3. Hint for Problem 3, Carroll problem 7.1.

Carroll 7.1 asks us to vary the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu h^{\mu\nu})(\partial_\nu h) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h)\right]$$

to construct the Einstein tensor, Eq. (7.8) of Carroll.

If you vary this Lagrangian in the most straightforward way possible, you will probably find that you get *almost* the correct Einstein tensor — you should get Eq. (7.8), but with the first two terms replaced with 2 times the first term. In other words, you don't get the symmetrization on μ and ν that should be obtained.

What is going on here? The issue is that the Lagrangian doesn't "know", *a priori*, that the tensor $h_{\mu\nu}$ is symmetric. This has a strong impact on the second term of the Lagrangian — it should be symmetric with respect to exchange of the indices ρ and σ, but isn't unless you somehow build in the knowledge we have of this symmetry.

There are two simple ways to address this:

a. Rewrite the Lagrangian to force this symmetrization:

$$(\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) \rightarrow \frac{1}{2}\left[(\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) + (\partial_\mu h^{\rho\sigma})(\partial_\sigma h^\mu{}_\rho)\right] \;,$$

or

b. Make sure that, in our variation, this symmetry is enforced. The way I did this was to note that I should have

$$\begin{aligned}
\frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\alpha\beta})} &= \frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\beta\alpha})} \\
&= \frac{1}{2}\left[\frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\alpha\beta})} + \frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\beta\alpha})}\right] \\
&= \frac{1}{2}\left[\delta^\gamma{}_\mu \delta^\alpha{}_\rho \delta^\beta{}_\sigma + \delta^\gamma{}_\mu \delta^\alpha{}_\sigma \delta^\beta{}_\rho\right] \;.
\end{aligned}$$

It shouldn't be too difficult to convince yourself that these methods are in fact equivalent.

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