.00	103.81	107.77	111.87	116.14	120.56	125.16	129.93	134.88	140.02	145.36	150.90	156.65
1		95.37	99.01	102.78	106.70	110.76	114.99	119.37	123.92	128.64	133.54	138.63	143.91
2			90.96	94.43	98.02	101.76	105.64	109.66	113.84	118.18	122.69	127.36	132.22
3				86.75	90.06	93.49	97.05	100.75	104.59	108.58	112.71	117.01	121.47
4					82.74	85.89	89.16	92.56	96.09	99.75	103.55	107.50	111.60
5						78.91	81.92	85.04	88.28	91.64	95.13	98.76	102.52
6							75.26	78.12	81.10	84.19	87.40	90.73	94.19
7								71.77	74.51	77.35	80.30	83.36	86.53
8									68.45	71.06	73.77	76.58	79.50
9										65.29	67.77	70.36	73.04
10											62.26	64.64	67.10
11												59.38	61.65
12													56.64

The ex ante probability (as of time 0) of any given state of the world i, j is given by:

$$prob(i, j) = \left(\frac{j!}{i!(j-i)!}\right) p^{(j-i)} (1-p)^{i}$$

where the symbol "!" indicates the "factorial" product operation: $x! = 1*2*3* \ldots *x$.

CD27.4 The Binomial Option Value Model

In this numerical example, the Period 12 underlying asset values and probabilities, indicated in the last column on the right in the previous two slides, give:

$$\mathbf{E_0[V_{12}] = E_0[V(1\ yr)] = \$104.05}$$

which is identical to: $V_0((1+r_V)/(1+y_V))^{12} = V_0(1+g_V)^{12} = \$100(1.0033)^{12} = \$104.05$,

And:

$$\mathbf{STD[V_{12}]/V_0 = 1\ yr\ Volatility = \pm 14.99\%}$$

which is very similar to the 15% simple annual volatility assumption.

(If you're curious, these statistics are found as follows…)

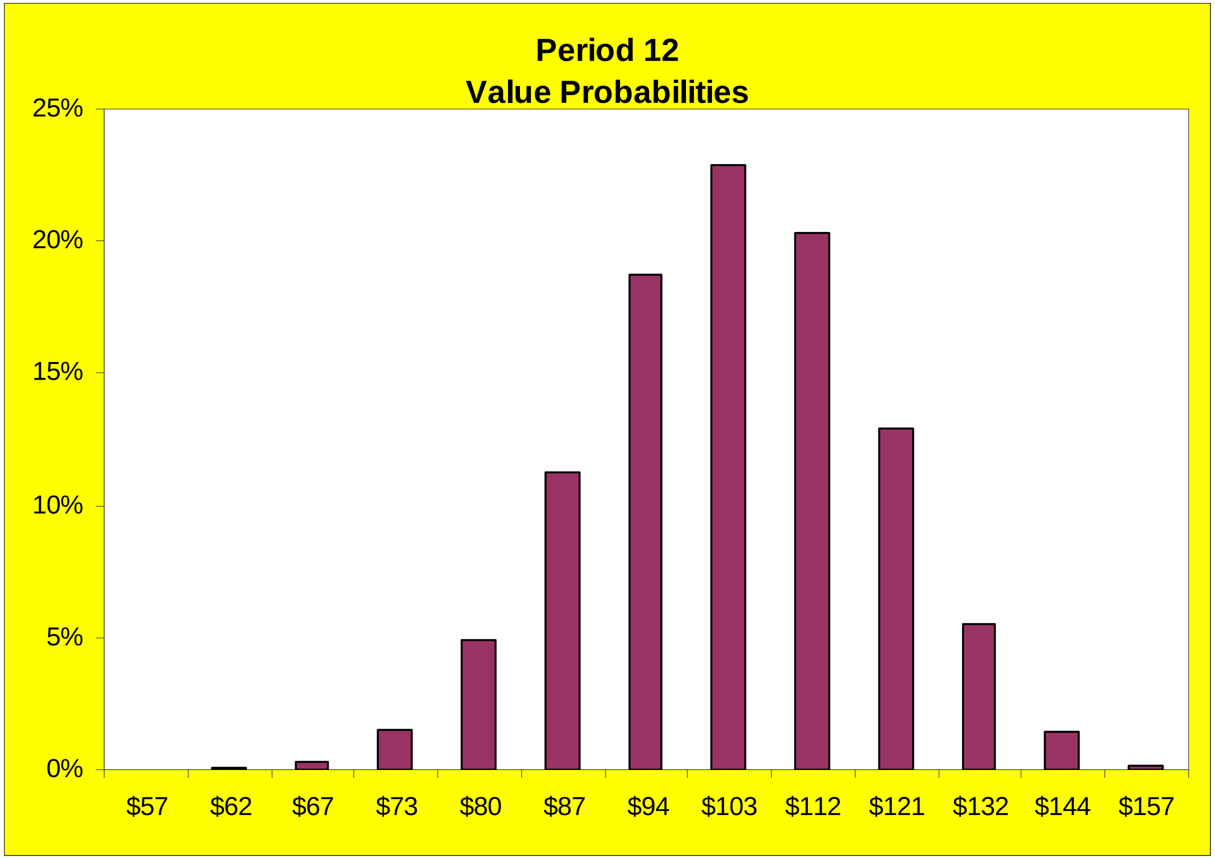
$$E_0[V_{12}] = \sum_{i=0}^{12} V_{i,12}\left(\frac{12!}{i!(12-i)!}\right)p^{(12-i)}(1-p)^i = \$104.05$$

where : $p = 0.5877$, *and* $V_{i,12}$ *are as found in the table.*

$$STD[V_{12}]/V_0 = \sqrt{\sum_{i=0}^{12}\left[\left(V_{i,12} - E_0[V_{12}]\right)^2\left(\frac{12!}{i!(12-i)!}\right)p^{(12-i)}(1-p)^i\right]} \Big/ V_0 = \pm 14.99\%$$

CD27.4 The Binomial Option Value Model

Value probabilities for the underlying asset (V) 12 periods into the future . . .



$$E_0[V_{12}] = \$104.05 = V_0(1.0033)^{12} = V_0(1+g_V)^{12}\ .$$

$$STD[V_{12}]/V_0 = \pm 14.99\% \approx 15\% = \sigma$$

CD27.4 The Binomial Option Value Model

For each node (cell) in the underlying asset value tree (the *"V Tree"*) described previously, there will also be associated a projected value of the *"exercise price"* for the option (the *construction cost* of the development project).

We label this cost *K*.

Assuming *K* grows risklessly at 2%/yr nominal (0.1667%/mo), the table of $K_{i,j}$ values giving construction costs corresponding to the previous $V_{i,j}$ values is as follows…

	Period ("*j*"):												"*n*" =
	0	1	2	3	4	5	6	7	8	9	10	11	12
"down" moves ("*i*"):	*K Value Tree:*												
0	80.00	80.13	80.27	80.40	80.53	80.67	80.80	80.94	81.07	81.21	81.34	81.48	81.61
1		80.13	80.27	80.40	80.53	80.67	80.80	80.94	81.07	81.21	81.34	81.48	81.61
2			80.27	80.40	80.53	80.67	80.80	80.94	81.07	81.21	81.34	81.48	81.61
3				80.40	80.53	80.67	80.80	80.94	81.07	81.21	81.34	81.48	81.61
4					80.53	80.67	80.80	80.94	81.07	81.21	81.34	81.48	81.61
5						80.67	80.80	80.94	81.07	81.21	81.34	81.48	81.61
6							80.80	80.94	81.07	81.21	81.34	81.48	81.61
7								80.94	81.07	81.21	81.34	81.48	81.61
8									81.07	81.21	81.34	81.48	81.61
9										81.21	81.34	81.48	81.61
10											81.34	81.48	81.61
11												81.48	81.61
12													81.61

CD27.4 The Binomial Option Value Model

How to value a call option using the model . . .

While the underlying asset value and exercise price trees are constructed going forward in time as described previously,

An option on the underlying asset is valued by working backward in time, starting at the right-hand edge of the tree (option expiration) and working back to the left.

The option can be valued one period at a time, at each node of the tree, based on the option values in the two subsequent possible nodes.

Starting in the last column (expiration period $j = n$) one works backwards in time ultimately to the present (time 0 at period $j = 0$).

Each valuation in each node (i , j cell in the table) is a simple 1-period binomial valuation using the certainty-equivalent present value model discussed previously.

CD27.4 The Binomial Option Value Model

The general formula and procedure for call option valuation is thus as follows . . .

***First*, let:**

$V_{i,j}$ = The ex-dividend underlying asset value in state-of-the-world *i* at time (period) *j*, as enumerated in the binomial value tree described previously (e.g., built property value).

K_j = The exercise price (construction cost) at time *j* (known for certain in advance). i.e., paying K_j in period *j* will produce in period *j* an asset worth $V_{i,j}$ at that time (instantaneous construction).

Then in the terminal period $j = n$ (in which the option expires), the call option is worth, in any given state whose underlying asset value is $V_{i,n}$:

$$C_{i,n} = Max(V_{i,n} - K_n, 0)$$

CD27.4 The Binomial Option Value Model

For example, look at the top two values in the terminal column $j = n = 12$ of our previous tree, $V_{0,12}$ = \$156.65 and $V_{1,12}$ = \$143.91 respectively.

Label the value of the call option in each of these nodes: $C_{0,12}$ and $C_{1,12}$.

Given that construction cost in month 12, K_{12}, is \$81.61 (see previous table), we thus have:

$$C_{0,12} = Max(\ \$156.65 - \$81.61,\ 0\ ) = \$75.04$$

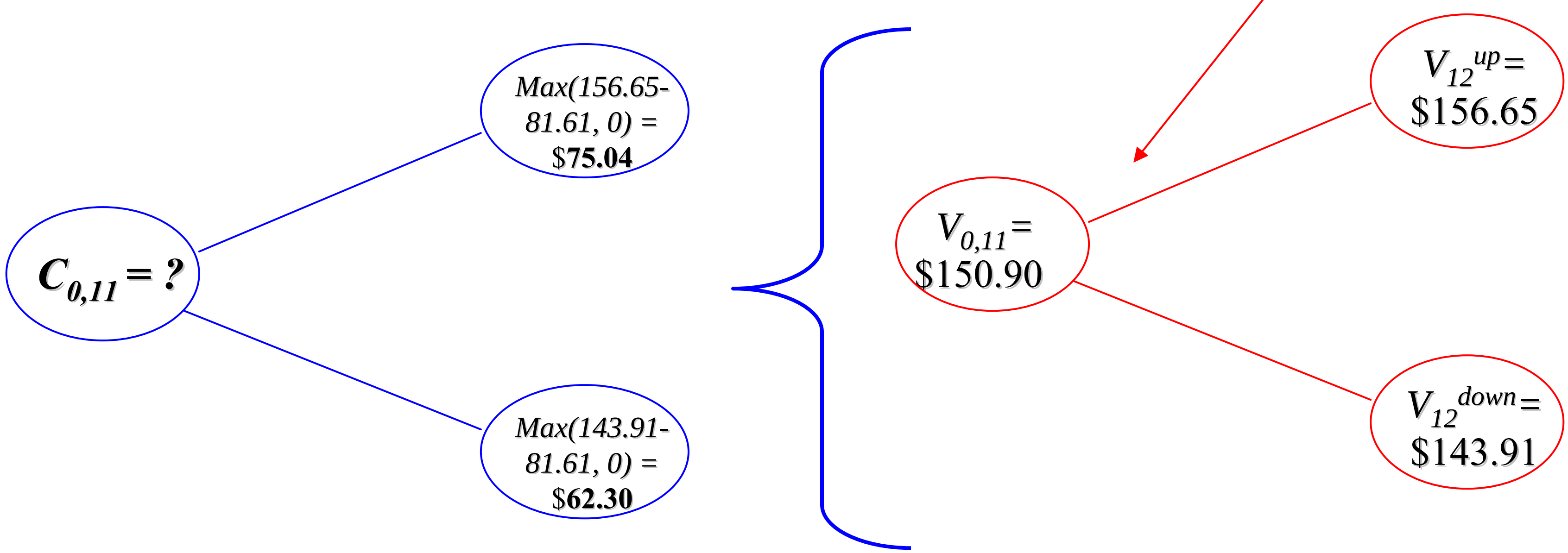
$$C_{1,12} = Max(\ \$143.91 - \$81.61,\ 0\ ) = \$62.30$$

Now consider the period $j = n\text{-}1$ state from which these two $j = n$ value states are each possible, and suppose (for now) that the option cannot be exercised prior to its maturity at period *n* (*"European Option"*)…

For example, for the \$156.65 and \$143.91 values in period 12, this would be the state in period 11 in our previous example where $V_{0,11}$ is worth \$150.90.

CD27.4 The Binomial Option Value Model

	V tree (net of payout, "ex dividend" values):												
	Period ("*j*"):												"*n*" =
	0	1	2	3	4	5	6	7	8	9	10	11	12
"down" moves ("*i*"):	V tree (net of payout, "ex dividend" values):												
0	100.00	103.81	107.77	111.87	116.14	120.56	125.16	129.93	134.88	140.02	145.36	150.90	156.65
1		95.37	99.01	102.78	106.70	110.76	114.99	119.37	123.92	128.64	133.54	138.63	143.91
2			90.96	94.43	98.02	101.76	105.64	109.66	113.84	118.18	122.69	127.36	132.22
3				86.75	90.06	93.49	97.05	100.75	104.59	108.58	112.71	117.01	121.47
4					82.74	85.89	89.16	92.56	96.09	99.75	103.55	107.50	111.60
5						78.91	81.92	85.04	88.28	91.64	95.13	98.76	102.52
6							75.26	78.12	81.10	84.19	87.40	90.73	94.19
7								71.77	74.51	77.35	80.30	83.36	86.53
8									68.45	71.06	73.77	76.58	79.50
9										65.29	67.77	70.36	73.04
10											62.26	64.64	67.10
11												59.38	61.65
12													56.64



CD27.4 The Binomial Option Value Model

The value of the call option in any state is a function of its two possible values in the subsequent period. The exact valuation formula is the same certainty-equivalence PV formula we have presented previously:*

$$C_t = CEQ[C_{t+1}] \big/ (1+r_f)$$

$$= \frac{E_t[C_{t+1}] - \left(C_{t+1}^{up} - C_{t+1}^{down}\right)\left(\dfrac{E[r_V] - r_f}{V_{t+1}^{up}\% - V_{t+1}^{down}\%}\right)}{1+r_f}$$

Substituting our previously-described binomial model parameters, this becomes:

$$C_{i,j} = \frac{\left(pC_{i,j+1} + (1-p)C_{i+1,j+1}\right) - \left(C_{i,j+1} - C_{i+1,j+1}\right)\left[\dfrac{r_V - r_f}{\left(1+\sigma\sqrt{T/n}\right) - 1\big/\left(1+\sigma\sqrt{T/n}\right)}\right]}{1+r_f}$$

where the probability *p* is as defined previously:

$$p = \frac{\left(1+r_V\right) - 1\big/\left(1+\sigma\sqrt{T/n}\right)}{\left(1+\sigma\sqrt{T/n}\right) - 1\big/\left(1+\sigma\sqrt{T/n}\right)}$$

CD27.4 The Binomial Option Value Model

In particular, for $C_{0,11}$, recalling our previous 1-year monthly numerical example input parameters of: r_V = 10%/12 = .833% , y_V = 6%/12 = 0.5% , r_f = 3%/12 = 0.25%, and σ = 15%, we obtain:

$$C_{0,11} = \left(\left(.5877C_{0,12} + .4123C_{1,12}\right) - \left(C_{0,12} - C_{1,12}\right)\left[\frac{r_V - r_f}{\left(1+\sigma\sqrt{1/12}\right) - 1/\left(1+\sigma\sqrt{1/12}\right)} \right] \right) \Big/ (1+r_f)$$

$$= \left(\left(.5877C_{0,12} + .412C_{1,12}\right) - \left(C_{1,12} - C_{2,12}\right)\left[\frac{0.833\% - 0.25\%}{104.33\% - 95.85\%} \right] \right) \Big/ 1.0025$$

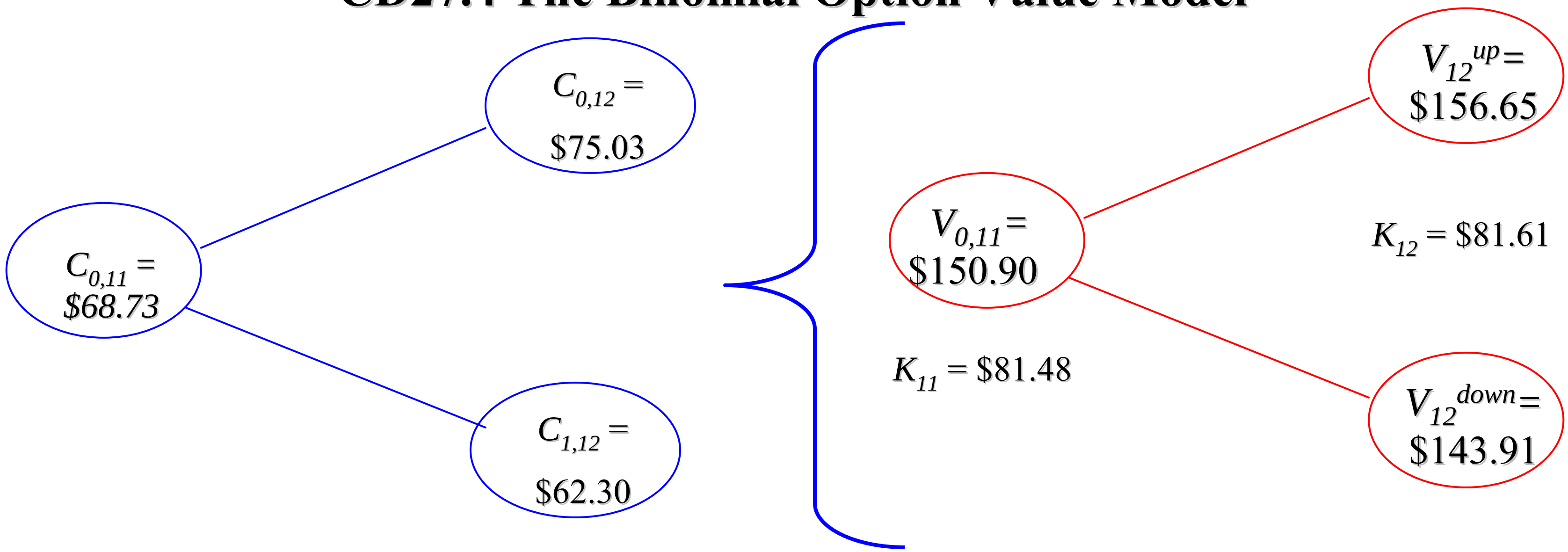
$$= \left(\left(.5877(75.04) + .4123(62.30)\right) - \left(75.04 - 62.30\right)\left[\frac{0.583\%}{8.48\%} \right] \right) \Big/ 1.0025$$

$$= \left((\$69.78) - (\$12.73)[0.0688] \right) / 1.0025$$

$$= \$68.91 / 1.0025$$

$$= \$68.73$$

CD27.4 The Binomial Option Value Model



$$C_{0,11}=\left(\left(pC_{0,12}+(1-p)C_{1,12}\right)-\left(C_{0,12}-C_{1,12}\right)\left[\frac{r_V-r_f}{\left(1+\sigma\sqrt{1/12}\right)-1\Big/\left(-\sigma\sqrt{1/12}\right)}\right]\right)\Big/(1+r_f)$$

$$=\left(\left(.5877(75.03)+.4123(62.30)\right)-\left(75.03-62.30\right)\left[\frac{10\%/12-3\%/12}{\left(1+15\%\sqrt{1/12}\right)-1\Big/\left(1+15\%\sqrt{1/12}\right)}\right]\right)\Big/(1+3\%/12)$$

$$=\$68.91/1.0025$$

$$=\$68.73$$

CD27.4 The Binomial Option Value Model

Repeating this process within each column for $i = 0, 1, 2, \ldots, j$, and then across columns from right to left for $j = 11, 10, 9, \ldots, 0$, we eventually obtain the value of the option as of the present time 0:

Note here the values we calculated in the previous slides.*

	Period ("*j* "):												"*n* " =
	0	1	2	3	4	5	6	7	8	9	10	11	12
"down" moves ("*i*"):	*Eurpoean Call Option Value Tree:*												
0	15.76	19.23	23.06	27.20	31.60	36.23	41.07	46.13	51.41	56.94	62.71	68.73	75.03
1		12.09	15.20	18.72	22.60	26.78	31.19	35.83	40.67	45.72	51.01	56.53	62.30
2			8.81	11.48	14.63	18.21	22.15	26.36	30.79	35.42	40.26	45.32	50.60
3				5.97	8.15	10.85	14.06	17.72	21.72	25.96	30.39	35.02	39.85
4					3.64	5.27	7.44	10.19	13.50	17.26	21.32	25.55	29.98
5						1.90	2.97	4.51	6.67	9.51	12.98	16.86	20.91
6							0.77	1.31	2.21	3.64	5.81	8.87	12.58
7								0.18	0.35	0.68	1.32	2.55	4.92
8									0.00	0.00	0.00	0.00	0.00
9										0.00	0.00	0.00	0.00
10											0.00	0.00	0.00
11												0.00	0.00
12													0.00

The European option is worth \$15.76 in the present time 0.

CD27.4 The Binomial Option Value Model

If the landowner can begin the development project (exercise the option) at <u>*any*</u> time (*"American option"*), then the value of the land in any state prior to option expiration is given by:

$$C_{i,j} = Max\left\{ V_{i,j} - K_{i,j},\ \frac{\left(pC_{i,j+1} + (1-p)C_{i+1,j+1}\right) - \left(C_{i,j+1} - C_{i+1,j+1}\right)\left[\dfrac{r_V - r_f}{\left(1+\sigma\sqrt{T/n}\right) - 1/\left(1+\sigma\sqrt{T/n}\right)}\right]}{1+r_f} \right\}$$

In our example, given that $K_{0,11}$ = \$81.48, then if the land can be developed immediately at any time, it is worth in state *0,11* :

$$C_{0,11} = Max(\$150.90 - \$81.48,\ \$68.73) = Max(\$69.42,\ \$68.73) = \$69.42$$

The flexibility to build the project at <u>*any time*</u> prior to the end of the year (*"American Option"* instead of *"European Option"*), makes the option worth more, namely, \$20 at time 0 (instead of \$15.76) . . .

CD27.4 The Binomial Option Value Model

Here is the complete *American Option* land value tree, assuming as before initial building value of: V_0 = \$100, initial construction cost: K_0 = \$80, (deterministic) construction cost growth rate of 2%/yr (0.167%/month), and that the right to ever develop the land expires after 1 year. From Excel:

	Period ("*j*"): 0	1	2	3	4	5	6	7	8	9	10	11	"*n*" = 12
"down" moves ("*i*"):	*LAND* Value tree:												
0	20.00	23.68	27.50	31.47	35.60	39.90	44.36	48.99	53.81	58.81	64.01	69.42	75.03
1		15.24	18.74	22.38	26.16	30.10	34.18	38.43	42.84	47.43	52.20	57.15	62.30
2			10.69	14.03	17.49	21.09	24.84	28.73	32.77	36.97	41.34	45.88	50.60
3				6.88	9.52	12.82	16.25	19.81	23.52	27.37	31.37	35.53	39.85
4					4.06	5.90	8.38	11.62	15.02	18.54	22.21	26.02	29.98
5						2.11	3.25	4.92	7.27	10.43	13.79	17.28	20.91
6							0.88	1.47	2.41	3.89	6.10	9.25	12.58
7								0.25	0.46	0.84	1.52	2.74	4.92
8									0.03	0.06	0.11	0.21	0.40
9										0.00	0.00	0.00	0.00
10											0.00	0.00	0.00
11												0.00	0.00
12													0.00

Note the value we just calculated, here:

An example Excel spreadsheet template for this binomial option model example is available for downloading from the course web site.

CD27.4 The Binomial Option Value Model

Note that, as we have presented it, the option (land) valuation formula appears to be a function of nine variables or parameters*:

$$C_{i,j} = C(V_{i,j}, K_{i,j}, r_V, r_f, y_V, g_K, \sigma, T, n)$$

In general, *ceteris paribus* (holding all other variables and parameters constant), option value *increases* as a function of:

$$V, r_f, \sigma, \text{ and } T.$$

And *decreases* as a function of:

$$K, y_V, \text{ and } g_K$$

Importantly (and very interestingly), note that the option value is:

***UNAFFECTED BY THE UNDERLYING ASSET REQUIRED RETURN*, r_V *OR THE GROWTH RATE,* g_V**

(holding constant the *current* underlying asset VALUE, $V_{i,j}$, and the payout rate y_V.)

CD27.4 The Binomial Option Value Model

Effect of Underlying Asset Volatility

This is the same option as before only we've increased the underlying asset volatility (σ) from 15% to 25%.

	Period ("*j*"):												"*n*" =
	0	1	2	3	4	5	6	7	8	9	10	11	12
"down" moves ("*i*"): *LAND* Value tree:													
0	20.16	26.55	33.55	41.02	49.00	57.52	66.63	76.34	86.72	97.80	109.63	122.26	135.74
1		13.86	18.90	25.22	32.15	39.55	47.45	55.88	64.89	74.51	84.79	95.75	107.46
2			8.89	12.66	17.62	23.91	30.76	38.08	45.90	54.26	63.17	72.70	82.87
3				5.15	7.76	11.40	16.31	22.60	29.39	36.63	44.37	52.64	61.47
4					2.56	4.14	6.55	10.08	15.02	21.30	28.02	35.19	42.85
5						0.99	1.75	3.04	5.19	8.62	13.79	20.01	26.66
6							0.24	0.46	0.91	1.78	3.48	6.81	12.58
7								0.01	0.02	0.04	0.08	0.16	0.32
8									0.00	0.00	0.00	0.00	0.00
9										0.00	0.00	0.00	0.00
10											0.00	0.00	0.00
11												0.00	0.00
12													0.00

Note the increase in value from \$20.00 to \$20.16

With 15% volatility, the option model called for optimal immediate exercise at time 0.

With 25% volatility, the model indicates that the option is more valuable held for speculation at time 0 instead of immediate exercise at that time.

CD27.4 The Binomial Option Value Model

Notice that the option value model not only values the option, but also indicates in which states of the world it is **optimal** to *exercise* the option (build the development project). Here are the valuation and optimal exercise trees for the option with **σ** back at the original **15%** . . .

	Period ("*j*"):												"*n*" =
	0	1	2	3	4	5	6	7	8	9	10	11	12
"down" moves ("*i*"):	*LAND* Value tree:												
0	20.00	23.68	27.50	31.47	35.60	39.90	44.36	48.99	53.81	58.81	64.01	69.42	75.03
1		15.24	18.74	22.38	26.16	30.10	34.18	38.43	42.84	47.43	52.20	57.15	62.30
2			10.69	14.03	17.49	21.09	24.84	28.73	32.77	36.97	41.34	45.88	50.60
3				6.88	9.52	12.82	16.25	19.81	23.52	27.37	31.37	35.53	39.85
4					4.06	5.90	8.38	11.62	15.02	18.54	22.21	26.02	29.98
5						2.11	3.25	4.92	7.27	10.43	13.79	17.28	20.91
6							0.88	1.47	2.41	3.89	6.10	9.25	12.58
7								0.25	0.46	0.84	1.52	2.74	4.92
8									0.03	0.06	0.11	0.21	0.40
9										0.00	0.00	0.00	0.00
10											0.00	0.00	0.00
11												0.00	0.00
12													0.00

	Period ("*j*"):												"*n*" =
	0	1	2	3	4	5	6	7	8	9	10	11	12
"down" moves ("*i*"):	Optimal exercise:												
0	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
1		exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
2			exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
3				hold	exer	exer	exer	exer	exer	exer	exer	exer	exer
4					hold	hold	hold	exer	exer	exer	exer	exer	exer
5						hold	hold	hold	hold	exer	exer	exer	exer
6							hold	hold	hold	hold	hold	exer	exer
7								hold	hold	hold	hold	hold	exer
8									hold	hold	hold	hold	hold
9										hold	hold	hold	hold
10											hold	hold	hold
11												hold	hold
12													hold

CD27.4 The Binomial Option Value Model

Here is the same option only with **σ = 25%** . . .

"down" moves ("*i*"):	Period ("*j*"): 0	1	2	3	4	5	6	7	8	9	10	11	"*n*" = 12
	LAND Value tree:												
0	20.16	26.55	33.55	41.02	49.00	57.52	66.63	76.34	86.72	97.80	109.63	122.26	135.74
1		13.86	18.90	25.22	32.15	39.55	47.45	55.88	64.89	74.51	84.79	95.75	107.46
2			8.89	12.66	17.62	23.91	30.76	38.08	45.90	54.26	63.17	72.70	82.87
3				5.15	7.76	11.40	16.31	22.60	29.39	36.63	44.37	52.64	61.47
4					2.56	4.14	6.55	10.08	15.02	21.30	28.02	35.19	42.85
5						0.99	1.75	3.04	5.19	8.62	13.79	20.01	26.66
6							0.24	0.46	0.91	1.78	3.48	6.81	12.58
7								0.01	0.02	0.04	0.08	0.16	0.32
8									0.00	0.00	0.00	0.00	0.00
9										0.00	0.00	0.00	0.00
10											0.00	0.00	0.00
11												0.00	0.00
12													0.00

"down" moves ("*i*"):	Period ("*j*"): 0	1	2	3	4	5	6	7	8	9	10	11	"*n*" = 12
	Optimal exercise:												
0	hold	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
1		hold	hold	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
2			hold	hold	hold	exer	exer	exer	exer	exer	exer	exer	exer
3				hold	hold	hold	hold	exer	exer	exer	exer	exer	exer
4					hold	hold	hold	hold	exer	exer	exer	exer	exer
5						hold	hold	hold	hold	hold	exer	exer	exer
6							hold	hold	hold	hold	hold	exer	exer
7								hold	hold	hold	hold	hold	exer
8									hold	hold	hold	hold	hold
9										hold	hold	hold	hold
10											hold	hold	hold
11												hold	hold
12													hold

Notice that with greater underlying asset volatility, option exercise is held back, less likely. *(Compare the “exer” cells in this table with the previous.)*

CD27.4 The Binomial Option Value Model

"down" moves ("*i*"):	Period ("*j*"): 0	1	2	3	4	5	6	7	8	9	10	11	"*n*" = 12
	Optimal exercise:												
0	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
1		exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
2			exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
3				hold	exer	exer	exer	exer	exer	exer	exer	exer	exer
4					hold	hold	hold	exer	exer	exer	exer	exer	exer
5						hold	hold	hold	hold	exer	exer	exer	exer
6							hold	hold	hold	hold	hold	exer	exer
7								hold	hold	hold	hold	hold	exer
8									hold	hold	hold	hold	hold
9										hold	hold	hold	hold
10											hold	hold	hold
11												hold	hold
12													hold

With 15% volatility, immediate exercise.

"down" moves ("*i*"):	Period ("*j*"): 0	1	2	3	4	5	6	7	8	9	10	11	"*n*" = 12
	Optimal exercise:												
0	hold	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
1		hold	hold	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
2			hold	hold	hold	exer	exer	exer	exer	exer	exer	exer	exer
3				hold	hold	hold	hold	exer	exer	exer	exer	exer	exer
4					hold	hold	hold	hold	exer	exer	exer	exer	exer
5						hold	hold	hold	hold	hold	exer	exer	exer
6							hold	hold	hold	hold	hold	exer	exer
7								hold	hold	hold	hold	hold	exer
8									hold	hold	hold	hold	hold
9										hold	hold	hold	hold
10											hold	hold	hold
11												hold	hold
12													hold

With 25% volatility, delay construction.

CD27.4 The Binomial Option Value Model

Risk & the OCC

What are the implications of the option model for the amount of investment risk, and the corresponding risk-adjusted discount rate (OCC) applicable to the land? . . .

For a finite-lived option, the risk and OCC will differ according to the "state-of-the-world" (the *V*, and *K* values, and the time until option expiration).

But recall that the certainty-equivalence valuation we are employing here allows us to *"back out"* what is the OCC once we have computed the option value. The formula to do this is obtained as follows:

$$C_{i,j} = \frac{CEQ[C_{j+1}]}{1+r_f} = \frac{E_j[C_{j+1}]}{1+r_f + E[RP_{C_{i,j}}]} = \frac{E_j[C_{j+1}]}{1+OCC_{i,j}},$$

$$\Rightarrow$$

$$1+OCC_{i,j} = \left(1+r_f\right)\frac{E_j[C_{j+1}]}{CEQ[C_{j+1}]}$$

Risk & the OCC

Thus, for any *i, j* state of the world in the binomial tree, we can compute the OCC (and the implied amount of investment risk indicated by the corresponding risk premium in the OCC).

Here is are the 1-period (monthly) OCCs for our previous numerical example (1-yr finite option with σ = 15% and the other parameters as before) **. . .**

	Period ("*j* "):											
	0	1	2	3	4	5	6	7	8	9	10	11
"down" moves ("*i*"):	1-Period Option Opportunity Cost of Capital:											
0	3.22%	2.84%	2.56%	2.35%	2.17%	2.03%	1.91%	1.81%	1.72%	1.65%	1.58%	1.52%
1		3.98%	3.39%	2.97%	2.66%	2.42%	2.23%	2.08%	1.95%	1.84%	1.75%	1.67%
2			4.94%	4.27%	3.58%	3.11%	2.77%	2.50%	2.30%	2.13%	2.00%	1.88%
3				5.82%	5.34%	4.62%	3.81%	3.27%	2.88%	2.59%	2.37%	2.19%
4					6.87%	6.40%	5.86%	5.03%	4.07%	3.45%	3.01%	2.69%
5						8.34%	7.85%	7.28%	6.61%	5.54%	4.38%	3.65%
6							10.46%	10.07%	9.57%	8.90%	7.92%	6.19%
7								13.54%	13.54%	13.54%	13.54%	13.54%
8									NA	NA	NA	NA
9										NA	NA	NA
10											NA	NA
11												NA
12												

Note that the OCC is mathematically indeterminate in states of the world where the option has a certain value of zero (in all future possible states of the world and hence in the current state).*

Risk & the OCC

Here are the per annum (effective annual rates) expected returns implied by the preceding periodic OCCs . . .

	Period ("*j* "):											
	0	1	2	3	4	5	6	7	8	9	10	11
"down" moves ("*i*"):	Option Opportunity Cost of Capital per Annum (EAR):											
0	46.2%	40.0%	35.5%	32.1%	29.4%	27.2%	25.5%	24.0%	22.7%	21.7%	20.7%	19.9%
1		59.8%	49.2%	42.1%	37.0%	33.3%	30.3%	28.0%	26.1%	24.5%	23.2%	22.0%
2			78.3%	65.2%	52.6%	44.4%	38.7%	34.5%	31.3%	28.8%	26.8%	25.1%
3				97.2%	86.7%	71.9%	56.6%	47.1%	40.6%	36.0%	32.4%	29.7%
4					121.9%	110.5%	98.1%	80.2%	61.4%	50.2%	42.8%	37.5%
5						161.6%	147.8%	132.4%	115.6%	91.1%	67.2%	53.8%
6							229.8%	216.2%	199.4%	178.3%	149.7%	105.5%
7								358.9%	358.9%	358.9%	358.9%	358.9%
8									NA	NA	NA	NA
9										NA	NA	NA
10											NA	NA
11												NA

Note: These OCCs are probably not very realistic (too high) for actual land investment, because of our assumption here of a 1-year finite life of the development option. In reality, land development rights typically do not expire at the end of a year. *(More on this shortly.)*

CD27.4 The Binomial Option Value Model

Risk & the OCC

How do these option OCCs compare to the *"Method 1"* ("canonical") Formula for the OCC of a development project, introduced in the Chapter 29 lecture notes? . . .

Recall that the Method 1 Formula is as follows (where T_C is the time required for construction):

$$E[r_C] = \left[\frac{(V_T - L_T)(1+E[r_V])^{T_C}(1+E[r_D])^{T_C}}{(1+E[r_D])^{T_C} V_T - (1+E[r_V])^{T_C} L_T} \right]^{(1/T_C)} - 1$$

Recasting this in our current nomenclature with $T_C = 1$ yr , this formula is:

$$E[r_{C_{i,j}}] = \left[\frac{(E_j[V_{j+1}] - K_{j+1})(1+r_f)}{(1+r_f)E_j[V_{j+1}] - (1+r_V)K} \right] - 1$$

This formula will in fact be equivalent to the option OCC previously computed in any state of the world where the option will definitely be exercised in the next period. *(Recall that the canonical formula assumes a definite commitment to go forward with the development project.)*

CD27.4 The Binomial Option Value Model

Comparison of "canonical" OCC versus actual option OCC *(previous numerical example, 1st 4 periods only)* **. . .**

"down" moves ("*i*"):	Period ("*j*"): 0	1	2	3	4
	1-Period Option Opportunity Cost of Capital:				
0	3.22%	2.84%	2.56%	2.35%	2.17%
1		3.98%	3.39%	2.97%	2.66%
2			4.94%	4.27%	3.58%
3				5.82%	5.34%
4					6.87%

"down" moves ("*i*"):	Period ("*j*"): 0	1	2	3	4
	"Canonical" ("Method 1") OCC Formula from Ch.29:				
0	3.22%	2.84%	2.56%	2.35%	2.17%
1		3.98%	3.39%	2.97%	2.66%
2			Not Valid	4.27%	3.58%
3				Not Valid	Not Valid
4					Not Valid

"down" moves ("*i*"):	Period ("*j*"): 0	1	2	3	4
	Optimal exercise:				
0	exer	exer	exer	exer	exer
1		exer	exer	exer	exer
2			exer	exer	exer
3				hold	exer
4					hold

Problems with the Binomial Model

There are two major technical problems with the Binomial Model:

1. **Discrete time & values (the real world is continuous).**
2. **Finite expiration of the option (land is perpetual).**

Both of these can have significant effects on the option value and optimal exercise decision characteristics.

The first problem (discreteness) can be addressed by making the periods of time very short ($m \rightarrow \infty$, $T/n \rightarrow 0$).

Perpetual expiration can be approximated by a long time horizon, but more accurate solution requires an entirely different type of model.

For modeling a simple option, sufficient for dealing with the *Wait Option*, there is a simple solution to this problem:

A model of perpetual option value in continuous time that includes the value of the option to delay construction as well as a solution to the decision problem of optimal development timing . . .

27.5 The Samuelson-McKean Formula Applied to Land Value as a Development Option

The simplest option valuation formula is also the first one developed (before *Black-Scholes*), and the one that is most relevant to land valuation and optimal development timing:

The Samuelson-McKean Formula

Developed by Nobel Prize winning economist Paul Samuelson and his mathematician partner Henry McKean, at MIT in 1965, as a model of a *"perpetual American warrant"*.

The Samuelson-McKean Model is consistent with the Binomial Model in that the latter would converge to the former if we could let $T \rightarrow \infty$ and also $T/n \rightarrow 0$. *(You can imagine how big a table this would require, since the binomial table has dimension nXn, with approximately $n^2/2$ elements in it .)*

To see how this works, recall our replicating portfolio model of option value…

	Today	Next Year	
Development Option Value	$C(0)=x$ "x" = *unkown val.*	$C(t)_{UP}=113.21-90$ **= $23.21**	$C(t)_{DOWN}$ **= 0** *(Don't build)*
Built Property Value	$V(0) = V_0 / (1+y_V)$ $= \$102.83/(1.09)$ $=\$94.34$	$V(t)_{UP}= \$113.21$	$V(t)_{DOWN}= \$78.62$
Bond Value	$B = \$51.21$	$B(t) = (1+r_f)B = \$52.74$	$B(t) = (1+r_f)B = \$52.74$
Replicating Portfolio: C(0) = (N)V(0) – B	$C(0) = (N)V(0) - B$ $= (0.7)\$94.34 - \$51.21 = \$12.09$	$C(t)_{UP} =$ $(0.7)113.21 - \$52.74$ **= $23.21**	$C(t)_{DOWN} =$ $(0.7)78.62 - \$52.74$ **= 0**

The Replicating Portfolio = *NV-B*, where: *N*=(Cu-Cd)/(Vu-Vd); *B*=(*N*Vd-Cd)/(1+r_f); and *V* = *V(0)* (not V_0) . . .

$$C_0 = \left(\frac{C_t^{UP} - C_t^{DOWN}}{V_t^{UP} - V_t^{DOWN}}\right) V(0) - \frac{\left(\frac{C_t^{UP} - C_t^{DOWN}}{V_t^{UP} - V_t^{DOWN}}\right) V_t^{DOWN} - C_t^{DOWN}}{1 + r_f} = \left(\frac{\Delta C}{\Delta V}\right) V(0) - B$$

Note that as we approach continuous time (periods get very short), N becomes like the derivative of the option value with respect to the underlying asset value: $N = \frac{dC}{dV}$

Thus: $C = NV - B, \Rightarrow dC = NdV - dB = \frac{dC}{dV}dV - dB$

We can also use the Taylor Series expansion from basic calculus (supplemented by some very advanced mathematics known as *"Stochastic Calculus"*) to approximate the change in value of a (perpetual) option over time as:

$$dC = \frac{dC}{dV}dV + \frac{1}{2}\frac{\partial^2 C}{\partial V^2}\sigma^2 V^2 dt$$

Combining our Replicating Portfolio formula $C = NV - B$ with the above, and looking at changes over time (returns), we see:

$$dC = \frac{dC}{dV}dV + \frac{1}{2}\frac{\partial^2 C}{\partial V^2}\sigma^2 V^2 dt = \frac{dC}{dV}dV - dB$$

$$\Rightarrow \quad dB = -\frac{1}{2}\frac{\partial^2 C}{\partial V^2}\sigma^2 V^2 dt$$

But we also know that the riskless bond value, B, given its Replicating Portfolio value of: $B = NV - C$, will change over time according to the riskfree interest rate, as:

$$dB = (B) r_f \; dt = \left(\frac{dC}{dV}V - C\right) r_f \; dt$$

Equating the above two expressions for dB, we obtain the following ordinary differential equation:

$$dB = -\frac{1}{2}\frac{\partial^2 C}{\partial V^2}\sigma^2 V^2 dt = \left(\frac{dC}{dV}V - C\right) r_f \; dt$$

$$\Rightarrow \quad \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 C}{\partial V^2} + r_f V \frac{dC}{dV} - r_f C = 0$$

$$r_f C = r_f V\left(\frac{dC}{dV}\right) + \frac{1}{2}\sigma^2 V^2\left(\frac{d^2C}{dV^2}\right)$$

The solution to this differential equation, combined with suitable boundary conditions (*$C(V=0) = 0$, $C(V=\infty) = V$*) and the conditions of optimal exercise (expected exercise timing so as to maximize the present value of the option), gives the *Samuelson-McKean Formula*.

This works as a model for land value because, like a perpetual American warrant, land never expires (*"perpetual"*), and can be developed at any time by its owner (exercise policy is *"American"*).

Actually, the equation presented above ignores dividends and assumes a constant exercise price. To allow (more realistically for construction projects) for the underlying asset to pay dividends (property net rent) and for construction costs to grow over time, some minor modifications must be made in the formula. These are reflected in the model presented on subsequent slides.

The Samuelson-McKean Model Applied to Land Value:

Let: *V* = Currently observable value of built property of the type that is the HBU for the land (underlying asset, what we have labeled V_0 , not *V(0)*).

σ = Volatility of (Std.Dev. of return to unlevered) *individual* built properties (= *"total risk"*, not just systematic or non-diversifiable risk, includes *idiosyncratic risk*: Typical range for real estate is 15% to 25% per year).

y_V = Payout ratio of the built property (current cash yield rate, like *cap rate* only net of capital improvement reserve, typical real estate values range from 4% to 12%).

y_K = Construction cost "yield" rate (= $r_f - g_K$, where g_K is the growth rate of construction costs, typically approximately equal to inflation).

The Samuelson-McKean Model Applied to Land Value:

Then the option value (and optimal exercise) formula has three steps:

(1) The *"option elasticity"* [*(dLAND/LAND)/(dV/V)*], η *("eta")*, is given by:

$$\eta = \{y_V - y_K + \sigma^2/2 + [(y_K - y_V - \sigma^2/2)^2 + 2y_K\sigma^2]^{1/2}\} / \sigma^2$$

(2) The option *critical value* (*"hurdle value"*) of the built property at and above which it is optimal to immediately exercise the option (develop the land), labeled *V , is:**

$$V^* = K\eta / (\eta - 1)$$

(3) The option (land) value is given by:

$$LAND = \begin{cases} (V^* - K)\left(\dfrac{V}{V^*}\right)^{\eta}, & if\ V \le V^* \\ V - K, & otherwise \end{cases}$$

Example:

$r_f = 3\%$, $y_V = 6\%$, $\sigma = 15\%$, $K = \$80$ (with y_K=1%, ➔2%growth) , $V = \$95$, ➔

$$\eta = \{y_V - y_K + \sigma^2/2 + [(y_K - y_V - \sigma^2/2)^2 + 2y_K\sigma^2]^{1/2}\} / \sigma^2$$

$$= \{.06-.01+.15^2/2+[(.01-.06- .15^2/2)^2+2(.01).15^2]^{1/2}\}/.15^2 = 5.60.$$

$$V^* = K[\eta/(\eta-1)] = \$80[5.6/(5.6-1)] = \$80(5.6/4.6) = \$80(1.22) = \$97.38.$$

$$LAND = (V^*-K)(V/V^*)^{\eta} = (\$97.38 - \$80)(\$95/\$97.38)^{5.6} = \$15.13.$$

In this example,

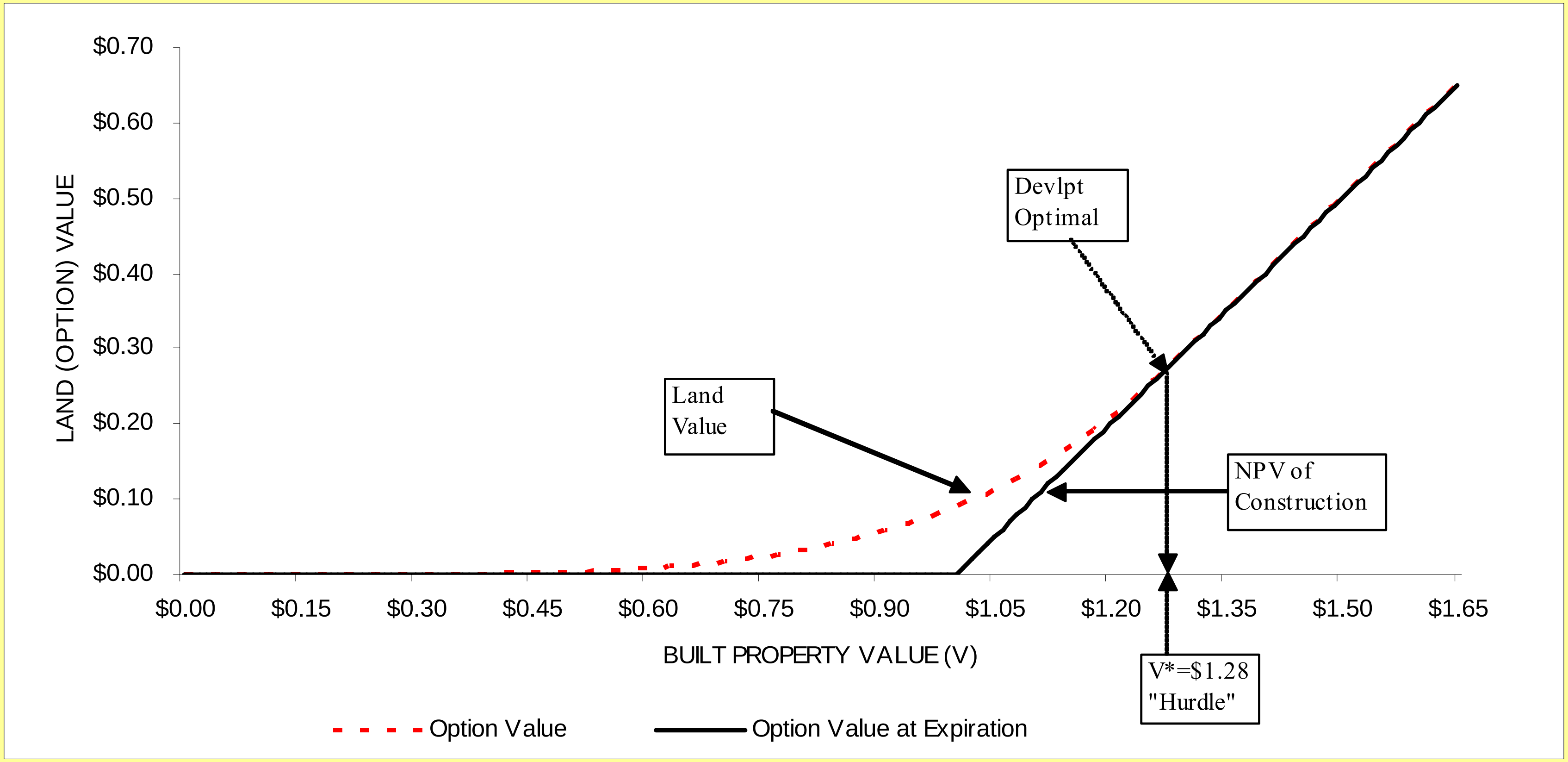
Option Elasticity = 5.60, Hurdle Benefit/Cost Ratio = 1.22,

Land Value = $0.16 per dollar of current built property value.

Note: In applying the Sam-McK Formula, in principle *V* and *K* should be defined based on the HBU that the site will ultimately be developed for (not necessarily what it could immediately be developed for).*

(Obviously you wouldn't memorize this formula! Use the downloadable file from course web site.)

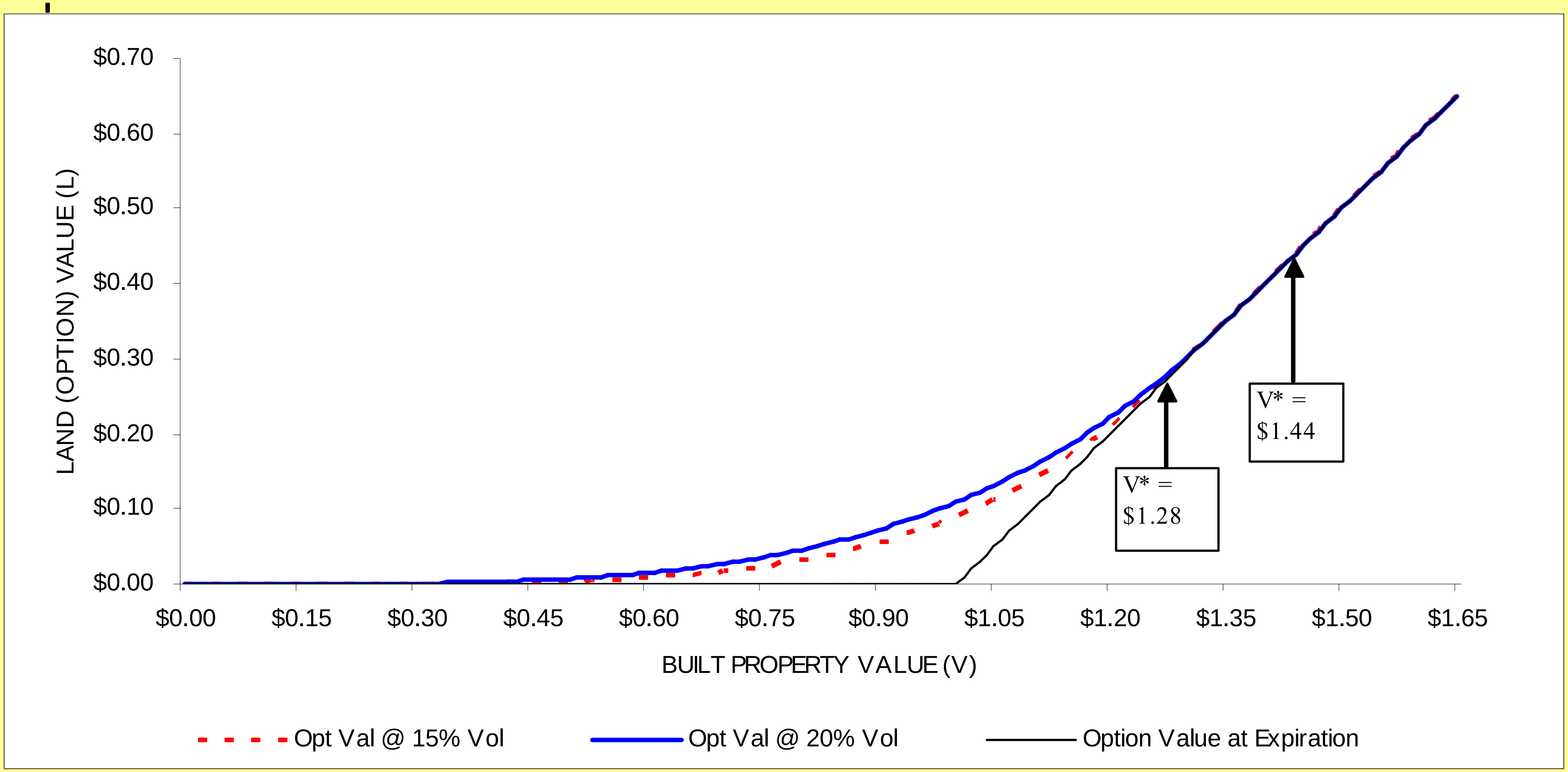
Here is a picture of what the Samuelson-McKean Formula looks like:



Parameters in above chart are: $\sigma = 15\%$, $y_V = 8\%$, $y_K = 5\%$.

Land value (*LAND*) is a monotonically increasing, convex function of the current HBU built property value (underlying asset value). Above the *hurdle* benefit/cost (*V/K*) ratio, the option should already be exercised, and its value is simply *V-K*.

Both the option value, and the hurdle *V/K* ratio, are *increasing* functions of the volatility (σ) and *decreasing* functions of the payout ratio (y).



Assuming: $y_V = 8\%$, $y_K = 5\%$.

The hurdle benefit/cost ratio, and the land value as a fraction of the construction cost, are *independent* of the *scale* of the site (in the sense of the size of the land parcel, holding HBU density constant).

Recall that with the Binomial Model there appeared to be a **"hurdle value"** of the underlying asset above which it is optimal to exercise (develop) . . .

V tree (net of payout, "ex dividend" values):

Period ("*j*"): "*n*" = 12

"down" moves ("*i*"):	0	1	2	3	4	5	6	7	8	9	10	11	12
0	100.00	103.81	107.77	111.87	116.14	120.56	125.16	129.93	134.88	140.02	145.36	150.90	156.65
1		95.37	99.01	102.78	106.70	110.76	114.99	119.37	123.92	128.64	133.54	138.63	143.91
2			90.96	94.43	98.02	101.76	105.64	109.66	113.84	118.18	122.69	127.36	132.22
3				86.75	90.06	93.49	97.05	100.75	104.59	108.58	112.71	117.01	121.47
4					82.74	85.89	89.16	92.56	96.09	99.75	103.55	107.50	111.60
5						78.91	81.92	85.04	88.28	91.64	95.13	98.76	102.52
6							75.26	78.12	81.10	84.19	87.40	90.73	94.19
7								71.77	74.51	77.35	80.30	83.36	86.53
8									68.45	71.06	73.77	76.58	79.50
9										65.29	67.77	70.36	73.04
10											62.26	64.64	67.10
11												59.38	61.65
12													56.64

Here (with 15% volatility) the hurdle value of V seems to be about $90 (until the end).

Period ("*j*"): "*n*" = 12

Optimal exercise:

"down" moves ("*i*"):	0	1	2	3	4	5	6	7	8	9	10	11	12
0	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
1		exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
2			exer	exer	exer	exer	exer	exer	exer	exer	exer	exer	exer
3				hold	exer	exer	exer	exer	exer	exer	exer	exer	exer
4					hold	hold	hold	exer	exer	exer	exer	exer	exer
5						hold	hold	hold	hold	exer	exer	exer	exer
6							hold	hold	hold	hold	hold	exer	exer
7								hold	hold	hold	hold	hold	exer
8									hold	hold	hold	hold	hold
9										hold	hold	hold	hold
10											hold	hold	hold
11												hold	hold
12													hold

With 15% volatility, immediate exercise.

With the Sam-McK Model there is an explicit formula for this hurdle value . . .

The *hurdle benefit/cost* ratio:

$$(V^*/K) = \eta / (\eta - 1)$$

is an interesting measure in its own right.

It tells you how much greater the anticipated completed new built property value (including land) must be than its construction cost (excluding land), in order for it to be optimal to stop waiting to develop, and immediately begin (instantaneous) construction.

Expressing this in terms of the *land value fraction* of the total development project value at the time of optimal development, the optimal land value fraction is given by the inverse of the elasticity:

$$\frac{V^* - K}{V^*} = 1 - \frac{K}{V^*} = 1 - \frac{\eta - 1}{\eta} = \frac{1}{\eta}$$

e.g., Elasticity = 3 ➔ Hurdle B/C Ratio = 1.5 ➔ Optimal Land Fraction = 33%.

Samuelson-McKean Implications for Optimal Development

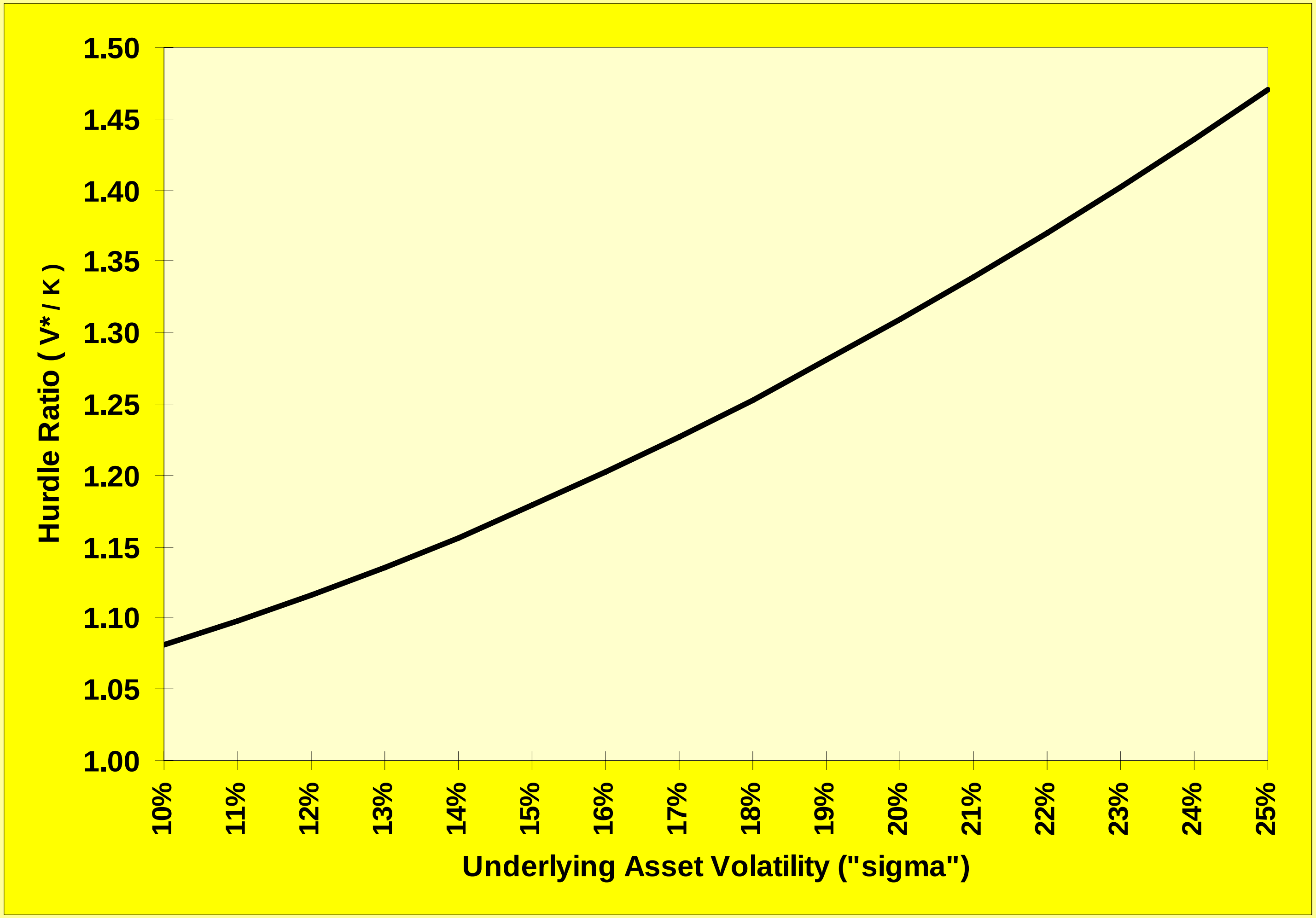
As noted, the option elasticity also determines the hurdle benefit/cost ratio, *V*/K*, at which it is optimal to immediately begin development whenever the current value of *V* and *K* equate to this ratio*:

$$\frac{V^*}{K} = \frac{\eta}{\eta - 1}$$

The hurdle benefit/cost ratio is thus an inverse function of the option elasticity: larger elasticity means a lower hurdle ratio.

Samuelson-McKean Implications for Optimal Development

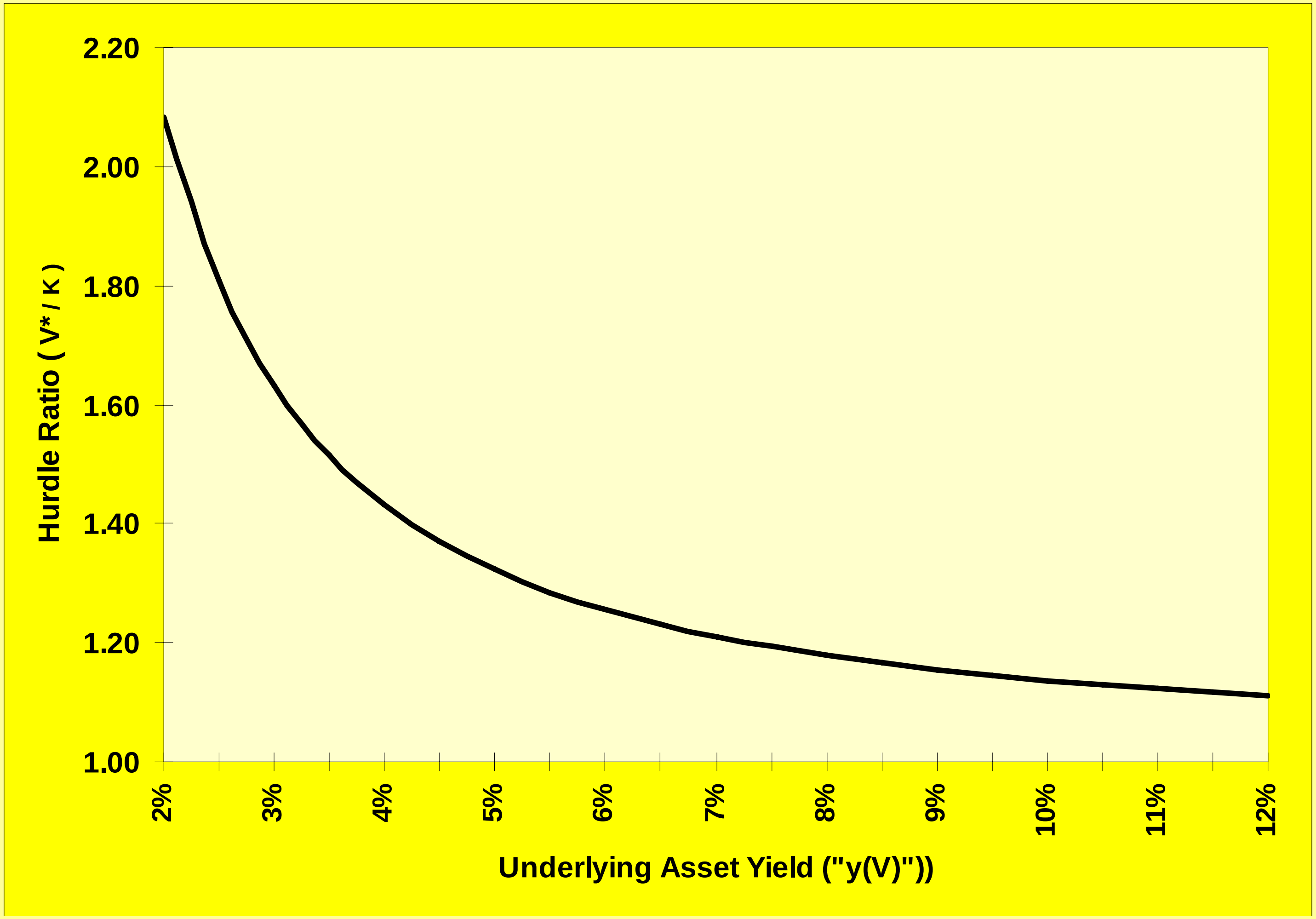
Hurdle Ratio (*V* / K*) as a function of underlying asset volatility (σ):



With: $y_V = 8\%$, $y_K = 2\%$.

Samuelson-McKean Implications for Optimal Development

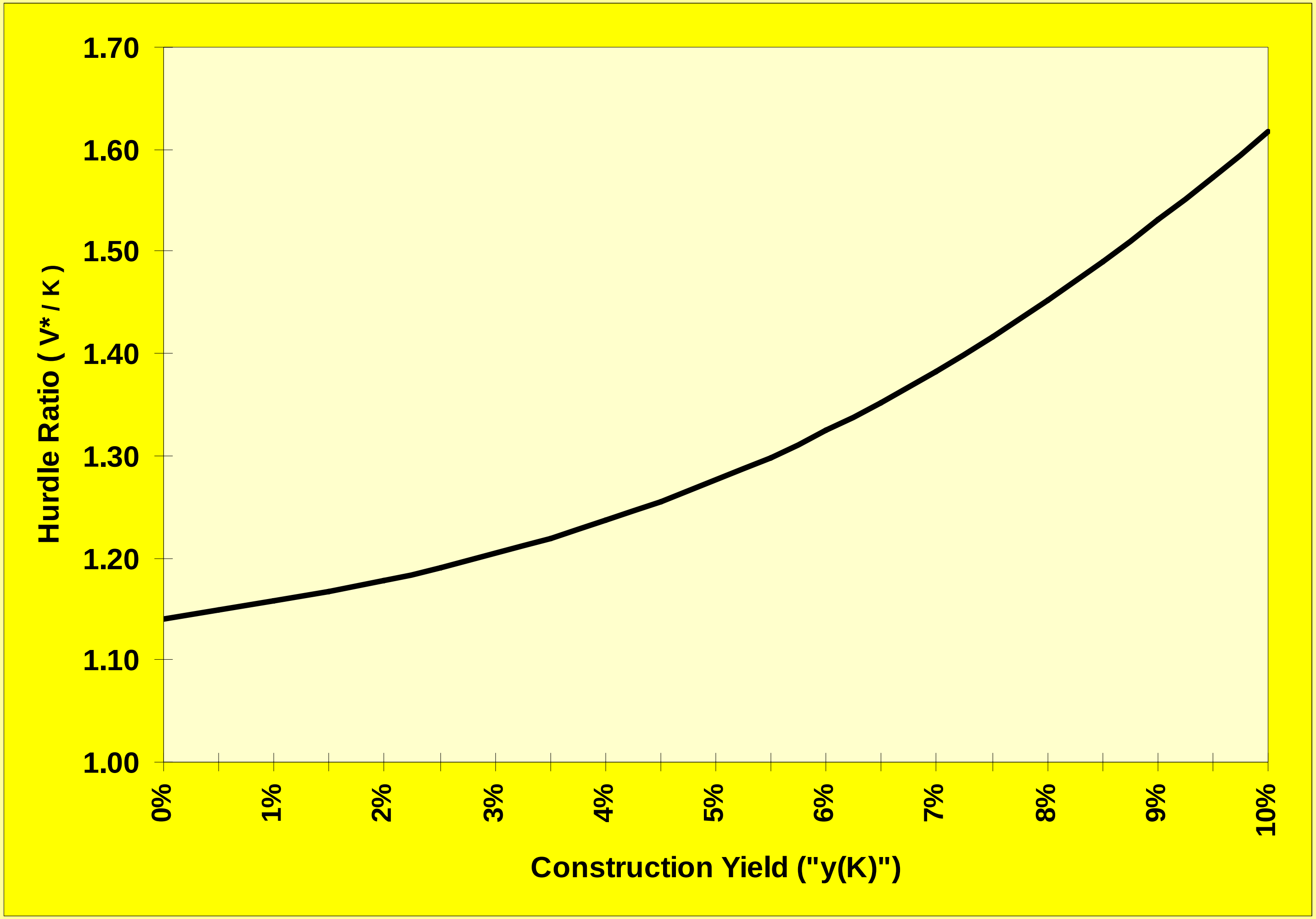
Hurdle Ratio (V^* / K) as a function of underlying asset yield (y_V):



With: $\sigma = 15\%$, $y_K = 2\%$.

Samuelson-McKean Implications for Optimal Development

Hurdle Ratio (V^* / K) as a function of construction yield ($y_K = r_f - g_K$):



With: $y_V = 8\%$, $\sigma = 15\%$.

The *hurdle benefit/cost* ratio (& *optimal land fraction)* is:

- **Greater the more volatile is the built property market (i.e., the more uncertainty there is in the future value of built properties):**
  - ***➔ As long as you hold the option unexercised, greater volatility gives you greater potential upside outcomes you can take advantage of while the option flexibility allows you to avoid the greater downside outcomes implied by the greater volatility.***
  - ***➔ Uncertain and volatile property markets will dampen development, as developers wait until they can get built property values (based on space market rents) sufficiently above the construction cost exclusive of land).***
- **Lower the greater is the current cash yield (akin to *cap rate*) being provided by built properties:**
  - ***➔ You <u>only</u> start to get the net rent the property can generate when the building is complete, so the greater the current yield, the greater the incentive to build sooner rather than later.***
  - ***➔ Land value (site acquisition cost) will be a smaller fraction of total development cost (including construction) in locations where built property values tend to grow slower (holding risk constant, lower "g" ➔ higher "y", as g+y=r, recalling Ch.9).***

Example:

In the U.S., land (site acquisition) is typically about 20% of the total development cost in most areas of the country, but often 50% in major metropolises on the East and West Coast. Why?...

Suppose *rf* = 5%, and property market volatility and payout rates differ as follows:

	Big East & West Coast Cities	Rest of U.S.
Property Mkt Volatility (σ)	20%	15%
Property Payout Rate (y)	5%	8%

Then the Samuelson-McKean Formula gives the following difference in land value fraction of total developed property value at the time of optimal development (based on the implied *V*/K* hurdle ratio):

	Big East & West Coast Cities	Rest of U.S.
*LAND/V** *@ V=V* (optimal dvlpt)*	46%	22%

The *option elasticity* measure, η, is also interesting in its own right.

Prior to the point of optimal exercise (when the land is still optimally held undeveloped for speculation), the elasticity tells the percentage change in land value resulting from a given percentage change in built property value (for the type of property that would be the HBU of the land).

For a "live" option (below the hurdle ratio) the elasticity is:

- **Independent of the size of the land parcel (for a given HBU density);**
- **Independent of the current value of the underlying asset (the state of the property market).***
- **A decreasing function of the volatility in the property market.**

The option elasticity relates the volatility (and risk) of the option (the undeveloped land investment) to the volatility (and risk) of the underlying asset (the built property market for the HBU of the site).

Assuming riskless construction costs:

$$\sigma_{LAND} = \eta\sigma_{V},$$

Where σ_{LAND} is the volatility of the undeveloped land.

Since the option return is perfectly correlated with the underlying asset return, the option elasticity can therefore also be used to relate the required expected investment return risk premium in undeveloped land to that in the HBU built property market:

$$RP_{LAND} = \eta RP_{V}.$$

Example:

Built property expected return $r_V = 8\%$, Cash yield $y_V = 6\%$

Riskfree interest rate = 4%, ➔ Built property $RP_V = 4\%$.

Construction yield $y_K = 2\%$.

(Which might be determined as the 4% riskfree rate minus a 2% likely construction cost growth rate: $y_K = r_f - g_K = 4\% - 2\% = 2\%$.)

If built property volatility σ = 15%, then: ($\sigma = .15, y_V = .06, y_K = .02$) ➔

$\eta = 4.9$.

Thus, $RP_{LAND} = \eta(RP_V) = 4.9(4\%) = 19.7\%$ ➔

Expected return (OCC) on land speculation investment =

$$r_f + RP_{LAND} = 4\% + 19.7\% = 23.7\%.$$

Based on the Samuelson-McKean assumptions, this required expected return for land speculation would hold no matter how big or small the land parcel (for a given HBU density), or what the current state of the built property market is, as long as σ, *y*, r_f, and RP_V remain the same. The Sam-McK Formula is a "*constant elasticity*" formula.

Samuelson-McKean Implications for Land OCC

As noted, the option elasticity, η , gives the ratio of the land risk to the underlying asset risk, hence the ratio of the land to underlying asset expected risk premium in the opportunity cost of capital (in the expected investment return):

$$\eta = \frac{E[RP_C]}{E[RP_V]}$$

We also noted that for a "live option" (not yet ripe for exercise) η is *independent of* both:

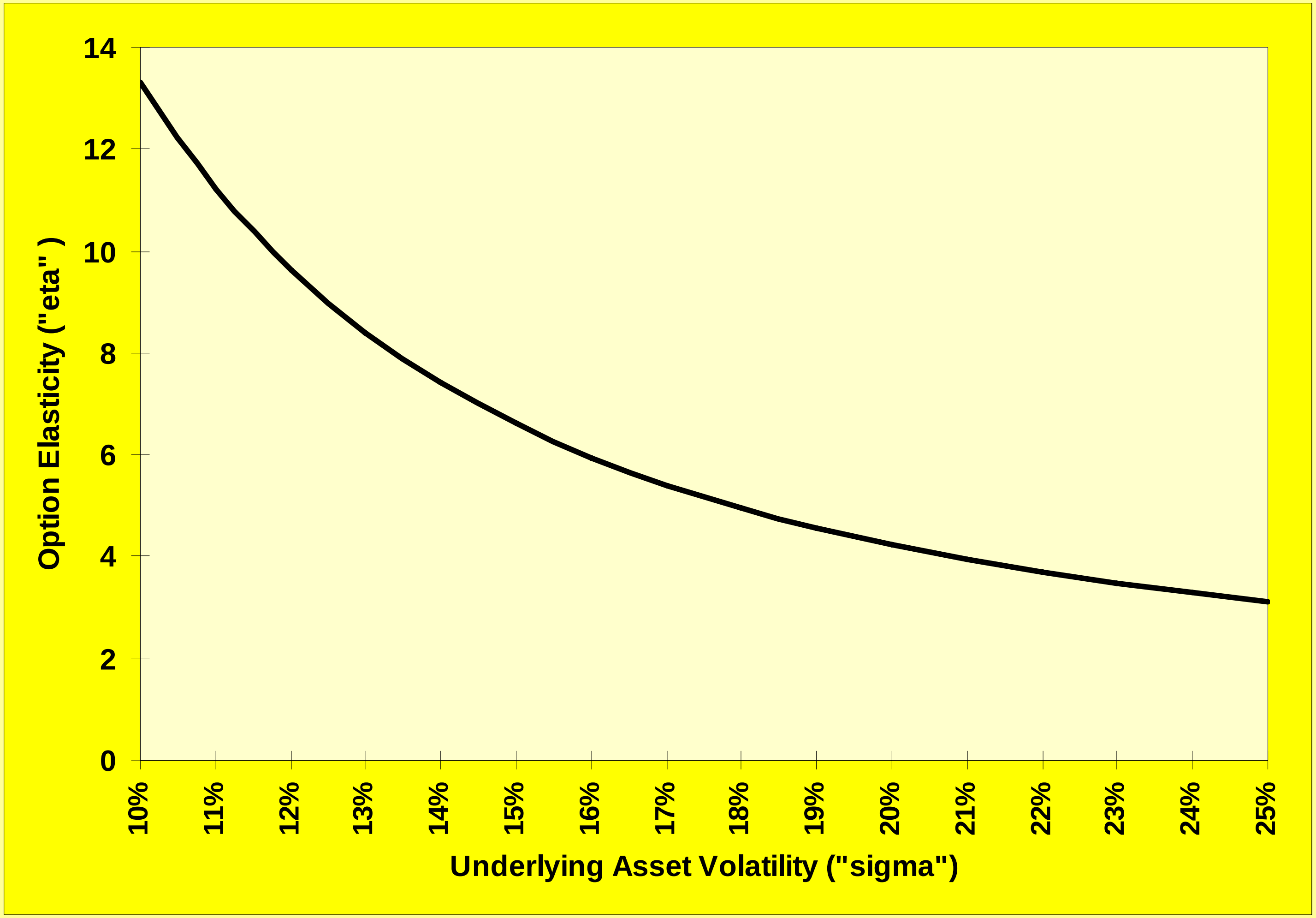
- **Scale (value of V or of K), and**
- **Current benefit cost ratio (amount of "operational leverage" in the construction project: V / K**

In fact, η is a function of only three variables: σ, y_V , and y_K .

This makes the option elasticity in the Samuelson-McKean Formula a very useful tool for understanding and quantifying land investment risk and return requirements.

Samuelson-McKean Implications for Land OCC

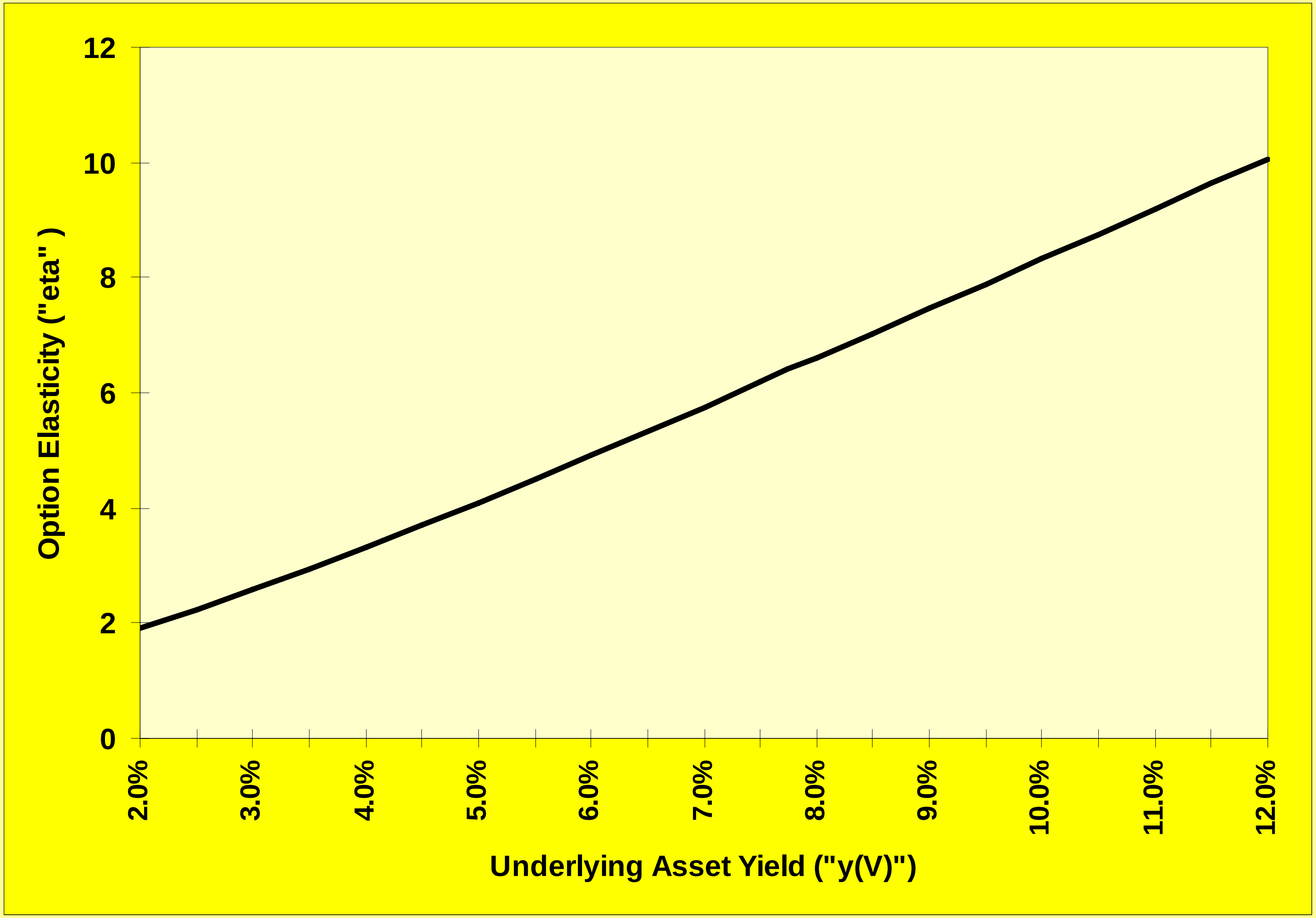
Option elasticity ($E[RP_C] / E[RP_V]$) as a function of underlying asset volatility (σ):



With: $y_V = 8\%$, $y_K = 2\%$.

Samuelson-McKean Implications for Land OCC

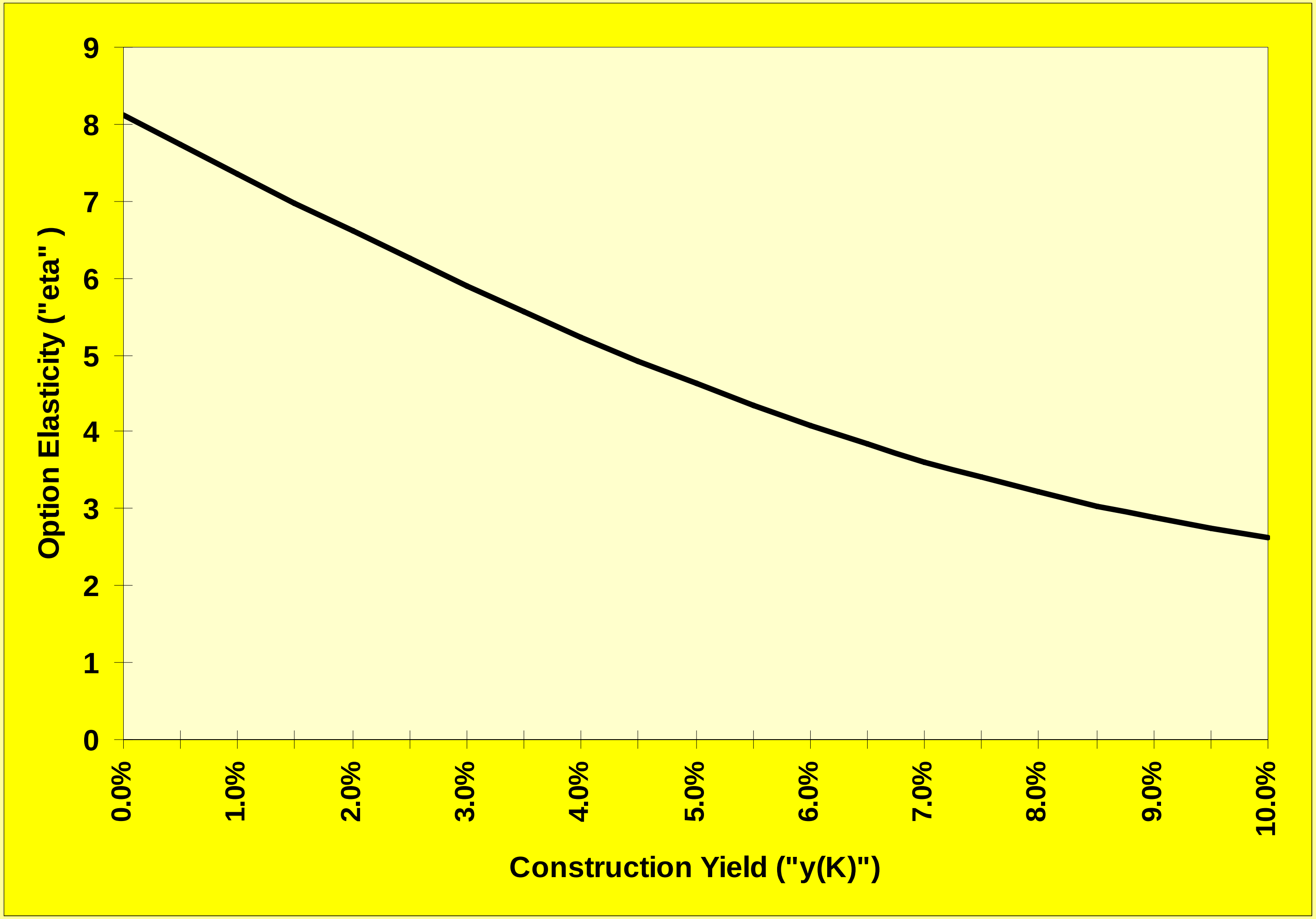
Option elasticity ($E[RP_C]$ / $E[RP_V]$) as a function of underlying asset yield (y_V):



With: $\sigma = 15\%$, $y_K = 2\%$.

Samuelson-McKean Implications for Land OCC

Option elasticity ($E[RP_C] / E[RP_V]$) as a function of construction yield ($y_K = r_f - g_K$):



With: $y_V = 8\%$, $\sigma = 15\%$.

CD27.5 Generalizing the Samuelson-McKean Model to allow for risky construction costs . . .

The Sam-McK Model can allow for risky construction costs by use of a simple transformation: Value the option *per dollar of construction cost*, as follows:*

1. **Divide the current underlying asset value by the current construction cost, replacing *V* in the formula with *V/K*, and replacing *K* in the formula with *1*.**

2. **In computing the "construction yield", y_K , use the expected return on an asset with market risk equivalent to that of the construction cost, instead of the riskfree rate. i.e., $y_K = r_K - g_K = r_f + E[RP_K] - g_K$.**

3. **In computing σ, use the volatility of a portfolio of the underlying asset minus the construction cost:**

$$\sigma = \sqrt{VAR[r_V] + VAR[r_K] - 2COV[r_V, r_K]}$$

(This transformation is attributed to Fisher and Margrabe .)*

Summarizing up to now:

- **The Binomial Model can handle:**
  - **Finite-lived development options (rights expire at a specified future time), "American" or…**
  - **"European" development options (construction prohibited prior to a given future point in time)**
- **The Samuelson-McKean Model can handle:**
  - **The simple *"Wait Option"* for a perpetual "American" development option (typical "land value" problem).**

It remains for us to address two important considerations:

- **Until now we have assumed instantaneous exercise: we need to consider the effect of construction time, aka *"time to build"* .**
- **The *"Phasing Option"*, in which the project is broken into sequential *phases* rather than building it all at once**

CD27.5 *Time to Build . . .*

With non-instantaneous construction, when you exercise the option to build in state of the world *i, j* , you don't get $V_{i,j} – K_j$.

You get the PV of the completed project:

$$(E_{i,j}[V_{j+TC}])/(1+r_V)^{TC} - K_{j+TC}/(1+r_f)^{TC}$$

where *TC* is the number of periods it will take to complete construction.

In the Binomial Model, if you exercise the option at time 0 when the underlying asset has current observable value V_0 , then if the time to build is 1 period you will get:

$$PV[V_1 - K_1] = \frac{E_0[V_1]}{1+r_V} - \frac{K_1}{1+r_f} = \frac{(p)uV_0/(1+y_V)+(1-p)d\,V_0/(1+y_V)}{1+r_V} - \frac{K_1}{1+r_f} = \frac{pV_{0,1}+(1-p)V_{1,1}}{1+r_V} - \frac{K_1}{1+r_f} = \frac{V_0}{1+y_V} - \frac{K_0}{1+y_K}$$

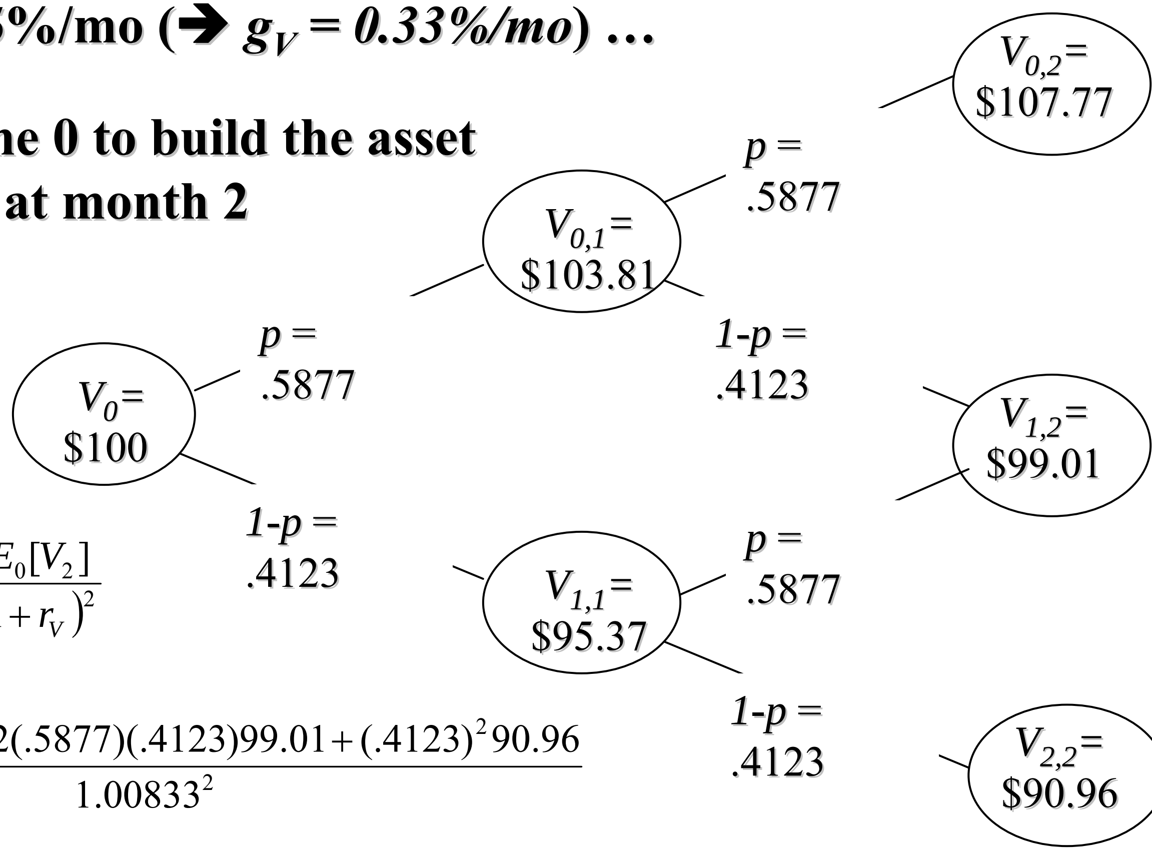
If the time to build is 2 periods, you will get:

$$PV[V_2 - K_2] = \frac{E_0[V_2]}{(1+r_V)^2} - \frac{K_2}{(1+r_f)^2} = \frac{p^2V_{0,2}+2(p)(1-p)V_{1,2}+(1-p)^2V_{2,2}}{(1+r_V)^2} - \frac{K_2}{(1+r_f)^2} = \frac{V_0}{(1+y_V)^2} - \frac{K_0}{(1+y_K)^2}$$

CD27.5 *Time to Build . . .*

For example, consider our previous underlying asset value tree and a time-to-build of 2 months with r_V = 10%/yr = 0.833%/mo, y_V = 6%/yr = 0.5%/mo (➔ g_V = *0.33%/mo*) …

A decision at time 0 to build the asset obtains an asset at month 2 that is worth at time 0: $99.01



$$PV[V_2] = \frac{(1+g_V)^2 V_0}{(1+r_V)^2} = \frac{E_0[V_2]}{(1+r_V)^2}$$

$$= \frac{(.5877)^2 107.77 + 2(.5877)(.4123)99.01 + (.4123)^2 90.96}{1.00833^2}$$

$$= \frac{(1.0033)^2 \$100}{(1.00833)^2} = \frac{\$100.66}{1.00833^2} = \$99.01 = \frac{\$100.00}{1.005^2} = \frac{V_0}{(1+y_V)^2}$$

Summarizing:

To account for time to build in the option model:

- **In any state *i, j* where the option could be exercised, replace the immediate exercise value $V_{i,j} - K_j$ with the present value as of time *j* of the exercise value *TC* periods later (where *TC* is the required construction time) : $PV_{i,j}\,[V_{j+TC} - K_{j+TC}]$, as defined in the previous slides.**

- **In the Samuelson-McKean Formula, replace the current value of the underlying asset, V_{ij} , with: $V_{ij} / (1 + y_V)^{TC}$, and replace the exercise price K_j with $K_j / (1 + y_K)^{TC}$.**

CD27.5 *Time to Build . . .*

The general effect of time-to-build is:

- **The value of the option is reduced below what it otherwise would be.**

- **The expected time until optimal exercise is increased beyond what it otherwise would be (hurdle value of V_t as measured by current observable price of pre-existing assets is increased):**

  - Condition of optimal exercise, where "V_t" is current observable price of identical pre-existing asset:

  - $\eta/(\eta-1) = (V_t /(1+y_V)^{TC}) / (K_t/(1+y_K)^{TC})$

  - ➔ $V_t = K_t[\eta/(\eta-1)]((1+y_V)/(1+y_K))^{TC}$,

  - and normally: $y_V > y_K$.

Old GM 1e 28.2.2 The Land Development Option Contrasted with Financial Options:

Distinguishing characteristics of the land devlpt option:

- **Perpetual (no expiriation):**
  - **➔ More flexibility (greater value),**
  - **➔ Only reason to exercise is to obtain operating cash flows.**
- ***"Time to Build"* (exercise not immediate):**
  - **➔ Can't observe exact at-completion mkt val of underl.asset at time exercise decision is made (added risk in exercise decision).**
- ***"Noisy"* value observation of (even current) mkt val of underl. asset. (*"thin mkt", recall Ch.12,* also adds to risk of exercise decision):**
  - **➔ Possibly heterogeneous information about *true* value of underlying asset (the to-be-built property): Some devlprs may be more knowledgable than others. *(➔ Wait longer until exercise.)***
- **Exercise *creates new real assets* that add to the supply side of the space market (affecting mkt val of all competing properties):**
  - **➔ Can increase risk of *not* exercising (option may effectively "expire" if demand is absorbed by competing devlpt projects).**

27.6 What the real option theory of land development can tell us about the *"overbuilding phenomenon"*. . .

What is the "overbuilding phenomenon"?...

The widely observed tendency for commercial real estate markets to periodically become "overbuilt", that is, characterized by *excess supply* (abnormally high vacancy, downward pressure on rents), due to excessive speculative development of new buildings.

Recall that in Chapter 2 we discussed an explanation for this "cyclicality" phenomenon using the *"4-Quadrant Diagram"*, based on the existence of ***myopic behavior*** (not just lack of perfect foresight, but some degree of *irrational expectations*) on the part of investors and developers in the system . . .

Real option theory offers several explanations for why/how overbuilding can be due to completely *rational* (i.e., profit-maximizing) behavior on the part of developers (landowners):

1. **"Cascades":** Noisy observations of the mkt values of the underlying assets (comparable built properties), combined with heterogeneous developer knowledge about the "true" value, causes a ***follow-the-leader*** type effect, in which developers wait longer than they otherwise would to develop, and then they all rush in as soon as the first (presumably most knowledgeable) developer reveals his knowledge by commencing development.

2. **"Lumpy supply & first out of the gate":** Economies of scale in building size, combined with finite user demand and the fact that option exercise creates real physical capital, leads to early exercise of the development option to preclude loss (expiration) of the option if a competitor builds first.

3. **"Long-term leasing option":** The cost of having empty space in a new building may be less than it first appears in space markets characterized by long-term leases, as it gives the landlord a ***leasing option***, that has value prior to its "exercise" (in the signing of a lease contract): Volatility in the rental mkt may bring better long-term lease deals in the future.