Physics 8.821: Problem Set 4 Solutions

1. **Inversions and Special Conformal Transformations**

    (a) Consider AdS_{d+1} in Poincare coordinates, with the AdS radius R set to 1:

    $$ds^2 = \frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2} \equiv \frac{dx^A dx_A}{z^2} \tag{1}$$

    where $x^A = (z, x^\mu)$ and we have introduced some new notation, $x^A x_A \equiv z^2 + x^\mu x_\mu$. Now I claim that the horribly violent inversion

    $$x^A \to \bar{x}^A = \frac{x^A}{x^B x_B} \tag{2}$$

    is actually an isometry of AdS, meaning that the metric in the barred coordinates takes precisely the same form as it did in the unbarred coordinates. This is not hard to verify, starting with the fact that

    $$d\bar{x}^A = \frac{1}{x^B x_B}\left(dx^A - 2\frac{x^A}{x^B x_B}x_C dx^C\right) \tag{3}$$

    It is now easy to show that

    $$ds^2 = \frac{d\bar{x}^A d\bar{x}_A}{\bar{z}^2} = \frac{(x^D x_D)^2}{z^2}\frac{1}{(x^B x_B)^2}\left(dx^A - 2\frac{x^A}{x^B x_B}x_C dx^C\right)\left(dx_A - 2\frac{x_A}{x^B x_B}x_C dx^C\right) \tag{4}$$

    Expanding out the bracket we obtain

    $$ds^2 = \frac{d\bar{x}^A d\bar{x}_A}{\bar{z}^2} = \frac{1}{z^2}\left(dx^A dx_A - 4\frac{(x_A dx^A)^2}{x^B x_B} + 4\frac{x_A x^A}{(x^B x_B)^2}(x_C dx^C)^2\right) = \frac{dx^A dx_A}{z^2} \tag{5}$$

    as claimed.

    (b) Perhaps the best way to show this is to consider the Jacobian of the coordinate transformation (2). We have the following relation

    $$\frac{\partial \bar{x}^A}{\partial x^B} = \frac{1}{x^C x_C}\left(\delta_B^A - 2\frac{x^A x_B}{x^B x_B}\right) \tag{6}$$

    Now consider the eigenvalues of this Jacobian. It is a $d+1$ dimensional matrix; to determine the eigenvalues consider hitting it with vectors y^A. There are d vectors $y_A^\perp$ for which $y_A^\perp x^A = 0$; for those we find that

    $$\frac{\partial \bar{x}^A}{\partial x^B}y_\perp^B = \frac{1}{x^C x_C}y_\perp^A, \tag{7}$$

i.e. that these eigenvalues of the Jacobian are *positive*. Now consider hitting it with the single vector $y_\parallel^A$ that is proportional to x^A, i.e. $y_\parallel^A = cx^A$; for that we have

$$\frac{\partial \bar{x}^A}{\partial x^B} y_\parallel^B = -\frac{1}{x^C x_C} y_\parallel^A, \tag{8}$$

i.e. there *one* negative eigenvalue. Thus the determinant of the Jacobian (which is the product of the eigenvalues) is *negative*.

Now to connect this inversion continuously to the identity (which obviously has positive Jacobian) we would somewhere need to pass through some transformation with Jacobian zero, which would not be invertible and so would not constitute a good coordinate transformation. Thus the inversion is not continuously connected to the identity.

(c) The space of points with $z = 0$ is mapped to itself under inversions. For the remaining coordinates x^μ we have

$$\bar{x}^\mu = \frac{x^\mu}{\eta_{\mu\nu} x^\mu x^\nu} . \tag{9}$$

Let us work out the transformation of the Minkowski metric under this coordinate transformation; we have for the Jacobian

$$\frac{\partial \bar{x}^\mu}{\partial x^\nu} = \frac{1}{x^\sigma x_\sigma} \left(\delta^\mu_\nu - 2 \frac{x^\mu x_\nu}{x^\sigma x_\sigma} \right) \tag{10}$$

Now direct computation shows

$$\bar{g}^{\mu\sigma} \equiv \eta^{\nu\alpha} \frac{\partial \bar{x}^\mu}{\partial x^\nu} \frac{\partial \bar{x}^\sigma}{\partial x^\alpha} = \frac{1}{(x^\alpha x_\alpha)^2} \eta^{\mu\sigma}, \tag{11}$$

where in the last equality we have expanded out the matrix product using (10) and combined terms; all terms not proportional to the metric cancel. Thus this is indeed a conformal transformation with conformal factor $(x^\alpha x_\alpha)^2$.

(d) Define the vector $b^A = (0, b^\mu)$, and denote by $\bar{x}^A$ the coordinates after the operation S has been performed. From the definition of S it is clear that the relation between $\bar{x}^A$ and the original coordinates is

$$\frac{\bar{x}^A}{\bar{x}^B \bar{x}_B} = \frac{x^A}{x^B x_B} + b^A \tag{12}$$

We would now like to solve explicitly for $\bar{x}^A$; to do this we contract $\bar{x}^A$ with itself to find an expression for $\bar{x}^A \bar{x}_A$,

$$\frac{1}{\bar{x}^A \bar{x}_A} = \frac{1}{x^A x_A} + 2 \frac{b^B x_B}{x^A x_A} + b^A b_A \tag{13}$$

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Putting this into the expression above and solving for $\bar{x}^A$ we find

$$\bar{x}^A = \frac{x^A + b^A x^C x_C}{1 + 2b^C x_C + (b^C b_C)(x^D x_D)} \tag{14}$$

This is the explicit expression for S on any point in the interior of AdS; it is easy to see that for $b \to 0$ this reduces to the identity. Note that our notation hides the fact that it is crucial that b^A point only along the Minkowski space directions; it cannot have a component along the z direction or this operation fails to be an isometry.

(e) Taking $z \to 0$ in (14) we find simply that on the Minkowski coordinates it reduces to

$$\bar{x}^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2} \tag{15}$$

where all dot products are taken with respect to the normal Minkowski metric; this is precisely the definition of a special conformal transformation.

2. **Geodesic motion in AdS**

We will work with AdS in global coordinates,

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega_{d-1}^2 \tag{16}$$

I have set the AdS radius $R \to 1$, as it drops out if we express everything in terms of the ratio to $\frac{E}{m}$.

For radial motion there is a single useful Killing vector, that for time translation; it tells us that

$$(1 + r^2)\dot{t} \equiv \mathcal{E} = \frac{E}{m} \tag{17}$$

is a constant of the motion (where an overdot denotes a derivative with respect to the proper time, $\cdot \equiv \frac{d}{d\tau}$). Note that to obtain the second equality we have assumed that the "energy" that the problem is referring to is that measured by a proper observer sitting stationary at $r = 0$.

We now use the normalization of the four velocity $\dot{x}^M \dot{x}_M = -1$ to write down the equation for radial motion,

$$\dot{r}^2 + (1 + r^2) = \mathcal{E}^2 \tag{18}$$

(a) The maximum radius r_m is at the turning point when $\dot{r} = 0$, i.e. when

$$r_{max} = \sqrt{\frac{E^2}{m^2} - 1} \ . \tag{19}$$

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(b) From (17) and (18) we find an expression for $\frac{dt}{dr}$,

$$\frac{dt}{dr} = \frac{\mathcal{E}}{(1+r^2)\sqrt{\mathcal{E}^2 - (1+r^2)}} \tag{20}$$

Integrating this over a round-trip we find

$$\Delta t = 2\int_0^{r_{max}} dr \frac{\mathcal{E}}{(1+r^2)\sqrt{\mathcal{E}^2 - (1+r^2)}} = 2\mathrm{ArcTan}\left(\frac{\mathcal{E}r}{\sqrt{\mathcal{E}^2 - 1 - r^2}}\right)\Bigg|_0^{r_{max}} = \pi \tag{21}$$

Thus the time taken is π.

3. **Geodesic distance**

We work with the AdS hyperboloid of unit radius

$$X_{-1}^2 + X_0^2 - \sum_{i=1}^d X_i^2 = 1 \ . \tag{22}$$

(a) Note that by an $SO(D,2)$ transformation we can put any two timelike separated points that are connected by a geodesic completely in the (X_{-1}, X_0) plane with all other coordinates 0, i.e. we may write A and B as

$$X_A = (1, 0, 0, \cdots) \qquad X_B = (a, \sqrt{1-a^2}, 0, \cdots) \tag{23}$$

Now the cross section of the hyperboloid with the (X_{-1}, X_0) plane is actually just a circle that is sitting in a $\mathbb{R}^2$ with *negative*-definite metric. Thus $s(A, B)$ is the distance *along* the circle, whereas $\sigma(A, B)$ is the (negative square of) the chordal distance between the two points of the circle. We then find from elementary geometry that

$$\cos(s(A, B)) = a \tag{24}$$

and

$$-\sigma(A, B) = (1-a)^2 + (1-a^2) = 2 - 2a \tag{25}$$

from which it trivially follows that

$$\cos(s(A, B)) = 1 + \frac{1}{2}\sigma(A, B) \tag{26}$$

(b) This is rather similar, except that for a spacelike geodesic we use one of the spacelike directions instead, i.e.

$$X_A = (1, 0, 0, \cdots) \qquad X_B = (a, 0, \sqrt{a^2 - 1}, \cdots) \tag{27}$$

Now the intersection of the hyperboloid with the X_{-1}, X_1 plane is a hyperbola, and to compute the proper distance along it we actually need to perform an integral

$$s(A, B) = \int_1^a da' \sqrt{-\left(\frac{dX_{-1}}{da'}\right)^2 + \left(\frac{dX_0}{da'}\right)^2} = \int_1^a da' \sqrt{\frac{1}{a'^2 - 1}} = \mathrm{ArcCosh(a)} \tag{28}$$

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The direct distance is

$$\sigma(A, B) = -(1 - a)^2 + (a^2 - 1) = 2a - 2 \tag{29}$$

and so we find the relation

$$\cosh(s(A, B)) = 1 + \frac{1}{2}\sigma(A, B) . \tag{30}$$

(c) Note that in our construction of timelike geodesics above, we seemed to find that all timelike geodesics would eventually return to their starting point after a time 2π, as they moved along a circle in the (X_0, X_{-1}) plane. In particular, we could not make $\sigma(A, B)$ more negative than -4, as is clear from (26).

Thus it seems that by following timelike geodesics we cannot actually reach points that are separated by a large timelike distance on the embedding space, even though they clearly exist on the hyperboloid (22).

4. **IR/UV connection in global AdS**

Let us compare the metric in global AdS

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega_{d-1}^2 \tag{31}$$

with that of the Poincare patch

$$ds^2 = r^2(-dt^2 + d\vec{x}^2) + \frac{dr^2}{r^2} \tag{32}$$

Note that r is not the same coordinate in these two equations, but in both cases the AdS boundary is towards $r \to \infty$ and the interior is towards $r \to 0$. The topology of the boundary is different; in the Poincare case the boundary is a flat slice of Minkowski space, but in the global AdS case the boundary is a sphere times a flat time direction. In both cases t measures the time on the boundary theory (although again it is not the same coordinate in these two equations).

There are some other clear differences between these two metrics; note that in the Poincare patch $g_{tt} \to 0$ as we approach the interior; for sufficiently small r we can have an arbitrarily small g_{tt}. Thus it seems there is some sort of a horizon at $r = 0$, and if we consider a local excitation at small r it can have an arbitrarily small energy when measured in terms of the time coordinate t. If we think about it, this is exactly what we expect from a CFT on a flat manifold; by the definition of a CFT, there should be excitations at arbitrarily low energy. Indeed one can compute (e.g.) the correlation functions of various operators in real-time; the answers that one gets have spectral density at arbitrarily low energies, provided the momentum $\vec{k}$ is also taken to be very low. In other words, we have a continuous spectrum.

The case is different in global AdS. Here we see that we ultimately end up with $g_{tt} \to 1$ as $r \to 0$. Thus it seems that we cannot get arbitrarily small energies; they eventually

saturate even if we go all the way into the interior. This is what we expect for a CFT on a sphere; various boundary conditions on the sphere mean that the momentum cannot be taken arbitrarily low, and so there should be a lowest energy state with $E \sim \frac{1}{l_s}$, where l_s is the radius of the sphere. Indeed one can compute (e.g.) the same correlation functions of operators in real-time; one finds instead a discrete spectrum of energies, with spacing (when measured in terms of boundary time t) in units of the radius of of the boundary sphere.

5. **Holographic principle: bound on the number of degrees of freedom**

(a) We know that the number of degrees of freedoms should be bounded by $\mathbb{N} = \frac{A}{4G_N}$. Now for the five dimensional theory we have

$$G_N^{(5)} = \frac{G_N^{(10)}}{V_5} \sim \frac{g_s^2 \alpha'^4}{R^5} \tag{33}$$

where we have used the fact that the radius of the 5-sphere is equal to the AdS radius R. Now evaluating $\mathbb{N}$ we find

$$\mathbb{N} \sim \frac{A}{4G_N} \sim \frac{R^3 L^3}{\delta^3} \frac{R^5}{g_s^2 \alpha'^4} \sim \frac{L^3}{\delta^3} \frac{R^8}{\alpha'^4} \frac{1}{g_s^2} \tag{34}$$

Now to express this in terms of field theory objects we use the fact that $\frac{R^8}{\alpha'^4} = 4\pi\lambda \sim g_{YM}^2 N$ and $g_s = g_{YM}^2$ to find

$$\mathbb{N} \sim \frac{L^3}{\delta^3} N^2 \tag{35}$$

This is precisely what we expect for a continuum large N gauge theory.

(b) Not much is different here; we can use for example the metric

$$\frac{ds^2}{R^2} = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-1}^2 \tag{36}$$

and repeat precisely the same steps as above, except that now the factor L^3 in the expression for the area is replaced by a dimensionless number related to the size of the unit 3-sphere. One finds

$$\mathbb{N} \sim \frac{1}{\delta^3} N^2 \tag{37}$$

which is what one expects for a CFT living on a unit sphere with δ a dimensionless cutoff of some sort.

(c) From (36) one finds that the volume inside a region of (large) radius r_c is

$$V(r_c) \sim R^4 \int^{r_c} dr \, r^2 \sim R^4 r_c^3 \tag{38}$$

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whereas the area is $A(r_c) \sim R^3 r_c^3$ (Note that the integral over dr in the volume contains an extra factor of $\frac{1}{r}$, meaning that the volume and the area have the same r-dependence at large r.) Thus the ratio $\frac{V}{AR}$ is indeed finite at large r_c.

However–and this is critical – the dependence on the AdS *radius* R is different between the volume and area. The precise agreement of the N^2 factor found above depends on the precise power of R appearing in the expression; it is the area that correctly reproduces the N^2, and not the volume, and thus this is still rather impressive.

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