15.401 Recitation

6: Portfolio Choice

Learning Objectives

- Review of Concepts
  - Portfolio basics
  - Efficient frontier
  - Capital market line
- Examples
  - XYZ
  - Diversification
  - Sharpe ratio
  - Efficient frontier

Review: portfolio basics

- A portfolio is a collection of N assets ($A_1, A_2, \ldots, A_N$) with weights ($w_1, w_2, \ldots, w_N$) that satisfy
  - $\sum_{i=1}^{N} w_i = 1$
- Each asset A_i has the following characteristics:
  - Return: $\tilde{r}_i$ (random variable)
  - Mean return: $\bar{r}_i$
  - Variance and std. dev. of return: σ_i^2, σ_i
  - Covariance with A_j: σ_{ij}

Review: portfolio basics

- The return of a portfolio is

$$\tilde{r}_p = \sum_{i=1}^{N} w_i \tilde{r}_i$$

- The mean/expected return of a portfolio is

$$E(r_p) = \bar{r}_p = \sum_{i=1}^{N} w_i \bar{r}_i$$

- The variance of a portfolio is

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}; \quad \sigma_p = \sqrt{\sigma_p^2}$$

- Note: $\sigma_{ii} \equiv \sigma_i^2;\ \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j$

Example 1: XYZ

	E(r)	Variance-Covariance		
		X	Y	Z
X	15%	0.090	0.125	0.144
Y	10%		0.040	-0.036
Z	20%			0.625

- What is the expected return and variance of a portfolio of …
  a. (X, Y) with weights (0.4, 0.6)?
  b. (X, Y, Z) with weights (0.2, 0.5, 0.3)?
  c. (X, Y, Z) with weights (1/3, 1/3, 1/3)?

Example 1: XYZ

- Answer:

a. $E\left(r_p\right)=12\%;\sigma_p^2=0.08880;\sigma_p=29.80\%$

b. $E\left(r_p\right)=14\%;\sigma_p^2=0.10133;\sigma_p=31.83\%$

c. $E\left(r_p\right)=15\%;\sigma_p^2=0.13567;\sigma_p=36.83\%$

Example 1: XYZ

- What is the minimum possible variance of a portfolio with only Y and Z?

	E(r)	Variance-Covariance		
		X	Y	Z
X	15%	0.090	0.125	0.144
Y	10%		0.040	-0.036
Z	20%			0.625

Example 1: XYZ

- Answer:

  Let (w, 1–w) be the weights for (Y, Z), then

$$\arg\min_{w}\left[w^2 \cdot 0.04 + 2w(1-w)(-0.036) + (1-w)^2 \cdot 0.625\right]$$

- First-order condition:

$$2w \cdot 0.04 + 2(1-2w)(-0.036) - 2(1-w) \cdot 0.625 = 0$$

$$w^* = 0.8969$$

- The minimum variance portfolio is

$$(0.8969, 0.1031)$$

Example 2: diversification

- Suppose that your portfolio consists of *N* equally weighted identical assets in the market, each of which has the following properties:
  - Mean = 15%
  - Std dev = 20%
  - Covariance with any other asset = 0.01
- What is the expected return and std dev of return of your portfolio if…
  - *N* = 2?
  - *N* = 5?
  - *N* = 10?
  - *N* = ∞?

Example 2: diversification

- Answer:
  - Expected return

$$E(r_p) = \sum_{i=1}^{N} \frac{1}{N} \cdot 0.15 = 0.15$$

  - Variance

$$\sigma(r_p) = \sum_{i=1}^{N} \frac{0.2^2}{N^2} + \sum_{i=1}^{N} \sum_{j \neq i} \frac{0.01}{N^2} = N\left(\frac{0.2^2}{N^2}\right) + N(N-1)\frac{0.01}{N^2}$$

$$= \frac{0.04}{N} + \left(1 - \frac{1}{N}\right)0.01 = 0.01 + \frac{0.03}{N}$$

Example 2: diversification

- Answer:
  - $N = 2$:

    $$E(r_p) = 15\%; \sigma_p^2 = 0.0250; \sigma_p = 15.81\%$$

  - $N = 5$:

    $$E(r_p) = 15\%; \sigma_p^2 = 0.0160; \sigma_p = 12.65\%$$

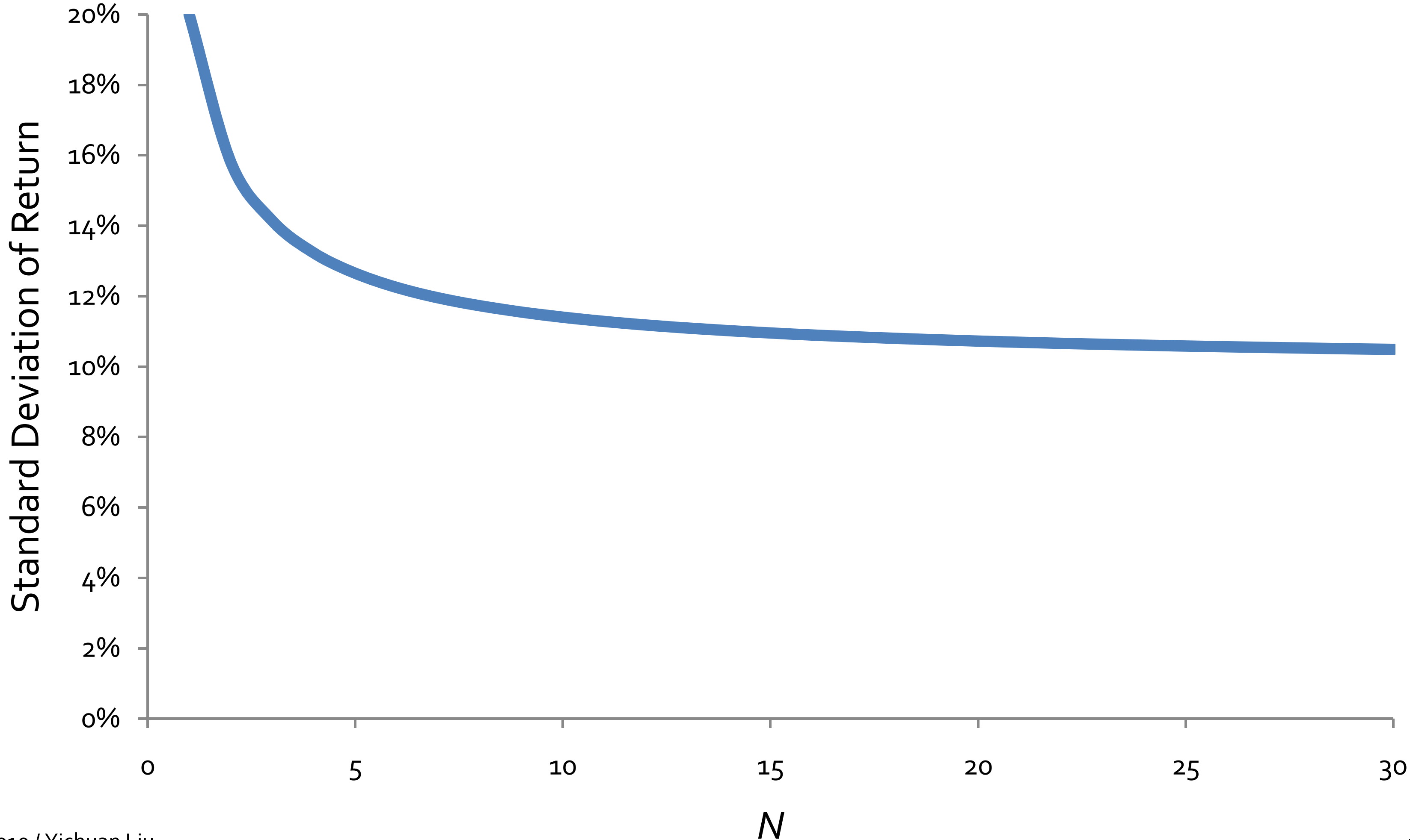
  - $N = 10$:

    $$E(r_p) = 15\%; \sigma_p^2 = 0.0130; \sigma_p = 11.40\%$$

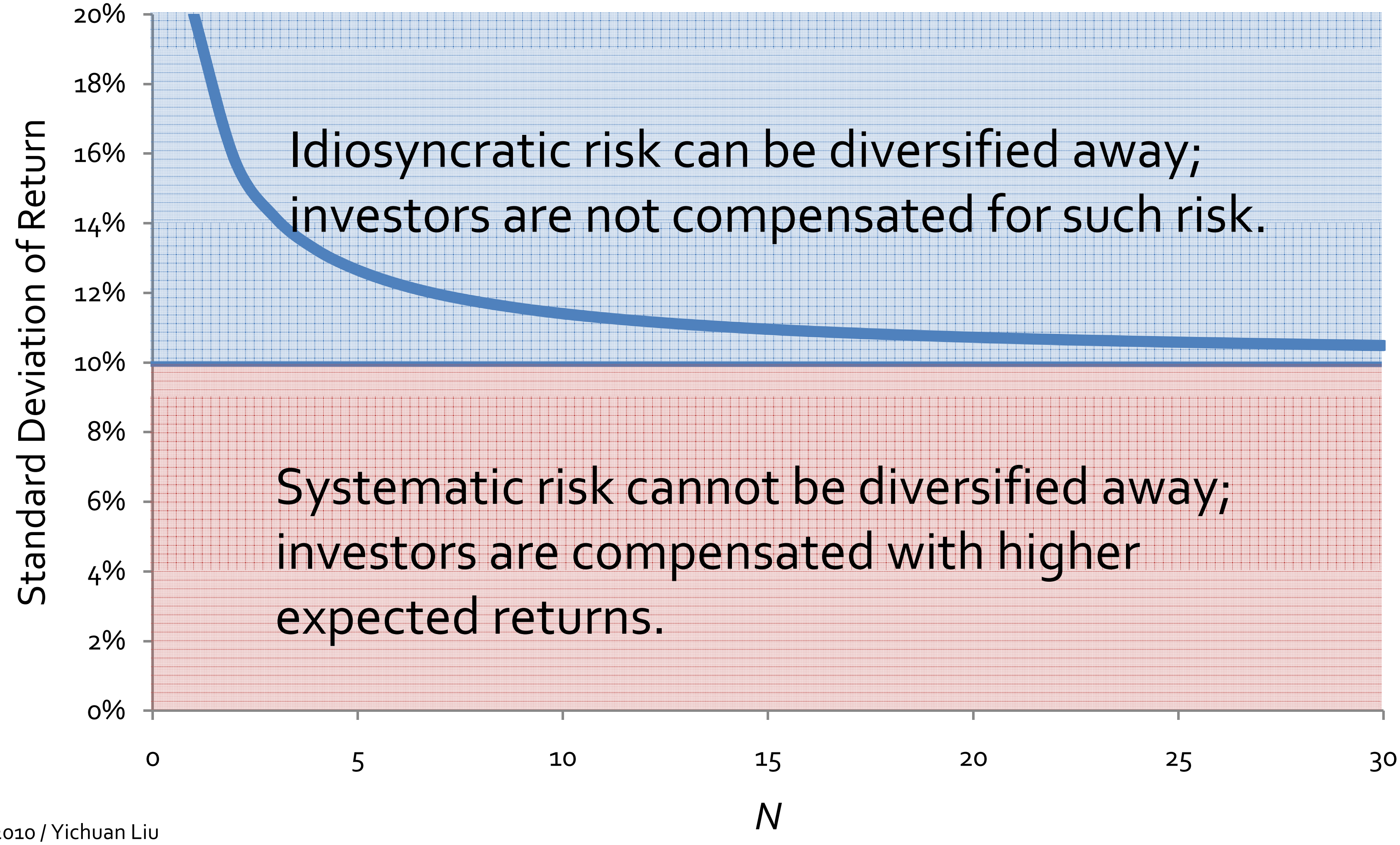
  - $N = \infty$:

    $$E(r_p) = 15\%; \sigma_p^2 = 0.0100; \sigma_p = 10.00\%$$

Example 2: diversification



Review: diversification

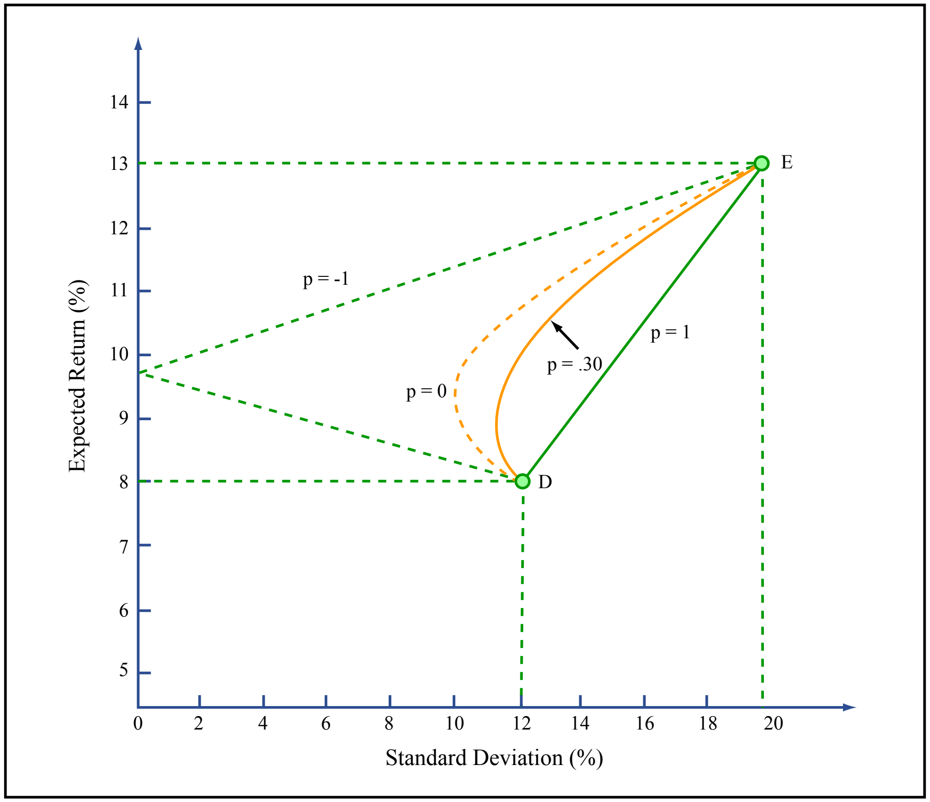


Review: efficient frontier

- Given two assets, we can form portfolios with weights (w, $1-w$). As we vary w, we can plot the **path** of the **mean return** and **standard deviation of return** of the resulting portfolio.
- The shape of the path depends on the correlation between the two assets.
- When the correlation is low, a large portion of asset return variation comes from idiosyncratic risk that can be diversified away.

Review: efficient frontier

- ρ = 1
  perfectly correlated
  no risk reduction potential
- -1 < ρ < 1
  imperfectly correlated
  some risk reduction potential
- ρ = -1
  perfectly negatively correlated
  most risk reduction potential



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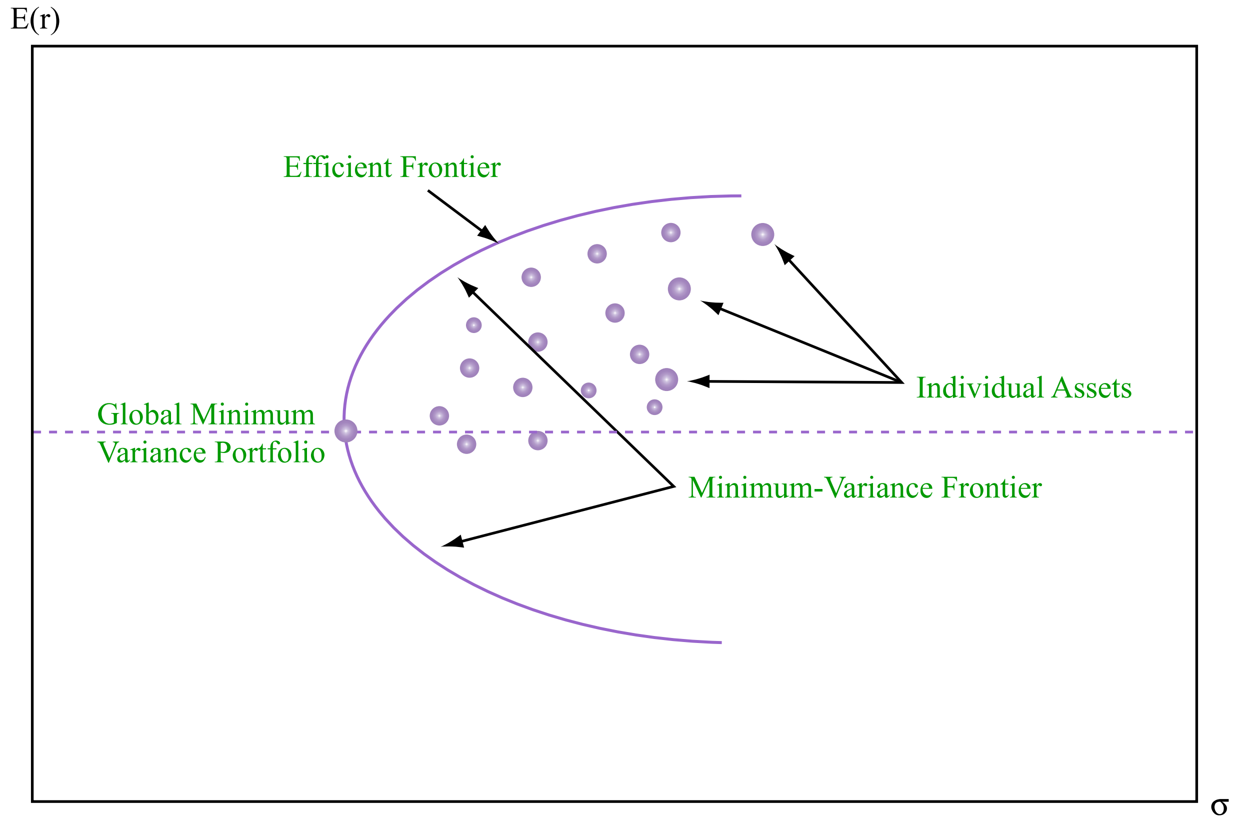
Review: efficient frontier

- We can repeat the previous exercise for *N* assets:



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Review: efficient frontier

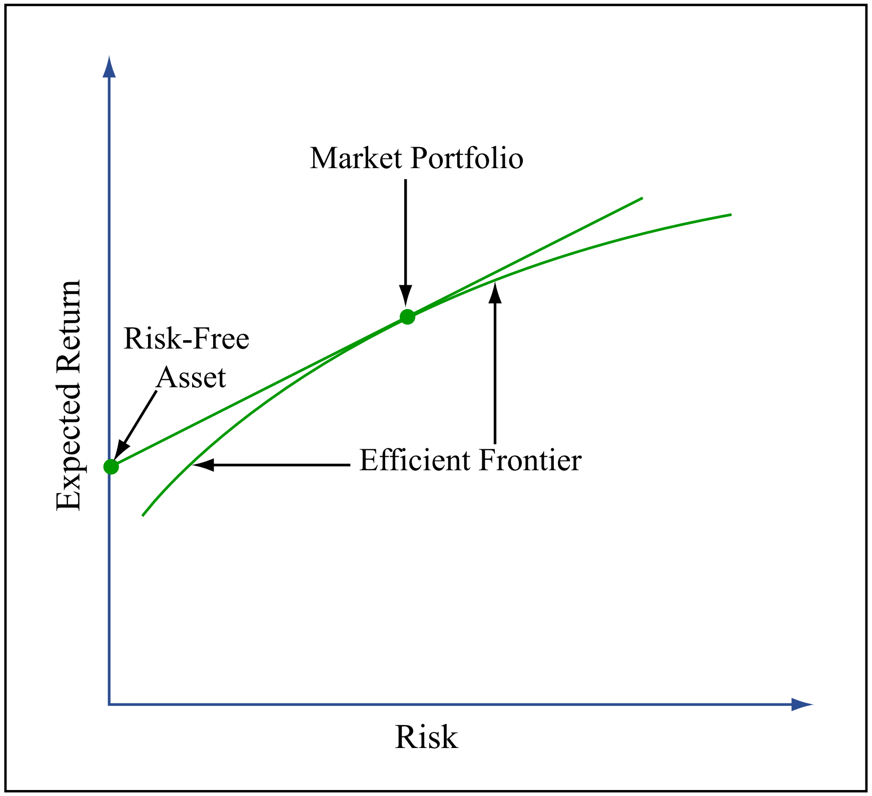
- The efficient frontier can be described by a function $\sigma^*(r_p)$, which minimizes the portfolio std dev given an expected return:

$$\sigma^*(r_p) \equiv \min_{\{w_i\}} \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}} \quad \text{s.t.} \quad \begin{cases} \sum_{i=1}^{N} w_i = 1 \\ \sum_{i=1}^{N} w_i \bar{r}_i = r_p \end{cases}$$

- Analytical solution for $\sigma^*(r_p)$ is possible but difficult to derive.

Review: capital market line

- Efficient frontier + risk-free asset = CML



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Example 3: Sharpe ratio

- The Sharpe ratio measures the reward-risk tradeoff of an asset or a portfolio. It is defined as

$$S = \frac{\bar{r} - r_f}{\sigma}$$

- The higher Sharpe ratio, the more desirable an asset / a portfolio is. Suppose r_f = 5%. What is the portfolio of (A, B) with the highest Sharpe ratio?

	E(r)	COV-VAR	
		A	B
A	15%	0.090	0.015
B	10%		0.040

Example 3: Sharpe ratio

- Answer:

$$\max_w S_p \equiv \max_w \frac{wr_A + (1-w)r_B - r_f}{\sqrt{w^2\sigma_A^2 + 2w(1-w)\sigma_{AB} + (1-w)^2\sigma_B^2}}$$

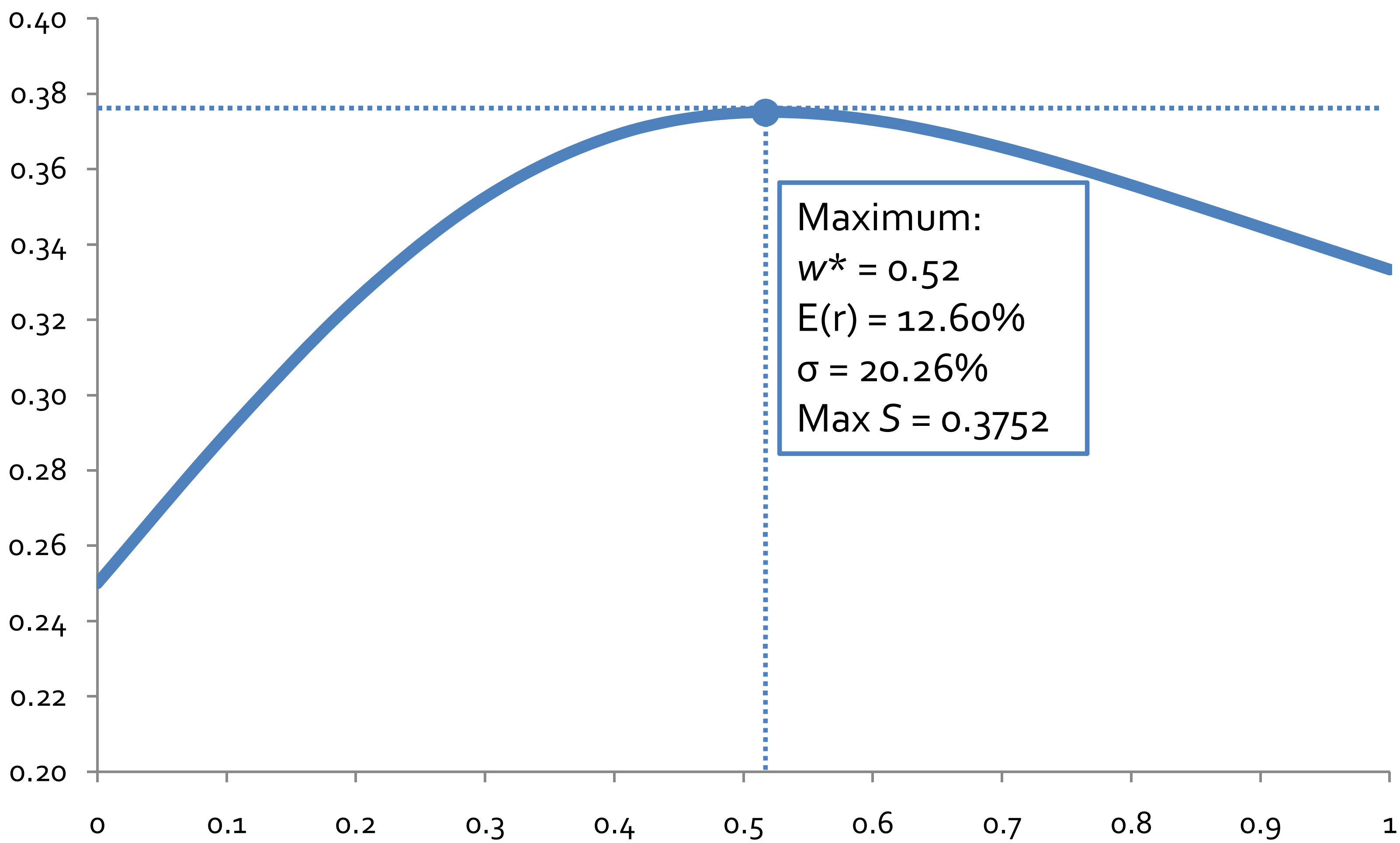
- Method 1: grid search
  1. Set up a grid for *w*, e.g., *w* = 0, 0.1, 0.2, …, 1.0
     The finer the grid, the more accurate the result
  2. Calculate the Sharpe ratio for each *w*
  3. Find the maximum Sharpe ratio.

Example 3: Sharpe ratio

- Method 1: grid search

w	$1-w$	$r_p - r_f$	σ_p	S_p
0	1	0.0500	0.2000	0.2500
0.1	0.9	0.0550	0.1897	0.2899
0.2	0.8	0.0600	0.1844	0.3254
0.3	0.7	0.0650	0.1844	0.3525
0.4	0.6	0.0700	0.1897	0.3689
0.5	**0.5**	**0.0750**	**0.2000**	**0.3750**
0.6	0.4	0.0800	0.2145	0.3730
0.7	0.3	0.0850	0.2324	0.3658
0.8	0.2	0.0900	0.2530	0.3558
0.9	0.1	0.0950	0.2757	0.3446
1	0	0.1000	0.3000	0.3333

Example 3: Sharpe ratio



Example 3: Sharpe ratio

- Method 2: Excel Solver

	A	B	C	D	E
1			E(r)	Asset A	Asset B
2				=B3	=B4
3	Asset A		0.15	0.09	0.015
4	Asset B	=1 –B3	0.1	0.015	0.04
5					
6			$r_p - r_f$	σ_p	S
7			=f	=g	=C7/D7

f: SUMPRODUCT(B3:B4, C3:C4) – 0.05
g: SQRT(B3*D2*D3+B3*E2*E3+B4*D2*D4+B4*E2*E4)

Solver

Set Target Cell:
E7

Equal To:
Max

By Changing Cell:
B3

Example 3: Sharpe ratio

- Method 2: Excel Solver

	A	B	C	D	E
1			E(r)	Asset A	Asset B
2				0.52	0.48
3	Asset A	0.52	0.15	0.09	0.015
4	Asset B	0.48	0.1	0.015	0.04
5					
6			$r_p - r_f$	σ_p	S
7			0.076	0.202583	0.375154

Example 3: Sharpe ratio

- Method 3: analytical solution
  - Full derivation:

$$\frac{\partial S}{\partial w} = \frac{(\bar{r}_A - \bar{r}_B)(\sigma_p^2)^{\frac{1}{2}} - \frac{1}{2}(\sigma_p^2)^{-\frac{1}{2}}(2w\sigma_A^2 + 2(1-2w)\sigma_{AB} - 2(1-w)\sigma_B^2)(\bar{r}_p - r_f)}{(\sigma_p^2)^{\frac{1}{2}}}$$

$$= \frac{(\bar{r}_A - \bar{r}_B)(w^2\sigma_A^2 + 2w(1-w)\sigma_{AB} + (1-w)^2\sigma_B^2) - (w\sigma_A^2 + (1-2w)\sigma_{AB} - (1-w)\sigma_B^2)(w\bar{r}_A + (1-w)\bar{r}_B - r_f)}{\sigma_p^2}$$

$$= 0$$

$$0 = (\bar{r}_A - \bar{r}_B)(w^2\sigma_A^2 + 2w(1-w)\sigma_{AB} + (1-w)^2\sigma_B^2) - (w\sigma_A^2 + (1-2w)\sigma_{AB} - (1-w)\sigma_B^2)(w\bar{r}_A + (1-w)\bar{r}_B - r_f)$$

$$= (\bar{r}_A - \bar{r}_B)(w^2\sigma_A^2 + 2w(1-w)\sigma_{AB} + (1-w)^2\sigma_B^2) - (w\sigma_A^2 + (1-2w)\sigma_{AB} - (1-w)\sigma_B^2)(w(\bar{r}_A - \bar{r}_B) + \bar{r}_B - r_f)$$

$$= (\bar{r}_A - \bar{r}_B)(w\sigma_{AB} + (1-w)\sigma_B^2) - (w\sigma_A^2 + (1-2w)\sigma_{AB} - (1-w)\sigma_B^2)(\bar{r}_B - r_f)$$

$$= [(\bar{r}_A - \bar{r}_B)\sigma_B^2 - (\sigma_{AB} - \sigma_B^2)(\bar{r}_B - r_f)] - [(\bar{r}_A - \bar{r}_B)(\sigma_B^2 - \sigma_{AB}) + (\sigma_A^2 - 2\sigma_{AB} + \sigma_B^2)(\bar{r}_B - r_f)]w$$

$$= [(\bar{r}_A - r_f)\sigma_B^2 - (\bar{r}_B - r_f)\sigma_{AB}] + [(\bar{r}_A - r_f)(\sigma_B^2 - \sigma_{AB}) + (\bar{r}_B - r_f)(\sigma_A^2 - \sigma_{AB})]w$$

$$w^* = \frac{(\bar{r}_A - r_f)\sigma_B^2 - (\bar{r}_B - r_f)\sigma_{AB}}{(\bar{r}_A - r_f)(\sigma_B^2 - \sigma_{AB}) + (\bar{r}_B - r_f)(\sigma_A^2 - \sigma_{AB})}$$

$$= 0.52$$

Example 3: Sharpe ratio

- Method 3: analytical solution
  - Result only:
    The general solution for the 2-asset Sharpe ratio maximization problem is

$$w^* = \frac{\left(\bar{r}_A - r_f\right)\sigma_B^2 - \left(\bar{r}_B - r_f\right)\sigma_{AB}}{\left(\bar{r}_A - r_f\right)\left(\sigma_B^2 - \sigma_{AB}\right) + \left(\bar{r}_B - r_f\right)\left(\sigma_A^2 - \sigma_{AB}\right)}$$

Example 4: efficient frontier

- Given the risky assets A and B in the previous question, what is the efficient frontier?

	E(r)	COV-VAR	
		A	B
A	15%	0.090	0.015
B	10%		0.040

- Given 5% risk-free rate, what is the capital market line?

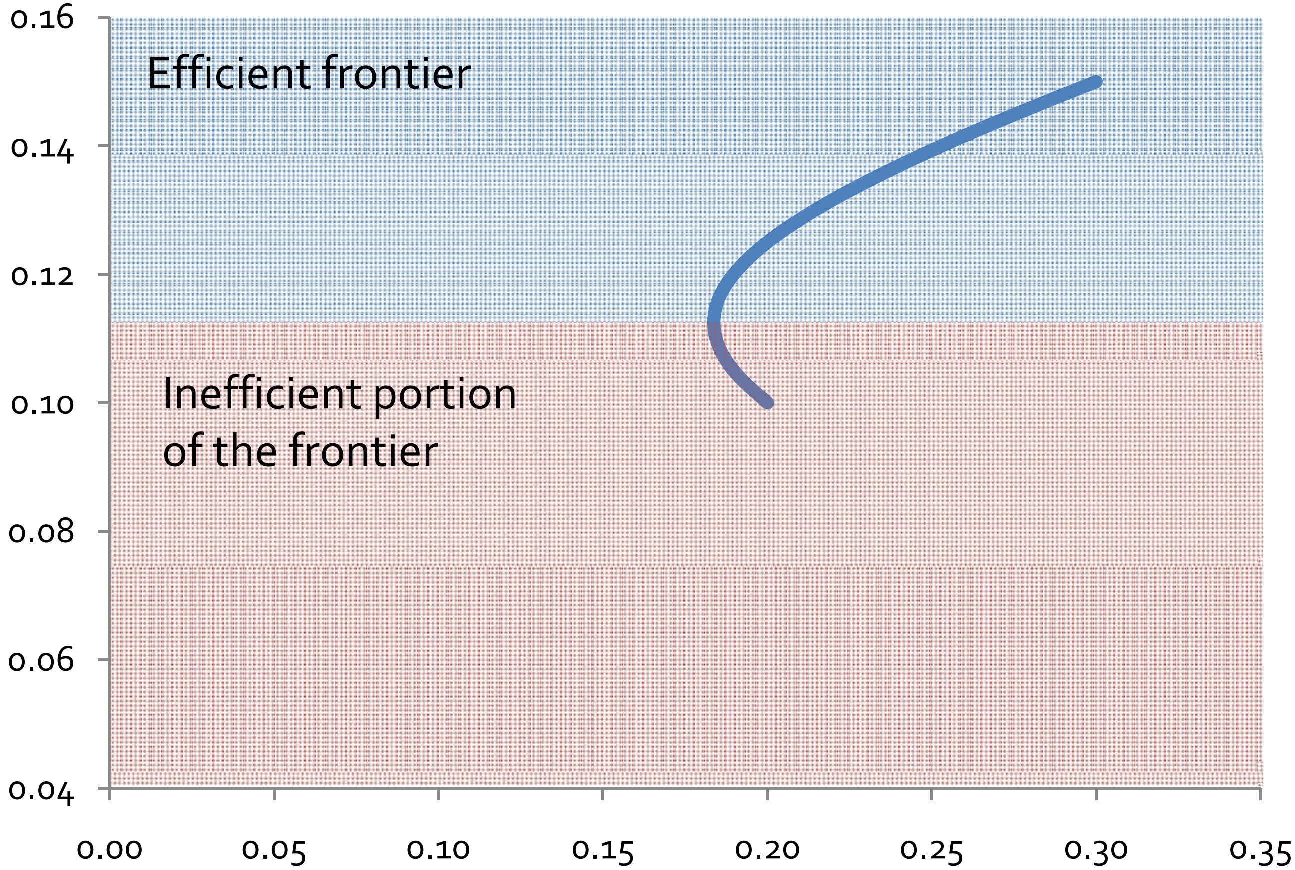
Example 4: efficient frontier

- Table from the previous question:

w	$1-w$	r_p	σ_p
0	1	0.1000	0.2000
0.1	0.9	0.1050	0.1897
0.2	0.8	0.1100	0.1844
0.3	0.7	0.1150	0.1844
0.4	0.6	0.1200	0.1897
0.5	0.5	0.1250	0.2000
0.6	0.4	0.1300	0.2145
0.7	0.3	0.1350	0.2324
0.8	0.2	0.1400	0.2530
0.9	0.1	0.1450	0.2757
1	0	0.1500	0.3000

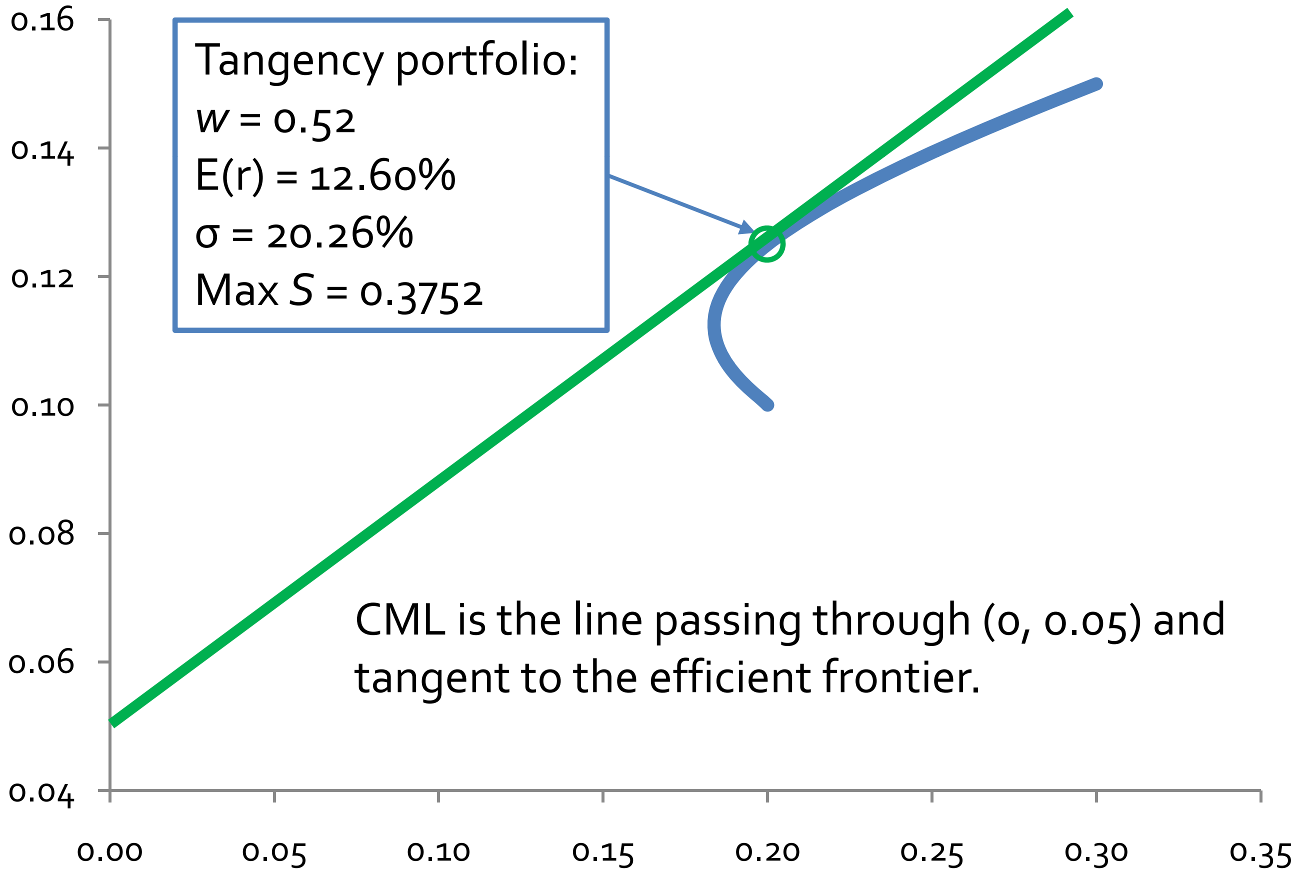
Example 4: efficient frontier

- Scatter plot of (r_p, σ_p) pairs:



Example 4: efficient frontier

- Capital market line:



Example 4: efficient frontier

- The moral of the story:
  - The CML is tangent to the efficient frontier at the **tangency portfolio**.
  - The tangency portfolio is the portfolio of risky assets that **maximizes the Sharpe ratio**.
  - The slope of the CML is the maximum Sharpe ratio.
  - Rational investors always hold **a combination of the tangency portfolio and the risk-free asset**. The proportion depends on investors' risk preferences.

Sneak Peak: CAPM

- The **tangency portfolio** is the **market portfolio**.
- An asset's **systematic risk** is measured by **beta**, which is defined as the **correlation** of its return and the market return, normalized by the variance of market return :

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$$

- Since investors are only compensated for **systematic risk**, asset return is an increasing function of beta:

$$E(\tilde{r}_i) = r_f + \beta_i(\tilde{r}_i - r_f)$$