The bose liquid - bosons with short ranged repulsive interactions.

Eg: Liquid He-4.

Model by the Hamiltonian

$$H = \int d^3x \; \hat{\psi}^\dagger(x)\left(-\frac{\nabla^2}{2m} - \mu\right)\hat{\psi}(x) + u\, \hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\psi}(x)$$

1^{st} term = kinetic energy

2^{nd} term = chemical potential

3^{rd} term = short ranged repulsive interaction.

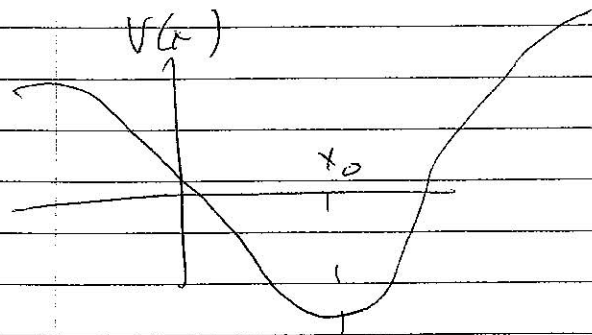
First analyse in an approximate way.

- assume that operators $\hat{\psi}$, $\hat{\psi}^\dagger$ may be simply treated as c-numbers & minimize H to find optimal value of ψ, $\psi^\dagger$.

Later one can analyse ~~the~~ role of small quantum fluctuations about the "classical minimum".

[Simpler example: Single ptcle QM with a Hamiltonian

$$H = p^2/2m + V(x)$$



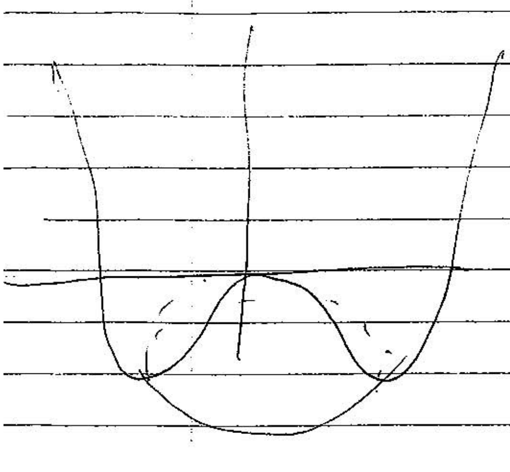
As an approximate solution, first find classical minimum x_0, then study QM of small oscillations about classical minimum

– qualitatively correct picture of ground state wavefn]

$$H_{\text{classical}} = \int d^3x \;\; \psi^* \left(-\nabla^2/2m - \mu\right)\psi + u|\psi|^4 .$$

Clearly $H_{cl.}$ minimized when $\psi(x) = \psi_0$ independent of x.

$$E/V = -\mu|\psi_0|^2 + u|\psi_0|^4$$



$\Rightarrow$ (Mexican hat potential).

Minimized by $\psi_0 = |\psi_0| e^{i\theta}$

$|\psi_0|^2 = \dfrac{+\mu}{2u}$, θ arbitrary (if $\mu > 0$)

This is an example of the phenomenon of spontaneously broken symmetry. To understand it better, first understand the symmetry properties of the system.

Clearly the Hamiltonian is invariant under translation/ spatial rotation, etc.

It is also invariant under
$$\hat{\varphi}(x) \to e^{i\alpha}\hat{\varphi}(x)$$
$$\hat{\varphi}^{+}(x) \to e^{-i\alpha}\hat{\varphi}^{+}(x).$$

with α independent of x.

Clearly this is because all the terms in the action have equal # of $\hat{\varphi}$'s and $\hat{\varphi}^{+}$'s.

(Eg: There are no terms like $\hat{\varphi}^{+2}\hat{\varphi}$, etc).

Thus this "U(1) phase rotation" symmetry is tied to conservation of total particle #

(i.e each term in the Hamiltonian involves no change in total particle # as it has equal # of $\hat{\varphi}$'s and φ^{+}'s).

In the classical ground state for $\mu > 0$, the system has a choice of picking any minimum at the bottom of the Mexican hat potential.

But any such minimum is not invariant under $\psi \to e^{i\alpha}\psi$; rather it transforms into a different equivalent minimum $\Rightarrow$ any choice of θ in $\psi_0 = |\psi_0| e^{i\theta}$ breaks the phase rotation symmetry.

Note that for any choice, θ is independent of x i.e in any one of these classical ground states, if you know θ in one region of space, you know θ arbitrarily far away - this is known as "long range order" (measurements in one region of space reveal some information about what is happening even very far away)

Small harmonic fluctuations about the classical minimum

Write $\hat{\psi}(x) = \psi_0 + \hat{\phi}(x)$

$\hat{\psi}^\dagger(x) = \psi_0 + \hat{\phi}^\dagger(x)$

(assume ψ_0 real for concreteness)

$$[\phi(x), \phi^\dagger(x')] = \delta(x-x').$$

Expand Hamiltonian to 2^{nd} order in $\hat{\phi}$:

$$H \approx E_{cl}^{gd} + \int d^dx \quad \hat{\phi}^\dagger(x)\left(-\nabla^2/_{2m}\right)\hat{\phi}(x) - \mu\hat{\phi}^\dagger\hat{\phi}$$
$$+ u\left(4\psi_0^2 \phi^\dagger\phi + \psi_0^2\left(\phi^2 + \phi^{\dagger 2}\right)\right)$$

E_{cl}^{gd} = Classical ground state energy

$$= -\mu\psi_0^2 + u\psi_0^4 = -\mu^2/_{2u} + \frac{\mu^2}{4u} = -\mu^2/_{4u}$$

$$\therefore H \approx \frac{-\mu^2}{4u} + \int d^dx \quad \phi^\dagger(x)\left(-\nabla^2/_{2m} - \tilde{\mu}\right)\phi(x)$$
$$+ \mu/_2\left(\phi^2 + \phi^{\dagger 2}\right)$$

where $\tilde{\mu} = \mu - 4u\psi_0^2$

$$= \mu - (4u)\frac{\mu}{2u} = -\mu.$$

$$\therefore H = \frac{-\mu^2}{4u} + \int d^dx \left[\phi^\dagger(x)\left(-\nabla^2/2m + \mu\right)\phi(x) + \mu/_2\left(\phi^2 + \phi^{\dagger 2}\right)\right].$$

To study go to Fourier modes

$$\phi(x) = \frac{1}{\sqrt{V}}\sum_k e^{ik\cdot x} \cancel{\phi}_k$$

$$\phi^\dagger(x) = \frac{1}{\sqrt{V}}\sum_k e^{-ik\cdot x} a_k^\dagger$$

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}$$

$$[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0.$$

Then $H = \frac{-\mu^2}{4u} + \sum_k a_k^\dagger\left(k^2/2m + \mu\right)a_k + \mu/_2\left(a_k a_{-k} + a_k^\dagger a_{-k}^\dagger\right).$

Aside

More generally consider

$$H = \sum_k t_k a_k^\dagger a_k + \Delta_k \left(a_k a_{-k} + a_k^\dagger a_{-k}^\dagger \right)$$

In the present context

$$t_k = \epsilon_k + \mu \quad , \quad \Delta_k = \mu/2 .$$

Diagonalize this by writing

$$a_k = u_k \gamma_k + v_k \gamma_{-k}^\dagger$$

$$a_k^\dagger = u_k \gamma_k^\dagger + v_k \gamma_{-k}$$

to find excitation energies.

To diagonalize this let

$$a_k = u_k \gamma_k + v_k \gamma^\dagger_{-k}$$

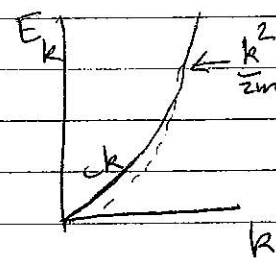
$$a^\dagger_k = u_k \gamma^\dagger_k + v_k \gamma_{-k} \quad \text{to express}$$

$$H = \text{const.} + \sum_k E_k \gamma^\dagger_k \gamma_k$$

This gives (see Prob 2 in HW 3),

$$E_k = \sqrt{\left(k^2/2m + \mu\right)^2 - \mu^2}$$

$$= \sqrt{\left(k^2/2m\right)\left(k^2/2m + 2\mu\right)} .$$



Thus the "normal" modes of small oscillation about the classical ground state can be labelled by momentum $\vec{k}$ & have excitation energy E_k.

As $k \to 0$, $E_k \sim k\sqrt{\mu/m}$

i.e the dispersion is linear & gapless $\left(E_k \to 0 \text{ as } k \to 0\right)$.

Aside

Diagonalization of ~~Smallea~~ Bogoliubov Hamiltonian.

$$H = \sum_{k \neq 0} \left[\epsilon_k a^\dagger_k a_k + \mu/2 \left(a_k a_{-k} + a^\dagger_k a^\dagger_{-k} \right) \right]$$

with $\epsilon_k = k^2/2m + \mu$.

To diagonalize let $a_k = u_k \gamma_k + v_k \gamma^\dagger_{-k}$

$$a^\dagger_k = u_k \gamma^\dagger_k + v_k \gamma_{-k}$$

$$[a_k, a^\dagger_{k'}] = \delta_{kk'}, \quad [a_k, a_{k'}] = 0, \quad \text{etc}$$

Also assume $[\gamma_k, \gamma_{k'}] = 0 = [\gamma^\dagger_k, \gamma^\dagger_{k'}]$

$$[\gamma_k, \gamma^\dagger_{k'}] = \delta_{kk'}.$$

For both sets of commutation relations to be satisfied,

~~let~~ assume γ commutators & examine a-commutators.

$[a_k, a_{k'}] =$ ~~$\sum_k \delta_{kk'} \ldots v_k$~~ $[a^\dagger_k, a^\dagger_{k'}] = 0$ clearly.

$$[a_k, a^\dagger_{k'}] = u_k^2 \delta_{kk'} - v_k^2 \delta_{kk'} = \delta_{kk'}$$

$$\Rightarrow u_k^2 - v_k^2 = 1$$

Let $u_k \equiv \cosh \alpha_k$, $v_k \equiv \sinh \alpha_k$.

$$H = \sum_{k \neq 0} \epsilon_k \left(u_k \gamma_k^\dagger + v_k \gamma_{-k} \right) \left(u_k \gamma_k + v_k \gamma_{-k}^\dagger \right)$$

$$+ \mu/_2 \left[\left(u_k \gamma_k + v_k \gamma_{-k}^\dagger \right) \left(u_k \gamma_{-k} + v_k \gamma_k^\dagger \right) + h.c \right]$$

$$= \sum_{k \neq 0} \epsilon_k \left[u_k^2 \gamma_k^\dagger \gamma_k + v_k^2 \gamma_{-k} \gamma_{-k}^\dagger \right]$$

$$+ \epsilon_k u_k v_k \left(\gamma_k^\dagger \gamma_{-k}^\dagger + \gamma_k \gamma_{-k} \right)$$

$$+ \mu/_2\, u_k v_k \left[\gamma_k \gamma_k^\dagger + \gamma_{-k}^\dagger \gamma_{-k} + h.c \right]$$

$$+ \mu/_2 \left[u_k^2 \gamma_k \gamma_{-k} + v_k^2 \gamma_{-k}^\dagger \gamma_k^\dagger + h.c \right]$$

Require that all $\gamma_k \gamma_{-k}$ terms have zero coefficient

$$\Rightarrow \epsilon_k u_k v_k + \mu/_2 \left(u_k^2 + v_k^2 \right) = 0$$

$$\Rightarrow 2\epsilon_k \cosh\alpha_k \sinh\alpha_k = -\mu \left(\cosh^2\alpha_k + \sinh^2\alpha_k \right)$$

$$\Rightarrow \epsilon_k \sinh 2\alpha_k = -\mu \cosh 2\alpha_k$$

$$\Rightarrow \tanh 2\alpha_k = \frac{-\mu}{\epsilon_k} = \frac{-\mu}{k^2/_{2m} + \mu}$$

Then $$H = \sum_{k \neq 0} \epsilon_k \left[u_k^2 \gamma_k^+ \gamma_k + v_k^2 \gamma_{-k}^+ \gamma_{-k} - v_k^2 \right]$$

$$+ \mu u_k v_k \left[1 + \gamma_k^+ \gamma_k + \gamma_{-k}^+ \gamma_{-k} \right]$$

$$= \text{const.} + \sum_{k \neq 0} \left[\epsilon_k \left(u_k^2 + v_k^2 \right) + 2\mu u_k v_k \right] \gamma_k^+ \gamma_k$$

$$\Rightarrow E_k = \epsilon_k \left(u_k^2 + v_k^2 \right) + 2\mu u_k v_k$$

$$\Rightarrow E_k = \epsilon_k \cosh 2\alpha_k + \mu \sinh 2\alpha_k$$

$$= (\cosh 2\alpha_k)\left(\epsilon_k + \mu \tanh(2\alpha_k)\right)$$

$$= (\cosh 2\alpha_k)\left(\epsilon_k - \frac{\mu^2}{\epsilon_k}\right) = (\cosh 2\alpha_k)\left(\frac{\epsilon_k^2 - \mu^2}{\epsilon_k}\right)$$

$$\tanh(2\theta_k) = \frac{-\mu}{\epsilon_k} \Rightarrow \frac{\cosh^2 2\alpha_k - 1}{\cosh^2 2\alpha_k} = \left(\mu/\epsilon_k\right)^2$$

$$\Rightarrow 1 - \frac{1}{\cosh^2 2\alpha_k} = \left(\mu/\epsilon_k\right)^2$$

$$\Rightarrow \frac{1}{\cosh^2 2\alpha_k} = 1 - \left(\mu/\epsilon_k\right)^2 \cancel{\Rightarrow \cosh 2\theta_k = \frac{1}{\sqrt{1 + \frac{\mu}{\epsilon_k}}}}$$

$$\cancel{\Rightarrow E_k = -\left(\sqrt{\frac{\epsilon_k}{\epsilon_k + \mu}}\right)\left(\frac{\epsilon_k^2 - \mu^2}{\epsilon_k}\right)}$$

$$\cancel{= -\left(\frac{\epsilon_k - \mu}{\epsilon_k}\right)\sqrt{\epsilon_k + \mu}}$$

$$\Rightarrow \cosh 2\theta_k = \frac{\epsilon_k}{\sqrt{\epsilon_k^2 - \mu^2}}$$

So that $E_k = \sqrt{\epsilon_k^2 - \mu^2}$.

Can also determine u_k, v_k in terms of ϵ_k & μ:

$$\cosh 2\theta_k = \cosh^2\theta_k + \sinh^2\theta_k = 2\cosh^2\theta_k - 1$$

$$= 2u_k^2 - 1 = \frac{\epsilon_k}{\sqrt{\epsilon_k^2 - \mu^2}} \equiv \epsilon_k / E_k$$

$$\Rightarrow u_k^2 = \frac{1}{2}\left(1 + \epsilon_k/E_k\right) \Rightarrow u_k = \sqrt{\frac{1}{2}\left(1 + \frac{\epsilon_k}{E_k}\right)}$$

$v_k = \sinh\theta_k$ satisfies $v_k^2 = \frac{1}{2}\left(-1 + \frac{\epsilon_k}{E_k}\right) \geq 0$

Note that $v_k < 0$ (so that $\tanh 2\theta_k < 0$)

$$\Rightarrow v_k = -\sqrt{\frac{1}{2}\left(\frac{\epsilon_k}{E_k} - 1\right)}.$$

For small k, $E_k \simeq \sqrt{k^2/2m\,(2\mu)} \simeq ck$ with $c = \sqrt{\mu/m}$.

$$\Rightarrow u_k \simeq \sqrt{\tfrac{1}{2}\left(1 + \frac{\mu}{ck}\right)} \simeq \sqrt{\frac{\mu}{2ck}} \quad \text{as } k \to 0$$

$$v_k \simeq -\sqrt{\frac{\mu}{2ck}} \quad \text{as} \quad k \to 0 .$$

$\therefore$ As $k \to 0$,

$$a_k \simeq \sqrt{\frac{\mu}{2ck}}\left(\gamma_k - \gamma^{+}_{-k}\right)$$

$$a^{+}_k \simeq \sqrt{\frac{\mu}{2ck}}\left(\gamma^{+}_k - \gamma_{-k}\right) .$$

To see the physical origin of this gapless linear spectrum more clearly, we will rederive it in different ways.

Clearly the existence of a manifold of degenerate classical gd states parametrized by θ ~~suggests~~ implies that a uniform phase rotation is a zero energy excitation.

This suggests that a "slow" phase change

$\psi(x t) \approx e^{i\theta(x)} \psi_0$ with θ a slowly varying function of x will ~~describe a mode~~ lead to an excitation with very low energy.

To make this more concrete, consider the real time coherent state path integral with action

$$S = \int d^d x\, dt \;\; i\psi^* \frac{\partial \psi}{\partial t} - \psi^*\left(-\frac{\nabla^2}{2m} - \mu\right)\psi - u|\psi|^4$$

Classical extremum is obtained by letting $\psi = \psi_0 e^{i\theta}$ independent of x.

As before, $\psi_0 = \mu/2u$

Note that $\psi_0 = \sqrt{\rho_0}$ where ρ_0 = density. (68)

To study small fluctuations, let $\psi = \sqrt{(\rho_0 + \delta\rho)}\, e^{i\theta(x,t)}$

$$\Rightarrow \psi \simeq \sqrt{\rho_0}\left(1 + \frac{\delta\rho}{2\rho_0}\right) e^{i\theta(x,t)}$$

$\delta\rho$ describes a small density fluctuation

θ is a fluctuation in the phase of ψ.

$$\text{Then } S = \int d^d x\, dt \; -(\rho_0 + \delta\rho)\frac{\partial\theta}{\partial t} + \frac{i}{2}\partial/_{\partial t}(\rho_0 + \delta\rho)$$

$$- \tfrac{1}{2m}\left[\left(\nabla\sqrt{\rho}\right)^2 + \left(\sqrt{\rho}\,\nabla\theta\right)^2\right]$$

$$+ \mu(\rho_0 + \delta\rho) - u(\rho_0 + \delta\rho)^2$$

with $\rho = \rho_0 + \delta\rho$ = density

Expand to quadratic order in ~~small quantities~~ $\delta\rho$.

$$\cancel{S \simeq \int d^d x\, dt \; -(\rho_0 + \delta\rho)\frac{\partial\theta}{\partial t}}$$

Also drop all terms that are total time derivatives.

Then $S \approx \int d^d x\, dt\ (-\delta\rho)\dfrac{\partial\theta}{\partial t} - \dfrac{1}{2m}\left[\left(\dfrac{\nabla\delta\rho}{2\rho_0}\right)^2 + \rho_0(\nabla\theta)^2\right]$

$$\cancel{-(2u\rho_0)\delta\rho} \quad -u(\delta\rho)^2$$

$$+ \underbrace{\text{const}}_{\text{drop}}.$$

Now vary w.r.t $\delta\rho$, θ to get equations of motion:

Vary w.r.t $\delta\rho$:

$$-\frac{\partial\theta}{\partial t} - 2u\,\delta\rho + \frac{\nabla^2\delta\rho}{4m\rho_0} = 0.$$

Vary w.r.t θ:

$$\frac{\partial\,\delta\rho}{\partial t} + \frac{\rho_0}{m}\nabla^2\theta = 0.$$

[Aside: $\Rightarrow$ Identify $\vec{J} = \rho_0/_m \vec{\nabla}\theta$ $\Rightarrow$ irrotational "superfluid" flow]

$\partial/_{\partial t}$(1st eqn) $\Rightarrow$ $-\dfrac{\partial^2\theta}{\partial t^2} = 2u\dfrac{\partial}{\partial t}\delta\rho - \dfrac{\nabla^2}{4m\rho_0}\,\partial/_{\partial t}\,\delta\rho$

$$\Rightarrow\ -\frac{\partial^2\theta}{\partial t^2} = -\frac{2u\rho_0}{m}\nabla^2\theta + \frac{\nabla^4}{4m}\theta$$

At long distances can drop ∇^4 compared to ∇^2

$$\Rightarrow \quad \partial^2\theta/\partial t^2 = \frac{2u\rho_0}{m}\nabla^2\theta$$

$\Rightarrow$ Wave equation with dispersion $\omega = ck$

with velocity $c = \sqrt{\frac{2u\rho_0}{m}}$

~~Thus the linear disp~~

How does $\delta\rho$ behave ?

At long distances $\quad \delta\rho = \frac{-1}{2u}\frac{\partial\theta}{\partial t} \Rightarrow \frac{\partial^2}{\partial t^2}\delta\rho = c^2\left(\nabla^2\delta\rho\right)$

~~The wave is an oscillation~~

$\Rightarrow$ The linear dispersing wave mode may be regarded either as a slow fluctuation of the phase θ of the boson field or equivalently as a slow fluctuation of the boson density about its mean value

($\Rightarrow$ regard as a sound wave).

An equivalent procedure is as follows:

Start with the approximate partition function defined by $S[\theta, \delta\rho]$:

$$Z = \int [D\theta\, D\delta\rho]\, e^{i\int d^dx\, dt \left[(-\delta\rho)\partial_t\theta - \frac{1}{2m}\rho_0(\nabla\theta)^2 - u(\delta\rho)^2\right]}$$

where I have dropped the $(\nabla\delta\rho)^2$ term as it is subdominant in the long distance limit.

Perform the Gaussian integral over $\delta\rho$ to get an effective action for θ.

$$Z = \int [D\theta]\, e^{i\int d^dx\, dt \left[\frac{(\partial_t\theta)^2}{4u} - \frac{\rho_0}{2m}(\nabla\theta)^2\right]}$$

$\Rightarrow$ Low energy effective action for θ-modes is

$$S_{eff}[\theta] = \frac{(\partial_t\theta)^2}{4u} - \frac{\rho_0}{2m}(\nabla\theta)^2.$$

Varying this w.r.t θ, directly get the sound wave equation of motion.

Note that all the microscopic information is contained in the 2 coefficients $\frac{1}{u}$ and ρ_0.

More generally write

$$S_{eff}[\theta] = \frac{K}{2}(\partial_t \theta)^2 - \frac{\rho_s}{2m}(\nabla\theta)^2$$

Later we will see that K may be interpreted as the compressibility and ρ_s as the superfluid density. (At $T=0$ in the Galilean invariant system we are considering $\rho_s = \rho_0 =$ total density).

The existence of such a gapless mode in the excitation spectrum is a rather general property of spontaneously broken continuous symmetry – the general result is known as Goldstone's theorem & the excitation mode is called the Goldstone boson.

~~Being gapless, the Goldstone mo~~ ~~con~~

(Eg: Spin waves in magnets, acoustic phonons in solids, etc.)

Being gapless the sound waves strongly affect the low-T, long distance, or low energy physics of the superfluid state.

Eg: ~~Eg: Lo~~ At low-T, the sound wave modes get populated according to the usual stat. mech. rules & affect the thermodynamic properties.

(See HW4 for an illustration in the specific heat of the superfluid).

As we will soon see, they also lead to depletion of the density in the condensate $\Rightarrow$ they form a "normal" fluid (this is a rough statement!).

At finite T, the superfluid density ρ_s < total density ρ.

Note that in the broken symmetry phase, to specify the macroscopic state of the system, we need to specify the value of $\langle \psi \rangle$ apart from all the usual thermodynamic variables (P, T, etc).

~~This extra variable that quantifies the amount of broken symmetry is known as the "order parameter".~~

In the superfluid context where the symmetry that is broken is the continuous $U(1)$ phase rotation symmetry, the fluctuations of the phase of the order parameter are slow & gapless at long wavelengths.

In the superfluid phase, we can essentially treat ~~classically~~ the operator ψ classically (& correct for fluctuations later as done earlier in these lectures).