Optimization Methods in Management Science

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Recitation 3

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At the end of this recitation, students should be able to:

1. Understand when an LP is: unbounded, degenerate, has multiple optimal solutions.
2. Find multiple optimal solutions of an LP, in case they exist.
3. Perform Phase I of the simplex method.

Problem 1

Consider the following simplex tableau:

Basic	x_1	x_2	x_3	s_1	s_2	Rhs
$(-z)$	1	2	5/4			0
s_1	2	1	1	1		6
s_2		2	1		1	4

As usual, empty cells contain a zero. Assume a maximization problem.

Part 1.A

Apply the simplex algorithm to compute an optimal solution. Always pivot in the column with largest reduced cost. Write down the optimal solution and optimal objective function value. Is the optimal solution that you found unique? Why?

Solution. Pivot column: x_2, pivot row: 2.

Basic	x_1	x_2	x_3	s_1	s_2	Rhs
$(-z)$	1		1/4		-1	-4
s_1	2		1/2	1	-1/2	4
x_2		1	1/2		1/2	2

Pivot column: x_1, pivot row: 1.

Basic	x_1	x_2	x_3	s_1	s_2	Rhs
$(-z)$				-1/2	-3/4	-6
x_1	1		1/4	1/2	-1/4	2
x_2		1	1/2		1/2	2

The optimal solution is $x_1 = 2, x_2 = 2$, with cost 6. It is not unique because x_3 has zero reduced cost in the final tableau, hence it could be pivoted in staying at the same objective function value.

Part 1.B

Describe *all* the optimal solutions to the problem. Use the final tableau of Part 1.A as a starting point. (Hint: the first step is to compute all optimal vertices. When we say "describe all solutions", we mean that you should find equations that describe them, not that you should write down all optimal points. That would be a very lengthy task, seeing as there is an infinite number of solutions.)

Solution. Starting from the optimal tableau, we pivot in x_3. The ratio test forces x_2 out of the basis. We obtain the following optimal tableau.

Basic	x_1	x_2	x_3	s_1	s_2	Rhs
$(-z)$				-1/2	-3/4	-6
x_1	1	-1/2		1/2	-1/2	1
x_3		2	1		1	4

This defines another optimal solution: $x_1 = 1, x_3 = 4$, with the same objective function value. We could pivot in x_2 again, but the ratio test would force x_3 out of the basis. Therefore, we would go back to the previous tableau and associated optimal solution. It follows that there are only two vertices of the feasible region that are optimal.

Since there are two distinct optimal vertices, the whole segment between them is optimal. We can express this segment as the convex combination of its two endpoints:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.$$

with $0 \leq \lambda \leq 1$. This can be rewritten as:

$$\begin{aligned} x_1 &= 1 + \lambda \\ x_2 &= 2\lambda \\ x_3 &= 4 - 4\lambda \end{aligned}$$

with $0 \leq \lambda \leq 1$.

An alternative way to find the equations describing all optimal solutions is to apply the Δ method to the optimal tableau of Part 1.A, with x_3 entering the basis. If $x_3 = \Delta$, we have:

$$\begin{aligned} x_1 &= 2 - \Delta/4 \\ x_2 &= 2 - \Delta/2 \\ x_3 &= \Delta, \end{aligned}$$

with $\Delta \geq 0$. Clearly the maximum value that Δ can take while leaving $x_1, x_2 \geq 0$ is 4. This system of equations with $0 \leq \Delta \leq 4$ describes the same segment as the system above for $0 \leq \lambda \leq 1$.

Problem 2

Consider the following simplex tableau:

Basic	x_1	x_2	x_3	x_4	x_5	Rhs
$(-z)$	0	-3	0	2	0	-6
x_1	1	-4		2		0
x_3		-6	1	3		2
x_5		-1			1	5

Assume that this is a maximization problem.

Part 2.A

What is the current basic feasible solution and its objective function value? Is the current bfs degenerate?

Solution. The current bfs is $x_3 = 2, x_5 = 5$. All the remaining variables have value 0, *including* x_1, which is basic. The objective function value is 6. The bfs is degenerate because $x_1 = 0$.

Part 2.B

Perform one pivot. What is the new basic feasible solution and its objective function value? Did anything change with respect to the previous bfs? And with respect to the previous tableau?

Solution. We pivot in s_1 and the ratio test tells us that we have to pivot out x_1 ($\min\{0/2, 2/3\}$ is clearly 0, thus we pivot on the coefficient "2"). We obtain:

Basic	x_1	x_2	x_3	x_4	x_5	Rhs
$(-z)$	-1	1	0	0	0	-6
x_4	1/2	-2		1		0
x_3	-3/2		1			2
x_5		-1			1	5

The new basic feasible solution is therefore $x_3 = 2, x_5 = 5$ and all the remaining variables have value zero. The objective function value is 6.

Nothing changed with respect to the previous bfs. We have exactly the same solution because we performed a *degenerate pivot.*

However the tableau has changed: we have different coefficients and a different set of basic variables.

Part 2.C

The tableau that you obtained after Part 2.B is not optimal because some reduced cost is positive. In fact, the problem is unbounded (why?). Give the equations describing a ray of the feasible region along which the objective function can be increased indefinitely.

Solution. The problem is unbounded because in the final tableau x_2 has positive reduced cost and all its coefficients in the rows are nonpositive. This means that x_2 can increased indefinitely to increase the objective function value. To describe the corresponding ray, we note that if we bring x_2 to the rhs of the equations that form the simplex tableau we obtain:

$$\begin{aligned} x_4 &= 0 + 2x_2 \\ x_3 &= 2 \\ x_5 &= 5 + x_2. \end{aligned}$$

Because x_2 is nonnegative, we can increase x_2 as much as we want without leaving the feasible region, and at the same time increasing the objective function value. The ray can therefore be described as:

$$\begin{aligned} x_2 &= \Delta \\ x_3 &= 2 \\ x_4 &= 2\Delta \\ x_5 &= 5 + \Delta \end{aligned}$$

for $\Delta \geq 0$.

Problem 3

Consider the following LP:

$$\left.\begin{array}{lrcl} \max & x_1 + 3x_2 & & \\ & 2x_1 - 2x_2 & = & 1 \\ & x_1 + x_2 & \geq & 5 \\ & x_1, x_2 & \geq & 0 \end{array}\right\} \qquad \text{(LP)}$$

Observe that a basic feasible solution is not readily apparent. We want to perform Phase I of the simplex method to find an initial basic feasible solution.

Part 3.A

Add surplus and artificial variables as needed to perform Phase I, then determine the Phase I objective function. Write the resulting LP and the corresponding initial Phase I simplex tableau. What is the basic feasible solution in this tableau?

Solution. We start by introducing a nonnegative surplus variable s_1 in the second constraint.

$$\left.\begin{array}{lrcl} \max & x_1 + 3x_2 & & \\ & 2x_1 - 2x_2 & = & 1 \\ & x_1 + x_2 - s_1 & = & 5 \\ & x_1, x_2, s_1 & \geq & 0. \end{array}\right\}$$

There is no isolated variable with a +1 coefficient in any row. We must introduce artificial variables s_2, s_3.

$$\left.\begin{array}{lrcl} \max & x_1 + 3x_2 & & \\ & 2x_1 - 2x_2 + s_2 & = & 1 \\ & x_1 + x_2 - s_1 + s_3 & = & 5 \\ & x_1, x_2, s_1, s_2, s_3 & \geq & 0. \end{array}\right\}$$

Next, we must determine the Phase I objective. We want to minimize the sum of the artificial variables: $\min s_2 + s_3$. In maximization form, this can be written as $\max -s_2 - s_3$. We can substitute s_2 and s_3 with their expression in terms of the original variables found through the problem's constraints. We obtain: $w = -s_2 - s_3 = (2x_1 - 2x_2 - 1) + (x_1 + x_2 - s_1 - 5) = (3x_1 - x_2 - s_1 - 6)$. This is the Phase I objective function. The Phase I LP is therefore:

$$\left.\begin{array}{rrcl} \max & 3x_1 - x_2 - s_1 - 6 & & \\ & 2x_1 - 2x_2 + s_2 & = & 1 \\ & x_1 + x_2 - s_1 + s_3 & = & 5 \\ & x_1, x_2, s_1, s_2, s_3 & \geq & 0. \end{array}\right\}$$

The corresponding simplex tableau is the following:

Basic	x_1	x_2	s_1	s_2	s_3	Rhs
$(-w)$	3	-1	-1			6
s_2	2	-2		1		1
s_3	1	1	-1		1	5

The basic feasible solution is $s_2 = 1, s_3 = 5$.

Part 3.B

Perform two iterations of the simplex algorithm on the tableau obtained at the end of Part 3.A. Is Phase I completed? If we now have a basic feasible solution for the original problem, explain why and write the solution. If we do not have a basic feasible solution for the original problem, explain why.

Solution. We start from this tableau:

Basic	x_1	x_2	s_1	s_2	s_3	Rhs
$(-w)$	3	-1	-1			6
s_2	2	-2		1		1
s_3	1	1	-1		1	5

We can now apply the simplex algorithm in the normal way. Pivot column: x_1, pivot row: 1.

Basic	x_1	x_2	s_1	s_2	s_3	Rhs
$(-w)$		2	-1	-3/2		9/2
x_1	1	-1		1/2		1/2
s_3		2	-1	-1/2	1	9/2

Pivot column: x_2, pivot row: 2.

Basic	x_1	x_2	s_1	s_2	s_3	Rhs
$(-w)$				-1	-1	0
x_1	1		-1/2	1/4	1/2	11/4
x_2		1	-1/2	-1/4	1/2	9/4

After two iterations, the tableau is optimal: all reduced costs are nonpositive. The final objective function value is 0. This means that we drove all the artificial variables out of the basis. The current basis is x_1, x_2, and the corresponding basic feasible solution is $x_1 = 11/4, x_2 = 9/4$. This is a basic feasible solution for the original problem. We can now ignore the columns corresponding to s_2 and s_3 and proceed with Phase II of the simplex tableau as usual.

Problem 4

(Note: this problem is more difficult than the rest, and will require some serious thinking. You will *not* be tested on Problem 4 in the upcoming quizzes, problem sets or midterms. However it is very useful for understanding some subtleties of Phase I of the simplex method.)

Suppose that we have an LP with three nonnegative variables x_1, x_2, x_3 to which we have to apply Phase I of the simplex method to find an initial basic feasible solution. The artificial variables are labeled s_1, s_2, s_3. After a few iterations of the simplex method in Phase I, we obtain the following optimal tableau with an objective function value of zero, where b is a parameter that will be specified later:

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Rhs
$(-w)$			-1	-3		-1	
x_1	1		2	1		2	4
x_2		1	1	-3		5	1
s_2			b	1	1	-1	

The artificial variable s_2 is still basic, therefore we do not have a basis for the original problem. But the objective value of Phase I is zero, therefore a bfs for the original problem exists. Our goal is to find a basis to start Phase II of the simplex method.

Part 4.A

Assume $b = -2$. How do we proceed in this case? Determine a basis for the original problem and the corresponding tableau that can be used to start Phase II (after restoring the original objective function, which is not relevant for the purposes of this problem). (Hint: try to perform a pivot to drive the artificial variable out of the basis; this pivot may be something that you are normally not allowed to do, but notice that the basis is degenerate.)

Solution. If $b = -2$, the tableau reads as follows:

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Rhs
$(-w)$			-1	-3		-1	
x_1	1		2	1		2	4
x_2		1	1	-3		5	1
s_2			-2	1	1	-1	

The troublesome row is the third one, where s_2 is still basic. However, because the Phase I objective function value is zero, all artificial variables *must* have value zero, as is the case. We can therefore pivot in x_3 and pivot out s_2, regardless of the fact that x_3 has negative reduced cost: the rhs value of the third constraint is zero, therefore the solution will not "move". Also note that we are pivoting on a negative element: usually, this is not allowed because dividing the constraint by a negative coefficient would make the rhs negative, but since the rhs is zero, there is no risk in this case. We perform the pivot and obtain:

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Rhs
$(-w)$			0	-7/2	-1/2	-1/2	
x_1	1			2	1	1	4
x_2		1		-5/2	1/2	9/2	1
x_3			1	-1/2	-1/2	1/2	

Now the basis contains only original variables. The corresponding basic feasible solution is $x_1 = 4, x_2 = 1, x_3 = 0$. We can drop the columns corresponding to s_1, s_2, s_3, restore the original objective function, and proceed with Phase II.

More generally, this method for driving an artificial variable out of the basis at the end of Phase I can always be used, as long as there is least one nonzero coefficient for the original variables in the row where the artificial variable is basic. In Part 4.B we examine what happens when this is not the case.

Part 4.B

Assume $b = 0$. How do we proceed in this case? Determine a basis for the original problem and the corresponding tableau that can be used to start Phase II restoring the original objective function. (Hint: write down the equation corresponding to the last row of the tableau, and observe that this is a linear combination of the original constraints. What does this row mean in terms of the original equations?)

Solution. If $b = 0$, the tableau reads as follows:

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Rhs
$(-w)$			-1	-3		-1	
x_1	1		2	1		2	4
x_2		1	1	-3		5	1
s_2				1	1	-1	

In this case, we cannot pivot out s_2 and pivot in x_3 because the corresponding element of the tableau is zero. However, the last equation in the tableau reads: $0x_1+0x_2+0x_3+s_1+s_2-s_3 = 0$. This means that by performing row operations we have obtained a linear combination of the original problem constraints that is identically zero, hence the constraints are linearly dependent, and one of them is redundant. It follows that we can simply drop the third row, together with the artifical variables. The tableau that we can use to start Phase II is:

Basic	x_1	x_2	x_3	Rhs
$(-w)$			-1	
x_1	1		2	4
x_2		1	1	1

The corresponding basic feasible solution is $x_1 = 4, x_2 = 1, x_3 = 0$.

In general, if all coefficients for the original variables in the row where the artificial variable is basic are zero, then the corresponding equation is redundant, and the initial constraint can be dropped.