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PROFESSOR: So today, what I want to do is to continue where we were last time with option pricing. As I promised you last time, having gone through the history of option pricing and the special role that MIT played, today I actually want to do some option pricing.

I want to show you a simple but extraordinarily powerful model for actually coming up with a theoretical pricing formula for options, and frankly, all derivative securities. So we're gonna actually do that in the space of about a half an hour, and then we're going to conclude. And I want to turn, then, to the next lecture, which is on risk and return. I want to now, after we finish option pricing, take on the challenge of trying to understand risk in a much more concrete way than we've done up until now.

OK, so let's turn to lecture 10 and 11. And I'd like you to take a look at slide 16. OK, this will be the first model of option pricing that any of you have ever seen.

You've all heard of Black-Scholes. We talked a bit about it last time. Frankly, this is a simpler version of option pricing that ultimately can actually be used to derive the Black-Scholes formula as well.

But the reason I love this model is because it is so simple that with only basic high school algebra, you can actually work out all of the analytics. So all of you already have the math that it takes to implement this formula, and even to derive the formula. But the underlying economics is extraordinarily deep, and so it's a wonderful way of sort of getting a handle on how these very complex formulas work.

So here's what we're going to do. We're going to simplify the problem in the following way. We're going to use-- the framework, by the way, is called the binomial option-pricing model that was derived by our very own John Cox, Steve Ross, and Mark Rubenstein of UC Berkeley. And although this is a simpler version of option pricing than Black and Scholes, it turns out that on the street, this is used much more commonly than the Black-Scholes formula.

So let me show you how it works. We're going to start with a very simple framework of one period option pricing, meaning we're going to focus on a stock that survives for two periods-- this period and the next period. And then we're going to consider the pricing of an option on

that stock that expires next period. We're going to figure out what the price is this period.

So we've got a stock XYZ, and let's suppose the current stock price is S_0 . And let's suppose that we have a call option on this stock with a strike price of K , and where the option expires tomorrow. And so tomorrow's value of the option is simply equal to C_1 , which is the maximum of tomorrow's stock price minus the strike, or 0, the bigger of those two. That's the payoff for the call option.

And the question that we want to attack is, what is the option's price today? In other words, what is C_0 ? So we draw a timeline, as I've told you, for every one of these problems. Draw a timeline just so that there's no confusion.

So tomorrow, the stock price is going to be worth S_1 , and the option price is just equal to C_1 , which is the payoff, since it expires tomorrow. And the payoff is just the maximum of S_1 minus K and 0.

And the object of our focus is to try to figure out what the value of the option is today. And so I'm going to argue that if we can figure out what it is today, based upon this, then we can actually generalize it in a very natural way to figure out what the price is for any number of periods in the future.

So how do we do that? Well, we first have to make an assumption about how the stock price behaves. As I mentioned last time, we need to say something about the dynamics of stock prices, and remember that Bachelier, that French mathematician that came up with a rudimentary version of an option pricing formula in 1900, he developed the mathematics for Brownian motion, or a random walk, for the particular stock price. So you have to assume something about how the stock price moves.

So what we're going to do in a simplified version is to assume that the stock price tomorrow is a coin flip. It's a Bernoulli trial. That's the technical term. So it either goes up or down.

And if it goes up, it goes up by a gross amount u . So the value of the stock tomorrow, S_1 , is going to be equal to u multiplied by S_0 . So if it goes up by 10% tomorrow, then u is equal to 1.1. Or it can go down by a factor of d tomorrow, and so if it goes down by 10%, then d is 0.9. So we're going to simply assert that this is the statistical behavior of stock prices.

Now, granted, this is a very, very strong simplification, but bear with me. After I derive the simple version of the pricing formula, I'm going to show you how to make it much, much more

complex. And the additional complexity will be really simple to achieve once we understand this very basic version.

Now, the probability of going up or down is not 50/50-- doesn't have to be 50/50. So I'm going to assert that it's equal to some probability of p and $1 - p$. So it either goes up by u or goes down with probability $1 - p$, and the amount that it goes up or down is given by u and d . And I'm going to assert that u is greater than d . Question?

AUDIENCE: So u and d are not the changes, but you're just assuming that, in your case, [INAUDIBLE].

PROFESSOR: Sorry, that they're--

AUDIENCE: It's not necessary to always use [INAUDIBLE] d , or can you assume--

PROFESSOR: Yeah, I'm assuming as a matter of normalization that u is greater than d . It doesn't matter. I mean, one thing has to be bigger than the other, so I just may as well assume that u is greater than d . Yeah, question?

AUDIENCE: Is it necessary to add up to 1 [INAUDIBLE]?

PROFESSOR: u and d don't have to add up to anything. That's right. p and $1 - p$ always add up to 1, right. So for example, if this is a growth stock that's really doing well, then u may be 1.1, 10%, and d may be 0.99. So in other words, when it goes down, it goes down by 1%. When it goes up, it goes up by 10%. So on average, when you multiply by p , $1 - p$, depending on what they are, you can get a stock that's got a positive drift.

If, on the other hand, you've got a stock that's declining in value, then it may end up that d is much smaller than u , and $1 - p$ is bigger than p , which means that you're more likely to be going down than you are going up. So it's pretty general. Yeah?

AUDIENCE: u is bigger than 1 and d is lower than 1, or you could [INAUDIBLE]?

PROFESSOR: It doesn't have to be. But typically you would think that in the up state, it's going to be bigger than 1, and the down state, it will be less than 1. But it doesn't have to be. What I'm going to normalize it to be is u is greater than d . And later on, we may make some other economic assumptions that I'll come to that will tell you a little bit more about what u and d are.

Now, if it's true that the stock price can only take on two values tomorrow, then it stands to reason that the option can only take on two values tomorrow. And those are the two values.

It's going to be C_u and C_d .

C_u is where the stock price goes up to u times S_0 . Therefore, the option's going to be worth $uS_0 - K$, 0, maximum of those two. And similarly, if it turns out that the stock price goes down tomorrow, then the option is worth this tomorrow. Two values for the stock tomorrow implies two values for the option tomorrow. Any questions about that?

OK, so having said that, we can now proceed to ask the question, given this simple framework, what should the option price today depend on? It's going to be a function of a bunch of parameters. So what should it depend upon?

Well, the parameters that are given are these-- the stock price today, the strike price, u and d , p , and the interest rate between today and tomorrow. Those are the only parameters that we have. These are it. This is everything.

It's going to turn out that with the simple framework that I've put down, we will be able to derive a closed-form analytical expression for what the option price has to be today-- C_0 . I'm going to do that for you in just a minute.

But it's going to turn out that that option pricing formula, that f of stuff, is going to depend on all of these parameters except for one. One of these parameters is going to drop out. In other words, one of these parameters is redundant.

And anybody want to take a guess as to what that parameter might be? What parameter do you think might not matter for pricing an option? Yeah, Terry.

AUDIENCE: The interest rate, the r ?

PROFESSOR: The interest rate. Well, that's a good guess, but that's not the case. That's what I would have guessed, because that seems to be the thing that should matter the least, given how important all of these other parameters are. Anybody want to take another guess? Yeah, Ken.

AUDIENCE: Today's stock price.

PROFESSOR: Today's stock price. That's another good guess.

[LAUGHTER]

Although that's not correct, because in both the case of the interest rate and today's stock price, you could ask the question, suppose the stock price were at \$1,000 versus \$10. That would matter, wouldn't it? Or if the interest rate were at 20% versus 1%, that should matter, shouldn't it? And it does. In fact, if you look at every single one of these parameters, none of them looks like they're unnecessary. It looks like all of them are required. Yeah, John.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Well, the strike price remains the same, but the thing is that the question is whether or not the value of the option depends on the strike price. And if the option is, for example, in the money or out of the money, you would expect that that would make a big difference.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Right. In fact, let me tell you that there is no good answer to this, because all of these parameters look like they belong. But I want to tell you that one of them will not. One of them will not be in here. And this is going to be a major source of both confusion and illumination for what really depends on-- what option pricing really depends on.

All right, so let me just show you how we're going to do this. Let me illustrate to you the method. And we're going to do this in the exact same way that we've priced virtually everything under the sun. We're going to use an arbitrage argument.

I'm going to construct a portfolio that will have the identical payoff to the option, and therefore if the portfolio has the exact same cash flows as the option, then the cost of constructing that portfolio has to be the price of the option. Moreover, if it's not, you're going to be very happy, because that will mean that there is an arbitrage opportunity. That is, there's money to be made. If this theory fails, then you're going to be able to get rich beyond your wildest dreams. So we're hoping for a violation of this.

So let's see how we do that. I want you to now forget about the option for a moment, and I want you to imagine that at time 0, we construct a portfolio consisting of stocks and riskless bonds, in particular Δ shares of stocks and B dollars of riskless bonds-- riskless in terms of default.

And the total cost of this portfolio today, time 0, is simply equal to the price per share times the number of shares of stock, so that's S_0 times Δ , plus the value of the bonds that I'm buying-- the market value of the bonds that I'm buying today, or selling. So B could be a

positive or negative number. Delta could be a positive or a negative number. And that's my cost today, time 0.

Now, I want to look at the payoff tomorrow for this portfolio. So V_1 is the payoff for the portfolio. That's what it's worth tomorrow. And V_1 is going to be given by the value of the stocks and the value of the bonds.

Now, the stocks are going to be-- there are two possibilities. Either the stock goes up or the stock goes down. And if it goes up, it'll be worth $u S_0$ times delta, and if it goes down it'll be worth $d S_0$ times delta. I don't know whether it'll go up or down, but whatever it does, this is the value tomorrow.

Now, what about my bond portfolio? Well, I bought B bonds, and r now is the gross rate of return, the gross interest rate, so it's a number like 1.03 or 1.05. And the reason that I'm switching notation is I'm following the notation used originally by Cox, Ross, and Rubenstein, so I apologize for the kind of cognitive dissonance that this may generate.

But this r , the way that Cox, Ross, and Rubenstein wrote it, was meant to be a gross rate of return. So you'll never see a $1 + r$, because this-- in their framework, because this already contains the 1. So just keep that in the back of your mind, and make a note of that. r is the gross interest rate. It's 1 plus the net interest rate, so it's a number like 1.03 for a 3% rate of return. Actually nowadays, it should be more like 1.01 for short-term interest rates, or less.

Now, you'll notice that whether or not stocks go up or down has no impact on your riskless borrowing. You're going to get r times B no matter what, or you're going to owe r times B if B was a negative number, in both cases, because it's riskless. It has nothing to do with whether or not stocks go up or down.

OK, now here's what I want you to do. I want you to select a specific amount of stocks and bonds at date zero in order to make two things true. I want you to select delta and B so as to satisfy these two equations. I want you to pick delta and B so that in the up state, you get C_u and in the down state you get C_d .

Now, what are C_u and C_d ? Remember what they are? They're the value of the call option in the up state and the down state. And you know that in advance. You know what those two possibilities are. You don't know which one's going to occur, but you know that if the up state occurs, it'll be C_u , and if the down state occurs, it'll be C_d .

So I want you to find two numbers, delta and B, that make those two equations true. Can you always do that? How do you know you can always do that?

OK, can you always find a delta and a B to make those two relationships true? Yeah.

AUDIENCE: You have two equations and two variables.

PROFESSOR: Ah, you have two linear equations in two unknowns. And from basic high school algebra, you know that unless those two linear equations are multiples of each other, you can always find one-- exactly one solution that satisfies those two equations in two unknowns. Kind of a handy feature about linear equations.

So as long as these two equations are said to be linearly independent-- that's a fancy way of saying that they're actually two different equations, they're not multiples of each other-- as long as these two equations are not multiples of each other, you can always find two numbers, delta and B, to make that true.

And here they are. Those are the two numbers, delta star and B star. I solved the equation for you, not that you couldn't do it on your own, but for convenience, there it is.

So let me tell you what we've done. We've put together a portfolio of stocks and bonds at date 0 such that at time 1, the value of this portfolio is always equal to the value of the call option in no matter what state of the world actually occurs. Well, by the principle of arbitrage, what this tells us is that the cost of putting together this portfolio that replicates the call option's cash flows, the value of that portfolio at date 0 must equal the price of a call option.

So we're done. The solution of what is the call option price at date 0, it's given by this formula right here. There it is-- a closed-form solution.

Now, before we beat up on it and say, gee, there's only two possibilities, life is more complicated than that, and also there's only one period, let's not-- let's not beat up on it just yet. Let's take a look to see whether or not this makes sense and whether we agree and understand that if, in fact, the assumptions are true, that this is indeed the price of an option. Because this is a pretty remarkable formula. It's a remarkable formula for its simplicity, and for the fact that we actually have been able to derive it explicitly.

Now, the other amazing thing is that there is a missing parameter here. Now you see what the

missing parameter is. What this formula doesn't depend on is the probability of the thing going up or down.

Now, that's astonishing. It's astonishing because what it says is that you and I, we can disagree on whether General Electric is going to go up tomorrow or down tomorrow, and yet we still are going to agree on what the value of a General Electric call option is tomorrow. That's a remarkable fact, and it has to do with a very deep, deep phenomenon going on in option pricing, which is that option pricing is all about pricing the relative magnitude of the security relative to the stock price. And once we understand the basic features of the stock price, like whether or not it can go up or down by u or d , that's more important than the actual probabilities of u and d .

So this expression-- and when you fill in for C_u and C_d , you can plug in for that maximum of uS_0 minus k , 0 , you'll see that there's no p in there as well. So any questions about this? Yeah.

AUDIENCE: You just said that we could disagree on what we-- if the stock will go up or down, [INAUDIBLE].

PROFESSOR: Yeah, the probability of it going up or down, right.

AUDIENCE: And what if we disagree on the actual number?

PROFESSOR: Then we will disagree on the option price. So we have to agree on the u and the d .

AUDIENCE: But if we--

PROFESSOR: Sorry-- yeah, the u and the d .

AUDIENCE: [INAUDIBLE] we will disagree the option price, or that's why this market is possible, because someone will think it will go up--

PROFESSOR: No, no, it's not the reason that the market will be possible. The possibility of the market actually does depend on whether or not there's a demand for this particular kind of payoff. But that doesn't necessarily hinge on the u or the d .

In other words, we can agree on the u and the d , but it turns out that you think that the price is going to go up, therefore, you want to have that kind of a call option bet. I think the price is going to go down, so I'm happy to sell it to you, because I think I'm going to get a good deal on it. So we disagree on the p . You think that there's a high p . I think it's a low p . That's what

drives the market.

And the beauty of this particular setup is that it tells you that you can actually agree on a price, but you have very different reasons for engaging in the transaction. And then you will have markets for this particular security.

Now, I want to go through and look at this formula and try to understand it. First of all, we see that this formula is a weighted average of the value of the call option in the up state and the down state. It's a weighted average. And this part inside the bracket you can think of as a weighted average of the outcome.

But then you discount it back to the 0th period using the one period interest rate. And again, remember this is not meant to be a perpetuity kind of expression. This r is a gross interest rate, so it is equivalent to our old 1 over 1 plus r , where the r that we used is the net interest rate. Here, because of the Cox, Ross, and Rubenstein notation, this is meant to be the gross interest rate. So this looks like a present value, because whatever is inside the bracket, you can think of as some weighted average of the value at date 1, and then this brings it back to date 0.

But now let's look at the weighted average. The weights r minus d and u minus d , those-- it turns out that this plus this adds up to 1, so indeed, it is a weighted average. When you multiply by θ and 1 minus θ , the weights add up to 1. You're basically taking a weighted average.

But I want to argue that it's more than just a simple weighted average. I'm gonna argue that these weights are always non-negative. So in fact, this looks like not just a weighted average, this looks an awful lot like a kind of an expected value, like a probability weighted average.

This looks like a probability. It's not a probability, but I want to argue that this number is always non-negative, and they add up to 1. So when you've got two numbers that are not negative and they add up to 1, you can interpret them as a probability.

Now, what's the argument for why this number is always going to be non-negative? The condition that's required for these numbers to be non-negative is that the interest rate r , the risk-free rate, is strictly contained in between u and d . So you've got u here, d here. r has to be in the middle. And when that's the case, then you've got these things looking like probabilities.

Now, the question is, is that a reasonable assumption? Is it reasonable to assume that d is

less than r is less than u ? Can anybody give me some intuition for why that makes economic sense? It has nothing to do with mathematics. The mathematics couldn't care less as to whether or not that inequality held. Brian?

AUDIENCE: If the downside was less than the rate, then you'd just automatically buy the security.

PROFESSOR: Right.

AUDIENCE: And if the upside was less than the risk-free rate, they'd you'd just go into the risk-free bills.

PROFESSOR: Right. That's exactly right. That's a very important economic insight. Let me go through that slowly.

So Brian, you said if r is less than the downside, then what happens in that case?

AUDIENCE: Then you'd want to buy the stock, the security.

PROFESSOR: If the stock in its worst possible state offers more than T-bills, why would you ever want to buy T-bills? In fact, you wouldn't. And if that were true, then what would happen to the price of T-bills? The price of T-bills--

AUDIENCE: It would go down.

PROFESSOR: It would go to 0. Nobody would hold it, and therefore the value of it would go to zero. It would not exist any longer. So if we're going to assume that there exists riskless borrowing, that can't be true. We can't have r over here.

Now, what about the other side, Brian? What happens if r is over here? What did you say?

AUDIENCE: Then you'd want to go into the risk-free.

PROFESSOR: Right. You would never hold the stock, because even in the best possible world for the stock, you would not be able to get as good a return as T-bills, in which case the value of the stock would go to zero, and therefore there'd be no more stocks in the economy. The only situation where you can have stocks and T-bills coexisting in this simple world-- the only case where that's true is if this inequality held.

That's the economics of this pricing formula. It has nothing to do with math. It's the economics. And the economics tells you that these things have to be non-negative. That's good, because

that suggests that the price of the call option at date 0 can never be negative, because these guys, C_u and C_d , are non-negative, and $1/r$ is non-negative.

So if it turns out that the weights can never be negative, then you know that you've got something that really is a pricing formula. You're never going to punch in some numbers and get out a formula that says this thing is worth minus 2. But more importantly, it suggests that there is a probability interpretation. But the probability is not the mathematical probability that matters. It is the economic probability.

And there is a term for this particular probability. This is known as the risk-neutral probabilities of the particular economy that we've created. And it turns out that these probabilities can be used to price not just options, but anything under the sun. So there's a very, very important property and very deep property that we can't go into here, but you'll cover in 15.437, about the so-called risk neutral probabilities.

But now we've got a formula here. This is a bona fide pricing formula. And the beauty of it is that if it is violated-- if it is violated but the assumptions are correct, then there is a way to create a money machine, an arbitrage, either by buying the cheap stuff and shorting the expensive, or vice versa, in the case where the signs are flipped.

So here's the argument. Suppose that C is greater than V . Then here's the arbitrage. Suppose it's less, and then you basically construct the opposite arbitrage. Therefore the cost of the option has to be equal to the value that we computed. Yeah, [INAUDIBLE].

AUDIENCE: Is this function sensitive [INAUDIBLE]

PROFESSOR: Well, I mean, you tell me. It's a convex combination of these two things. So in that sense--

AUDIENCE: [INAUDIBLE]

PROFESSOR: That's right, exactly. Yeah.

AUDIENCE: And this is not going to [INAUDIBLE]

PROFESSOR: Yeah.

AUDIENCE: The u and d are determined by the market--

PROFESSOR: No.

AUDIENCE: [INAUDIBLE]

PROFESSOR: No. That's a modeling assumption. So in advance, we agree what u and d are. Now in a minute, I'm going to start relaxing all of these assumptions. But before we do that, I want make sure we all agree on what this says. Yeah.

AUDIENCE: So the p is missing, as you said.

PROFESSOR: Yes.

AUDIENCE: But isn't that-- isn't that embedded in C_u and C_d , because you rely on a market price for C_u and C_d ?

PROFESSOR: No, there is no market price for C_u and C_d . Let's go back and take a look at what C_u and the C_d are. That's not a market price. This is not a market price.

C_u is basically the outcome of u times S_0 minus K , and C_d is the outcome of d S_0 minus K , 0 . That's not a market price. We have to agree in advance on what the possible outcomes are. But once we agree on those outcomes, everything follows from that.

There's no market price here. The only market price is S_0 . That is the market price. That is determined today. But fortunately, that market price we observe. We can see it.

Now, there is a link between the market price, u and d , p , because if the stock price today is worth $\$20$, and tomorrow we say that there are two possibilities, either it's $\$30$ or $\$10$, then that tells you that p sort of has to be somewhere around 0.5 . We may disagree. I may think it's 0.55 . You may think it's 0.45 , whatever. But when we aggregate all of our expectations, we come up with $\$20$ for the stock today.

So it's all related. It's in there. But we don't need to make an assumption explicitly for what p is. That is the power of this kind of option pricing approach. [INAUDIBLE]

AUDIENCE: [INAUDIBLE] the reason you get the market here is we agree on everything except that I think that the higher [INAUDIBLE].

PROFESSOR: Yeah.

AUDIENCE: --it will go up, and you think it will go down [INAUDIBLE].

PROFESSOR: Right.

AUDIENCE: What if we fundamentally don't agree on u and d ?

PROFESSOR: Oh, then we have a problem. We need to assume a particular u and d that we can agree on. So let me turn to that now. Let me turn to the extension of this.

So what I've derived is a one-period pricing model-- very, very simple. It turns out that you can do a multiperiod pricing model. And this multiperiod generalization is given by this.

What is that multiperiod generalization? Basically you have-- let me see if I have the diagram here-- the multiperiod generalization is simply that you now have a bunch of possibilities, and you are figuring out what the price of the option is at date 0 when it pays off at date capital N , or lowercase n in this case-- n periods.

And you can use exactly the same arbitrage argument that I just showed you, but it's a little bit more complicated now because you've got multiple branches. But it's still, at every step of the way, a binomial or a Bernoulli trial. And so in a multiperiod setting, you get a binomial tree.

Now, the reason that this is such a powerful extension is that nowhere have I specified what a period is. I just said it's a period, today versus tomorrow. But it could be today versus three minutes from now, or three femtoseconds from now, or three years from now. I haven't specified.

So if you say we can't agree on a u and a d , fine, let's not agree on a u and a d . Let's agree that between now and five minutes from now, there are 256 possible outcomes for the stock price. Do you agree on that? You think we can agree on that? Is that something that's easy to agree on? Well, if that's the case, then all I need to do is to have enough steps between now and five minutes from now to have 256 possibilities.

And by the way, I chose that number specifically as a power of 2 because with these kinds of branches, it's actually very easy to be able to get that kind of a tree, with that many branches. So now you see that the u and the d , that's not relevant, because we can make it as small as you would like. If you would like to have it really, really fine, I can get it down to double precision, 32 decimal places, by basically taking one period to be a millisecond.

And this binomial option pricing formula will apply exactly in the same way. It turns out that when you let the number of periods go to infinity, and at the same time, you control the u and

the d and make them smaller and smaller and smaller and smaller so as to be able to get a tree that is reasonably realistic, you know what you get? You get the Black-Scholes formula. The pricing formula that you get is a solution to this parabolic partial differential equation with the following boundary conditions.

And so using the simple binomial two-step kind of process, when you let it go to infinity and you shrink the probabilities and the u and the d to make it more and more refined, you get the Black-Scholes formula. This is something that Black and Scholes never, never contemplated. So this is a completely different approach that allows you to reach the exact same conclusion, which is a startling one.

Now, as I told you at the beginning, when people apply option pricing formulas, most of the time they do not do this. They do not solve the heat equation. What they do is that. They do a binomial tree.

The reason is because in order to solve these PDEs, except in a very, very small number of textbook example cases, you can't solve this analytically anyway. You can't get a formula. You have to solve it numerically. And so if you're going to go to the trouble of solving these differential equations numerically, you may as well just do the binomial option pricing formula, because that's numerical as well. And it's a lot simpler computationally to be able to do that binomial tree.

By the way, for those of you computing fans who like to think about parallel processing, these kinds of binomial trees are extraordinarily easy to parallelize. So if you thought about the old days, where you had a connection machine that was developed by Danny Hillis, you had 64,000 processors in parallel. You can actually make use of that by implementing a binomial tree.

Nowadays we've got grid computing. The most recent advance is to be able to use both hardware and software to do distributed computing. The binomial tree is ideally suited for being able to do that. So you can evaluate extraordinarily complex derivatives very, very quickly using this kind of a framework.

So you're not giving up a lot by the u and the d , because we can make the u and the d as fine as possible so that ultimately we would all say, yeah, enough. I agree, all right, leave me alone. I don't want any more binomial trees. This is complicated enough. 256 of them over a five-minute interval is enough for all practical purposes. Yeah.

AUDIENCE: [INAUDIBLE] if that cannot be solved, why was it so important?

PROFESSOR: Oh, no, this can be solved. The solution of this equation is the Black-Scholes formula. What I said cannot be solved is when you have a more complicated security.

So for example, the option pricing formula that we looked at with the simple plain vanilla call and put option, that's relatively straightforward. But think about something like a mortgage-backed security that has all sorts of conversion features and knockout features, and other types of legal restrictions, as well as certain rights and requirements. Then it's not so easy. It looks much more complicated.

For example, this particular coefficient that multiplies this second derivative ends up being a highly non-linear function, not just a quadratic. Or this piece here becomes a nonlinear function, or the boundary conditions are kind of weird. In that case, you can't solve it analytically. You have to use numerical methods to solve it.

AUDIENCE: [INAUDIBLE]

PROFESSOR: This is just the arbitrage condition that says that the solution C will give you a null arbitrage price for the call option. So the equivalent of this PDE, partial differential equation, is-- go back-- is this, the simultaneous equation up there and down here, and then this expression that says that the price of the option has to be given by this particular portfolio. That's what the PDE looks like in continuous time, or when you have an infinite number of time steps.

So it is not-- that's absolutely a good question, because this is solvable. But very quickly, when you change the terms of a contract, it turns out that it's very hard to model. Yes.

AUDIENCE: Question about the random walk.

PROFESSOR: Yes.

AUDIENCE: Can you just briefly mention how that feeds into the final answer, and how it will change things if it's--

PROFESSOR: Well, the random walk hypothesis is implicit in here, because I've got a coin toss. And the coin toss is independent period by period. If the coin toss is not independent, then that's the wrong formula. In other words, you don't have a simple binomial distribution if you don't have IID coin

tosses.

The random walk is basically the assumption of IID coin tosses-- independently and identically distributed. That's what IID stands for-- IID coin tosses. So that's where the Bachelier assumption came in. In order for Bachelier to derive the heat equation, or some variant of the heat equation, he was implicitly assuming that what happens in one period for the stock has no bearing on what happens in next period. If stock prices are correlated over time, then these formulas do not work. You need a different kind of formula.

It's actually not that far off. You can derive an expression for an option pricing formula with correlated returns. In fact, professor Wang and I published a paper, I think it's maybe close to 10 years ago, where we worked out that case. But up until then, most people assumed that stock prices are not correlated, so the Brownian motion or random walk idea fit in very nicely with this binomial.

If they're correlated, then you no longer have IID Bernoulli trials, you have a Markov chain, and you have to use Markov pricing in order to be able to get this formula. If you're interested in this, I urge you to take 15 437, because that's where we go into it in much more depth. Yeah.

AUDIENCE: [INAUDIBLE] use the binomial coin to value options, and we see a range of prices.

PROFESSOR: Yeah.

AUDIENCE: So how do we approach that? Do we take some kind of average? Is this common, or do we receive a specific u and a d each time? I mean, I imagine it could be a range [INAUDIBLE].

PROFESSOR: No, no. So the way that you would apply this is that you would, first of all, pick the number of periods that are appropriate to the problem at hand. So if you have got an option that's expiring in three months, then typically, if you did it on a daily basis or an hourly basis, that would be more than enough. And then you would assume that there would be a u and a d in order to match the approximate outcomes that you would expect. And then out of that, you would actually get a number.

So this, this C_0 , when you plug in all of these parameters, you actually get a number, like \$30.25. That's the price of the option. And of course if you change the parameters, you change the strike price, the interest rate changes, the u and the d changes, that will change

the value of the option price as well.

AUDIENCE: [INAUDIBLE] every now and then, [INAUDIBLE] to receive a range from you, and a range--

PROFESSOR: No, no, no. What you do is you start off with an assumption for what u and d exactly are. Not a range, but actually if it goes up, it goes up by 1.05. If it goes down it goes down by 0.92. Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Oh, well, it varies depending on the particular instrument that you're trying to price. So-- well, no, what I mean is options on what stock? So in other words, with any kind of option pricing formula, you actually have to calibrate these parameters. So you have to figure out what the interest rate is, and then typically what is done is you assume a particular grid, and then use a u and a d that will capture all the elements of that grid.

So for example, let's assume that u is 25 basis points plus 1, and d is 1 minus 25 basis points. So that means you can capture stock price movements that go up by 25 basis points or down, and you assume a number of n in order to get that tree to be as fine as you would like for the particular time that you're pricing it at.

So in other words, if I use 25 basis points and n equal to 1, that means that I can capture a situation where, at maturity, the stock price goes up or down by 25 basis points. If I now go four periods, then I can capture a situation where the stock price goes up by 1% or down by 1% in 25-basis-point increments. And if I want more refinements, then I keep going, let n get bigger and bigger and bigger. And then whatever that is, that final number of nodes will be the possible stock price values.

AUDIENCE: [INAUDIBLE] historical data on the specific stock to--

PROFESSOR: You would use historical data. You would use historical-- because the way you calibrate this is you can show that the expected value-- so the expected value of S_1 is just equal to the probability of u S_0 plus 1 minus probability of d S_0 . So you've got the expected value. Calculate the variance of S_1 , and you'll get another expression with u and d and p , and then you simply use historical data to match the parameters and pick them so that they give you a reasonable approximation to reality.

AUDIENCE: [INAUDIBLE] doesn't continue to behave as the history--

PROFESSOR: Yes.

AUDIENCE: --so the options--

PROFESSOR: Yeah.

AUDIENCE: --don't match.

PROFESSOR: Absolutely. That's always the case, isn't it? In other words, if you don't have IID, you're going to get a problem. But remember, it doesn't depend upon the p . And so in that sense, if there's a change in p , as long as the u and the d are appropriate, you'll still be able to capture the value of the option. Question.

AUDIENCE: I'm trying to figure out the analogy with the [INAUDIBLE].

PROFESSOR: Yeah.

AUDIENCE: So I understand how it works in temperature. What would be here that [INAUDIBLE].

PROFESSOR: Let me-- let me not talk about that now, because I suspect that while you may be interested and a couple of other people, we probably don't have everybody being physicists here. So we'll talk about that afterwards. And also, that's something that, again, in 437, they may touch upon. But I want to keep moving along, because this is already more complicated than the nature of what I want to cover in this course. So let me get back to you on that, but we can talk about it afterwards. Any other questions about this? Yeah.

AUDIENCE: I have a question about volatility, and how it is going to play in the equation. Like for example, I have two scenarios. They all, in three months, could go up or down by u or d . But the volatility of those two scenarios vary dramatically.

PROFESSOR: Right.

AUDIENCE: So how does--

PROFESSOR: How does volatility enter into this. That's a good question. Well, what do you think volatility is captured by in this simple Bernoulli trial?

AUDIENCE: The difference between u and d .

PROFESSOR: Exactly, exactly. Volatility is a measure of the spread between u and d . Holding other things

equal-- by that, I mean holding the current stock price equal, holding the probability p equal-- so fixing that, as I increase the spread between u and d , I'm increasing the volatility. And if there's one thing that we see that matters is the spread between u and d .

So if the spread between u and d increases, that actually will have an impact on this formula, and you have to work out the effects, which is a very easy thing to do. You can even do this in a spreadsheet. But you can show that as the volatility increases, the value of the call option is actually increasing. So take a look at that, and you'll see that it behaves the way that we think it should. OK, other questions?

OK, well, so I'll leave it at this point, which is to say that the derivatives literature is huge. And it has really spawned a number of different not only securities, but also different methods for hedging and managing your portfolios, to the point where really, derivatives are everywhere.

And there are some examples that I've given you here, but this is an area which is considered rocket science because of the analytics that are so demanding. So this is a natural area for students here at MIT to be involved in, but it's certainly not the only area.

And ultimately, what's important about derivatives is not just the pricing and the hedging, but rather the application. So the fact that we spend a fair bit of time at the beginning of this lecture talking about payoff diagrams, that wasn't just for completeness. That really is one of the most important aspects, is how you use options in order to tailor the kinds of risks and return profiles that you'd like to have. And now that you know how to price them, you can have a very clear sense of whether or not they are appropriate from a risk-return tradeoff. But they are very different, as you can see, from the securities that we've done.

However, having done it, having now priced options and other derivatives, which are really relatively straightforward extensions, we've now been able to price virtually 99% of all the securities that you would ever run into. We've done stocks. We've done bonds. We've done futures, forwards, and now options, so there really isn't any other kind of financial security out there that you could possibly come across that you don't know how to price.

You may not realize it yet, and the purpose of the second half of the course is to introduce risk and show you how to use all of these methods to price all of the other securities that you will come into contact with. And then, of course, in 402 and other finance courses, you'll see that

much more closely.

So for example, a revolving credit agreement, a sinking fund debt issue, a credit default swap, an interest rate swap-- all of these securities are mixtures of the securities that we've seen till now. And the pricing method in all of these cases is exactly the same, which is identify the cash flows, come up with another portfolio that has the same cash flows, but where you know how to construct it, therefore the price of that security has to be equal to the price of the thing that you're trying to value.

That's the basic principle in virtually all financial pricing applications. So once you understand these concepts, you can literally price anything under the sun, and all you need between now and then is practice, practice, practice in doing that.

All right, so that wraps up the lecture on derivatives. And now I want to turn to risk and reward, because up until now, we've really talked about risk in an indirect way, and I want to talk about it in a much more direct fashion by looking at measures of risk.

So what I want to do now is to turn to a little bit of statistical background to talk about risk and return. I want to motivate it first, and then give you the measures that we're going to use for capturing risk and return, and then apply it to stocks, and get a sense of what kinds of anomalies are out there that we should be aware of.

And then I'm going to take these measures, and then tell you how to come up with the one number that I've had to put off for the first half of the semester, which is the cost of capital-- the required rate of return, the risk-adjusted rate of return. We are now going to get to a point where we can actually identify what that number is, and how to make that risk adjustment. So that's where we're going.

Now, to give you a quick summary of where we are, as I told you, we've priced all of these different securities. But underlying all of these prices is a kind of a net present value calculation where we're taking some kind of a payoff or expected payoff and discounting it at a particular rate, and we need to figure out what that appropriate rate of return is.

I've said before that that rate of return is determined by the marketplace. But what we want to know is how. How does the market do that? Because unless we understand a little bit better what that mechanism is, we won't be in a position to be able to say that the particular market that we're using is either working very well or completely out to lunch and crazy. So we need to

deconstruct the process by which the market gets to that.

In order to do that, we have to go back even farther and peel back the onion and ask the question, how do people measure risk, and how do they engage in risk taking behavior? So we have to do a little bit more work in figuring out these different kinds of measures, and then talking explicitly about how individuals actually incorporate that into their world view.

Along the way, we're going to ask questions like, is the market efficient, and how do we measure the performance of portfolio managers? This past year, the typical portfolio manager has lost about 30% to 40%. That's a pretty devastating kind of return. And in that environment, if you found a portfolio manager that ended up losing you 10%, you might think, gee, that's pretty good.

Does that really make sense? Is it ever the case that we want to congratulate a portfolio manager for losing money for us? We have to answer that question in the context of how you figure out what an appropriate or fair rate of return is. So that's what we're going to be doing.

Now, to do that, I need to develop a little bit of new notation. And so the notation that I'm going to develop is to talk about returns that are inclusive of any kind of distributions, like dividends. So when I talk about the returns of equities, I'm going to be talking explicitly about a return that includes the dividend.

And so the concept that we're going to be working on, for the most part, for the next half of this course is the expected rate of return. We obviously will be talking about realized returns, but from a portfolio management perspective, we're going to be focusing not just on what happened this year or what happened last year, but we're going to be focusing on the average rate of return that we would expect over the course of the next five years.

We're going to be looking at excess returns, which is in excess of the net risk-free rate, little r_f . And what we refer to as a risk premium is simply the average rate of return of a risky security minus the risk-free rate. So the excess return you can think of as a realization of that risk premium. But on average over a long period of time, the number that we're going to be concerned with most is this risk premium number, the average rate of return minus the risk-free rate.

Over the course of the last 100 years or so, US equity markets have provided an average rate of return minus the risk-free rate on the order of 7%. That's pretty good, but that's a long-run

average. The realized excess rate of return this year is horrible, so I'm not even going to talk about what that number is, but it's bad. But do you see the difference between this year's rate of return versus the long-run average? And we can talk about both of them, but we're going to use different techniques for each.

So the technique for talking about the statistical aspects of returns will be from the language of statistics. We're going to talk about the expected rate of return. I'm going to use the Greek letter mu to denote that.

We're gonna also talk about the riskiness of returns, which I'm going to use the variance and the standard deviation to proxy for. So the variance is simply the expected value of the squared excess return. That gives you a sense of the fluctuations around the mean.

And the standard deviation is the square root of the variance. And we use the standard deviation simply because that's in the same units. It's in units of percent per year, whereas the variance is in units of percentage points squared per year, so it's a little easier to deal with the standard deviation.

And those concepts are the theoretical or population values of the underlying securities that we're going to look at. We also want to look at the historical estimates, and the historical estimates are given by the sample counterparts. So this is the sample mean, the sample variance, and the sample standard deviation.

You should all remember this from your DMD class. But if not, we'll have the TAs go over it during recitation. You can also look in the appendix of Brealey, Myers, and Allen, and they'll provide a little review about this.

Now, there are lots of other statistics, and the only one that I'm gonna spend time on is the correlation. There's the median instead of the mean. You can look at skewness, which way the distribution leans. But what we're going to look at in just a little while is correlation, which is how closely do the returns of two investments move together.

If they move together a lot, then we say that they're highly correlated, or co-related. And if they don't move together a lot, they're not very highly correlated. And in some cases, if they move in opposite directions, we say that they're negatively correlated.

So correlation, as most of you already know, is a statistic that's a number between minus 1 and 1, or minus 100% and 100%, that measures the degree of association between these two

securities. We're going to be making use of correlations a lot in the coming couple of lectures to try to get a sense of whether or not an investment is going to help you diversify your overall portfolio, or if an investment is only going to add to the risks of your portfolio.

And you can guess as to how we're going to measure that. If the new investment is either zero correlated or negatively correlated with your current portfolio, that's going to help in terms of dampening your fluctuations. But if the two investments move at the same time, that's not only going to not help, that's going to actually add to your risks. And you don't want that, at least not without the proper reward. So that's a brief preview of how we're going to use these statistics.

And you get some examples here about what correlation looks like. Here I've plotted four different scatter graphs of the return of one asset on the x-axis and the return of another asset on the y-axis. And the dots represent those pairs of returns for different assumptions about correlation.

So the upper left-hand scatter graph is a graph where there's no correlation. The correlation is zero. The scatter graph on the lower left is where there's very high positive correlation-- 80% correlation between the two. And the scatter graph on the lower right is where there's a negative 50% correlation.

So we're going to use correlation, along with mean and variance, to try to put together good collections of securities-- in other words, good portfolios of securities. And by doing that, we're going to show that we can actually construct some very attractive kinds of investments using relatively simple information. But at the same time, we're going to use that insight to then deconstruct how to come up with the appropriate risk adjustment for cost of capital calculations.

Now, there's a review here about normal distributions and confidence intervals, and I'd like you to go over that, either on your own or with the TAs during recitations. We're going to be using these kinds of concepts to try to measure the risk and return of various different investments.

Here's an example of General Motors' monthly returns. That's a histogram in blue, and the line, the dark line, is the assumed normal distribution that has the same mean and variance. And you could see that it looks like it's sort of a good approximation, but there are actually little bits of extra probability stuck out here and stuck out here that don't exactly correspond to

normal.

In other words, the assumption of normality would say that the probability of getting a return of minus 15% is relatively low, then getting a return less than minus 20% is exceedingly low. But the reality is different. There are risks of having much lower returns in the data. And after this year, I can tell you that these tails are going to be fatter. So this is meant to be an approximation, not reality.

The approximation is what we're going to go over in this course, and in the very last lecture, I want to tell you how good that approximation is. And then I'm going to tell you about a number of courses you might want to take that focus on getting that last 5% right. So 95% of the distribution is captured by what I'm going to teach you in this course, but if you want to get the other 5% right-- and by the way, if you're going into investments as a profession, it's all about that 5%-- then you'll want to take 15 433, investments.

So with that as the basic preamble, let me tell you what I'm going to talk about next time, since we're almost out of time. What we're gonna do next time is I'm going to talk about the US stock market. I'm gonna talk about volatility, about predictability, and then I'm going to talk a bit about the notion of efficient markets, and try to describe to you what kinds of properties we expect from typical investments.

And we're actually going to go through some numbers. I'm gonna show you some examples of basic statistics for the stock market that will give you a sense of how things have behaved over the last 50 years. And what you'll get a sense of is that in some cases, there is a lot of predictability. There are certain things that we can count on.

For example, these are stock market returns from 1946 to 2001. This is monthly data, monthly returns of the S&P 500 over a fairly long period of time. And this might sort of look like a typical person's EKG over the last few weeks. Not surprisingly, there were periods where we had some pretty bad returns. We're going to see another one of these things as well over the more recent period.

But when you look at this thing, you then begin to appreciate that what we're living through now, while it's bad and it's scary, it's not at all unusual or completely unheard of. There are periods in the stock market where we've seen really big swings. And by the way, this is just the US. If I had shown you some emerging market returns, it would go off the screen. So we're going to talk about that next time.

And out of all of this chaos, we're going to distill a very important relationship. We're going to ultimately come up with a simple linear equation that shows you how to make that risk adjustment between the expected return and the underlying risk of a portfolio. So that's coming up, and we'll do that on Wednesday. All right, see you then.