Physics 8.821: Problem Set 2 Solutions

1. **Hawking temperature from analytic continuation**

   We are studying a set of black hole metrics of the form

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + ... \tag{1}$$

   where $f(r)$ and $h(r)$ have zeroes at the horizon $r = r_0$ and everything else is regular there. Now consider switching to a coordinate system with a reparametrized radial coordinate ρ which is the proper distance from the horizon, i.e.

$$d\rho = \frac{1}{\sqrt{h(r)}}dr \qquad \rho(r = r_0) = 0 \tag{2}$$

   Integrating this in a small neighbourhood of $r = r_0$ we find the relation between ρ and r close to the horizon

$$\rho = \frac{2}{\sqrt{h'(r_0)}}\sqrt{r - r_0}\ . \tag{3}$$

   Now we note that as $f(r)$ has a zero at $r = r_0$ we can write it in terms of the ρ coordinate as

$$f(r \to r_0) \sim f'(r_0)(r - r_0) = \frac{1}{4}\rho^2 h'(r_0)f'(r_0). \tag{4}$$

   Now consider analytically continuing to Euclidean time τ via $t = -i\tau$. Putting all of these ingredients together, the metric close to the horizon becomes

$$ds^2 = \rho^2\left(\frac{\sqrt{h'(r_0)f'(r_0)}}{2}d\tau\right)^2 + d\rho^2 + ... \tag{5}$$

   Now staring at this metric we realize that it is actually very familiar; this is the metric of *flat 2D Euclidean space in polar coordinates* $ds^2 = d\rho^2 + \rho^2 d\theta^2$, with the radial coordinate being ρ and the role of the angle θ being played by the coordinate $\sqrt{h'(r_0)f'(r_0)}\tau/2$. However, we know from our familiarity with flat space that θ must be a periodic coordinate with the identification $\theta \sim \theta + 2\pi$, as otherwise the space has a conical singularity at the origin. This means that τ must be similarly periodic,

$$\tau \sim \tau + \frac{4\pi}{\sqrt{h'(r_0)f'(r_0)}}\ . \tag{6}$$

   But we know how to interpret a periodicity in Euclidean time; it means that we are studying an ensemble at a finite temperature T_H, where the period is the inverse temperature, i.e.

$$T_H = \frac{\sqrt{h'(r_0)f'(r_0)}}{4\pi}, \tag{7}$$

   as claimed in the problem set.

2. **Kerr-Newman metric**

(a) The horizon is essentially defined to be the outermost zero of $\Delta(r)$; setting $\Delta(r) = 0$ we find the two solutions

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2} \tag{8}$$

of which the outer root r_+ is the location of the horizon. To find the area of the horizon we write down the induced metric on a sphere of constant r,

$$ds^2 = \gamma_{ab} dx^a dx^b = \frac{\Sigma}{\rho^2} \sin^2 d\phi^2 + \rho^2 d\theta^2 \tag{9}$$

The area is $A = \int d\theta d\phi \sqrt{\gamma}$ which trivially evaluates to

$$A = 4\pi (r_+^2 + a_+^2), \tag{10}$$

where we have used the fact that $\Delta(r_+) = 0$.
From the formula for T_H in (7) and the explicit form of the metric we find

$$T_H = \frac{1}{4\pi} \frac{\Delta'(r_+)}{\sqrt{\Sigma}} = \frac{2(r_+ - M)}{A} \tag{11}$$

Finally, we turn to the angular velocity Ω, which may be slightly unfamiliar. In Section 6.6 of [1] the meaning of the angular velocity of the event horizon is discussed. It can be defined as *the minimum angular velocity of a particle at the horizon.* The limiting trajectory of a massive particle is equal to that of a photon; setting $ds^2 = 0$ on such a trajectory and attempting to find the *minimum* angular velocity we set motion in all other directions to 0 to find at the horizon

$$\frac{d\phi}{dt} = \omega(r_+) \tag{12}$$

thus we have

$$\Omega = \omega(r_+) = \frac{a(r_+^2 + a^2)}{\Sigma} = \frac{4\pi a}{A} \ . \tag{13}$$

(b) To saturate the inequality and arrive at an extremal black hole we set $M = a^2 + Q^2$. Note that at the extremal point the two roots of $\Delta(r)$ in (8) are *merging*; precisely at extremality we obtain a double zero of $\Delta(r)$ at the value $r_0 = r_+ = r_- = M$. It is clear from here that the temperature (11) is going to 0 at the extremal point.
However the entropy S of the black hole is

$$S = \frac{A}{4} = \pi(M^2 + a^2) \tag{14}$$

at extremality; essentially the surface area of the horizon and thus the entropy is nonzero even if the temperature is zero.[1]

[1] But by holography presumably most black holes are dual to field theory states in some sense; how can a field theory state have a finite entropy even if it has a vanishing temperature? Discuss.

(c) The extremal case with zero angular momentum $a = 0$ (and thus $M = Q$) is rather simple:

$$ds^2 = -\frac{(r-Q)^2}{r^2}dt^2 + \frac{r^2}{(r-Q)^2}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{15}$$

To compute the distance to the horizon (located at $r = Q$) we calculate the integral

$$\int^Q dr\sqrt{g_{rr}} = \int^Q \frac{r}{r-Q}dr, \tag{16}$$

which is clearly logarithmically divergent as we approach the horizon. Thus at extremality the horizon is *infinitely far away*; what has appeared to fill this space? To answer this cleanly let us zoom in to the near-horizon limit by considering the following *scaling limit*

$$(r-Q) = \lambda\zeta \qquad t = \frac{\tau}{\lambda} \tag{17}$$

in which we take $\lambda \to 0$ while holding the new coordinates ζ, τ fixed. Note that this has the effect of focusing our attention near the horizon at $r = Q$. It is interesting to note that in some sense we are also focusing our attention to variation on time scales that are very long in terms of the original t coordinate; thus the near-horizon limit is in some sense also a small-energy limit, a fact whose importance cannot be overstated.

In any case, we find for the metric

$$ds^2 = -\frac{\zeta^2}{Q^2}d\tau^2 + \frac{Q^2}{\zeta^2}d\zeta^2 + Q^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{18}$$

up to finite λ corrections. This is the metric of $\mathrm{AdS}_2 \times S^2$, where the AdS_2 has AdS radius Q (which happens to coincide with the radius of the S^2.)

3. (a) The first law of black hole mechanics (which we are trying to verify) in this case reads as

$$dM = TdS + \Phi dQ + \Omega dJ \tag{19}$$

where the electric potential $\Phi = \frac{4\pi Q r_+}{A}$ and all other quantities are defined above. Verifying this is a straightforward exercise in writing out differentials. I found it most convenient to write everything in terms of da, dM and dQ, in which case we find

$$\begin{aligned} dS &= \frac{4\pi(r_+ - M)}{A}\left(r_+ dM + a da + \frac{r_+}{r_+ - M}\left(MdM - a da - QdQ\right)\right) && (20) \\ dJ &= adM + Mda && (21) \end{aligned}$$

Assembling things together in the combination on the right-hand side of (19) it is now easy to see that everything cancels out except for a term in dM:

$$TdS + \Phi dQ + \Omega dJ = \frac{4\pi}{A}\left(r_+^2 + a^2\right)dM = dM, \tag{22}$$

just as expected. The rather nontrivial structure of the cancellations should give you some respect for the rich structure that is hidden inside solutions such as the Kerr metric.

(b) Consider first the black hole with mass M and angular momentum J. Then by the usual formula we have for the initial entropy of the black hole

$$S = \pi(r_+^2 + a^2) = 2\pi\left(M^2 + M\sqrt{M^2 - a^2}\right) \tag{23}$$

where $a \equiv \frac{J}{M}$ as above. Now imagine some classical process happens to cause the black hole to lose all of its angular momentum; at the end it has a different mass M_f and a final entropy S_f which is just

$$S_f = 4\pi M_f^2 \tag{24}$$

However by the Second Law we know that in any classical process $S_f \geq S_i$, meaning that we have

$$\left(\frac{M_f}{M}\right)^2 \geq \frac{1}{2}\left(1 + \sqrt{1 - \frac{a^2}{M^2}}\right) \tag{25}$$

This is a fundamental inequality on the ratio of initial and final masses; we see that the final mass cannot drop arbitrarily low. The largest fractional change occurs when the black hole was initially extremal, i.e. $a = M$, in which case we find

$$\left(\frac{M_f}{M}\right)^2 \geq \frac{1}{2}, \tag{26}$$

corresponding to a fractional change of $1 - \frac{1}{\sqrt{2}} = 29\%$.

(c) Let us consider the various inequalities; the black hole has charge Q and mass M, and the particle we are dropping into it has charge q and mass m. We require that

$$1 < \frac{q}{m} < \frac{M}{Q} \tag{27}$$

Clearly this is a tighter and tighter window as we get closer and closer to extremality of the black hole, i.e. taking Q to M. Let us now see if we can ever push the black hole into extremality by dropping in a particle. I will in fact assume that we pick the most dangerous (i.e. highest $\frac{q}{m}$ ratio) particle that we can subject to the constraint (27). This particle has charge

$$q = \frac{M}{Q}m \tag{28}$$

After we drop it in, the black hole has charge $Q' = Q + q$ and $M' = M + m$, and we find in this most dangerous case

$$\frac{Q'}{M'} = \frac{Q}{M}\frac{1 + \frac{M^2}{Q^2}\frac{m}{M}}{1 + \frac{m}{M}} \tag{29}$$

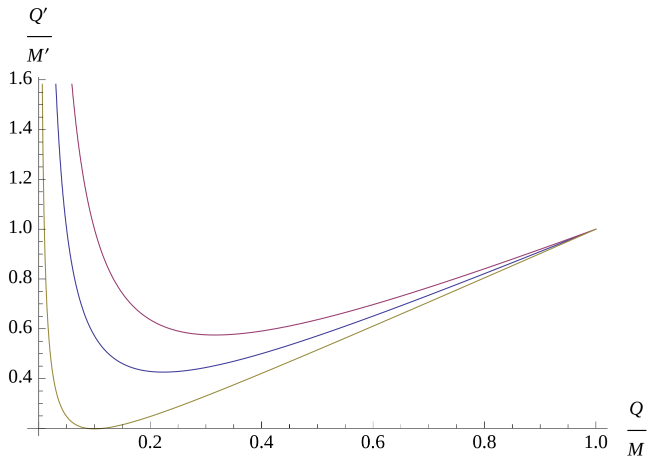

Figure 1: Plot of (29) for various (small) values of $\frac{m}{M}$; any $\frac{Q}{M} < 1$ will take us to $\frac{Q'}{M'} < 1$, except for spurious solutions at small $\frac{Q}{M}$, discussed below.

Have we reached extremality? To get some insight into this let us assume we were close to extremality to begin with and expand $\frac{Q}{M}$ near 1,

$$\frac{Q'}{M'} = 1 - \left(1 - \frac{Q}{M}\right) \frac{1 - \frac{m}{M}}{1 + \frac{m}{M}} + \dots \tag{30}$$

As any sensible particle should have $\frac{m}{M} < 1$ the second term has a definite sign, and we see that provided $\frac{Q}{M} < 1$ then we also have $\frac{Q'}{M'} < 1$; thus we cannot reach extremality in any single step (and thus in any finite number of steps, since this proof makes no assumption about how close $\frac{Q}{M}$ was to 1 when we started).

However staring at (29) (or Figure 1) we may be uneasy; after all, I expanded it near $\frac{Q}{M} \sim 1$, but the condition $\frac{Q'}{M'} = 1$ has multiple solutions; in fact, it looks like if $\frac{Q}{M} \ll 1$ we *could* reach extremality in one step. Indeed, if we drop in a particle whose mass is $m = Q$, it appears we could bound to extremality in one giant leap while still remaining within the bound (27). Does this make sense?

It should seem physically clear that it does not, but let us carefully understand why. In our construction such a particle mass with $m = Q$ will have charge $q = M$ from (28). Thus it has a charge that is (in Planck units) equal to the mass of a *black hole*. This is not an elementary particle; it should be considered a gravitating object in its own right, and is actually a naked singularity in general relativity. Of course we can add a naked singularity to an near-extremal black hole to get an extremal black hole; but that is not the physics question that we are answering. The asymmetry betwee n elementary particles and black holes is evident from the very fact thatwe are studying objects with $q > m$, which means that they cannot be large enough to be gravitating (a criteria which boils down to $m, q \ll 1$ in the Planck units we have.)

4. **A gas of radiation and maximal entropy bound**

This problem involves raising many things to many awkward powers. To avoid repeatedly writing things like $\left(\frac{4\pi}{3}\right)^{\frac{7}{4}}$ that do not help understanding I am going to completely neglect all order 1 constants throughout (i.e. $2\pi = 1$). I will also set $\hbar = k_B = c = 1$. I will *not* set G_N to 1, although in these units I do have $G_N = l_P^2 = m_P^{-2}$

(a) From scale invariance we have for the energy density of the system

$$\rho = Z\alpha T^4 \tag{31}$$

where α is an $O(1)$ dimensionless constant that will be set to 1 from now on. Thus the total energy is

$$E \sim ZR^3T^4 \ . \tag{32}$$

From elementary thermodynamics we have for the total entropy

$$S = \frac{4}{3}\frac{E}{T} \ . \tag{33}$$

Now eliminating T using (32) we have

$$S_{gas} \sim Z^{\frac{1}{4}}(ER)^{\frac{3}{4}} \ . \tag{34}$$

(b) The system is about to form a black hole if the radius of the box is equal to the Schwarzschild radius of the energy that the box contains, i.e.

$$R = R_s(E) \sim G_N E \sim l_P^2 E \tag{35}$$

The entropy of such a black hole is

$$S_{bh} \sim \frac{R^2}{G_N} \sim (El_P)^2 \tag{36}$$

Let us now rewrite the entropy of the gas using (35) to write E in terms of R at threshold:

$$S_{gas} \sim Z^{\frac{1}{4}}\left(\frac{R}{l_P}\right)^{\frac{3}{2}} \tag{37}$$

Thus for the ratio we find

$$\frac{S_{gas}}{S_{bh}} \sim Z^{\frac{1}{4}}\sqrt{\frac{l_P}{R}} \tag{38}$$

Indeed, the ratio of the two entropies is very small at reasonable box sizes $R \gg l_P$. Thus it seems rather difficult to violate the entropy bound in this way.

(c) In this case the box will form a black hole once the energy stored in the box is equal to the energy of a Schwarzschild black hole of the same radius, i.e.

$$G_N R_M^3 \rho \sim R_M \qquad \to \qquad R_M^2 \sim \frac{1}{l_P^2 \rho} \ . \tag{39}$$

In this case we rewrite everything in terms of the energy density of the box, to find

$$\frac{S_{gas}}{S_{bh}} \sim Z^{\frac{1}{4}} \left(\rho l_P^4\right)^{\frac{1}{4}} \tag{40}$$

Thus indeed provided that the energy density $\rho \ll l_P^{-4} \sim m_P^4$ we find that the ratio is again miniscule.

(d) We can violate the bound by taking

$$Z \sim \left(\frac{R}{l_P}\right)^2 \tag{41}$$

in the first case (38) or

$$Z \sim \left(\frac{m_P^4}{\rho}\right) \tag{42}$$

in the second case (40). This number of species is of course utterly ridiculous. As an example, let us see how many photon species we would need to violate the bound using photons at room temperature $T = 300K$. This corresponds to an energy density of $(10^{-3}\text{eV})^4$; plugging this into (42) we need around $Z \sim 10^{96}$ species to saturate the bound.

5. **Wilson loop in the large N limit**

(a) In this problem we seek to establish a relation between $W_F(C)$ and $W_A(C)$, the Wilson loops in the fundamental and adjoint respectively. I would like to provide a somewhat low-brow treatment of this that involves the bare minimum of math. Note that for both the adjoint and fundamental we have

$$W(C) = \langle \mathrm{Tr}\, U(C) \rangle, \tag{43}$$

where $U(C)$ is an element of $U(N)$ that is then averaged over various gauge field configurations, and the difference between the two lies solely in which representation the trace is taken in.

Let us then back up for a minute and consider what it *means* to take a trace. Imagine we have some matrix M that acts on some vector space V. To take a trace we construct a basis for the space $\{v^{(i)}\}$. We should probably normalize the $\{v^{(i)}\}$ in some nice way with an inner product,

$$\langle v^{(i)}, v^{(j)} \rangle = \delta^{ij} \tag{44}$$

We then define the trace to be

$$\mathrm{Tr}(M) \equiv \sum_i \langle v^{(i)}, M v^{(i)} \rangle \tag{45}$$

We now go through this procedure for our case. First consider the fundamental representation of $U(N)$. Here we have matrices U_{ab} that act on a vector space that is just $\mathbb{C}^N$; thus the vectors $v^{(i)}$ defined above can be conveniently taken to be unit vectors, i.e. in components $v^{(i)}_b = \delta^i_b$. The notation is clear, hopefully: (i) labels which unit vector I am talking about, and b is which component. They both run from 1 to N). In this case we find that the trace is

$$\mathrm{Tr}_F(U) = \sum_a U_{aa} \tag{46}$$

Now let us repeat the same exercise for the adjoint. Here $U(N)$ acts in a more interesting way; given an $N \times N$ complex matrix A_{ab} and an element $U \in U(N)$, the action of U on A is given by

$$(UA)_{ab} \equiv U^*_{ea} U_{fb} A_{ef} \tag{47}$$

Thus our vector space V is the set of $N \times N$ matrices. Now we use this definition to figure out the trace. First, we take our inner product on this space to be the normal trace on matrices, i.e.

$$\langle A, B \rangle \equiv \mathrm{Tr}\, A^\dagger B = \sum_{ab} A^*_{ba} A_{ba} \tag{48}$$

This seems reasonable. Now we need a basis; I pick it to be

$$V^{(ij)}_{ab} = \delta^i_a \delta^j_b \tag{49}$$

where ab labels components and (ij) has a total of N^2 labels that tell me which basis element I am talking about. Now I use the definition (45) to take the trace of some element $U \in U(N)$:

$$\begin{aligned} \mathrm{Tr}_A(U) &= \sum_{(ij)} \langle V^{(ij)}, U V^{(ij)} \rangle & (50) \\ &= \sum_{abij} \delta^i_b \delta^j_a U^*_{eb} U_{fa} \delta^i_e \delta^j_f = \sum_{ef} U^*_{ee} U_{ff} & (51) \end{aligned}$$

Thus we conclude that for any $U(N)$ element U we have

$$\mathrm{Tr}_A(U) = \mathrm{Tr}_F(U) \mathrm{Tr}_F(U^\dagger) \tag{52}$$

In the large N limit this works for $SU(N)$ as well. If we were at finite N and were more careful presumably we would need to worry about the fact that $SU(N)$

actually acts only on the space of $N \times N$ matrices with fixed determinant, there are only $N^2 - 1$ of them, we should constrain the basis (49) somehow, etc. etc. In our context we have then

$$W_A(C) = \langle \mathrm{Tr}_A U \rangle = \langle \mathrm{Tr}_F U \mathrm{Tr}_F U^\dagger \rangle \tag{53}$$

where U is some path-ordered exponential of A_μ,

$$U = P \exp \left(ig \oint_C dx^\mu A_\mu(x) \right) \ . \tag{54}$$

Up till now we have only done mathematics; let us now apply some properties of large-N gauge theories. We are studying the product of two single-trace operators. Each of these operators U is nonlocal; however this is not terribly important, as by expanding the integrals they can be written as a series of integrals over correlation functions of local operators. The key fact here is that at large-N a product of single-trace operators factorizes, i.e.

$$\langle \mathrm{Tr}_F U \mathrm{Tr}_F U^\dagger \rangle = \langle \mathrm{Tr}_F U \rangle \langle \mathrm{Tr}_F U^\dagger \rangle \tag{55}$$

With (53) we thus find

$$W_A(C) = W_F(C) W_F(C)^\dagger, \tag{56}$$

with $W_F(C) = \langle \mathrm{Tr}_F U \rangle$. This is the relation we seek.

(b) In a confining theory a Wilson loop behaves (at large distances) like the exponential of the *area* of the loop, i.e.

$$W(C) \sim \exp(-T A(C)), \tag{57}$$

where T a quantity with units of mass squared that can be thought of as the tension of a confining string connecting two infinitely heavy quarks. If we call the tension of the string between fundamental quarks T_F and that between *adjoint* quarks T_A, then (56) tells us that in such a confining theory,

$$T_A = 2T_F \tag{58}$$

Let us now try to understand this intuitively. In terms of representation theory we can imagine an adjoint quark as one fundamental q and one anti-fundamental quark $\bar{q}$. Consider two adjoint quarks; in calculating the force between them one can imagine one fundamental string stretching from fundamental to anti-fundamental, q_1 to $\bar{q}_2$ and another fundamental string from anti-fundamental to fundamental, $\bar{q}_1$ to q_2. We do not expect any strings to connect a fundamental quark with a fundamental quark, as there is no confining force trying to bind them into a color-singlet meson.[2] Thus we have two strings and the net attractive force (and thus the net string tension) between two adjoint quarks is twice that between two fundamental quarks, leading to (58).

[2] Of course if we had N fundamentals confinement would try to bind them into a baryon, and so there is *some* force between any two fundamentals; however it seems reasonable to expect this to be suppressed in the large-N limit, a fact I will not try to prove.

References

[1] S. M. Carroll, "An Introduction to General Relativity: Spacetime and Geometry," Addison Wesley, San Francisco 2004.