WOLFGANG KETTERLE: So, good afternoon. Let me start by presenting something I learned today during lunch. I had lunch with three colleagues and we discussed entanglement. And, well, I'm telling you the story a little bit differently, but the question which came up is you have two harmonic oscillators. One harmonic oscillator is in the ground state, one is in the excited state.

And then you have the harmonic oscillators where the excitation is here and there. Is that an entangled state or not? It's a real question. I want to think about it.

AUDIENCE: What's the physical nature of these oscillators?

AUDIENCE: Can you distinguish these?

WOLFGANG KETTERLE: OK. Who's working with ion types? There's a few people I know, a few ion types here at MIT. If this is ion-- what is a good ion? [INAUDIBLE]?

AUDIENCE: [INAUDIBLE].

WOLFGANG KETTERLE: Strontium plus, OK. So if it's strontium plus, which can be in the quantum state v equals 1 or v equals 0, and you have another strontium plus ion in a second ion [INAUDIBLE]-- and it can be in the [INAUDIBLE] first excited state-- isn't it wonderfully entangled? Two systems far away. You can manipulate them, you have two particles, and then two quantum states. It fulfills with flying colors all the qualities you want to see of an entangled state.

But now I can say, if I call this harmonic oscillator, I have two optical fibers. And you know, a single-mode fiber defines a harmonic oscillator, namely the electromagnetic field in the single mode. And if I put one excitation into this fiber, I have a photon here and here I have zero photon. Or here, I have one photon and I have zero photons.

So if you look at the fiber, you would say the fiber has two states. It can have an excitation or not. And if you use now that approach, you would say, well, I have an entangled state between two fibers. However, I can realize exactly that state by having a beam splitter, by having one photon coming on the beam splitter, and then the photon is coupled into one of the fibers.

So in other words, there is an ambiguity which I want to point out between whether you have a state of one particle-- one photon into two fibers-- or whether you have two fibers which can be in two different states. So there is often an ambiguity, what do you call the state and what do you call the particle? So at our lunch conversation it came up in a somewhat different context, but people would actually say that splitting a photon-- sending one photon across a beam splitter fulfills, if you interpret it in that way, fulfills the definition of entanglement.

I know our wiki page and Professor Schwan would probably not agree, but welcome to the frontier of research where different people have different opinions and certain definitions are still being worked out. But there is one thing which is real. We have here two different harmonic oscillators, ions oscillating in a harmonic oscillator potential or photons in a single mode. I remember from 15 years ago, there were papers that show if you put the ion in a cavity or such, you can really transfer the quantum state from the single photon from this harmonic oscillator-- the harmonic oscillator of the electromagnetic field-- to the harmonic oscillator of the ion.

So if we want to hold onto the definition, which I'm not sure if you should that this is a single photon, a single particle cannot be entangled, well I know and people reminded me at lunch the there is a protocol to [INAUDIBLE] map the single photon in the two modes after the beam splitter onto an entangled state of ions. So if you focus here on the photon and say a single particle cannot be entangled, you're facing the problem that there is a protocol which would transfer something which you call not entangled to something which is entangled.

I hope you enjoy that. It's not clear what is the particle and what is the excitation.

There's really ambiguity. Any questions?

OK, so that's what part of my lunch was about. I thought since I wanted to remind you of the schedule. In a few hours I will actually be heading to Europe and say there for next week, so therefore no classes next week. We've already made up for one class with the Friday class, and the week later we also have three classes and we've made up for the week.

I've just put down here for you the PSET schedule. I kept the Monday due date so there is one PSET every week. We have wonderful material for those PSETs. The due date is now for this week not the day of the class, but you will find a way to get your homework delivered or email it to the TA.

Any questions about schedule? I think that has been clearly announced. There was a question in the last class about if you have a density matrix, if any arbitrary density matrix can be regarded as a partial trace of a pure state. I was not immediately clear-- I thought it was, but I wasn't sure. But the proof is so simple that it fits on three lines.

You can simply take a density matrix, you can double up the Hilbert space, and define our pure state in a Hilbert space of twice the dimension, and you immediately see that this density matrix is the partial trace of a pure state. So therefore, if you want, you can always say the density matrix is sort of entangled-- you are entitled to the opinion that the density matrix always originated into a pure state, but the entanglement [INAUDIBLE] to the other subsystem have been broken. And I showed you last class that if you are one of the Bell states, the most entangled states in a fully-entangled state between two particles, but you just look at one particle, one particle isn't just a random states with a density matrix which is a unity matrix.

Now let's get back to where we left it on Monday. I introduced to you-- and I just want to remind you of that, because [INAUDIBLE] I introduced to you this famous phenomenon of Hong-Ou-Mandel interference which is the following situation. If you have 50/50 beam split and you have two identical photons, you will never get single photons out.

The photons are bosons. They want to be together. And so you will always get two photons, but you don't throw on which side of the beam splitter. Now, this is important because we can now use the Hong-Ou-Mandel interferences effect to entangle atoms. So the element we will take from this beam splitter is the following.

We will have atoms emit photons. But then we create entanglement probablistically. When these photons have passed through beam splitters-- and I tell you everything about it-- and both detectors make a click, if both detectors make a click you know for sure that the two photons at the input of the beam splitter were not identical. Because if they were identical, only one detector receives light.

So we will actually used this home interference to project out photon states where the two photons are not identical because they have difference polarization. Any questions? Yes?

AUDIENCE: Does [INAUDIBLE] include the phase. Like if you were in phase lab with those photons, would it be still?

WOLFGANG KETTERLE: Frequency very important. Polarization important, timing is important. A single photon has no phase. It's a global phase.

And if you have many single photons, your many photon state is a product state of those single photon states. And all of the individual phases-- if you want to hold onto this concept for a second-- just become one big multiplicity phase. Yes, Nancy?

AUDIENCE: [INAUDIBLE]?

WOLFGANG KETTERLE: Oh, everything [INAUDIBLE]. easy experiment. But I think the beam splitter is probably not-- I mean, beam splitter is a piece of wonderfully polished optics. And in Germany and elsewhere, people have really learned how to do exquisite optics. I don't think you're limited by optics right now.

What I would think is difficult, if you have two photons on a beam splitter, they have to be in the same spatial mode. So if you have to fibers and the fibers are not fully aligned, or you have another mode which is [INAUDIBLE], or if you have a lens which is distorting your Gaussian mode, you mix other modes in so all this creates non-distinguishability. But I think equality of optics is probably the least of your concerns.

So actually for today, because I'm out of town next week, I want you to have something to think about it, I already pre-wrote some of the slides. So I can go a little bit faster. Give me feedback. If you think I'm going too fast, I will not do it again. But I felt I can probably have a reasonable speed by making annotations to what I've already prepared.

So the situation is the following, we want to entangle two atoms. We have two atoms which are both in the excited state in the scatter light. And they can go to two different ground states. Think about two different type of ion states. I call them U and D, Up and Down.

And when because of selection rules, one state can be reached with one polarization. The other state can be reached with the other polarization. And I call this polarization H and V, horizontal and vertical.

In reality, this will be circular polarization, so you may want to transform from linear to circular after the fact. But for just the discussion right now, I assume that horizontal polarization leads to one state, vertical polarization leads to the other state. So therefore, if you have two atoms-- and Chris Monroe in Michigan did the experiment where he really had two different vacuum chambers with two different ions, they both emitted photons, and then the photons came together on a beam splitter. So that's what you should sort of have in mind, two ions here and there.

They both emit a photon. And after the have both emitted a photon, each ion is in a superposition before you make any measurements. It can be in up with a horizontal photon or it can be in down with a vertical photon. So that is the state of the ion.

And so we have two of those. And then you want to detect photons. And the goal now is by using beam splitters and all those tricks, that the detection of photons and the outcome of a measurement will project this product state of two atoms with their photon into a bell state for the atoms. So measurement on photons can take two atoms which were completely remote, had nothing to do with each other, and suddenly they're in a Bell state. And this is done by the probabalistic measurement.

And so let's-- so lets develop it. Our initial state is UH plus DV. This is system one. And we have the direct product with our second ion trap apparatus which has done the same. The ion has emitted a photon and we don't know anything about it at this point.

And now it is important, if you want to have entanglement you cannot go and measure the polarization, because this would project an individual ion into a state and would not have entangled it. So what we are doing is we are allowing the two photons, one photon here, another photon here, they come together at a beam splitter. Actually, before I do it let me just perform the product here.

So what we get is-- well, it's just multiplying it out. UU for the atoms, HH for the photons. UDHV plus DUVH, plus DDVV. So what you want to do is we want to send this beam-- the photons of course, the atoms stay put-- onto a beam splitter.

Now, I can do it more rigorously by keeping all the terms until the end of the calculation. But I hope it's rather obvious that we want to have it-- if we want to send it on a beam splitter and then we want to detect that both detectors make click. That requires the photons to be distinguishable. So therefore, those two terms will not contribute because the photons have the same polarization.

So we are now using a protective measurement. We want to find out what is the wave function of the atoms. Measurement of one photon in the modes A and B, which are the output modes of our beam splitter.

So it needs one more line. The output of the beam splitter is-- so I'm only focusing now on the two terms which can give rise to two clicks at the two different detectors. So I will factor out UD.

And now we have the two photons hitting the beam splitter and each photon goes into a 50/50 superposition. So I take know this horizontal photon. And since we don't mess around with the polarization, it's still a horizontal photon, but the photon can be now in mode B or in mode A in one of the output modes of the beam splitter. And I label that as 0,1 plus 1,0.

Now for the second photon, it had a vertical polarization, so it's a vertical photon. The second photon is coming onto the beam splitter in the other input port, because we have taken the two photons from two different experiments, from two different experimental setups, and now we combine them so that we'll also be now in a superposition of 0,1 and 1,0. But you know that one mode transforms with a plus sign and the other mode transforms at the beam splitter with a minus sign. So that's how you should look at those terms.

And of course, from of this here, from this term we get something very analogous, 0,1 plus 1,0. And for the other spatial mode, the minus sign from the beam splitter, 1,0. OK, now we are done. Now we are detecting photons.

So we want to look at this wave function and ask, what happens if in port A-- the output port-- we find one horizontal photon, and in B it must be vertical because the photons have to be distinguishable. So where do we have one photon in mode A? In horizontal it is here. And here it is this product with the minus sign where we have-- no, sorry, the plus sign.

Here we have one photon in the vertical, so this is with a plus sign. But we have a second possibility, one photon in mode A horizontal happens here with a minus sign. And one photon in mode B vertical happens here.

So in other words, form the term which gives rise to this measurement process, we have a plus UD here, and because of the minus sign, a minus DU. So therefore, if you would detect the photon with the polarization at this point, which we don't do, this polarization gives rise to UD minus DU. We have a second possibility to have a click at each detector, and this is when we reverse the two polarizations.

But in this case, you can just add the wave function. You get the atomic state, the sign doesn't matter. So you have to measurements, two possibilities in polarization, but they both have the same outcome. They generate this state, so the result of all that is I've shown you that the two atoms are left now in-- and now I put back in the correct normalization-- in one of the Bell states.

So whenever you detect the two single photons in coincidence on both counters, at that moment you know what you have is an entangled state, a Bell state of atoms. Now, the big issue here is efficiency. Your photon detectors are not highly efficient.

And I was sort of only looking at combinations where we have this double detection. Most components of the wave function will give the Hong-Ou-Mandel effect. The two photons go in one way. So therefore, you have to prepare your system, have photons emitted, and in most of the cases you will not get the state you want, so you have to keep on trying.

And usually those experiments are heavily limited by the very, very dismal success probability of creating the state you want. And when you want to extend it, not just to two atoms entangled, to three atoms and four atoms entangled, then you get a smaller number to the power n for your efficiency. So this probabalistic preparation doesn't scale up, but it is a very simple scheme, it's a very powerful scheme, and it has been used to create teleportation of atomic states and do some tests of Bells inequality and the EPR Paradox.

And here is the abstract of a paper. So-- well, this was just 2008, so just a few years ago. And what you see here is that here are one meter apart two [INAUDIBLE] ions.

They emit light. The light goes through an optical fiber. And now, after the beam splitter, when you detect one photon you don't know which ion has emitted the photon.

So this is exactly the setup which I described. Of course, in an optical fiber you have the two good polarizations are horizontal and vertical in the polarization maintaining fiber. So therefore, the light which originally was emitted in a circular base is--

angular momentum selection rules-- transformed into linearly polarized light using the [INAUDIBLE]. Any questions? Yes, Nikki?

AUDIENCE: [INAUDIBLE] create the initial state [INAUDIBLE] atom [INAUDIBLE] one state [INAUDIBLE] make sure that the ions state is a pure state, not a mixed state, [INAUDIBLE]?

WOLFGANG KETTERLE: Good question. So first of all, if you think about it you will discover more and more experimental challenges. What people must have used there is sort of a storm laser pulse that you have more than overkill to make sure that with a very, very short time window both ions are excited.

The ions are in a pure state, and then they emit a photon. And if you have one system prepared which can emit a photon, but it has a branching ratio of 50/50, if it's an isolated system, it will have a superposition state of photon in one polarization-- let's say a sigma plus photon going to a magnetic quantum number state m equals plus 1 and a sigma minus photon going to m equals minus 1. And this is a pure state.

The mixture only comes if you're not careful. If you have a magnetic field and you don't shield your magnetic field well, or you have some magnetized materials and you were not aware of it, then that means that you get different phase shifts which you can't calculate for, so now you have a random phase. You don't know it, and if you have ignorance you have to trace out or you have to average over the phase. And then your pure state becomes a density matrix.

But the quantum mechanical process itself of a particle having different branching ratios is a pure system. It undergoes unitary time evolution, and it stays in a pure state. And the state which is populated by two ions, both emitting a photon simultaneously, is exactly the state I've written down for you. But you're absolutely right that decoherence is an issue. You have to be very careful with magnetic fields. You probably want to work with atoms which are non-magnetic. So you want to where the spin is only a nuclear spin which is much less sensitive with a magnetic field, and so on. Other questions?

AUDIENCE: So as soon as both detectors click, then [INAUDIBLE] so that you can no longer make use of that entanglement?

WOLFGANG KETTERLE: No, no wait. Look here. We have a state which has atoms in a state and two photons.

So this state has four particles. We detect two particles which are the photons. And the atoms are untouched. The atoms are then afterwards in an entangled state.

So we have a bigger system, we do a measurement of part of the system, and we have arranged things in the skillful way that the moment we know the outcome of the measurement is such and such, we know in which quantum state the rest of the system is. And this protocol means that the atoms are left after the outcome of the measurement is such and such, the atoms are left in a pure Bell state.

OK I've used the word Bell, Bell states so often, I think it's time to talk about what Mr. Bell is famous for, namely the Bell's Inequality. So I said already in the introduction for the quantumness of light and entanglement that it is the EPR Paradox in the Bell's inequalities which a lot of people, including myself, think is the deeper essence of quantum physics. It really shows that quantum physics is not just classical physics with wave character, it goes way beyond it.

So I want to demonstrate that with Bell's inequality. And the formulation which is very simple, and I want to present it here, is an inequality, which is-- I mean the name. Many Bell's inequalities now, you have different states, different detectors, and you can derive Bell's inequalities which are then violated by quantum physics. And what I want to present here is the CHSH inequality. That's Clauser, Horn, Shimony, and--

AUDIENCE: Holt.

WOLFGANG KETTERLE: Yes, thank you. So the situation is the following. We have something which decays, something which emits two photons. Maybe an atom in an excited state, and it does a click clack, a two photon cascade. One photon goes to Bob, one goes to Alice.

Or I always try to stress similarities between light and atoms. We discussed the experiment where you take a mercury molecule, you dissociate it, and then Bob and Alice each get an atom. And you can say these photons has a polarization or the atom has a spin.

But Bob and Alice can now use different Stern Gerlach figures in x and y, at obscure angles, circular bases, I mean you name it. But what happens is because it's a spin up, spin down, horizontal, vertical polarization, it's a two-level system which is immediate. And after a Stern Gerlach filter, you have only two combinations. We call it plus and minus.

So in its most general form, what we assume is that Bob does measurements in a basis, if you think. Think about a spatial orientation of a Stern Gerlach filter called S, where the outcome is plus minus 1, t where the outcome is plus minus 1, and Alice has her own choices. And we want to assume now that this is everything is classical probability. That if you just write down this expression-- don't ask me where it comes from.

This is probably something which people wrote up after they found something interesting and try to prove it or simplify it. You just write down QS, RS, RT minus QT. You can then rewrite it by factoring out S and T.

And now the next step is because Q and R are-- they're are in each measurement 1 or minus 1, either this is 0 or that is 0. No sorry, one is 0, one is 2. And therefore, this kind of funny combination of letters for every measurement is either plus or minus 2.

Now we do many measurements, and the probability for a certain outcome that the variable Q-- there's a certain probability that the outcome is Q, so you sort of use this pretty much probabilistic thing that the system has a certain probability to B. And this is of course the assumption, the probability is that the particular comes with a certain probability in the state q, r, s, t. Of course you should scream, q, r, s, t commute, but this is classical now. They don't commute with each other quantum mechanically, but [INAUDIBLE] in a moment.

So then you simply put-- by multiplying each event with its portability, you put now brackets around it. These are expectation values, and what you have is an inequality that this expression, which is the correlation between certain measurements between the quantity QS, RS, and so on is smaller or equal than 2. I mean it's very, very basic. And that's why I wrote it down.

I don't want to spend a lot of time on it. It's just classical probabilistic reasoning. So now what are we doing quantum mechanically?

We want to prove that quantum mechanics cannot be reduced to this classical reasoning. So we want to show that this inequality is violated. We have a source or entangled photons. We've talked about how we can entangle photons. And now we measure the quantities Q, R, S, and T.

And just to give you an example about the many possible choices, Q can be a linear polarizer and R can be a circular polarizer. S can be a linear polarizer at 45 degree, and T can be a [INAUDIBLE] plate followed by a linear polarizer at 45 degrees. Well, that sounds like many trig functions, but we're not going into it.

You have to choose your polarization. There's a certain scheme that linear polarization is not orthogonal to circular polarization, so there's a certain theme behind it. And if you would work it out by just looking at entangled photons, the entangled photons are in a state, let's say HV plus VH. They are correlated in polarization or spin up, spin down with the down spin [INAUDIBLE] state. And you can just work out what is the polarization when you detect it.

And what happens is that a simple but tedious calculation says that all those quantities QS, RS, and and RT are equal. Here we have a minus sign and they are all 1 over square root 2. And that means that instead of the classical inequality, that this quantity is smaller than 2.

We find that this is 2 times square root 2. And the fact that this is larger than 2 has been experimentally confirmed with larger and larger precision. Actually, the person who gave this [? CUA ?] seminar a week ago, [INAUDIBLE], was part of the team with [INAUDIBLE] who did one of the very, very first measurements violations of Bell's inequality in the '80s some 30 years ago. So I mean, this happened rather recently.

So the math is trivial, the result seems trivial. It just shows that the world is quantum mechanically and not classical. And a lot of papers have been written and discussions have been had about what does it really mean? What does it mean about the world? What does it tell us about the world?

So so the implication from the established violation of Bell's inequality and the CHSH inequality is that when we assumed that the state has definite values of Q, R, S, T before observation, we have to give up that. Or we have to give up that a measurement performed by Alice does not influence Bob's experiment, Bob's measurement.

So in other words, what I formulated here is the locality principle that what happens in one location cannot influence what happens in the other location. Well you can say maybe there's some secret channel sending [INAUDIBLE] the speed of light. But people [INAUDIBLE] to create links to put the detector so far away that even a [INAUDIBLE] at the speed of light could not have influenced the other measurement. So there has been after the first Bell's inequality experiment, there have been a series of experiments to avoid all of those loopholes that there is some unknown secret communication between the two detectors.

The first one is the principal of-- sometimes I think a more philosophical word than physical word, "reality". That those quantities are real and they should exist before we measure them. So the outcome of Bell's inequality is-- the violation of Bell's inequality is, at least one of them, at least one of these principles-- reality-- at least one of these principles does not hold in nature. So either reality or locality are violated. Any questions?

And maybe just to connect it to what I've said before, one reason you saw that here in the title, Bell's inequality with two remote atomic [INAUDIBLE]. Those experiments got a lot of attention because Bell's inequality is something all physicists talk about. And when you do it with photons, there is always the detection loophole. You don't have perfect photon to detectors.

And some people said, well, maybe the violation of Bell's inequality comes only from the photons we detect. The undetected photons would make up for the violation of Bell's inequality, and Bell's inequality is not violated. I mean, that almost sounds like that nature wants to fool us. The photons team up and say the photons which are detected behave very differently from the photons which are not detected.

But these are at least logical loopholes. And graduate students spend half of their Ph.D., or maybe several graduate students spend their whole Ph.D. In building such an experiment which is now-- you get a famous paper out of it. And your research has attention, so you know.

I didn't sleep so much, so maybe I'm not too serious now. But this is great research. I mean, this is really pushing the limit of our understanding of quantum physics. And Chris Monroe is a wonderful physicist. And we are in the same [INAUDIBLE].

So if you have a Bell state of atoms, and I told you how you get it also with a lousy efficiency but you get a Bell state. If you now do a measurement of those Bell state and you measure violation of Bell's inequality, atoms, they don't run away. You can detect the atoms with 100% probability. You can shine a laser light on them, hundreds, thousands, millions of photons until they have scattered enough light that you know 100%, I've detected the atom in it's quantum state. So Bell states with atoms, one real world application of them is to test violations of Bell's inequality with atoms which do not fall into the detection loophole.

Our next unit is partially motivated to show you what all this entanglement can do for us. So I mentioned that entanglement is a resource. It's something useful like energy is a resource. And so maybe ultimately there should be a stock market, and you can buy so and so many bits of entanglement. And you have to pay for it because there's something you can do.

And let me introduce what can be done with it. If you have one photon at frequency omega, and you have an observation time-- you do a measurement over time T, then the precision at which you measure the frequency is fundamentally limited by, you can say time energy uncertainty by the Fourier theorem, that the uncertainty and the frequency of a single measurement is 1 over the time T you had to detect or to measure the frequency of the photon.

But now we have n photons. And that means that the uncertainty in the measurement, which is now small delta omega, is delta omega for a single photon. But you know, when you do n measurements, and average n measurements, you gain by the square root of n. And this is regarded as the fundamental shot noise limit of measurements.

But now assume we can do something fancy. We can make a super photon. We take our n photons of frequency omega and make one photon of frequency n omega. Now we have only one photon. The frequency uncertainty of the measurement is delta omega, but this is now the uncertainty off the n times more energetic photon. So therefore, if you're interested in the quantity omega, which is the frequency of the single photon, we have now made an improvement over the standard shot noise limit by square root n.

So everybody follows? So that's just a [INAUDIBLE] experiment. If you can take n photons-- I've told you how we've talked about the [INAUDIBLE] oscillator, how we can pump a crystal. And in your homework assignment you do a nice calculation with a Hamiltonian. One big photon in, it breaks into two photons.

The reverse process is frequency dot doubling. So if you would not measure n photons individually, but first ate up their frequency by making a photon of n times the frequency and then look a single photon, you have now, you measure all the photons together. And therefore, your accuracy improves by a factor of n and not just by a factor of square root n.

So that tells us something. If we do something with the photons, if we entangle them, there is a possibility to vastly improve the standard limit of measurements.

And so people who are really interested in it are people who push the limits of precision, people who build atomic clocks and want to get the last little bit of accuracy which is possible. So they have already exhausted all technical possibilities, and the next thing is now, well maybe entanglement and sort of subtleties of quantum physics can come to their help.

So this shows you what is possible. But now I want to tell you how it can be done. So what I first want to do is, before we can improve over the fundamental shot noise limit, I have to show you how the fundamental shot noise limit naturally emerges.

So I want to introduce to you beam splitter interferometers. Interferometers are there to measure phase shifts. You split a beam, you combine it, and if there's a phase shift you notice if. You get interference fringes.

And I want to show you that very naturally to standard quantum limit, the shot noise emerges. But then we are ready to look at our description of the interferometer and say, where can we now change the rules to put in more quantum-ness or entanglement and eventually get higher precision. Since the standard quantum limit is well know, I want to rather quickly go over that.

So the goal is that we want to measure a phase shift. And my first simple deviation of the phase shift is that in this picture of the quasi probabilities, the coherent state is a circle like that. The width of the circle in natural units is 1, but the radius of that is alpha of the coherent state, which is square root n.

So now if you ask how well with this uncertainty can I observe the phase of the photon which circulates in this quasi probability plane, you find that the phase is 1 over square root n. Or based on your homework assignment number one, you can say we have some Heisenberg uncertainty between photon number and the phase. And in a coherent state, the standard deviation in the photon number is square root n, and you again get the standard quantum limit.

Let's now obtain the standard quantum limit from a real measurement device, because this is what we want to generalize for entangles, [INAUDIBLE] state, and all the special things. So an interferometer is the following. It consists of two beam splitters.

After the second beam splitter we have two detectors. And the quality we will measure will be the difference of the two photo currents. And if you don't do any phase shift, we know the two beams splitters are just the identity transformation. But now we put in a phase shift, and the question is, what is the smallest phase shift we can measure? What is the accuracy measuring this phase?

We have discussed at length all the elements. So we have a beam splitter, a phase shifter, and a second beam splitter. You know that the phase shifter is a rotation in z, the beam splitter is a rotation in-- was it X or Y? Looks like I think Y because of the i's.

Just a warning here, in this section I use a beam splitter which uses a different phase convention. But they are all equally apart from some phases. So by multiplying the three matrices, we get the transformatrix for the [INAUDIBLE] interferometer.

So the output CD is this matrix here which is a simple rotation times the input state. And our measurement is the difference in photon numbers in the two output modes. So it's D dega D minus C dega C.

So what we can do now is we can obtain C and D from the input modes A and B. So therefore we can now express our signal in terms of the input modes. When we know what we put into the interferometer, because we know all its elements we know what we can get out. And this is done here.

And the phase which appeared in the rotation matrix, the phase shift in the interferometer appears now as a cosine phi and sine phi contribution. And we have sort off- the cosine and sine phi have two operators as a pre-factor. In one case, it's A dega A minus B dega B. In the other case it's a cross-term, A dega B plus B dega A,

Now we will be more specific what is signal and noise. I just want to tell you in the standard way of operating the interferometer you have only one beam in the mode A. You take a beam, you split it, you recombine it. B is nothing or the vacuum. It may introduce some noise.

But if you have a lot of intensity in the beam A, it is this term which dominates, and you find fringes as a function of the phase shift cosine phi. Whereas what comes here is sort of more the vacuum mode. It gives rise to noise terms.

So if you ask, what is the expectation value for x, this x operator, it varies co-sinusoidally. And there is a special point for a phase shift of 90 degrees when we have the steepest slope and the highest sensitivity. So these are sort of all just setting the stage. Which we are interested in is, if we now have an input state of light-- and as you know, there's all of this noise. There's the fundamental noise of the vacuum, or coherent state, of Heisenberg's Uncertainty. And this means when we do repeated measurements we will have a variance in the phase. And we want to know, what is the fundamental limit on the standard deviation in the measurement of the phase.

Well, the standard deviation in the phase is nothing else than the standard deviation of what we measure in divided by how sensitive it is to the phase. So by taking this expression for M, M is an operator X plus cosine phi plus something times sine phi. We can now evaluate this expression.

This will actually appear several times today, and it's also a key question to one of your homework problems. So this is sort of what defines the accuracy of the interferometer. And what we have here are expectation values of operators. And now we can-- and that's what we will do for the rest of this lecture. We will look at this expression for different inputs states. Coherent state, single photons, [INAUDIBLE] light, entangled state. So that's what you're going to do.

So if you take the derivative of dn d phi, the cosine becomes a sine. A minus sign, the sine becomes a cosine. And if you specialize to the situation, which is where the slope is very steep, a 90 degree phase shift around this point, our phase sensitivity is given by the expectation value of the variance of the operator y divided by the operator x.

So now we are ready to plug things in. The first thing is of course the coherent state. For the coherent state, our input to the beam splitter, one is a coherent state, the other one is a vacuum in mode A and B.

We had expressions-- let me just scroll back-- for x and y, expressed by the operators a, b, a dega b dega. So therefore we can calculate that now. And x is nothing else than the number of photons in the coherent state. And this is our signal. And y is 0.

This was the noise term. And the variance in y-- we're just doing commutators-- is given by n. So therefore, if we calculate the square root of y square over x for the coherent state input, we take the square root of y squared, which is square root n, we divide by n, we obtain the standard shot noise limit. Sure, what else should we expect?

So now we know how to use the formalism. We can now apply it too-- we can go from the most classical state of light, the coherent state, to what I regard at least for a single mode the most quantum state, namely a single photon. Remember when we talked about the G2 function, G2 function can only be smaller than 1 for non-classical states. And for the single photon it's 0. That's the biggest violation of the classical equation that G2 has to be larger or equal than 1.

So now we're really dealing with a quantum system. And the question is, what will we get for the single photon? So for the single photon we want to use the dual-rail representation. We want to use the powerful formalism we have used.

After the beam splitter, the photon can be in one mode or the other mode. We call this in the dual-rail representation the logical 0 or the logical 1. So in this representation, when we start with a photon in one input mode, this is the logical 0, the beam splitter is directing the photons into one mode or the other mode, but this is a single qubit rotation.

The phase shifter is another single qubit operation. The second beam splitter is another operation. So by just using the rotation, we'll find out what is the output state. So we start with the logical 0 with the photon in one mode.

The beam splitter creates a superposition state. Remember, we have the interferometer, and in the lower arm we put in a phase. So therefore, the lower arm gets multiplied with a phase shift. And then it goes again through a beam splitter, which is just another rotation, and with that we obtain the output state.

And so the output state is now a superposition of the logical 0 and the logical 1, and we've picked up a phase shift. So no we ask-- we have the beam splitter, phase shift, recombiner, and now we are asking, what is the probability to detect the photon in one of the two output modes. We just put a counter there.

And this probability is nothing else than the probability to have a logical 1, because a logical 1 in the dual-rail representation means the photon is in one mode. The logical 0 would be the other mode. I felt if I would spend three or four times as much time on it, it wouldn't increase your understanding. You may have to sit down and look through it, but it's just really putting matrices together, [? single ?] photons, mapping it. Every step is trivial and is what we have done in a different context before.

So the result is that the probability of finding the [? single ?] photon in our detector has a cosine phi factor. So that's how we measure phase. Our counter, the probability that photons arrive. If phi is 0, the probability is one.

If phi is 180 degrees, all the photons go to the other mode. I mean, that's what you'd expect an interferometer to do. But now comes the interesting question. How precise can be measured the phase?

Remember, in the coherent state, the coherent state had fluctuations in the photon number of square root n, and this gave rise to the standard quantum limit. A single photon is a single photon. There is no noise. We always measure a single photon if you would simply detect the number of photons in the input state.

But we are not doing that. We have sent the photon through an interferometer in order to measure the phase. And now we have-- and this is a result of this, I would say, trivial calculation that this is now the probability to observe a photon.

Well, there is not a whole lot we can learn from one photon, so we run the experiment n times. And what we get is a binomial distribution, just a coin toss with probability P. What is the variance in P? It's not a Poissonian distribution. It is a binomial distribution.

Of course, if the probability is 1 you detect all your photons. There is no uncertainty. If the probability is 0 you detect 0 with 0 uncertainty.

So therefore, the expression for the variance of the binomial distribution is p times 1 minus p. If not, I admit I had to refresh my memory about the binomial distribution yesterday evening. So therefore, when we repeat the experiment many times and we have sent into our interferometer m photons and we measure n clicks. And n over m is our measurement for the probability.

This measurement of the probability has noise. And the noise is a variance of the standard deviation of the binomial distribution. This expression for P put into the variance of the binomial distribution, and a little bit of triganomic manipulation gives us the result on the right-hand side. And we want to know what is the uncertainty in the phase. I mean, this is what we want to measure.

Well, the uncertainty in the phase is the uncertainty in our measurement, which is delta P. And then we have to divide by how sensitive the probability is with respect to the phase. So we take this expression of the probability as a function of the phase, take a derivative, and then by doing the ratio of this over that we find 1 over square root n.

So it doesn't make any difference if you put the n photons into a coherent beam and run our interferometer or if you go through the great pain of preparing each photon in a non-classical flux state and just operating the interferometer with single photons. In both cases do we obtain the standard quantum limit 1 over square root n.

Now we are really ready to see how can we improve on the shot noise? It seems that unless we do something special, we will always get square root n. I've already shown you at the beginning that's the shot noise limit is not fundamental. Just take this thought experiment that you take the n photons through a highly nonlinear process, you create one photon of frequency n omega, and then you have more precision of this measurement. So there must be a way to have a precision which scales 1 over n and not 1 over square root n.

The mathematical augment related to our treatment of the interferometer is the following, our signal was A dega A minus B dega B. Well, if one of the input mode dominates, A dega A is the [? photon ?] [? number. ?] So this sort of looks like the [? photon ?] [? number ?].

But if we now find a scheme-- and I will show you that this is possible-- that this cos term A dega B plus B dega A, which is some form of quantum noise, as we will see, is 0, well what do we get now? So let's hope that maybe by some squeezing-- I will show you several versions which show you how quantumness can give us more than short noise. And one example will be that by squeezing light.

We've learned already that squeezed light can suppress the noise. If you squeeze light, something which comes from the modes where we apply squeezed vacuum has been reduced. I mean, it's what we discussed already. So the best we can do is that the noise term is 0.

So what do we have now? If we have a signal which is finite and the noise is 0, what is our precision of measurement? Well, it first looks like the signal to noise ratio is infinite. But it's not quite that, because if our signal is x, it's a photon number. And the signal x was sensitive to the phase by cosine phi.

The sensitivity of our measurement of the quantity m with phase is n times sine phi. And this is smaller or equal than n, where the equal sign is obtained for 90 degrees. So therefore, we have a [? photon ?] number n. We may absolutely know what the initial [? photon ?] number is. But now we want to measure whether the phase is different from pi over 2. And the smallest change in our signal which we can resolve is that we get one photon less.

So that's the smallest resolvable change due to the [? photon ?] nature of our detection. And that implies that the smallest phase shift we can resolve is 1 over n. So this is more a thought experiment, now looking at the math and seeing what is the best signal?

A dega A can never be larger than [? photon ?] numbers, so that's a resource. We put in energy in the terms of photons, and then the best we can do is that we don't have any noise. And then we are simply limited by the fact that when we deviate from the maximum output signal because there is a phase shift in the interferometer, our sensitivity comes in grains, is grainy by the [? photon ?] number.

So I'm not telling you how we achieve to get y equals 0. The question is only that at least the math seems to make it possible. So now we ask how to achieve sub-shot noise precision. So here we know this is at least mathematically possible.

So we want to improve this interferometer, and the question is we have to change something. This interferometer, [INAUDIBLE] we've seen [INAUDIBLE] state is not giving us any better result, so we have three options now. We can do something fancy in the input state, use entangled state. We can use some very fancy beam splitters, or we can do something special to the way how we read out the interferometer. And all those three are possibilities. We can put in some extra quantumness into each of the three steps. OK

AUDIENCE: [INAUDIBLE]?

WOLFGANG KETTERLE: OK, the question is can we do something with the phase shift?

AUDIENCE: [INAUDIBLE].

WOLFGANG I think if we would do something with the phase shift, we would do-- you know, we

KETTERLE: would change apples with oranges, because we want to compare different interferometers by measuring the same thing. And maybe as an experimentalist say, what I want to be able to measure I take a very, very thing glass plate which just makes a very small phase shift, and I'm now comparing different interferometers. And when I put the glass plate in and pull it out, I want the person who reads out the interferometer to see a change. So that's how I want to compare the interferometers.

Actually, the question you are raising was in another way also discussed today at lunch with three of my wonderful colleagues. Namely what we discussed was some recent papers, one of them published in *Nature*, which claims precision better than Heisenberg. I mean, what I'm sort of indicating to you is Heisenberg should be the best which can be achieved. How can we do even better than Heisenberg?

Well, and this was actually related, Nancy, to your question. If you want to measure magnetization and you use some nonlinear physics where the photons, where the magnetization involved affects the photon field by some higher power of what you measure, you really change what you measure. Maybe I'm not expressing it clearly.

If you measure magnetization and you have a certain quantum limit in measuring the magnetization, that's one thing. But if you now bring another material close to your magnetization and this material goes through a phase shift, you sort of amplify by a physical process by a nonlinear Hamiltonian what you want to measure. And then of course you can-- depending on what kind of nonlinear process you're using-- you can get a signal which scales tremendously with the number of photons you put into your system. So in other words, there are loopholes like this, and some of them led to very fleshy papers.

And our lunch discussion was that some of it is really completely trivial. And some papers who claim that they have seen a scaling of the precision with photon number which is better than Heisenberg, not 1 over n, maybe 1 over n square, that some of those papers were purely classical and the only quantum character of this paper was the name of Heisenberg in the title. So it's not related to any uncertainty relation, but that's not what we want to discuss. Let me just spend quickly-- let me see, yup.

We should the week with something interesting and not just shot noise. So what I want to use now is I want to replace the beam splitter by something which involves Bell states. So as the beam splitter we do a massive creation of bell states. It's our entangler. And our second beam splitter is a Bell analyzer. It is a disentangler.

So what I mean is the following, we will actually just put one photon into this input beam, and we will only read out one channel. So all these here are only auxiliary modes. This is how I make a special quantum beam splitter. And what we need for this description is essentially two gates.

We need a single qubit. The single qubit is the [INAUDIBLE] which we have already discussed. And in the dual-rail representation where its photon can be in one of the two modes, the [INAUDIBLE] is simply connecting the two modes in this way. And we discussed that the [INAUDIBLE] can be described by a beam splitter with a phase shift.

So in other words, we need one element which is a beam splitter with a phase shift. But this is only acting on one qubits. And now we want to connect qubits with a controlled NOT gate. To remind you, controlled NOT gate is something where the photons stay where they are. But if the control bit is 1, it flips the target bit. If the control bit is 0, nothing happens to the target bit.

And we discussed already that we can realize one qubit with one interferometer. These are the dual-rails. We always have one photon in two states, this mode or that mode. And know the other qubit, one of the rails can go through the nonlinear Kerr medium. If the bit is in c through the phase shift in the nonlinear Kerr medium, it flips the [INAUDIBLE] down there, and this the controlled knot.

So now we want to use two qubits. So I'm talking about, in our fancy entangler I'm just talking about the first two rails here. So we have those two rails.

The [INAUDIBLE] puts us into a superposition of the logical CO and 1. And now after the c-not gate, so we are in a superposition of-- just give me a second-- we start here with 1. Sorry, we start here with 1.

AUDIENCE: Excuse me.

WOLFGANG KETTERLE: Yeah? I'm wrapping up. This is my last.

AUDIENCE: One [INAUDIBLE] minus 1 [INAUDIBLE].

WOLFGANG KETTERLE: The 0 [INAUDIBLE] in 0 plus 1. And let me just finish that. This is the target, year. Sorry, it's correct.

We start with 0, we start with 0. So what we have here is the product state of 0 plus 1 upstairs and 0 downstairs. But if we have a 1, we flip the 0 to 1.

So therefore, what we have now is the state 00 plus 1 1 over square root 2. And then we apply our phase shifter. And what we get out of this state is 00 plus e to the 2i phi times 1 1 over square root 2.

So now by using sort of-- and now I should stop. I know people are waiting. But by using two of those, I suddenly have multiplied the phase by 2, so something is now sensitive to 2 phi. And if I use n such, more and more entanglement, I will show you-- no class next week, but in the following week-- that we suddenly have a term in our quantum state which is e to the n phi. And this gives us effect of an in precision. OK, have a good rest of the week and we meet again in a week and a half.