

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Physics

Physics 8.901: Astrophysics I

Spring Term 2006

PROBLEM SET 7

**Due:** Thursday, April 20 in class

**Reading:** Read “How Massive Stars End Their Life” by Heger et al. (2003, *Astrophys. J.*, 591, 288), handed out in lecture. You will also find it useful to read Sections 2.6–2.8 and 2.11.2 in HK& T, as well as Chapter 18 in Shapiro & Teukolsky, *Black Holes, White Dwarfs, & Neutron Stars*. You should also read Chapters 5 and 12 in Shapiro and Teukolsky (handed out in lecture).

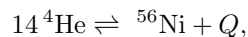
1. **Energy released in supernova core collapse.** (Problem 2.12 from Hansen, Kawaler, & Trimble.)

Estimate the gravitational potential energy released in the collapse of a  $1.2M_{\odot}$  core from an initial density of  $10^9 \text{ g cm}^3$  to a final density of  $10^{15} \text{ g cm}^3$ .

2. **Nuclear statistical equilibrium.** (Adapted from Hansen, Kawaler, & Trimble, Problem 6.7).

In the normal course of evolution of a massive star, the end products of nuclear burning are elements in the iron region of nucleon number. If the temperatures get high enough, the radiation field is capable of initiating photodisintegration, and all the iron-peak elements end up as individual nucleons. This can happen on such rapid time scales that the abundances of nuclei (as functions of temperature and density) can be calculated approximately as if the gas were in chemical equilibrium using a version of the Saha equation.

To look at this in a very simplified way, consider a gas composed only of  $^{56}\text{Ni}$  and  $^4\text{He}$  where the “chemical reaction” between them is



where you may compute the  $Q$ -value from the mass excesses,

$$\begin{aligned} (M - A)c^2 &= 2.42 \text{ MeV for } ^4\text{He} \\ (M - A)c^2 &= -53.9 \text{ MeV for } ^{56}\text{Ni}. \end{aligned}$$

In equilibrium, the chemical potentials obey the relation

$$14\mu_{\text{He}} = \mu_{\text{Ni}}.$$

Since the nuclei are non-degenerate, the number density of species  $i$  (either He or Ni) obey the Maxwell-Boltzmann distribution,

$$n_i = g_i \left( \frac{m_i kT}{2\pi\hbar^2} \right)^{3/2} \exp \left( \frac{\mu_i - m_i c^2}{kT} \right).$$

- (a) Set up an equivalent of Saha equation for the reaction, pretending that you are dealing with atoms and ions and assuming that both nuclei are in their ground states. To do this, substitute the Maxwell-Boltzmann relation for the number densities into the equilibrium relation for the

chemical potentials, and derive an equation whose left hand side is an appropriate ratio of number densities. Let the statistical weights be unity (which is appropriate since the ground state spins are zero).

Note that there will be two differences from the Saha equation you encountered earlier in the term: First, there are only two species so there will only be two different  $n$ 's. Second, the multiple He nuclei in the reaction will result in a nonlinear dependence on  $n_{\text{He}}$ . The  $Q$ -value plays the role of the ionization energy.

- (b) Re-cast your Saha equation so that the unknowns are the mass fractions  $X_4$  and  $X_{56}$  where  $X_4 + X_{56} = 1$ .
  - (c) Fix the density to be  $\rho = 10^7 \text{ g cm}^{-3}$  and solve for  $X_4$  and  $X_{56}$  for temperatures in the range  $4.5 < T_9 < 6.5$ , where  $T_9$  is the temperature in units of  $10^9 \text{ K}$ .
  - (d) Plot your results for the mass fractions versus temperature. At what temperature is  $X_4 = X_{56}$ ?
3. **Trajectories near a black hole.** You should read Chapter 5 and Sections 12.1–12.5 of Shapiro & Teukolsky (handed out in lecture) before doing this problem, in which you will study particle trajectories near black holes by numerically integrating the equations of motion for both massive test particles and photons near a Schwarzschild black hole (Equations 12.4.13–15 and 12.5.1–3 in Shapiro & Teukolsky). You should treat the equations as ordinary differential equations. Do *not* attempt to solve them by direct analytic integration. For Equation 12.4.13, there is a singularity at the turning points that makes numerical integration tricky. One can avoid this problem by taking an additional time derivative and rearranging appropriately.

Define your units such that  $c = G = M = 1$ . What are then the units of time and length, scaled to  $M = 10M_{\odot}$ ? Compute trajectories for the following cases:

- (a) Compute the radial infall ( $l = 0$ ) of a massive particle into the black hole. Plot  $r$  as a function of both coordinate time  $t$  (time as measured by an observer at infinity) and proper time  $\tau$ , thus reproducing the results illustrated in Figure 12.1 of Shapiro & Teukolsky). Verify explicitly that energy is conserved in your numerical integration.
- (b) For a massive test particle, construct examples of the following types of orbits (defined on page 345 in Section 12.4 of Shapiro & Teukolsky): (i) an unbound orbit (how close can you get to the black hole without being captured?); (ii) a “capture” orbit (check the definition of this in the reading); (iii) a bound orbit (what is their shape?); (iv) a stable circular orbits; (v) an unstable circular orbit (Where does the particle go? You might try perturbing it slightly to make it go the other way). Verify numerically that the innermost stable circular orbit is at  $r = 6M$ .
- (c) Construct light rays emitted in many different directions from a point near the horizon (choose  $r/M$  between 2.1 and 2.5). Verify numerically Equation 12.5.15 and the result of Exercise 12.14 from Shapiro & Teukolsky, both illustrated in Figure 12.6.