Chapter 3: Duality Toolbox

MIT OpenCourseWare Lecture Notes

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Lecture 23

So far, we have discussed the thermal boundary theory on $\mathbb{R}^{d-1}$, which is dual to a black brane, *i.e.* horizon with topology $\mathbb{R}^{d-1}$. One can also consider the same boundary theory on S^{d-1} at a finite temperature. For a CFT on $\mathbb{R}^{d-1}$, T is the only scale, which provides the unit of energy scale. This implies that physics at all temperatures are the same, *i.e.* related by a scaling. For a CFT on S^{d-1}, which has a size R, then physics will depend on the dimensionless number RT, and can have nontrivial physics depending on T. Here are some important features:

1. A thermal gas is allowed in a thermal AdS. If we write in global AdS_{d+1}:

$$ds^2 = -\left(1+\frac{r^2}{R^2}\right)dt^2 + \frac{dr^2}{1+\frac{r^2}{R^2}} + r^2 d\Omega_{d-1}^2 \qquad (r \in (0,\infty)) \tag{1}$$

If we rotate the time to be Euclidean, $t \to -i\tau$, we must require a periodicity, $\tau \sim \tau + \beta$. The local proper size of τ-circle is $\sqrt{1+r^2/R^2}\beta \geq \beta$, which is perfectly defined, as long as β is not too small, say $\beta \sim \sqrt{\alpha'}$.

2. The black hole solution is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{d-1}^2 \tag{2}$$

where

$$f = 1 + \frac{r^2}{R^2} - \frac{\mu}{r^{d-2}} \tag{3}$$

where μ related to black hole mass. The horizon is located at $r = r_0$ where $f(r_0) = 0$. The temperature is given by

$$\beta = \frac{4\pi}{f'(r_0)} = \frac{4\pi r_0 R}{d r_0^2 + (d-2)R^2} \tag{4}$$

Notice here is a β_{max} for black hole solution, which corresponds to T_{min}. Furthermore, for any $T > T_{min}$, we can have two black hole solutions as shown in the picture below, where the small black hole has negative specific heat since $r_0 \downarrow \implies T \uparrow$ whereas the big black hole has positive specific heat since $r_0 \uparrow \implies T \uparrow$.

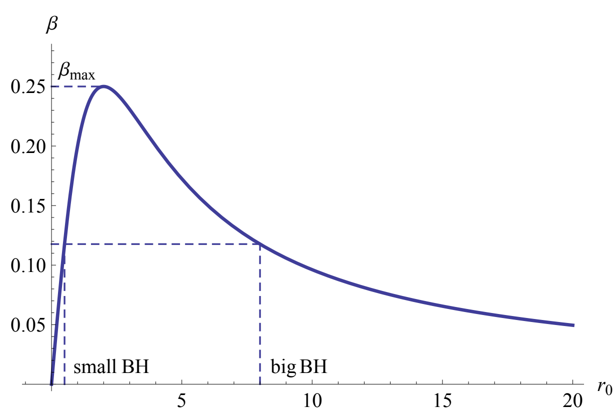

Figure 1: Temperature of different black holes

3. One thus finds: (i) $T < T_{min}$: only thermal AdS (TAdS); (ii) $T > T_{min}$: three possibilities: TAdS, big black hole (BBH) and small black hole (SBH). What does this mean? Indeed, three possible gravity solutions implies three possible phases for a CFT on S^{d-1}, which are determined by the minima of free energy. Recall $e^{-\beta F} = Z_{CFT} = Z_{gravity} = \int D\Phi e^{S_E[\Phi]} \sim e^{S_E[\Phi_c]}$, we can write the free energy of CFT in terms of classical gravity solution:

$$F = -\frac{1}{\beta} S_E[\Phi_c] \tag{5}$$

Thus we need to evaluate the Euclidean action for the three solutions and find the one with largest S_E. This also follows from the saddle-point approximation itself:

$$Z_{gravity} = e^{S_E}|_{TAdS} + e^{S_E}|_{BBH} + e^{S_E}|_{SBH} \tag{6}$$

where clearly the solution with largest S_E dominates.

4. We know $S_E \propto \frac{1}{G_N} \sim O(N^2)$. For TAdS, it is $O(N^0)$ from classical thermal graviton gas as it differs from global AdS only in global structure. For two black hole solutions, one can show that $S_E(BBH) > S_E(SBH) \sim O(N^2)$. Hence SBH will not dominate anyway. There exists a temperature $T_c (T_c > T_{min})$ such that (i) $T < T_c$, $S_E(BBH), S_E(SBH) < 0$, TAdS dominates; (ii) $T > T_c$, $S_E(BBH) > 0$ and dominates. This means the system experiences a *first order phase transition* at T_c since the free energy jumps from $O(N^0)$ to $O(N^2)$ (derivative of F is not continuous) to go from $TAdS$ to BBH, which is called Hawking-Page transition. To find the $S_E|_{BH}$, one may encounter divergence and need renormalization, which can be done by either subtracting covariant local counterterms at the boundary or subtracting the value of pure AdS. A short cut is

$$S = \frac{w_{d-1} r_0^{d-1}}{4G_N} \tag{7}$$

where w_{d-1} is the area of unit $(d-1)$-sphere and $r_0 = r_0(\beta)$. Integrate over

$$S = -\frac{\partial F}{\partial T} = -\frac{\partial F}{\partial r_0}\frac{\partial r_0}{\partial T} \tag{8}$$

to get

$$F = \frac{w_{d-1}}{16\pi G_N}\left(r_0^{d-2} - \frac{r_0^d}{R^2}\right) \tag{9}$$

where the integral constant is chosen such that $F = 0$ for $r_0 = 0$. Thus $F_{BH} > 0$ if $r_0 < R$ and $F_{BH} < 0$ if $r_0 > R$. The critical temperature is $\beta_c = \beta(r_0 = R) = \frac{2\pi R}{d-1}$.

5. Since physics only depends on RT, large R at fixed T is the same as large T at fixed R. So a CFT on $\mathbb{R}^{d-1}$ where $R \to \infty$ always corresponds to the high temperature phase, described by a black hole.

6. Physics reasons for Hawking-Page transitions. Consider $2N^2$ free harmonics oscillators with same frequency $\omega = 1$. It can be described by two matrices A and B, each containing N^2 harmonic oscillators, whose Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}Tr\dot{A}^2 + \frac{1}{2}Tr\dot{B}^2 - \frac{1}{2}TrA^2 - \frac{1}{2}TrB^2 \tag{10}$$

The spectrum density with respect to energy is roughly

$$D(E) \sim O(N^0) \qquad \text{for } E \sim O(N^0) \tag{11}$$

$$D(E) \sim e^{O(N^2)} \qquad \text{for } E \sim O(N^2) \tag{12}$$

For temperature $\beta \sim O(N^0)$, then the partition function

$$Z = \int dE e^{-\beta E} D(E) \tag{13}$$

naively contains most contribution from $E \sim O(N^0)$. However for $E \sim O(N^2)$, those contributions are

$$\int dE e^{-\#\beta N^2} e^{\# N^2} \tag{14}$$

which means when β is large, T is small, then $e^{-\beta E}$ dominates whereas when β is sufficiently small, T is large, such that $\log D(E) - \beta E > 0$, $O(N^2)$ states dominate and $Z \sim e^{O(N^2)}$. We thus expect a phase transition at some point going from $F \sim O(N^0)$ to $F \sim O(N^2)$ as we raise the temperature. This discussion can be generalized to a CFT, say $\mathcal{N} = 4$ SYM, on a sphere. Expand all fields in terms of harmonics on S^{d-1}, then we will have $O(N^2)$ harmonic oscillators, which (i) have different frequencies (ii) interact with each other (iii) form $SU(N)$ singlets as physical states. Nevertheless, the qualitative picture above survives. Finally, we conclude:

$$\begin{aligned} TAdS &\iff \text{states with } E \sim O(N^0) \\ BBH &\iff \text{states with } E \sim O(N^2) \end{aligned}$$

and Hawking-Page transition becomes first order in $N \to \infty$ limit. Sometimes, it is also called "deconfinement" transition.

3.2.2: FINITE CHEMICAL POTENTIAL

$\mathcal{N} = 4$ SYM has $SO(6)$ global symmetry. We can choose *e.g.* one of the $U(1)$ subgroup and turn on a chemical potential for that $U(1)$. In statistical physics, grand canonical ensemble is defined as

$$\Xi = Tr(e^{-\beta H - \beta\mu Q}) \tag{15}$$

where Q is the conserved charge for $U(1)$. In field theory, this corresponds to deforming the action by

$$\int d^4x \mu J^0 \tag{16}$$

On gravity side, we should then turn on the non-normalizable modes for the gauge field A_μ dual to J^μ, *i.e.*

$$\lim_{z\to 0} A_0(z, x) = \mu \tag{17}$$

The bulk geometry dual to the boundary theory at a finite chemical potential can then be found by solving Einstein-Maxwell system with boundary condition (17). Metric should still be normalizable. The ansatz is

$$ds^2 = \frac{R^2}{z^2}(-f(z)dt^2 + d\vec{x}^2) + \frac{R^2}{z^2} g(z)dz^2 \tag{18}$$

and

$$A_0(z) = h(z) \qquad h(0) = \mu \tag{19}$$

The solution is charged black hole in AdS which is characterized by a T and μ.

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