Midterm Solutions

1 Problem 1

Suppose a random variable X is such that $\mathbb{P}(X > 1) = 0$ and $\mathbb{P}(X > 1 - \epsilon) > 0$ for every $\epsilon > 0$. Recall that the large deviations rate function is defined to be $I(x) = \sup_\theta(\theta x - \log M(\theta))$ for every real value x, where $M(\theta) = \mathbb{E}[\exp(\theta X)]$, for every real value θ.

(a) show that $I(x) = \infty$ for every $x > 1$.

Since $\mathbb{P}(X > 1) = 0$, we have

$$M(\theta) = \int_{-\infty}^{1} \exp(\theta x) dP_X(x) \leq \exp(\theta) P(X \leq 1) = \exp(\theta)$$

We therefore obtain $-\log M(\theta) \leq -\theta$, and conclude that for $x > 1$

$$I(x) = \sup_\theta(\theta x - \log M(\theta)) \geq \sup_\theta \theta(x-1) = \infty$$

(b) show that $I(x) < \infty$ for every $\mathbb{E}[X] \leq x < 1$.

since $x \geq \mathbb{E}[X]$, we have that

$$I(x) = \sup_\theta(\theta x - \log M(\theta)) = \sup_{\theta \geq 0}(\theta x - \log M(\theta))$$

Now, take any $\epsilon > 0$ such that $x < 1 - \epsilon$, and note that

$$M(\theta) = \int_{-\infty}^{1} \exp(\theta x) dP_X(x) \geq \int_{1-\epsilon}^{1} \exp(\theta x) dP_X(x) \geq \exp(\theta(1-\epsilon))\mathbb{P}(X > 1-\epsilon)$$

Therefore $-\log M(\theta) \leq -\theta(1-\epsilon) - \log \mathbb{P}(X > 1 - \epsilon)$, and we obtain

$$I(x) = \sup_{\theta \geq 0}(x\theta - \log M(\theta)) \leq \sup_{\theta \geq 0}((x-(1-\epsilon))\theta - \log \mathbb{P}(X > 1-\epsilon)) \leq -\log \mathbb{P}(X > 1-\epsilon) < \infty$$

(c) show that $\lim_{\epsilon \to 0} \mathbb{P}(1 - \epsilon \leq X \leq 1) = 0$. Show that $I(1) = \infty$.

For any $\epsilon > 0$,

$$\begin{aligned} M(\theta) = \int_{-\infty}^{1} \exp(\theta x) dP_X(x) &= \int_{-\infty}^{1-\epsilon} \exp(\theta x) dP_X(x) + \int_{1-\epsilon}^{1} \exp(\theta x) dP_X(x) \\ &\leq \exp(\theta(1-\epsilon))\mathbb{P}(X < 1-\epsilon) + \exp(\theta)\mathbb{P}(1-\epsilon \leq X \leq 1) \\ &\leq \exp(\theta(1-\epsilon))(1 + \mathbb{P}(1-\epsilon \leq X \leq 1)(\exp(\theta\epsilon) - 1)) \\ &\leq \exp(\theta(1-\epsilon))(1 + \mathbb{P}(1-\epsilon \leq X \leq 1)\exp(\theta\epsilon)) \end{aligned}$$

Let $f(\theta, \epsilon)$ denote the quantity above. For any $\theta \geq 0$ and $\epsilon > 0$, we have $M(\theta) \leq f(\theta, \epsilon)$, and we obtain that

$$I(1) \geq \sup_{\theta \geq 0, \epsilon > 0} \theta - \log f(\theta, \epsilon)$$

Take $\theta = \frac{1}{\epsilon} \log(\frac{1}{\cdot}\mathbb{P}(X \geq 1 - \epsilon))$, so that $1 + \mathbb{P}(1 - \epsilon \leq X \leq 1) \exp(\theta\epsilon) = 2$. We obtain that $-\log f(\theta(\epsilon), \epsilon) = -\theta(1-\epsilon) - \log 2$, and so $\theta - \log f(\theta(\epsilon), \epsilon) \geq \theta\epsilon - \log 2$. Finally, note that $\theta\epsilon = \log(\frac{1}{\mathbb{P}(1-\epsilon \leq X \leq 1)})$ goes to ∞ as ϵ goes to zero.

2 Problem 2

Recall the following one-dimensional version of the large Deviations Principles for finite state Markov chains. Given an N-state Markov chain $X_n, n \geq 0$ with transition matrix $P_{i,j}$, $1 \leq i, j \leq N$ and a function $f : \{1, ..., N\} \to \mathbb{R}$, the sequence $\frac{S_n}{n} = \frac{\sum_{1 \leq i \leq n} f(X_i)}{n}$ satisfies the Large Deviations Principle with the rate function $I(x) = \sup_\theta(\theta x - \log \rho(P_\theta))$, where $\rho(P_\theta)$ is the Perron-Frobenius eigenvalue of the matrix $P_\theta = (e^{\theta f(j)} P_{i,j}, 1 \leq i, j \leq N)$.

Suppose $P_{i,j} = \pi_j$ for some probability vector $\pi_j \geq 0$, $1 \leq j \leq N$, $\sum_j \pi_j = 1$. Namely, the observations X_n for $n \geq 1$ are i.i.d. with the probability mass function given by π. In this case we know that the large deviations rate function for the i.i.d. sequence $f(X_n)$, $n \geq 1$ is described by the moment generating function of $f(X_n)$, $n \geq 1$. Establish that the two large deviations rate functions are identical, and thus the LDP for Markov chains in this case is consistent with the LDP for i.i.d. processes.

Proof. We have that P_θ is

$$P_\theta = \begin{pmatrix} \exp(\theta f(X_1))\pi_1 & \cdots & \exp(\theta f(X_N))\pi_N \\ \vdots & \cdots & \vdots \\ \exp(\theta f(X_1))\pi_1 & \cdots & \exp(\theta f(X_N))\pi_N \end{pmatrix}$$

Let $v = [1, ..., 1]^T$ and $M(\theta) = \mathbb{E}[\exp(\theta f(X_1))]$. Then we have that

$$P_\theta v = M(\theta) v$$

Since P_θ has rank 1 and $M(\theta) > 0$, we have that $M(\theta)$ is the Perron-Frobenius eigenvalue of P_θ. Thus, we have

$$I(x) = \sup_\theta(\theta x - \log \rho(P_\theta)) = \sup_\theta(\theta x - \log M(\theta))$$

□

3 Problem 3

(a) Suppose, X_n, $n \geq 0$ is a martingale such that the distribution of X_n is identical for all n and the second moment of X_n is finite. Establish that $X_n = X_0$ almost surely for all n.

Proof. Let $\mathcal{F}_n$ be a filtration to which the martingale X_n, $n \geq 0$ is adapted. Then for any $n \geq 1$, by tower property, we have

$$\begin{aligned}
\mathbb{E}[(X_n - X_0)^2] &= \mathbb{E}[X_n^2] + \mathbb{E}[X_0^2] - 2\mathbb{E}[X_0 X_n] \\
&= \mathbb{E}[X_n^2] + \mathbb{E}[X_0^2] - 2\mathbb{E}[\mathbb{E}[X_0 X_n|\mathcal{F}_0]] \\
&= \mathbb{E}[X_n^2] + \mathbb{E}[X_0^2] - 2\mathbb{E}[X_0\mathbb{E}[X_n|\mathcal{F}_0]] \\
&= \mathbb{E}[X_n^2] + \mathbb{E}[X_0^2] - 2\mathbb{E}[X_0^2] \\
&= 0
\end{aligned}$$

by the fact that X_n, $n \geq 0$ has the same distribution and that X_n has finite second moment. Thus, we have $X_n = X_0$ almost surely.

□

(b) An urn contains two white balls and one black ball at time zero. At each time $t = 1, 2, ...$ exactly one ball is added to the urn. Specifically, if at time $t \geq 0$ there are W_t white balls and B_t black balls, the ball added at time $t+1$ is white with probability $\frac{W_t}{W_t+B_t}$ and is black with the remaining probability $\frac{B_t}{W_t+B_t}$. In particular, since there were three balls at the beginning, and at every time $t \geq 1$ exactly one ball is added, then $W_t + B_t = t + 3$, $t \geq 0$. Let T be the first time when the proportion of white balls is exactly 50% if such a time exists, and $T = \infty$ if this is never the case. Namely $T = \min\{t : \frac{W_t}{W_t+B_t} = \frac{1}{2}\}$ if the set of such t is non-empty, and $T = \infty$ otherwise. Establish an upper bound $\mathbb{P}(T < \infty) \leq \frac{2}{3}$.

Proof. First, we will establish that $\frac{W_t}{W_t+B_t}$ is a martingale. Let $\mathcal{F}_t$ be a filtration to which $\frac{W_t}{W_t+B_t}$ is adapted. Then we have that

$$\mathbb{E}\left[\left|\frac{W_t}{W_t + B_t}\right|\right] \leq 1$$

and that

$$\begin{aligned}
\mathbb{E}[\frac{W_t}{W_t + B_t}|\mathcal{F}_t] &= \frac{W_t}{W_t + B_t}\frac{1 + W_t}{1 + W_t + B_t} + \frac{B_t}{W_t + B_t}\frac{W_t}{1 + W_t + B_t} \\
&= \frac{W_t(1 + W_t + B_t)}{(W_t + B_t)(1 + W_t + B_t)} = \frac{W_t}{W_t + B_t}
\end{aligned}$$

Thus, $\frac{W_t}{W_t+B_t}$, $t \geq 0$ is a martingale. Since $|X_{t\wedge T}| \leq 1$, the optional stopping theorem gives that X_T is almost surely well defined random variable and $E[X_T] = E[X_0]$. Thus, we have

$$E[X_T] = \frac{1}{2}\mathbb{P}(T < \infty) + \alpha\mathbb{P}(T = \infty) = \mathbb{E}[X_0] = \frac{2}{3}$$
$$\Rightarrow \frac{1}{2}\mathbb{P}(T < \infty) + \alpha(1 - \mathbb{P}(T < \infty)) = \frac{2}{3} \tag{1}$$

where $0 \leq \alpha \leq 1$ is the fraction $\frac{W_t}{W_t+B_t}$ for $t \to \infty$ and it exists by Martingale convergence theorem. By (1), we have that $\mathbb{P}(T < \infty) < 1$, thus

$$\alpha = \frac{\frac{2}{3} - \frac{1}{2}\mathbb{P}(T < \infty)}{1 - \mathbb{P}(T < \infty)} \Rightarrow 0 \leq \frac{\frac{2}{3} - \frac{1}{2}\mathbb{P}(T < \infty)}{1 - \mathbb{P}(T < \infty)} \leq 1 \Rightarrow \mathbb{P}(T < \infty) \leq \frac{2}{3}$$

□

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