Truss Design and Convex Optimization

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1 Outline

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2 Physical Laws of a Truss System

A **truss** is a structure in $d = 2$ or $d = 3$ dimensions, formed by n nodes and m bars joining these nodes. Figure 1 shows an example of a truss.

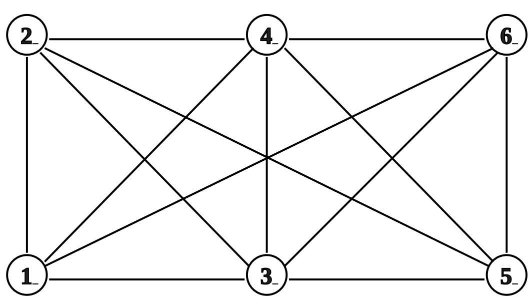

Figure 1: Example of a truss in $d = 2$ dimensions, with $n = 6$ nodes and $m = 13$ bars.

Examples of trusses include bridges, cranes, and the Eiffel Tower.

The data used to describe a truss is:

- a set of nodes (given in physical space)
- a set of bars joining pairs of nodes with associated data for each bar:
  - the length L_k of bar k
  - the Young's modulus E_k of bar k
  - the volume t_k of bar k
- an external force vector F on the nodes

Nodes can be *static* (fixed in place) or *free* (movable when the truss is stressed). The allowable movements of the nodes defines the degrees of freedom (dof) of the truss. We say that the truss has N degrees of freedom. Of course, $N \leq nd$.

Movements of nodes in the truss will be represented by a vector u of *displacements*, where $u \in \Re^N$.

The external force on the truss is given by a vector $F \in \Re^N$.

Displacements of the nodes in the truss cause internal forces of compression and/or expansion to appear in the bars in the truss. Let f_k denote the internal force of bar k. The vector $f \in \Re^m$ is the vector of forces of the bars.

Figure 2 shows an example of a truss problem. In the figure, there is a single external force applied to node 3 in the direction indicated. This will result in internal forces along the bars in the truss and will simultaneously cause small displacements in all of the nodes.

In Figure 2, nodes 1 and 5 are fixed, and the other nodes are free. To simplify notation we will label the 13 different bars by the nodes that hey link to: in general the bar k will join nodes i and j. The bars will be: 12, 13, ..., 56.

The internal force f_k of bar k can be positive or negative:

- If $f_k \geq 0$, bar k has been expanded and its internal force counteracts the expansion with compression, see Figure 3.

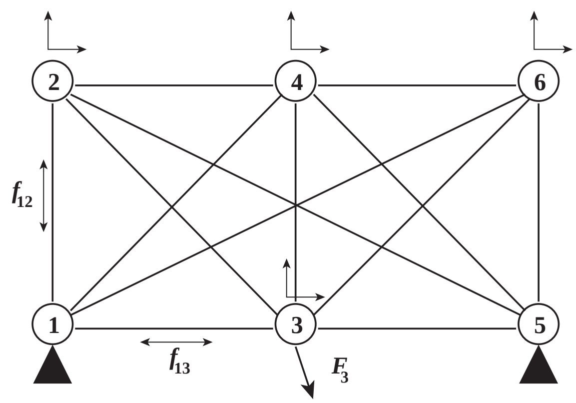

Figure 2: A truss problem.

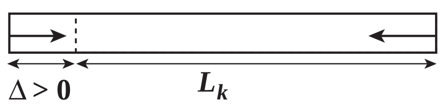

Figure 3: A bar under expansion.

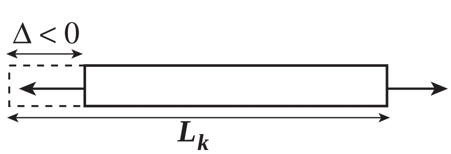

Figure 4: A bar under compression.

- If $f_k \leq 0$, bar k has been compressed and its internal force counteracts the compression with expansion, see Figure 4.

2.1 Physical Laws

2.1.1 Force Balance Equations

In a static truss the internal forces will balance the external forces in every degree of freedom. That is, the forces around each free node must balance. This is a law of conservation of forces. (This is akin to material balance equations in networks, for example.)

For example, consider the balance of forces on node 3:

$$\begin{array}{lllllll} x \text{ coordinate:} & -f_{13} & -f_{23}\cos(\pi/4) & & +f_{35} & +f_{36}\cos(\pi/4) & = -F_{3x} \\ y \text{ coordinate:} & & +f_{23}\sin(\pi/4) & +f_{34} & & +f_{36}\sin(\pi/4) & = -F_{3y} \end{array}$$

For the entire truss we write N linear equations that represent the balance of forces in each degree of freedom.

In matrix notation this can be written as

$$Af = -F$$

where A is an $N \times m$ matrix, and where each column of A, denoted as a_k, is the projection of the bar onto the degrees of freedom of the nodes that bar k meets.

In our example, we have:

$$
\begin{array}{ccccccccccccc}
12 & 13 & 14 & 16 & 23 & 24 & 25 & 34 & 35 & 36 & 45 & 46 & 56
\end{array}
$$

$$
A = \begin{pmatrix}
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 1 & \frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 1 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 1 & 0 \\
0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & -\frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -1 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -1
\end{pmatrix}
\begin{array}{c}
2_x \\ 2_y \\ 3_x \\ 3_y \\ 4_x \\ 4_y \\ 6_x \\ 6_y
\end{array}
$$

2.1.2 Constitutive Relationship between Forces and Distortion of a Bar

If bar k has length L_k and cross-sectional area A_k^c, Young's modulus E_k, and its length is changed by Δ_k, then the internal force f_k is given by:

$$f_k = E_k A_k^c \frac{\Delta_k}{L_k} \ .$$

However, for our purposes it will be easier to work instead with the volume t_k of bar k, which is:

$$t_k = A_k^c L_k \ .$$

We therefore can write the constitutive relationship as:

$$f_k = \frac{E_k}{L_k^2} t_k \Delta_k \ .$$

2.1.3 Distortion and Displacements

Displacements in the nodes will cause the lengths of the bars to change. Suppose that $L_{12} = L_{13} = L_{35} = 1.0$, with all other bars measured proportionally. In Figure 5, we show a displacement of:

$$u = (-\epsilon, -\epsilon, 0, 0, 0, 0, 0, 0) \ .$$

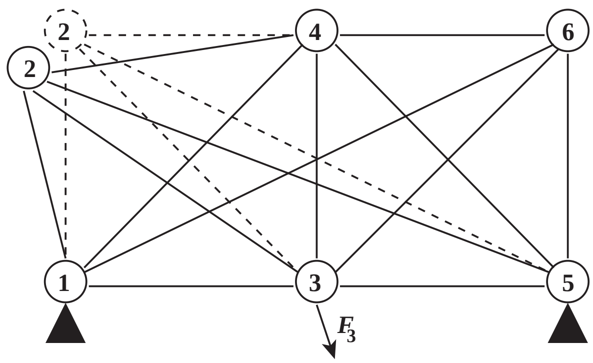

Figure 5: Example of a small displacement in a node.

Table 1 shows the new lengths of the bars under the displacement $u = (-\epsilon, -\epsilon, 0, 0, 0, 0, 0, 0)$. The third column of the table shows the linearization (first-order Taylor approximation) of the lengths of the bars, and the last column shows the computation of the inner product:

$$(a_k)^T u \ .$$

Note that if ϵ is small, then the distortion of bar k is nicely approximated by the linear expression:

$$\Delta_k = -(a_k)^T u \ .$$

Bar k	Old Length	New Length	Linearized Length	Linearized Change Δ_k	$(a_k)^T u$
12	1	$\sqrt{1+2\epsilon(\epsilon-1)}$	$1-\epsilon$	$-\epsilon$	ϵ
13	1	1	1	0	0
14	1	1	1	0	0
16	$\sqrt{5}$	$\sqrt{5}$	$\sqrt{5}$	0	0
23	$\sqrt{2}$	$\sqrt{2+2\epsilon^2}$	$\sqrt{2}$	0	0
24	1	$\sqrt{1+2\epsilon(\epsilon+1)}$	$1+\epsilon$	ϵ	$-\epsilon$
25	$\sqrt{5}$	$\sqrt{5+2\epsilon(\epsilon+1)}$	$\sqrt{5}+\frac{1}{\sqrt{5}}\epsilon$	$\frac{1}{\sqrt{5}}\epsilon$	$-\frac{1}{\sqrt{5}}\epsilon$
34	1	1	1	0	0
35	1	1	1	0	0
36	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	0	0
45	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	0	0
46	1	1	1	0	0
56	1	1	1	0	0

Table 1: Changes in the lengths of the bars under nodal displacement

We therefore write the internal force on bar k due to a feasible displacement u as:

$$f_k = -\frac{E_k}{L_k^2} t_k (a_k)^T u$$

Let the matrix B be the following diagonal matrix:

$$B = \begin{pmatrix} \frac{L_1^2}{E_1 t_1} & & 0 \\ & \ddots & \\ 0 & & \frac{L_m^2}{E_m t_m} \end{pmatrix} \quad , \quad B^{-1} = \begin{pmatrix} \frac{E_1 t_1}{L_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{E_m t_m}{L_m^2} \end{pmatrix} .$$

Then we can write the above relationship as:

$$f = -B^{-1} A^T u \ ,$$

which is:

$$Bf + A^T u = 0 \ .$$

2.1.4 Equilibrium Conditions and Compliance

Now we can write the physical laws of the truss system as:

$$\begin{array}{llll} Af & & = -F & \text{(conservation of forces)} \\ Bf & +A^T u & = 0 & \text{(forces, distortions, and displacements)} \end{array}$$

The *compliance* of the truss is the work (or energy) performed by the truss, which is the sum of the forces times displacements:

$$\frac{1}{2} F^T u \ .$$

We really do not need the fraction "$\frac{1}{2}$", but we will keep it for convenience.

Notice that the larger the compliance, the more work or displacement that the truss undergoes under the load F. Ideally, we would like the compliance to be a small number.

Combining the equilibrium conditions, we write:

$$\begin{aligned} F &= -Af \\ &= AB^{-1}A^T u \\ &= Ku \end{aligned}$$

where the matrix K is defined to be:

$$K = AB^{-1}A^T \ .$$

The matrix K is called the *stiffness matrix* of the truss. Note that K is an SPSD matrix.

Using this, we can write the compliance as:

$$\frac{1}{2}F^T u = \frac{1}{2}u^T K u \ .$$

Note from this that the compliance will always be a nonnegative quantity, as intuition suggests. More generally, of course, we would expect the stiffness matrix K to be SPD, and so the compliance will generally be a positive quantity.

2.1.5 Solving the Equations and Computing the Compliance

We solve the equation system:

$$Ku = F$$

to obtain the nodal displacements u, and then obtain the forces on the bars by computing:

$$f = -B^{-1}A^T u \ .$$

For each bar k, this last expression is simply:

$$f_k = -\frac{E_k}{L_k^2} t_k (a_k)^T u \ .$$

The compliance is then computed as:

$$\frac{1}{2}F^T u = \frac{1}{2}u^T K u \ .$$

2.2 Viewing the Equilibrium Conditions as the Solution to an Optimization Problem

We can also view the truss equilibrium conditions as the solution to an optimization problem. Consider the following optimization problem:

$$
\begin{aligned}
\text{OP}: \quad \text{minimize}_{\tilde{f}} \quad & \sum_{k=1}^{m} \frac{1}{2} \, \frac{1}{t_k} \, \frac{L_k^2}{E_k} \, \tilde{f}_k^2 \\
\text{s.t.} \quad & \sum_{k=1}^{m} a_k \tilde{f}_k = -F \; .
\end{aligned}
$$

Problem OP seeks a force vector $\tilde{f}$ that satisfies the force balance equations:

$$
-F = \sum_{k=1}^{m} a_k \tilde{f}_k = A\tilde{f} \; ,
$$

and that minimizes the weighted sum of squares of the forces:

$$
\sum_{k=1}^{m} \frac{1}{2} \, \frac{1}{t_k} \, \frac{L_k^2}{E_k} \, \tilde{f}_k^2 \; .
$$

Problem OP is a convex quadratic problem. The KKT optimality conditions for this problem are:

$$
\sum_{k=1}^{m} a_k f_k = -F
$$

$$
\frac{L_k^2}{t_k E_k} \, f_k + a_k^T u = 0 \quad \text{for} \quad k = 1, \ldots, m.
$$

Notice that we can rewrite these optimality conditions as:

$$
\begin{aligned}
Af &= -F \\
Bf + A^T u &= 0 \; ,
\end{aligned}
$$

which are precisely the equilibrium conditions for the truss system. This shows that the truss system is nature's solution to an optimization problem.

The optimal objective function value of the optimization problem OP is:

$$\sum_{k=1}^{m} \frac{1}{2} \, \frac{1}{t_k} \, \frac{L_k^2}{E_k} \, f_k^2 = \frac{1}{2} f^T B f = -\frac{1}{2} f^T A^T u = -\frac{1}{2} u^T A f = \frac{1}{2} F^T u \ ,$$

which is the compliance of the truss. Therefore we see that the optimal objective value of OP is the compliance of the truss system.

3 The Truss Design Problem

Let us now consider the volumes t_k of the bars k to be design parameters that we wish to determine. We can write the system of equations:

$$F = Ku$$

as:

$$\begin{aligned} F &= Ku \\ &= AB^{-1}A^T u \\ &= \sum_{k=1}^{m} (a_k) \left(B^{-1} A^T u \right)_k \\ &= \sum_{k=1}^{m} -(a_k) f_k \\ &= \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T u \\ &= K(t) u \end{aligned}$$

where the stiffness matrix is now written as:

$$K(t) = \sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \ .$$

Note in this expression that $K = K(t)$ is a weighted sum of rank-one matrices (weighted by the volumes t_k). Put another way, $K = K(t)$ is a weighted sum of outer-product matrices.

In designing the truss, we must choose the volumes t_k of the bars subject to linear constraints on the volumes of the bars:

$$\begin{aligned} Mt &\leq d \\ t &\geq 0 \,, \end{aligned}$$

where typically these constraints include upper and lower bounds on the volumes of certain bars, as well as an overall cost or volume constraint:

$$\sum_{k=1}^{m} t_k \leq V \ .$$

The criterion in truss design is to choose the volumes t_k of the bars so as to minimize the compliance of the truss, namely:

$$\text{minimize}_{t,u} \ \ \frac{1}{2} \, F^T u \ .$$

Such a truss will be the most resistant to the external force F.

The single-load truss design problem can be stated as:

$$\begin{aligned} (\text{TDP}_1): \quad \text{minimize}_{t,u} \quad & \frac{1}{2} \, F^T u \\ \text{s.t.} \quad & K(t)u = F \\ & Mt \leq d \\ & t \geq 0 \\ & u \in \Re^N, t \in \Re^m. \end{aligned}$$

which is the same as:

$$
\begin{array}{lll}
(\mathrm{TDP}_1): & \mathrm{minimize}_{t,u} & \frac{1}{2}\ F^T u \\
 & \mathrm{s.t.} & \left[\sum_{k=1}^{m} t_k \frac{E_k}{L_k^2}(a_k)(a_k)^T\right] u = F \\
 & & Mt \le d \\
 & & t \ge 0 \\
 & & u \in \Re^N, t \in \Re^m .
\end{array}
$$

The decision variables here are u and t. However, one should really think of the decision variables as t only, since once t is chosen, u will be determined by the solution to the system of equations:

$$
\left[\sum_{k=1}^{m} t_k \frac{E_k}{L_k^2}(a_k)(a_k)^T\right] u = F\ .
$$

Note that as written, the truss design problem TDP is *not* a convex problem.

4 A Convex Version of the Truss Design Problem

Recall that for given volumes t_k on the bars of the truss, that the compliance of the truss is the optimal objective value of the quadratic problem:

$$
\begin{array}{lll}
\mathrm{OP}: & \mathrm{minimize}_{\tilde{f}} & \sum_{k=1}^{m} \frac{1}{2}\ \frac{1}{t_k}\ \frac{L_k^2}{E_k}\ \tilde{f}_k^2 \\
 & \mathrm{s.t.} & \sum_{k=1}^{m} a_k \tilde{f}_k = -F\ .
\end{array}
$$

The truss design problem is then to choose the values of the volumes $t = (t_1, \ldots, t_m)$ on the bars so that the optimal solution of OP (which we have shown is the compliance of the truss) is minimized, subject to the constraints on t:

$$\begin{aligned} Mt &\leq d \\ t &\geq 0 \,. \end{aligned}$$

We write this all as:

$$\begin{aligned} (\text{TDP}_2): \quad \text{minimize}_{f,t} \quad & \sum_{t_k > 0} \frac{1}{2} \frac{1}{t_k} \frac{L_k^2}{E_k} f_k^2 \\ \text{s.t.} \quad & \sum_{t_k > 0} a_k f_k = -F \\ & Mt \leq d \\ & t \geq 0 \\ & f \in \Re^m, t \in \Re^m. \end{aligned}$$

Notice here that we have made two modifications to the optimization problem. First, we have modified the summation in the objective function, so that bars with zero volume ($t_k = 0$) are no longer counted, since they will not exist. Second, we changed the notation, using f instead of $\tilde{f}$. Because the summations with the $t_k > 0$ is rather awkward mathematically, we further re-write TDP_2 as follows:

$$
\begin{aligned}
(\mathrm{TDP}_2): \quad \text{minimize}_{f,t,s} \quad & \frac{1}{2}\sum_{k=1}^{m} s_k \\
\text{s.t.} \qquad & \sum_{k=1}^{m} a_k f_k = -F \\
& Mt \leq d \\
& \tfrac{L_k^2}{E_k}\ f_k^2 \leq t_k s_k \ \ \text{for } k = 1, \ldots m \\
& t \geq 0, \ \ s \geq 0 \\
& f, t, s \in \Re^m .
\end{aligned}
$$

In this problem, we have replaced the quantity

$$\frac{1}{t_k}\ \frac{L_k^2}{E_k}\ f_k^2$$

with the new variable s_k subject to the condition:

$$\frac{L_k^2}{E_k}\ f_k^2 \leq t_k s_k \ .$$

We can further re-write this now as:

$$
\begin{aligned}
(\mathrm{TDP}_2): \quad \text{minimize}_{f,t,s} \quad & \frac{1}{2}\sum_{k=1}^{m} s_k \\
\text{s.t.} \quad & Af = -F \\
& Mt \le d \\
& \tfrac{L_k^2}{E_k}\, f_k^2 \le t_k s_k \quad \text{for } k = 1, \ldots m \\
& t \ge 0, \;\; s \ge 0 \\
& f, t, s \in \Re^m .
\end{aligned}
$$

This last problem has a linear objective function, and linear constraints except for the constraints:

$$
\frac{L_k^2}{E_k}\, f_k^2 \le t_k s_k \ .
$$

However, it is pretty easy to show that the constraints:

$$
\frac{L_k^2}{E_k}\, f_k^2 \le t_k s_k, \quad t_k \ge 0, \quad s_k \ge 0
$$

describe a convex region, and so this formulation is actually a convex optimization problem.

5 Second-Order Cone Optimization

A second-order cone optimization problem (SOCP) is an optimization problem of the form:

$$
\begin{aligned}
\mathrm{SOCP}: \quad \min_x \quad & c^T x \\
\text{s.t.} \quad & Ax = b \\
& \|Q_i x + d_i\| \le \left(g_i^T x + h_i\right) \ , \quad i = 1, \ldots, k \ .
\end{aligned}
$$

In this problem, the norm $\|v\|$ is the standard Euclidean norm:

$$\|v\| := \sqrt{v^T v} \ .$$

The norm constraints in SOCP are called "second-order cone" constraints.

Notice first that SOCP is a convex problem, since the function:

$$\|Q_i x + d_i\| - \left(g_i^T x + h_i\right)$$

is a convex function.

Notice also that linear optimization is a special case of SOCP, if we set $Q_i = 0, d_i = 0, h_i = 0$, and g_i to be the i^{th} unit vector for $i = 1, \ldots, n$.

Also notice that any convex quadratic constraint can be converted into a second-order constraint. To see this, suppose we have a constraint:

$$\frac{1}{2} x^T Q x + q^T x + r \leq 0 \ ,$$

where Q is SPSD. We can factor

$$Q = M^T M$$

for some matrix M, and then write our constraint as:

$$\left\| \left(\frac{1}{\sqrt{2}} M x \ \ , \ \ \frac{q^t x + r + 1}{2} \right) \right\| \ \leq \ \frac{-q^t x - r + 1}{2} \ .$$

If you square both sides and collect terms, you will see that this is indeed equivalent to the original convex quadratic constraint.

Finally, note that we can always formulate a convex quadratic problem as an SOCP, since we can create a new variable x_{n+1} and write our quadratic problem as:

$$
\begin{array}{ll}
\min_{x,x_{n+1}} & x_{n+1} \\
 & \\
\text{s.t.} & \\
 & \vdots \\
 & \\
 & \frac{1}{2}x^T Q x + q^T x \le x_{n+1} \ ,
\end{array}
$$

which can be further re-written as an SOCP.

6 Truss Design and Second-Order Cone Optimization

We now present a second-order cone optimization problem that is equivalent to the truss design problem TDP_2. Recall this version of the truss design problem:

$$
\begin{array}{lll}
(\text{TDP}_2): & \text{minimize}_{f,t,s} & \displaystyle \frac{1}{2}\sum_{k=1}^{m} s_k \\
 & & \\
 & \text{s.t.} & Af = -F \\
 & & \\
 & & Mt \le d \\
 & & \\
 & & \frac{L_k^2}{E_k}\, f_k^2 \le t_k s_k \quad \text{for } k = 1, \ldots m \\
 & & \\
 & & t \ge 0, \ \ s \ge 0 \\
 & & \\
 & & f, t, s \in \Re^m .
\end{array}
$$

Let us now make the simple change of variables:

$$\begin{aligned} w_k &= \tfrac{1}{2}t_k + \tfrac{1}{2}s_k \\ y_k &= -\tfrac{1}{2}t_k + \tfrac{1}{2}s_k \end{aligned}$$

Then $s_k = w_k + y_k$ and $t_k = w_k - y_k$, and so:

$$t_k s_k = (w_k - y_k)(w_k + y_k) = w_k^2 - y_k^2,$$

and so the constraint

$$\frac{L_k^2}{E_k}\ f_k^2 \le t_k s_k$$

becomes:

$$\frac{L_k^2}{E_k} f_k^2 \le w_k^2 - y_k^2\ .$$

We can rearrange this and take square roots to obtain:

$$\sqrt{\frac{L_k^2}{E_k}\ f_k^2 + y_k^2} \le w_k\ ,$$

which we can in turn write as:

$$\left\| \left(y_k, \frac{L_k}{\sqrt{E_k}} f_k \right) \right\| \le w_k\ .$$

Therefore we can re-write TDP_2 as:

$$\begin{aligned} \text{(CTDP):}\quad \text{minimize}_{f,w,y}\quad & \frac{1}{2}e^T(w+y) \\ \text{s.t.}\quad & Af = -F \\ & \left\| \left(y_k, \tfrac{L_k}{\sqrt{E_k}} f_k \right) \right\| \le w_k\ , \quad k = 1,\ldots,m \\ & M(w-y) \le d \\ & w, y, f \in \Re^m\ . \end{aligned}$$

Notice that CTDP is a second-order cone problem. The objective function and the first and third set of constraints are linear equations and inequalities. The second set of constraints are second-order-cone constraints, since the LHS is the norm of a 2-dimensional vector:

$$\|(v_1, v_2)\| := \left\| \left(y_k \quad , \quad \left(\frac{L_k}{\sqrt{E_k}} \right) f_k \right) \right\| \ ,$$

and the RHS is the linear expression:

$$w_k.$$

Proposition 6.1 *Suppose that (f, w, y) is a feasible or optimal solution of CTDP. Let:*

$$t = w - y \quad \textit{and} \quad s = w + y.$$

Then (f, t, s) is the corresponding feasible or optimal solution of TDP_2.

6.1 Other Formulations and Formats for the Truss Design Problem

We have just seen how to formulate the truss design problem as the convex optimization problem TDP_2 and more specially as the second-order cone optimization problem CTDP. In addition to these two formulations of the problem, there is a way to formulate the truss design problem as an instance of a more general type of convex problem called a "semi-definite optimization problem" (SDO for short). The topic of SDO and its application to truss design is discussed herein in Sections 8 and 9.

Finally, when the only constraints on the volume variables $t_1, \ldots, t_m$ are nonnegativity conditions and a total-volume constraint of the form:

$$\sum_{k=1}^{m} t_k \leq V \ ,$$

then the truss design problem can actually be converted to a linear optimization problem. This is shown herein in Section 10.

7 Some Computational Results

In this section we report some computational results on solving truss design problems. We solved three different types of truss design problems. The first type is a basic bridge design, and is illustrated in Figure 6. In the figure, the arrows collectively comprise the force vector F applied to the structure, and the circles (in the left lower and right lower corners) are the fixed nodes of the structure. The stars in the figures are the other nodes in the truss structure. Figure 7 shows the set of bars that can be used to construct the bridge. These bars were generated by considering bars between any two nodes whose "distance" from one another was three nodes or less.

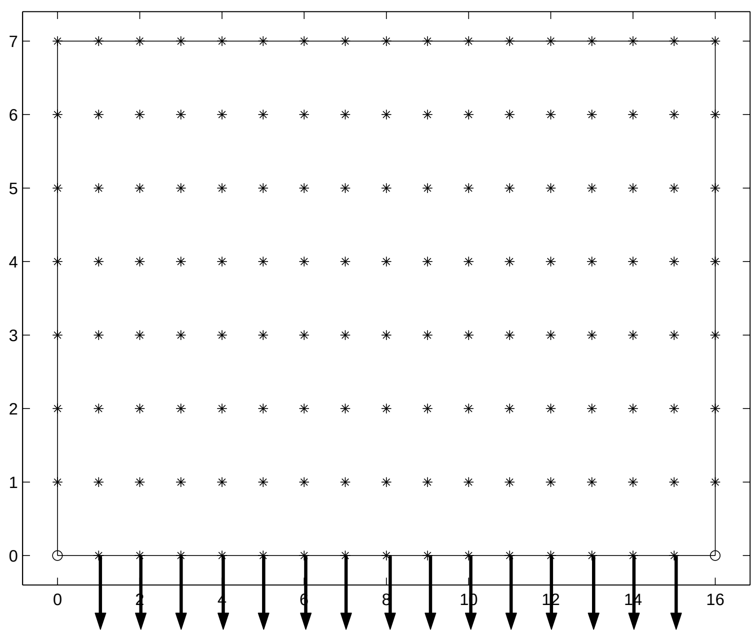

Figure 6: The forces and nodes for the basic bridge design model.

The second type of problem that we solved is the problem of designing a hanging support for a load such as a traffic sign, and is illustrated in Figure

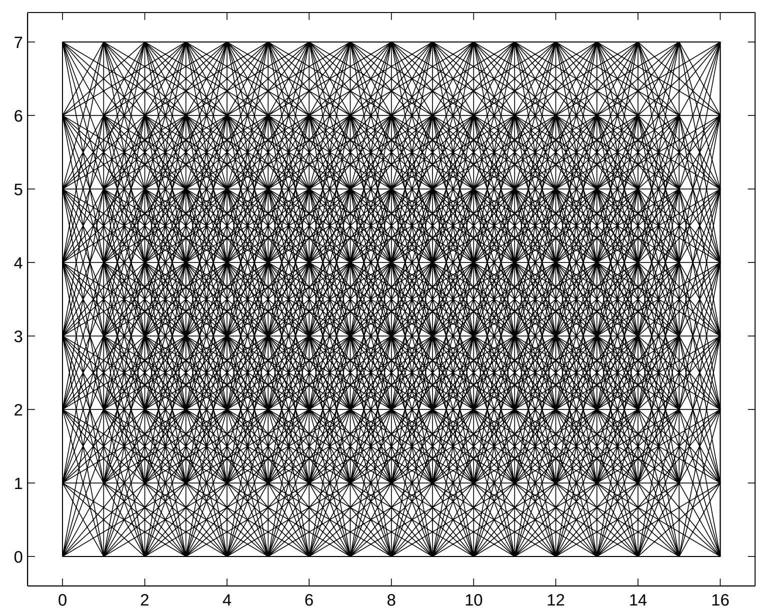

Figure 7: The set of possible bars for the basic bridge design problem.

8. The circles on the left part of the figure correspond to the fixed nodes, and the arrow on the right side is a force that will be applied (the gravitational force of the traffic sign).

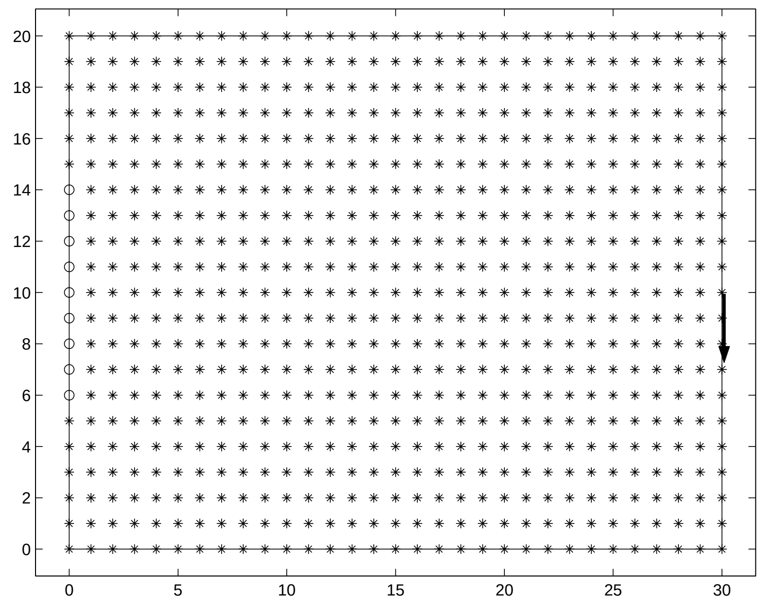

Figure 8: The forces and nodes for the hanging sign design model.

The third type of problem that we solved is the problem of designing a crane to handle a gravitational load, and is illustrated in Figure 9. The circles on the bottom of the figure correspond to the fixed nodes, and the arrow on the right side is a force that will be applied (the gravitational force of the traffic sign).

For the problems shown in Figures 6, 8, and 9, the truss design model solutions are shown in Figures 10, 11, and 12. In these figures, the thickness of the image of the bars is drawn proportional to the volumes of the bars in the optimal solutions.

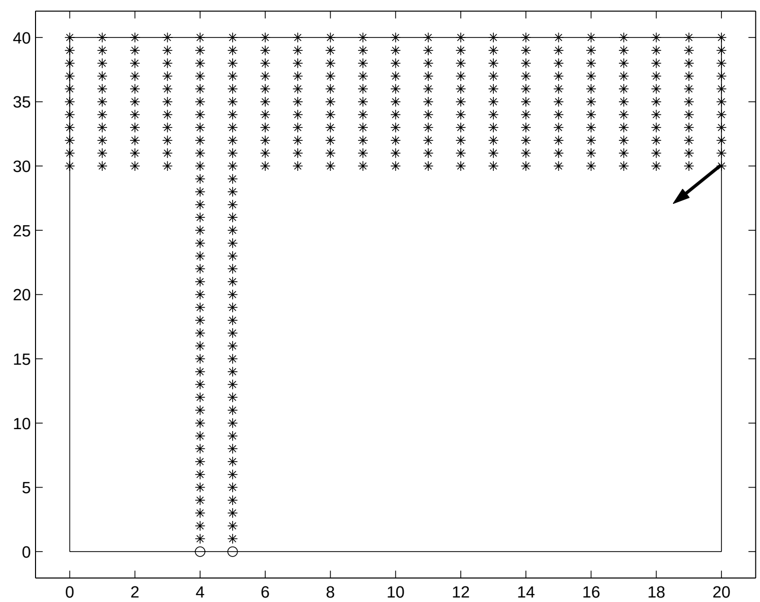

Figure 9: The forces and nodes for the crane design model.

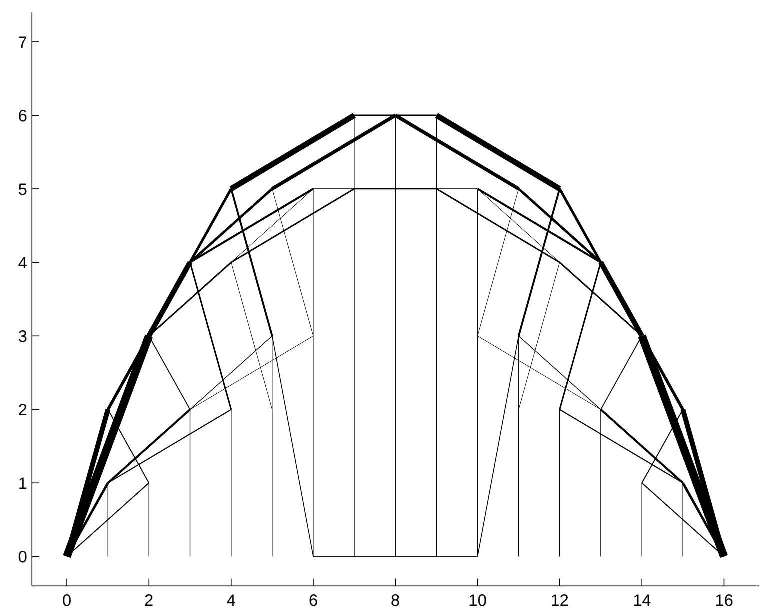

Figure 10: Optimal solution to the basic bridge design problem.

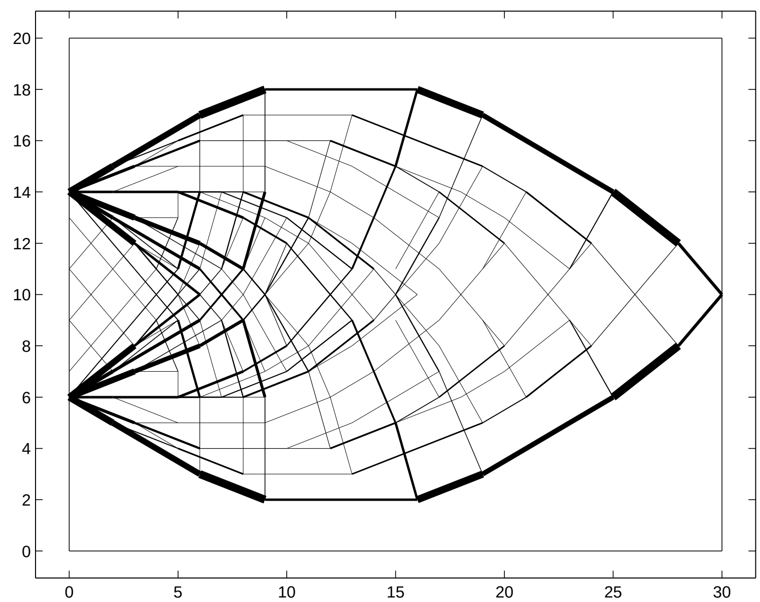

Figure 11: Optimal solution to the hanging sign design problem.

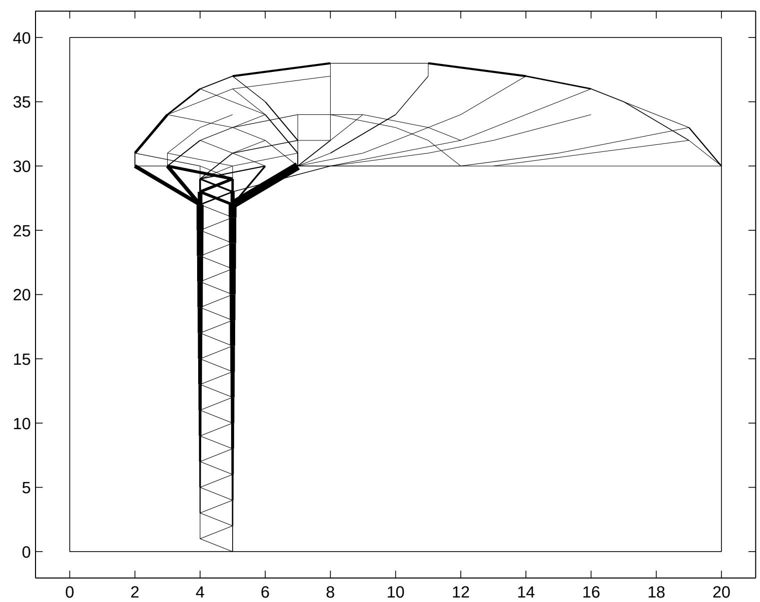

Figure 12: Optimal solution to the crane design problem.

Notice in Figure 10 that the basic bridge design does not automatically include horizontal bars for a "road surface" on the base of the structure. In order to force there to be such a road surface, we must add lower bounds on the volumes of those bars on the base of structure. If we add such lower bounds, the optimal bridge design is as shown in Figure 13.

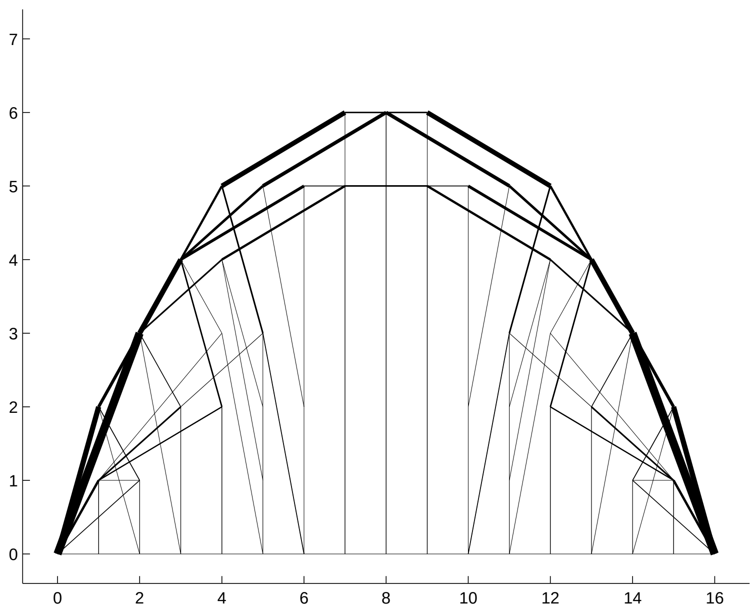

Figure 13: Optimal solution of the bridge design problem, with lower bounds on the "road surface" bar volumes.

7.1 Details of Problems Solved

For the bridge design problem, we included lower-bound constraints on the volumes of the road-surface bars as well as a constraint on the total volume of the structure. With the lower-bound constraints in the model, the model must be solved either as a second-order cone problem (SOCP) or as a semi-definite optimization problem (SDO). For the hanging sign design problem

Name	Size of grid	Nodes n	Arcs m	Degrees of freedom N	Maximum Volume
Bridge-16×7	16×7	136	1,547	268	1,600
Bridge-20×10	20×10	231	2,878	458	2,000
Bridge-30×10	30×10	341	4,368	678	4,000
Sign-10×20	10×20	231	2,878	444	1,000
Sign-20×30	20×30	651	9,058	1,280	2,000
Sign-30×40	30×40	1,271	18,438	2,512	3,000
Crane-10×20	10×20	96	818	188	2,000
Crane-20×40	20×40	291	3,188	578	5,000
Crane-30×40	30×40	401	4,678	798	10,000

Table 2: Dimensions of the truss design problems.

as well as for the crane design problem, the only constraint on the volumes was a constraint on the total volume of the truss structure. This made it possible to solve these problems using linear optimization.

We solved three different instances of each of the bridge, hanging sign, and crane design problems, with different numbers of nodes corresponding to different meshes. Table 2 shows details of the models that we solved. The right-most column in the table shows the value of the volume constraint on the volume of the truss structure.

Table 3 shows the number of variables and the number of inequalities in these problems, and Table 4 shows the number of variables and the number of equations/constraints/matrix dimension of these problems.

Table 5 shows the compliance of the optimal truss design for the nine problems that were solved.

Table 6 shows the iterations of the interior-point algorithms that were used to solve the truss design problems. For the linear problems and the second-order cone problems, we used LOQO to solve the model. For the semi-definite optimization models, we used an SDO algorithm called `SDPa`. The running times of these methods is shown in Table 7.

Problem	LP Model		SOCP Model	
	Variables	Inequalities	Variables	Inequalities
	$2m$	$2m$	$3m$	$m+1+\text{LBs}$
Bridge-16×7	†	†	4,641	1,564
Bridge-20×10	†	†	8,634	2,899
Bridge-30×10	†	†	13,104	4,399
Sign-10×20	5,756	5,756	8,634	2,879
Sign-20×30	18,116	18,116	27,174	9,059
Sign-30×40	36,876	36,876	55,314	18,439
Crane-10×20	1,636	1,636	2,454	819
Crane-20×40	6,376	6,376	9,564	3,189
Crane-30×40	9,356	9,356	14,034	4,679

Table 3: Number of variables and inequalities in the truss design problems. †The linear optimization model cannot be used for the modified bridge design model.

Problem	LP Model		SOCP Model		SDO Model
	Variables	Equations	Variables	Constraints	Matrix Dimension
	$2m$	N	$3m$	$N+m+1+\text{LBs}$	$N+m+2$
Bridge-16×7	†	†	4,641	1,832	1,685
Bridge-20×10	†	†	8,634	3,357	3,111
Bridge-30×10	†	†	13,104	5,077	4,711
Sign-10×20	5,756	444	8,634	3,323	3,111
Sign-20×30	18,116	1,280	27,174	10,339	9,711
Sign-30×40	36,876	2,512	55,314	20,951	19,711
Crane-10×20	1,636	188	2,454	1,007	916
Crane-20×40	6,376	578	9,564	3,767	3,481
Crane-30×40	9,356	798	14,034	5,477	5,081

Table 4: Number of variables and equations in the truss design problems. †The linear optimization model cannot be used for the modified bridge design model.

Problem	LP	SOCP	SDP
Bridge-16×7	†	27.43365	27.15510
Bridge-20×10	†	52.58314	52.10850
Bridge-30×10	†	132.46038	‡
Sign-10×20	0.77279	0.77293	0.77279
Sign-20×30	2.52003	2.52056	‡
Sign-30×40	4.08573	4.08681	‡
Crane-10×20	28.67827	28.67840	28.67753
Crane-20×40	286.24854	286.24954	286.19340
Crane-30×40	596.73177	596.73324	596.63740

Table 5: Compliance of the optimized truss design for the truss design problems. †The linear optimization model cannot be used for the modified bridge design model. ‡The data to run the SDO model could not be prepared for this instance due to memory restrictions.

Problem	LP	SOCP	SDO
Bridge-16×7	†	38	36
Bridge-20×10	†	55	43
Bridge-30×10	†	47	‡
Sign-10×20	18	61	37
Sign-20×30	21	47	‡
Sign-30×40	24	53	‡
Crane-10×20	17	52	34
Crane-20×40	31	158	58
Crane-30×40	31	144	60

Table 6: Number of iterations of the interior-point algorithm to solve the truss design problems. †The linear optimization model cannot be used for the modified bridge design model. ‡The data to run the SDO model could not be prepared for this instance due to memory restrictions.

Problem	LP	SOCP	SDO
Bridge-16×7	†	93.90	257.01
Bridge-20×10	†	460.96	1,088.09
Bridge-30×10	†	904.86	‡
Sign-10×20	3.33	513.37	1,081.54
Sign-20×30	32.08	4,254.08	‡
Sign-30×40	111.03	26,552.74	‡
Crane-10×20	0.52	41.12	446.98
Crane-20×40	6.46	1,299.12	33,926.20
Crane-30×40	10.67	1,599.85	95,131.49

Table 7: Running time (in seconds) of the interior-point algorithm to solve the truss design problems. †The linear optimization model cannot be used for the modified bridge design model. ‡The data to run the SDO model could not be prepared for this instance due to memory restrictions.

8 Semi-Definite Optimization

If S is a $k \times k$ matrix, then S is a symmetric positive semi-definite (SPSD) matrix if S is symmetric ($S_{ij} = S_{ji}$ for any $i, j = 1, \ldots, k$) and

$$v^T S v \geq 0 \ \text{ for any } \ v \in \Re^k.$$

If S is a $k \times k$ matrix, then S is a symmetric positive definite (SPD) matrix if S is symmetric and

$$v^T S v > 0 \ \text{ for any } \ v \in \Re^k, v \neq 0.$$

Let $\mathcal{S}^k$ denote the set of symmetric $k \times k$ matrices, and let $\mathcal{S}^k_+$ denote the set of symmetric positive semi-definite (SPSD) $k \times k$ matrices. Similarly let $\mathcal{S}^k_{++}$ denote the set of symmetric positive definite (SPD) $k \times k$ matrices.

Let S and X be any symmetric matrices. We write "$S \succeq 0$" to denote that S is symmetric and positive semi-definite, and we write "$S \succeq X$" to denote that $S - X \succeq 0$. We write "$S \succ 0$" to denote that S is symmetric and positive definite, etc.

Remark 1 $\mathcal{S}^k_+ = \{S \in \mathcal{S}^k \mid S \succeq 0\}$ *is a convex set in* $\Re^{k^2}$.

Proof: Suppose that $S, X \in \mathcal{S}^k_+$. Pick any scalars $\alpha, \beta \geq 0$ for which $\alpha + \beta = 1$. For any $v \in \Re^k$, we have:

$$v^T(\alpha S + \beta X)v = \alpha v^T S v + \beta v^T X v \geq 0,$$

whereby $\alpha S + \beta X \in \mathcal{S}^k_+$. This shows that $\mathcal{S}^k_+$ is a convex set.

q.e.d.

A semi-definite optimization problem is an optimization problem of the following type:

$$\begin{array}{lll} \text{SDO}: & \text{minimize}_y & b^T y \\ & \text{s.t.} & C + \sum_{i=1}^m y_i A_i \succeq 0 \\ & & My \geq g \ . \end{array}$$

where the matrices $C, A_1, \ldots, A_m$ are symmetric matrices. One convenient way of thinking about this problem is as follows. Given values of the m scalar variables $y_1, \ldots, y_m$, the objective is to minimize the linear function:

$$\sum_{i=1}^m b_i y_i \ .$$

The constraints of SDO state that the variables $y = (y_1, \ldots, y_m)$ must satisfy the linear inequalities:

$$My \geq g$$

as well as the condition that the matrix S, defined by:

$$S := C + \sum_{i=1}^{m} y_i A_i \ ,$$

must be positive semi-definite. That is,

$$S := C + \sum_{i=1}^{m} y_i A_i \succeq 0.$$

We illustrate this construction with the following example:

$$\begin{aligned} \text{SDO}: \quad & \text{minimize}_{y_1,y_2} \quad 11y_1 + 19y_2 \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{pmatrix} + y_1 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{pmatrix} = S \succeq 0 \\ & 3y_1 + 7y_2 \leq 12 \\ & 2y_1 + y_2 \leq 6 \ . \end{aligned}$$

which we can rewrite in the following form:

$$\begin{aligned} \text{SDO}: \quad & \text{minimize} \quad 11y_1 + 19y_2 \\ \text{s.t.} \quad & \begin{pmatrix} 1 + 1y_1 + 0y_2 & 2 + 0y_1 + 2y_2 & 3 + 1y_1 + 8y_2 \\ 2 + 0y_1 + 2y_2 & 9 + 3y_1 + 6y_2 & 0 + 7y_1 + 0y_2 \\ 3 + 1y_1 + 8y_2 & 0 + 7y_1 + 0y_2 & 7 + 5y_1 + 4y_2 \end{pmatrix} \succeq 0 \\ & 3y_1 + 7y_2 \leq 12 \\ & 2y_1 + y_2 \leq 6 \ . \end{aligned}$$

Remark 2 *SDO is a convex minimization problem.*

Proof: The objective function of SDO is a linear function, which is convex. Suppose that $\bar{y}$ and $\tilde{y}$ are two feasible solutions of SDO, and let $y = \alpha\bar{y} + \beta\tilde{y}$, where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Then:

$$S := C + \sum_{i=1}^{m} \bar{y}_i A_i \succeq 0$$

and

$$X := C + \sum_{i=1}^{m} \tilde{y}_i A_i \succeq 0 \ .$$

Therefore,

$$\begin{aligned} C + \sum_{i=1}^{m} y_i A_i &= C + \sum_{i=1}^{m} \left(\alpha\bar{y} + \beta\tilde{y}\right) A_i \\ &= \alpha \left(C + \sum_{i=1}^{m} \bar{y}_i A_i \right) + \beta \left(C + \sum_{i=1}^{m} \tilde{y}_i A_i \right) \\ &= \alpha S + \beta X \succeq 0 \ . \end{aligned}$$

This shows that the feasible region of SDO is a convex set.
q.e.d.

Semi-definite optimization is a unifying model in optimization. Linear optimization, quadratic optimization, second-order cone optimization, as well as certain other convex optimization problems, can all be shown to be special cases of semi-definite optimization. Furthermore, semi-definite optimization has applications that span convex optimization, discrete optimization, and control theory. In fact, semi-definite optimization may very well become the canonical way that optimizers will think about optimization in the next decade.

9 Truss Design and Semi-Definite Optimization

We now present an SDO problem that is equivalent to the truss design problem TDP. This problem is:

$$
\begin{aligned}
\text{(STDP):}\quad \text{minimize}_{t,\theta}\quad & \theta \\
\text{s.t.}\quad & \begin{pmatrix} \theta & F^T \\ F & \left[\sum_{k=1}^{m} t_k \frac{E_k}{L_k^2}(a_k)(a_k)^T\right] \end{pmatrix} \succeq 0 \\
& Mt \le d \\
& t \ge 0 \\
& \theta \in \Re, t \in \Re^m \ .
\end{aligned}
$$

Notice that STDP is a semi-definite optimization problem. The equivalence of TDP and STDP is a consequence of the following two propositions:

Proposition 9.1 *Suppose that (t, u) is a feasible solution of TDP. Let:*

$$\theta = F^T u \ .$$

Then (t, θ) is a feasible solution of STDP, and $\theta = F^T u$.

Proposition 9.2 *Suppose that (t, θ) is a feasible solution of STDP. Then there exists a vector u for which (t, u) is feasible for TDP, and in fact:*

$$F^T u \le \theta \ .$$

Together, Propositions 9.1 and 9.2 demonstrate that any solution of STDP translates into a solution to TDP. Therefore, in order to solve TDP, we can solve STDP for (t^*, θ^*) and then solve the following linear equation system for u^*:

$$\left[\sum_{k=1}^{m} \frac{t_k^* E_k}{L_k^2} a_k a_k^T \right] u^* = F \ .$$

As a means to prove Propositions 9.1 and 9.2, we first show the following:

Remark 3 *Given a vector v, a square matrix M, and a scalar θ, then:*

$$Q := \left[\begin{array}{cc} \theta & v^T \\ v & M \end{array} \right] \succeq 0$$

if and only if:

$$M \succeq 0 \ , \ \textit{there exists} \ \ u \ \textit{satisfying} \ Mu = v \ , \ \textit{and} \ \theta \geq v^T u \ .$$

Proof of Remark 3: To see why this is true, let us first suppose that $Q \succeq 0$. Then clearly $M \succeq 0$, since M is formed by a subset of the components that form Q. Now, let us suppose that there is no u such that $Mu = v$. This implies that there exists λ such that $\lambda^T M = 0$ and $\lambda^T v > 0$. Then:

$$(-\epsilon, \lambda^T) \left[\begin{array}{cc} \theta & v^T \\ v & M \end{array} \right] \left(\begin{array}{c} -\epsilon \\ \lambda \end{array} \right) = \theta\epsilon^2 - 2v^T\lambda\epsilon + \lambda^T M \lambda = \theta\epsilon^2 - 2v^T\lambda\epsilon < 0$$

for ϵ small enough, which is a contradiction. Therefore there exists u such that $Mu = v$. Furthermore, for such a u, we have:

$$0 \leq (-1, u^T) \left[\begin{array}{cc} \theta & v^T \\ v & M \end{array} \right] \left(\begin{array}{c} -1 \\ u \end{array} \right) = \theta - 2v^T u + u^T M u = \theta - v^T u$$

which shows that $\theta \geq v^T u$.

Now let us prove the reverse of these implications. Suppose that $M \succeq 0$, there exists u satisfying $Mu = v$, and $\theta \geq v^T u$. Since $M \succeq 0$ we only have to prove that:

$$\delta(x) := (-1, x^T) \begin{bmatrix} \theta & v^T \\ v & M \end{bmatrix} \begin{pmatrix} -1 \\ x \end{pmatrix} = \theta - 2v^T x + x^T M x \geq 0 \quad \forall x \ .$$

But notice that $\delta(x)$ is a convex quadratic function of x (since in particular $M \succeq 0$), and therefore

$$\nabla \delta(x) = -2v + 2Mx = 0$$

is a necessary and sufficient condition for the unconstrained minimization of $\delta(x)$. Since u satisfies this condition, it is true that

$$\delta(x) \geq \delta(u) = \theta - 2v^T u + u^T M u = \theta - v^T u \geq 0 \quad \forall x \ .$$

q.e.d.

Now let us prove the propositions.

Proof of Proposition 9.1: Suppose that (t, u) is a feasible solution of TDP. Let:

$$\theta = F^T u \ ,$$

and let

$$M = \left[\sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \right] \ ,$$

and let $v = F$. Then

$$Mu = \left[\sum_{k=1}^{m} t_k \frac{E_k}{L_k^2} (a_k)(a_k)^T \right] u = F \ ,$$

and notice as well that

$$M \succeq 0$$

because M is the nonnegative sum of rank-one SPSD matrices. From Remark 3,

$$\begin{bmatrix} \theta & F^T \\ F & \left[\sum_{k=1}^{m} t_k \frac{E_k}{L_k^2}(a_k)(a_k)^T\right] \end{bmatrix} = \begin{bmatrix} \theta & v^T \\ v & M \end{bmatrix} \succeq 0 \ .$$

Therefore (t, θ) is feasible for STDP with objective function value $\theta = F^T u$.
q.e.d.

Proof of Proposition 9.2: Suppose that (t, θ) is a feasible solution of STDP. Then from Remark 3, there exists a vector u for which:

$$\left[\sum_{k=1}^{m} t_k \frac{E_k}{L_k^2}(a_k)(a_k)^T\right] u = F \ ,$$

and $\theta \geq F^T u$. Therefore (t, u) is feasible for TDP and $\theta \geq F^T u$.
q.e.d.

10 Truss Design and Linear Optimization

Consider the following special, but not unusual, instance of the truss design problem:

$$
\begin{aligned}
\text{(TDP):}\quad \text{minimize}_{t,u} \quad & F^T u \\
\text{s.t.} \quad & \left[\sum_{k=1}^{m} t_k \frac{E_k}{L_k^2}(a_k)(a_k)^T\right] u = F \\
& \sum_{k=1}^{m} t_k \leq V \\
& t \geq 0 \\
& u \in \Re^N, t \in \Re^m.
\end{aligned}
$$

In this problem, the only constraint on the volumes of the bars t_k is a volume constraint limiting the total volume of the bars to not exceed the given value V. Substituting the notation:

$$
K(t) := \left[\sum_{k=1}^{m} t_k \frac{E_k}{L_k^2}(a_k)(a_k)^T\right] ,
$$

we can also conveniently write our problem as:

$$
\begin{aligned}
\text{(TDP):}\quad \text{minimize}_{t,u} \quad & F^T u \\
\text{s.t.} \quad & K(t)u = F \\
& \sum_{k=1}^{m} t_k \leq V \\
& t \geq 0 \\
& u \in \Re^N, t \in \Re^m.
\end{aligned}
$$

In this section, we show that when TDP has this particularly simple form, then it can be solved via linear optimization. Before doing so, we first write down a dual problem associated with TDP in this form:

$$\text{(DTDP):}\quad \text{maximize}_{v,z} \quad -2F^T v - Vz$$

$$\text{s.t.} \quad \left(a_k^T v\right)^2 \leq \tfrac{L_k^2}{E_k} z \ , \qquad k = 1, \ldots, m$$

$$y \in \Re^N .$$

In order for DTDP to be an honest dual of TDP, we now present a weak duality result:

Proposition 10.1 *Suppose that (t,u) is feasible for TDP and that (v,z) is feasible for DTDP. Then:*

$$F^T u \geq -2F^T v - Vz \ .$$

Proof: For $k = 1, \ldots, m$, we have

$$t_k \geq 0 \quad \text{and} \quad \frac{E_k}{L_k^2}(a_k^T v)^2 \leq z.$$

Multiplying and summing terms, we obtain:

$$v^T K(t) v = \sum_{k=1}^{m} \frac{t_k E_k}{L_k^2} \ (v^T a_k) \ (a_k^T v) \leq \left(\sum_{k=1}^{m} t_k \right) z \leq Vz.$$

Also,

$$\begin{aligned} 0 \leq (u+v)^T K(t)(u+v) &= u^T K(t) u + 2u^T K(t) v + v^T K(t) v \\ &= F^T u + 2v^T F + v^T K(t) v \\ &\leq F^T u + 2v^T F + Vz. \end{aligned}$$

Therefore $F^T u \geq -2F^T v - Vz$.
q.e.d.

Consider the following pair of primal and dual linear optimization models:

$$
\begin{aligned}
\text{(LP):}\quad \text{minimize}_{f^+,f^-} \quad & \sum_{k=1}^{m} \tfrac{L_k}{\sqrt{E_k}} \left(f_k^+ + f_k^-\right) \\
\text{s.t.} \quad & A(f^+ - f^-) = -F \\
& f^+ \geq 0 \ , \ \ f^- \geq 0 \\
& f^+, f^- \in \Re^m .
\end{aligned}
$$

$$
\begin{aligned}
\text{(LD):}\quad \text{maximize}_y \quad & -F^T y \\
\text{s.t.} \quad & -\tfrac{L_k}{\sqrt{E_k}} \leq a_k^T y \leq \tfrac{L_k}{\sqrt{E_k}} \ , \qquad k = 1, \ldots, m \\
& y \in \Re^N .
\end{aligned}
$$

We will prove the following important result that shows how to use solutions of LP and LD to construct solutions to TDP:

Proposition 10.2 *Suppose that $(\bar{f}^+, \bar{f}^-)$ is an optimal solution of LP and that $\bar{y}$ is an optimal solution of LD, and consider the following assignment of variables:*

$$
\begin{aligned}
R &= \sum_{k=1}^{m} \frac{L_k}{\sqrt{E_k}} \left(\bar{f}_k^+ + \bar{f}_k^-\right) \\
\bar{t}_k &= \frac{V}{R} \frac{L_k}{\sqrt{E_k}} \left(\bar{f}_k^+ + \bar{f}_k^-\right) \qquad k = 1, \ldots, m, \\
\bar{u} &= -\frac{R}{V} \bar{y} \\
\bar{v} &= \frac{R}{V} \bar{y} \\
\bar{z} &= \frac{R^2}{V^2}
\end{aligned}
$$

Then $(\bar{t}, \bar{u})$ solves TDP and $(\bar{v}, \bar{z})$ solves DTDP.

Proof: Note that by the complementary slackness property of linear optimization, we have:

$$\begin{aligned} a_k^T \overline{y} \overline{f}_k^+ &= \frac{L_k}{\sqrt{E_k}} \ \overline{f}_k^+, \quad k = 1, \ldots, m, \\ a_k^T \overline{y} \overline{f}_k^- &= \frac{-L_k}{\sqrt{E_k}} \ \overline{f}_k^-, \quad k = 1, \ldots, m. \end{aligned}$$

We first show that $(\bar{t}, \overline{u})$ is feasible for TDP. Note that $\bar{t} \geq 0$ and $\sum_{k=1}^{m} \bar{t}_k = \frac{V}{R} R = V$. Also

$$\begin{aligned} K(\bar{t})\overline{u} &= -\frac{R}{V} K(\bar{t})\overline{y} \\ &= -\frac{R}{V} \sum_{k=1}^{m} \frac{\bar{t}_k E_k}{L_k^2} \ a_k a_k^T \overline{y} \\ &= -\frac{R}{V} \ \frac{V}{R} \sum_{k=1}^{m} \frac{L_k}{\sqrt{E_k}} \ \frac{E_k}{L_k^2} \ (\overline{f}_k^+ + \overline{f}_k^-) a_k a_k^T \overline{y} \\ &= -\sum_{k=1}^{m} a_k \ \frac{\sqrt{E_k}}{L_k} (\overline{f}_k^+ a_k^T \overline{y} + \overline{f}_k^- a_k^T \overline{y}) \\ &= -\sum_{k=1}^{m} a_k \ (\overline{f}_k^+ - \overline{f}_k^-) = -A(\overline{f}^+ - \overline{f}^-) = F. \end{aligned}$$

We next show that $(\overline{v}, \overline{z})$ is feasible for DTDP. To see this, observe that for $k = 1, \ldots, m$, we have:

$$(a_k^T \overline{v})^2 = \frac{R^2}{V^2} \ (a_k^T \overline{y})^2 \leq \frac{R^2}{V^2} \ \left(\frac{L_k}{\sqrt{E_k}} \right)^2 = z \ \frac{L_k^2}{E_k},$$

and so $(\overline{v}, \overline{z})$ is feasible.

Finally, we show that these solutions exhibit strong duality:

$$-2F^T \overline{v} - V \overline{z} = \frac{-2R}{V} \ F^T \overline{y} - \frac{VR^2}{V^2} = \frac{2R^2}{V} - \frac{R^2}{V} = \frac{R^2}{V} = \frac{R}{V} R = \frac{R}{V}(-F^T \overline{y}) = F^T \overline{u}.$$

q.e.d.

11 Extensions of the Truss Design Problem

- **Multiple Loads.** We consider a finite set of external loads on the truss:

$$\mathcal{F} = \{F_1, F_2, \ldots, F_J\} \subset \mathbb{R}^N \ .$$

In this case we might consider solving two types of design problems. The first is to design the truss structure conservatively, so as to minimize the maximum compliance:

$$\begin{array}{lll} \text{minimize}_t & \max_{j=1,\ldots,J} \left\{ F_j^T u_j \right\} & \\ Mt \le d, & & \\ t \ge 0 & \text{s.t.} & K(t)u_j = F_j. \end{array}$$

Alternatively, we might consider the "average-case" design problem: let λ_j denote the relative frequency or importance associated with the truss structure undergoing the external force F_j, where $\sum_{j=1}^{J} \lambda_j = 1.0$. The following optimization model minimizes the average compliance of the truss structure:

$$\begin{array}{lll} \text{minimize}_t & \sum_{j=1}^{J} \lambda_j F_j^T u_j & \\ Mt \le d, & & \\ t \ge 0 & \text{s.t.} & K(t)u_j = F_j. \end{array}$$

- **Self-weights.** The truss design problem that we have formulated presumes that the truss structure itself is not affected by its own weight. To correct for this, we can simply add an external force corresponding to the gravitational force associated with bar k, and linearly proportional to t_k for $k = 1, \ldots, m$. Let us denote by $g_k \in \mathbb{R}^N$ the vector that projects the gravitational force of bar k onto the appropriate nodes.

Then we can write the TDP of a truss under a single external load vector F as:

$$\begin{array}{lll} \text{(TDP):} & \text{minimize}_{t,u} & \left(F + \sum_{k=1}^{m} t_k g_k\right)^T u \\ & \text{s.t.} & K(t)u = \left(F + \sum_{k=1}^{m} t_k g_k\right) \\ & & Mt \leq d \\ & & t \geq 0 \\ & & u \in \Re^N, t \in \Re^m. \end{array}$$

It is straightforward to convert this problem into a convex problem, just as for the base case considered earlier.

- **Reinforcement.** In this problem, we are given an existing truss structure, and we must determine how to strengthen it. To solve this problem, we simply add lower bounds on the volumes of the existing bars equal to the current volumes of the bars:

$$\begin{array}{lll} \text{(TDP):} & \text{minimize}_{t,u} & F^T u \\ & \text{s.t.} & K(t)u = F \\ & & Mt \leq d \\ & & t \geq 0 \\ & & t_k \geq \bar{t}_k \ , \ k = 1, \ldots, m \\ & & u \in \Re^N, t \in \Re^m. \end{array}$$

- **Robustness of Solutions.** The optimal solution of the TDP might yield a truss design that is not rigid for a different external force than the one used, and could change considerably even under a small change in the external force. To obtain a robust solution, we must consider multiple external loads that will contain the possible loads that the truss will be subject to. This can be modeled by convex optimization as well.

- **Buckling Constraints.** The truss design problem that we have presented ignores the fact that if a bar is under great compression it might actually collapse instead of counteracting that external force with its internal force. For this we have to add lower bounds on the allowable internal forces in terms of the design variables t_k and the geometry of the bars. These constraints cause the resulting problem to lose its convex structure.

- **Other Considerations . . . .**