MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.901: Astrophysics I **Spring Term 2006**

PROBLEM SET 8

Due: Thursday, April 27 in class

Reading: Read Chapters 9 and 10 in Shapiro & Teukolsky, *Black Holes, White Dwarfs, & Neutron Stars.*

1. **Incompressible fluid model for a neutron star.** Consider a nearly incompressible fluid as material from which to construct a neutron star. Describe the fluid as having a constant density throughout the star, $\rho = \rho_0$. This is equivalent to supposing that the pressure and density are related as $P \propto \rho^\gamma$, where $\gamma \to \infty$.) To compute the structure of a neutron star, general relativistic corrections to the stellar structure equations must be made. The relativistic version of the equation of hydrostatic equilibrium, called the *Oppenheimer-Volkoff equation*, is given by

$$\frac{dP}{dr} = \frac{-G[M(r) + 4\pi r^3 P/c^2][\rho + P/c^2]}{r^2 \left[1 - \frac{2GM(r)}{rc^2}\right]},$$

where M and ρ refer to the total mass-energy and its density. This can be combined with $M(r) = \int dr\, 4\pi r^2 \rho$.

(a) Integrate the Oppenheimer-Volkoff equation to show that the pressure as a function of radius is

$$P(r) = \rho_0 c^2 \frac{\left[\left(1 - \frac{2GMr^2}{R^3c^2}\right)^{1/2} - \left(1 - \frac{2GM}{Rc^2}\right)^{1/2}\right]}{\left[3\left(1 - \frac{2GM}{Rc^2}\right)^{1/2} - \left(1 - \frac{2GMr^2}{R^3c^2}\right)^{1/2}\right]}.$$

Take $\rho_0 = (3/4\pi)MR^{-3}$, where M and R are the mass and radius of the neutron star, respectively.

(b) Show that, in order for the pressure to remain finite, R must be greater than $(9/8)(2GM/c^2)$.

2. **Internal structure of neutron stars.** In this problem, you will solve numerically for the equilibrium structure of a neutron star. Detailed discussions of the technique are given by Arnett & Bowers (1977, *Astrophys. J. Suppl.*, **33**, 415) and Lattimer & Prakash (2001, *Astrophys. J.*, **550**, 426). These papers can be found in the library, or online via the NASA Astrophysics Data System (`http://ads.harvard.edu`, click on "Search References").

(a) Use Figure 4 of Arnett & Bowers (1977) or Figure 1 of Lattimer & Prakash (2001) to choose a plausible power-law equation of state of the form $P = K\rho^\gamma$. To do this, draw a single ("average") straight line through equation of state models A through G in Arnett & Bowers or the equivalent models (*not* the strange quark matter models denoted by "SQM") in Lattimer & Prakash. You will use your power-law model down to arbitrarily low densities extending below the lower limits of the figures in these papers. Note that one can convert between the number density in baryon fm^{-3} plotted in Lattimer & Prakash and mass density of g cm^{-3} using $m_b = 1.7 \times 10^{-24}$ g and 1 fm=10^{-13} cm.

(b) Consider a range of central densities, $14 < \log \rho_c < 16.5$ (g cm^{-3}), uniformly spaced in $\log \rho_c$. For each of these, integrate the Oppenheimer-Volkoff equation for hydrostatic equilibrium in general relativity,

$$\frac{dP}{dr} = \frac{-G[M(r) + 4\pi r^3 P/c^2][\rho + P/c^2]}{r^2 \left[1 - \frac{2GM(r)}{rc^2}\right]},$$

to find the run of density as a function of radial coordinate r. To do this, you can directly integrate $d\rho/dr$ as determined from your power-law equation of state and $dM/dr = 4\pi r^2 \rho$.

(c) Plot neutron star mass versus central density of your range of models. What is the maximum mass $M_{\max}$ of a neutron star for the assumption that your chosen pressure law is correct?

(d) Plot radius versus central density for your range of models ($M < M_{\max}$ only).

(e) Plot mass versus radius for your models ($M < M_{\max}$ only).

(f) Repeat steps (b) and (c) for the following hybrid equation of state:

$$P = \rho c^2 \quad \text{for } \rho > 10^{14.6} \text{ g cm}^{-3}$$
$$P = K\rho^{5/3} \quad \text{for } \rho < 10^{14} \text{ g cm}^{-3}$$

where $K = 5.5 \times 10^9$ (cgs) is the appropriate constant for a non-relativistic Fermi gas of neutrons. [Note that $P = \rho c^2$ corresponds to the causality limit, since it gives a sound speed $c_s = (dP/d\rho)^{1/2}$ equal to c. See Section 9.3 and 9.5 of Shapiro & Teukosky for further discussion.] For densities between 10^{14} g cm^{-3} and $10^{14.6}$ g cm^{-3}, use a simple linear interpolation between the pressures given by the above expressions.

3. **Maximum rotation rate of a pulsar.** Estimate the maximum rotation rate for a neutron star before it breaks up.

(a) Find an expression for the minimum rotation period, $P_{\min}$, of a neutron star of a function of its mass M and radius R. Simply estimate the rotation at which a small mass parcel at the neutron star surface, near the equator, would experience centrifugal and gravitational forces of the same magnitude.

(b) Evaluate $P_{\min}$ for a neutron star with $M = 1.4M_\odot$ and $R = 10$ km. For comparison, the fastest known millisecond pulsar is PSR J1748-2446ad, which has a spin period of 1.4 ms (Hessels et al. 2006, Science, 311, 1901).

(c) Newton studied the equatorial bulge of a homogeneous fluid body of mass M that is *slowly* rotating with angular velocity ω. He proved that the equatorial radius R_e, polar radius R_p, and mean radius R_m are related by

$$\frac{R_e - R_p}{R_m} = \frac{5\omega^2 R_m^3}{4GM}.$$

Use this to estimate the equatorial and polar radii for a $1.4M_\odot$ neutron star rotating at twice the minimum rotation period you found in part (b).

4. **Pulsar spin-down properties.** Consider a pulsar with spin period $P = 2\pi/\omega$ that is losing rotational kinetic energy and thus spinning down.

(a) For magnetic dipole radiation, $\dot{\omega} = -k\omega^3$. For the case where k is a constant, show that the magnetic field strength $B \propto \sqrt{P\dot{P}}$.

(b) For a more general braking index n, where $\dot{\omega} = -k\omega^n$, show that $n = \ddot{\omega}\omega/\dot{\omega}^2$.

(c) Show that a good estimate for the age of a pulsar is

$$\tau = \frac{|P/\dot{P}|_{\rm final}}{(n-1)} \left[1 - \frac{P_{\rm initial}^{(n-1)}}{P_{\rm final}^{n-1}}\right]$$

(d) Derive an expression for the spin-down time scale of a pulsar with a braking index of 3 in terms of B_{12} (the magnetic field strength in units of 10^{12} G) and $P_{\rm s}$ (the rotation period in seconds).

5. **Dispersion of pulsar radio pulses in the interstellar medium.**

(a) Show that the index of refraction of a plasma is given by

$$n = \sqrt{1 - (\omega_p/\omega)^2},$$

where $\nu_p = \omega_p/2\pi$ is the plasma frequency (with $\omega_p^2 = 4\pi n_e e^2/m_e$ in terms of electron number density n_e) and $\nu = \omega/2\pi$ is the frequency of the radio waves.

(b) What is the phase velocity at frequency ν?

(c) What is the group velocity at frequency ν?

(d) Show that a pulsar pulse observed near radio frequency ν is delayed (compared to, say, optical or X-ray pulses emitted at the same time) by

$$\Delta t = (\text{constant}) \left(\frac{\nu}{400 \text{ MHz}}\right)^{-2} \int (n_e/0.01 \text{ cm}^{-3})(dx/1 \text{ kpc}) \text{ s},$$

where n_e has been scaled in units of 0.01 electrons/cm^3, the distance in kpc, and the observing frequency in units of 400 MHz. The integral is over the distance from the pulsar to the Earth. Assume that $\nu \gg \nu_p$ always. Evaluate the constant in the above expression.