# Optimal Inventory Policies for Assembly Systems under Random Demands

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**This summary presentation is based on: Rosling, K. "Optimal Inventory Policies for Assembly Systems Under Random Demand."** ***Operations Research*** **43 (6), 1989.**

# Main Result

- Remodel an assembly system as a series system

(See Figure 1 on page 566 of the Rosling paper)

(See Figure 2 on page 571 of the Rosling paper)

- Simple re-order policies are optimal

# Main Result

- Assembly system:
  - Ordered amounts available after a fixed lead time
  - Random customer demands only for the end product
  - Assumptions on cost parameters
- Under assumptions and restriction on initial stock levels, assembly system can be treated as a series system
- Optimal inventory policy – can be calculated by approach in Clark and Scarf's paper (1960)*

*Clark, A.J. and Herbert Scarf. "Optimal Policies for a Multi-echelon Inventory Problem." *Management Science* 6 (1960): 475-90.

# Relevant Literature

- Clark and Scarf (1960) – derive optimal ordering policy for pure series system
- Fukuda (1961) – include disposal of items in stock
- Federgruen and Zipkin (1984) – generalize Clark and Scarf approach to stationary infinite horizon case

# Model

- N items (components, subassemblies, the end item)
- Each non-end item has exactly one successor
  - product networks forms a tree rooted in the end item
- Exactly one unit of each item required for the end item
- Notation:

$s(i)$ = unique *immediate* successor of item *i*=1…N; $s(1)=0$

$A(i)$ = the set of *all* successors of item *i*

$P(i)$ = the set of immediate predecessors of item *i*

$B(i)$ = the set of *all* predecessors of item *I*

$l_i$ = number of periods (lead-time) for assembly (or delivery) of item *i*

# Model

- At the beginning of a time period:
  1. Outstanding orders arrive and new ordering decisions made
  2. Old backlogs fulfilled and customer demands occur (for the end period)
  3. Backlog and inventory holding costs incurred

# Notation

$\xi_t$ = iid demand in period *t* for the end item with density $\phi(\cdot)$ and distribution $\Phi(\cdot)$

$\lambda$ = $E[\xi_t]$, expected value of $\xi_t$

$X_{it}$ = echelon inventory position of item *i* in period *t before* ordering decision are made ( = inventory on hand + units in assembly/order - units backlogged)

$Y_{it}$ = echelon inventory position of item *i* in period *t after* ordering decisions are made; $Y_{it} \geq X_{it}$

$Y_{it} - X_{it}$ = amount ordered for item *i* in period *t;* arrives after $l_i$ periods

# Notation: contd..

- $X_{it}^{l}$ = echelon inventory on hand of item *i* in period *t* before ordering decisions are made but after assembles arrive

  $= \quad Y_{i,t-l_i} - \sum_{k=t-l_i}^{t-1} \xi_k$

- $Y_{kt} \leq X_{it}^{l}$ if $i \in P(k)$ cannot order more than at hand

  (no intermediate shortage)

# Model: Cost parameters

$H_i$ = unit installation holding cost per period of item *i*

$h_i$ = unit echelon holding cost per period of item *i*

$$h_i = H_i - \sum_{k \in P(i)} H_k$$

$p$ = unit backlogging cost per period of the end item

$\alpha$ = period discount factor $0 < \alpha \leq 1$

- Cost in period *t*

$$\underbrace{\sum_{i=2}^{N} H_i (X_{it}^{l} - X_{s(i)t}^{l})}_{\text{Installation Holding Cost}} + \underbrace{H_1 \cdot \text{Max}(0, X_{1t}^{l} - \xi_t) + p \cdot \text{Max}(0, \xi_t - X_{1t}^{l})}_{\text{Holding cost for end item/ Backlogging cost}}$$

# Model: Cost

- Alternate Formulation:

(see page 567 of Rosling paper, left hand column)

- Using $X^{l}_{i,t+l_i} - \xi_{t+l_i} = Y_{it} - \sum_{s=1}^{t+l_i} \xi_s$
- Total Expected Cost over an infinite horizon:

(see equation 1 on page 567)

$\Phi_1^{l+1}(\cdot)$ : convolution of $\Phi(\cdot)$ over $(l_1 + 1)$ periods

# Long-Run Inventory Position

- $M_i$ : total lead-time for item *i* and all its successors $M_i = l_i + \sum_{k \in A(i)} l_k$

- (See page 567, part 2 “Long-Run Inventory Position)

# Long-Run Balance

- Assembly system is in *long-run balance* in period t iff for *i=1,..,N-1*

$$X_{it}^{M-\mu} \leq X_{i+1,t}^{M-\mu} \text{ for } \mu = 1,\ldots,M_i - 1$$

- Inventory positions equally close to the end item increase with *total lead-time*
- Satisfied trivially if $(i+1) \in P(i)$

# Assumptions on Cost parameters

- $h_i > 0$ for all $i$
  - All echelon holding costs positive
- $\sum_{i=1}^{N} h_i \cdot \alpha^{-M_{s(i)}} < p + H_1$
  - Better to hold inventory than incur a backlog

# Long-run Inventory position

**Lemma 1**: (See page 568 of Rosling paper)

**Lemma 2:** (See page 568 of Rosling paper)

# Long-run Inventory position

**Theorem 1**: "Any policy satisfying Lemmas 1 and 2 leads the system into long-run balance and keeps it there. This will take not more than $M_N+1$ periods after accumulated demand exceeds $Max_i\, X_{i1}$."

**Proof**: Outline

- $X_{it} \le X^{L}_{i+1,t}$ for all $t \ge q(i)$ by Lemma 1
- $X^{M-\mu}_{it} = Y_{iq} - \sum_{r=q}^{t-1} \xi_r \le X^{L}_{i+1,q} - \sum_{r=q}^{t-1} \xi_r = X^{M-\mu}_{i+1,t}$

  for $t = q(i) + M_i - \mu$
- long-run balance for *i* for $t \ge q(i) + M_i$
- Upper bound *q(i)*

# Equivalent Series System

**Theorem 2**: If the Assumptions hold and system is initially in long-run balance, then optimal policies of the assembly system are equivalent to those of a pure series system where:

- *i* succeeds item *i+1*
- lead-time of item *i* is $L_i$
- holding cost $h_i \leftarrow h_i \cdot \alpha^{l_i - L_i}$

**Proof:** Cost function

$$\operatorname*{Min}_{\mathrm{Y}} \mathrm{E}\left\{ \sum_{\mathrm{i}=1}^{\infty} \alpha^{t-1} \cdot \left( \sum_{i=1}^{N} \alpha^{L_i} \cdot (h_i \alpha^{l_i - L_i}) Y_{it} + \alpha^{L_1} \cdot (p + H_1) \int_{Y_{it}}^{\infty} (\xi - Y_{it}) \phi_1^{L+1}(\xi) d\xi \right) \right\}$$

$$+ \text{Constant}$$

# Equivalent Series System

Easy to show, using Theorem 1,:

$$X_{it} \leq X_{kt}^{M-M_i} \text{ for all } i, t \text{ and } k \in P(i)$$

Hence, using Lemma 1,

$$X_{it} \leq Y_{it}^{*} \leq X_{i+1,t}^{L}$$

Use this constraint in Problem **P.**

# New Formulation for **P**

$$\underset{\mathrm{Y}}{\mathrm{Min}}\ \mathrm{E}\left\{\sum_{\mathrm{i=1}}^{\infty}\alpha^{t-1}\cdot\left(\sum_{i=1}^{N}\alpha^{L_i}\cdot(h_i\alpha^{l_i-L_i})Y_{it}+\alpha^{L_1}\cdot(p+H_1)\int_{Y_{it}}^{\infty}(\xi-Y_{it})\phi_1^{L+1}(\xi)d\xi\right)\right\}$$

$$+\,\mathrm{Constant}$$

such that

$$X_{it}\le Y_{it}\le X_{i+1,t}^{L}\ \text{for all}\ i,t$$

where

$$X_{i+1,t}^{L}=Y_{i+1,t-L}-\sum_{s=t-L}^{t-1}\xi_s$$

and

$$X_{it}=Y_{i,t-1}-\xi_{t-1}$$

# Equivalent Series System

***Corollary 2***: There exist $S_i$*'s* such that the following policy is optimal for all *i* and *t*

$$
\begin{aligned}
Y_{it}^{*} &= \mathrm{Min}(S_i, X_{i+1,t}^{L}) && \text{if } X_{it} \le S_i \\
Y_{it}^{*} &= X_{it} && \text{if } X_{it} \ge S_i
\end{aligned}
$$

$S_i$ – obtained from Clark and Scarf's (1960) procedure

- Critically dependent on initial inventory level assumption (long-run balance initial inventory levels)
- Generally optimal policy by Schmidt and Nahmias (1985)

# Equivalent Series System

**Corollary 3**: If $L_i$=0, then

- Optimal order policy with $S_i = S_{i-1}$
- $i$ and $i-1$ can be aggregated
- lead time of aggregate $L_{i-1}$
- holding cost coefficient $h_{i-1} + h_i \alpha^{-L_{i-1}}$

# General Assumption on Costs

- Generalized Assumption

(See page 571, section 4)

- Allowed to have $h_i \leq 0$ for some $i$
- Examples: Meat or Rubber after cooking/ vulcanization
  - Components more expensive to store than assemblies
  - May have negative echelon holding costs

# Generalized Assumption

- Aggregation procedure to eliminate *i* for which $h_i \leq 0$
- Leads to an assembly system satisfying the original assumption

# Practical Necessity of Generalized Assumption

1. If $h_i \cdot \alpha^{-M_{s(i)}} + \sum_{k \in B(i)} h_k \cdot \alpha^{-M_{s(k)}} < 0 \text{ for some } i$
   – Minimal cost of **P** is unbounded
2. If $h_i \cdot \alpha^{-M_{s(i)}} + \sum_{k \in B(i)} h_k \cdot \alpha^{-M_{s(k)}} = 0 \text{ for some } i$
   – item *i* is a free good, hence predecessors of *i* may be neglected
3. If assumption (ii) not satisfied
   – production eventually ceases

# Summary

- Assembly system:
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# Comments

- Series analogy does not work for:
  - Non stationary holding/production costs
  - Non-zero setup costs