Problem Set 5

Due Wednesday, November 18, 2009; 4pm

Problem 1 (Monomial orderings).

(a) Show that an order relation $\succ$ on $\mathbb{Z}^n_+$ is a monomial ordering if and only if

  (i) $\succ$ is a total ordering on $\mathbb{Z}^n_+$,

  (ii) $\alpha \succ \beta$ and $\gamma \in \mathbb{Z}^n_+$ imply $\alpha + \gamma \succ \beta + \gamma$, and

  (iii) every strictly decreasing sequence in $\mathbb{Z}^n_+$, $\alpha(1) \succ \alpha(2) \succ \alpha(3) \succ \cdots$, eventually terminates.

(b) Prove that the lex ordering is a monomial ordering.

(c) Use Dickson's Lemma to prove that an order relation $\succ$ on $\mathbb{Z}^n_+$ that satisfies (i) and (ii) is a monomial ordering if and only if $\alpha \succeq 0$ for all $\alpha \in \mathbb{Z}^n_+$.

(d) Let $f, g \in k[x_1, \ldots, x_n]$ be nonzero polynomials. Show that $\text{multideg}(fg) = \text{multideg}(f) + \text{multideg}(g)$. Also show that, if $f+g \neq 0$, then $\text{multideg}(f+g) \leq \max(\text{multideg}(f), \text{multideg}(g))$.

Problem 2 (Monomial ideals). If $I = \langle x^{\alpha(1)}, \ldots, x^{\alpha(s)} \rangle$ is a monomial ideal, prove that a polynomial f is in I if and only if the remainder of f on division by $x^{\alpha(1)}, \ldots, x^{\alpha(s)}$ is zero.

Problem 3 (Gröbner bases).

(a) Let G be a minimal Gröbner basis for a polynomial ideal $I \neq \{0\}$. We say that $g \in G$ is *reduced* for G if no monomial of g is in $\langle \text{LT}(G \setminus \{g\}) \rangle$. For $g \in G$, let g' be the remainder of g on division by $G \setminus \{g\}$. Set $G' := (G \setminus \{g\}) \cup \{g'\}$.

  (i) Prove that G' is a minimal Gröbner basis for I.

  (ii) Show that g' is reduced for G'.

  (iii) Use (i) and (ii) to suggest an algorithm for computing a reduced Gröbner basis.

(b) Consider the ideal $I = \langle 3x - 6y - 2z, 2x - 4y + 4w, x - 2y - z - w \rangle$. Compute the reduced Gröbner basis with respect to lex order with $x > y > z > w$, and use it to solve the corresponding system of linear equations.

Problem 4 (Gröbner basis algorithm for 0/1-integer programming). Solve the problem

$$\begin{aligned} \min \quad & 3x_1 - 2x_2 - 4x_3 + 7x_4 + x_5 + 2x_6 \\ \text{subject to} \quad & 8x_1 + 4x_2 - 3x_3 - 5x_4 + 2x_5 - x_6 = 7 \\ & x_1, \ldots, x_6 \in \{0, 1\} \end{aligned}$$

by using the algorithm described on Page 263 of the textbook. (You may use computer algebra software to compute Gröbner bases, if you want to.)

Problem 5 (Total dual integrality)**.**

(a) Consider the linear system

$$\begin{aligned} x_1 + 2x_2 &\leq 6, \\ 2x_1 + x_2 &\leq 6, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Show that it is not totally dual integral (TDI), yet it defines an integer polytope P. Find a representation $Ax \leq b$ of P, with A and b integral, such that $Ax \leq b$ is TDI.

(b) Let $Ax \leq b$ be a rational system of linear inequalities. Show that there exists a positive integer t such that $\frac{1}{t}Ax \leq \frac{1}{t}b$ is totally dual integral.

(c) Let $G = (V, E)$ be a $2k$-connected undirected graph.[1] Orient the edges of G arbitrarily, yielding the directed graph $D = (V, A)$. Consider the following linear system:

$$\sum_{a \in \delta^-(U)} x_a - \sum_{a \in \delta^+(U)} x_a \leq |\delta^-(U)| - k \text{ for } \emptyset \subset U \subset V, \quad 0 \leq x_a \leq 1 \text{ for } a \in A. \tag{1}$$

Here $\delta^-(U)$ ($\delta^+(U)$) denotes the set of arcs of D entering (leaving) U.

(i) Show that if (1) has an integral solution x, then G can be oriented so as to become a k-connected digraph.[2] (Hint: Reverse the orientation on the arcs of D with $x_a = 1$.)

So it suffices to show that (1) has an integral solution to prove the following theorem:

A $2k$-connected undirected graph can be oriented so as to become a k-connected directed graph.

(ii) Show that (1) has a feasible (fractional) solution.

(iii) Show that (1) is TDI. (You may use the following fact: Let $D = (V, A)$ be a digraph, and let $\mathcal{F} \subseteq 2^V$ be a family of subsets of V with the following property: if $T, U \in \mathcal{F}$, then $T \subseteq U$ or $U \subseteq T$ or $T \cap U = \emptyset$ or $T \cup U = V$. Then the matrix M with entries $M_{U,a} = +1$ if a enters U, $M_{U,a} = -1$ if a leaves U, and $M_{U,a} = 0$ otherwise, for $U \in \mathcal{F}$, $a \in A$, is totally unimodular.)

[1] An undirected graph is $2k$-connected if each nonempty proper subset of vertices is entered by at least $2k$ edges.
[2] A directed graph is k-connected if each nonempty proper subset of the vertices is entered by at least k arcs.

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