**PROFESSOR:** Good afternoon. We continue our discussion of quantum states of light. We talked at length about coherent state, and when you talk about quantum states of light, each mode of the electromagnetic field is an harmonic oscillator.

We also encountered, naturally, the number states. And we realized-- yesterday, actually, in the last class-- that those number states have non-classical properties. For instance, they have a g2 function, the second order correlation function, which is smaller than 1, which is impossible for classic light, as you're proving in one of your homework assignment.

So at that point, we have encountered coherent states, which are as close as possible to classical states. And we have found the number states as non-classical states.

Well, are there other interesting states? I wouldn't ask you this question if the answer would not be yes, and this is what we want to discuss today. We want to talk about non-classical states of light, which we can engineer, actually, in the laboratory, by sending laser light through nonlinear crystals. Those go by the name, squeezed states.

Just to give you the cartoon picture, in our two-dimensional diagram, with the quasi-probabilities, we have coherent states, where the area of this disk, delta x delta p, is h-bar over two. It's uncertainty limited.

What we can do we now, is-- we cannot go beyond this. This is the fundamental limit of quantum physics. However, we can take this circle and we can squeeze it. We can squeeze it horizontally, we can squeeze it into an elongated vertical shape, or we can squeeze it at any angle. That's what we call, squeezed states.

And those states have non-classical properties. They are important for metrology they are important for teleportation. There are lots and lots of reasons why you want to know about them.

But again, as so often, I feel I cannot convey to you the excitement of doing squeezing in the quantum domain. And many, many physicists now, they hear about squeezing just in the quantum domain. But I want to start with classical squeezing. I will actually show you video of an experiment on classical squeezing. You can see squeezing with your own eyes.

But this is just sort of to set the stage, to also get a feel of what squeezing is. And then we'll do quantum mechanical squeezing. But maybe-- tongue in cheek-- I would say, since classical harmonic oscillators and quantum harmonic oscillators have a lot in common, the step from classical squeezing to quantum of mechanical squeezing is actually rather small.

It's nice to squeeze light. It's nice to have those non-classical states. But the question is, how can you detect it? If you can't detect it, you can't take advantage of it. And the detection has to be face-coherent. I will tell you what that is. And it goes by the name, homodyne detection.

And finally, we can take everything we have learned together, and discuss how, in the laboratory, teleportation of a quantum state is done. There is a nice teleportation scheme, and I want to use that as an example that the language and the concepts I've introduced are useful. Concepts like, squeezing operator, displacement operator-- those methods allowing us to, in a very clear way, discuss schemes which lead to teleportation. That's the menu for today.

Let's start with classical squeezing.

For squeezing, we need an harmonic oscillator, means for parabolic potential, we have potential v of x. And then we study the motion of-- that should be x squared-- the motion of a particle in there.

Before I even get started any equation, let me explain what the effect of squeezing will be about. If you have an harmonic oscillator, you have, actually, the motion of a pendulum. It has two quadrature components, the cosine motion and the sine motion. And they are 90 degrees out of phase.

What happens now is, if you parametrically drive the harmonic oscillator-- you modulate the harmonic oscillator potential-- it's to omega. I will show you mathematically, it's very, very easy to show, that depending on the phase of the drive, you will actually exponentially amplify the sine motion, and exponentially damp the cosine motion. Or if you change, vice versa. So by driving the system, you can amplify one quadrature component, and exponentially die out the other quadrature component. And that is called, classical squeezing.

Let's do the math. It's very simple. Our equation of motion has the two solutions I've just mentioned. It has a solution with cosine omega 0, and one with sine omega 0 t. And we have two coefficients. The cosine is called, c. The sine coefficient is called, s. I have to call it c 0, because I want to call that c and s.

So what we have here is, we have the two quadrature components of the motion in an harmonic oscillator. And graphically, we need that for the electromagnetic field, as well. When we have our two axes, like, you know, the complex plane for the cosine of probabilities, I call one the s-axis. One is the c-axis.

That's just something which confuses me. If you have only one-- just give me one second. Cosine-- Yeah.

If you have only cosine motion, the s component is 0, and the harmonic oscillator would just oscillate here. If you have only a sine component, you stay on the x-axis.

And now, if you have an equal amount of cosine and sine, then you can describe the trajectory to go in a circle. OK.

This is just the undriven harmonic oscillator. I don't want to dwell on it any longer. But what we are doing now is, we are adding a small parametric drive.

Parametric drive means we modulate the spring constant, or we replace the original harmonic potential, which was this, by an extra modulation term. So we have a small parameter, epsilon. And as I pointed out, the modulation is at twice the resonance frequency.

Now we want to solve the equation of motion for the harmonic oscillator, using this added potential. The way how we want to solve it is, we assume epsilon is very small. So if the pendulum is swinging with cosine omega t, it will take a while for the epsilon term-- for the small term-- to change the motion.

So therefore, we assume that we can actually go back and use our original solution. And assume that over a short term, the epsilon term is not doing anything. So for a short time, it looks like an harmonic oscillator with a sine omega 0, and cosine omega 0 t oscillation.

But over any longer period of time, the small term will have an effect. And therefore, the coefficients c of t, c, and s are no longer constant, but change as a function of time.

We want to solve, now, the equation of motion. That means we use this, here, as our ansatz. And we calculate the second derivative. We assume that the coefficients c and s are changing slowly. Therefore, the second derivative of c and s can be neglected.

By taking the derivative of the second derivative of the cosine term and the sine term, of course we simply get, minus omega 0 squared, x of t.

And now we have the second-order derivatives. Since we neglect the second-order derivative of c and s, the other terms we get when we take the second derivative is, first derivative of c times first derivative of cosine. First derivative of s times first derivative of sine. So we get two more terms, which are, minus omega 0 c dot, times sine omega 0 t. Plus omega 0 s dot, times cosine omega 0 t.

This is the second derivative of our ansatz for x. This has to be equal to the force provided by the potential. So taking the potential--

We need, now, the derivative of the potential, for the potential of use across this line. The first part is the unperturbed harmonic oscillator, which gives us simply, omega 0 squared times x.

And the second term, due to the parametric drive, is 2 sine omega 0 t. And now, for x, we use our ansatz for x, which is the slowly-changing amplitude c times cosine omega 0 t, plus s times sine omega 0 t.

Those two terms cancel out.

So now we have products of trig function. Sine 2 omega times cosine omega. Well, you know if you take the product of two trig functions, it becomes a trig function of the sum or the difference of the argument.

So if you take sine 2 omega 0 times cosine omega 0, and we use trigonometric identities, we get an oscillation at 3 omega 0, which is 2 plus 1. And one at the difference, which is omega 0.

Let me write down the terms which are of interest to us. Namely, the ones at omega 0. So let me factor out epsilon omega 0 squared over 2. Then we have the term c times sine omega 0 t, plus s times cosine. And then we have terms at 3 omega 0, which we are going to neglect.

Now we compare the two sides of the equations. We have sine omega 0 term. We have cosine omega 0 term. And the two sides of the equations are only consistent if the two coefficients of the sine term, and the sine term, are the same.

So therefore, we obtain two equations. One for c dot, one for s dot. And these are first-order differential equations. The solution is clearly an exponential.

But one has a plus sign, one has a minus sign. So the c component, the c quadrature component, is exponentially amplified with this time constant. Whereas the sine component is exponentially de-amplified.

This finishes the mathematical discussion of classical squeezing. We find that s of t, and c of t, are exponential functions. In one case, it's exponentially increasing. In the other case, it is exponentially decreasing. And that means that, well, if we go to our diagram, here-- and let's assume we had an arbitrary superposition of cosine and sine amplitude.

This is cosine. This is sine. We had sort of a cosine oscillation, and a sine oscillation. Which means that, as a phasor, the system was moving on an ellipse.

If the sine component is exponentially de-amplified, and the cosine component is exponentially amplified, that means whatever we start with is squashed horizontally, is squashed vertically. And is amplified horizontally. In the end, it will become a narrow strip. So this is classical squeezing.

You may want to ask, why did I neglect the 3 omega 0 term. Well, I have to, otherwise I don't have a solution. Because I have to be consistent with my approximations.

So what I did here is, I had an equation where I have the clear vision that the solution has a slowly varying c and s coefficient. And then I simply use that. I take the second-order derivative, and I have only Fourier components with omega 0, the sine, and cosine.

Now I've made an approximation, here. For the derivative of the potential, the first line is exact. But in order to match the approximation I've done on the other side, I can only focus on two Fourier components resonant with omega 0, which I have here.

So in other words, the 3 omega 0 term would lead to additional accelerations. Which I have not included in the treatment. So it's consistent with the ansatz. It's consistent with the assumption that we have resonant oscillations with a slowly changing amplitude. There will be a small [INAUDIBLE] for your omega 0, but it will be small. Any questions about that?

Let me then show you an animation of that. Classroom files.

[VIDEO PLAYBACK]

-We have Dave Pritchard, professor of physics at MIT, demonstrating what squeezing is. Right now, we see a wave that's going around in a circle. What's next? What's going to happen now, Professor Pritchard?

-Well, if we drive it in twice the basic period, then we will parametrically amplify one quadrature component, and we will un-amplify the other one. So now I'm going to start doing that. And then you notice that its motion turns into an ellipse. We've amplified this quadrature component, but we've un-amplifed that one. And that's squeezing.

[END VIDEO PLAYBACK]

**PROFESSOR:** Feel free to try it at home.

[LAUGHTER]

**PROFESSOR:** Actually, you may start to think about this demonstration. What he has shown was, when you have a circular pendulum which goes in a circle or an ellipse, and you start pulling on the rope with a certain phase, that one quadrature component will be de-amplified. The other one will be amplified. And as a result, no matter what the circular or the elliptical motion was, after driving it for a while, it will only swing in one direction. And this is the collection you have amplified.

There is one thing which should give you pause. I have discussed, actually, a single harmonic oscillator. What Dave Pritchard demonstrated was actually two harmonic oscillators. The harmonic oscillator has an x motion and a y motion. However, you can say, this was just sort of a trick for the demonstration, because when you have a circular motion, initially, you have the sine omega and the cosine omega 0 component present simultaneously.

And you can see what happens to the sine and the cosine component in one experiment. So in that sense, he did two experiments at once. He showed what happens when you have, initially, a sine component, and what happens when you initially have a cosine component.

OK. So we know what classical squeezing is. And what we have learned, also-- and this helps me now a lot to motivate how we squeeze in quantum mechanics-- you have realized that what is really essential here is, to drive it to omega 0.

What we need now to do squeezing in the quantum domain, if we want to squeeze light, we need something at 2 omega 0.

So let's now squeeze quantum mechanically. Go back here.

The second sub-section is now, squeezed quantum states.

What we want to discuss is, we want to discuss a quantum harmonic oscillator. We want to have some form of parametric drive at 2 omega 0. And this will result in squeezed states.

Now, what does it require, if you want to bring in 2 omega 0? Well, let's not forget our harmonic oscillators are modes of the electromagnetic field. If you now want to couple a mode of the electromagnetic field, at 2 omega 0, with our harmonic oscillator at omega 0, we need a coupling between two electromagnetic fields.

So therefore, we need nonlinear interactions between photons. So this was a tautology. We need nonlinear physics, which leads to interactions between photons. Linear physics means, each harmonic oscillator is independent. So we need some nonlinear process six which will be equivalent to have interactions between photons.

The device which we will provide that is an optical parametric oscillator. I could spend a long time explaining to you how those nonlinear crystals work. What is the polarization, what is the polarizability, how do you drive it, what is the nonlinearity.

But for the discussion in this class, which focuses on fundamental concepts, I can actually bypass it by just saying, assume you have a system-- and this is actually what the optical parametric oscillator does, is you pump it with photons at 2 omega 0. And then the crystal generates two photons at omega 0. Which of course, is consistent with energy conservation. And if you fulfill some phase-matching condition, it's also consistent with momentum conservation. But I don't want to go into phase-matching at this point.

Technically, this is done as simple as that. You have to pick the right crystal. Actually, a crystal which does mixing between three photon fields cannot have inversion symmetry, otherwise this nonlinear term is 0.

What you need is a special crystal. KDP is a common choice. And this crystal will now do for us the following. You shine in laser light. Let's say, at 532 nanometer, a green light. And then this photon breaks up into two photons of omega 0. This is how it's done in the laboratory.

The piece of art is, you have to pick the right crystal. It has to be cut at the right angle. You may have to heat it, and make sure that you select the temperature for which some form of resonant condition is fulfilled, to do that. But in essence, that's what you do. One laser beam, put in a crystal, and then the photon is broken into two equal parts. And these are our two photons at omega 0.

OK. I hope you enjoy the elegance-- we can completely bypass all the material physics by putting operators on it. We call this mode, b. And we call this mode, a. So the whole parametric process, the down conversion process of one photon into two, is now described by the following Hamiltonian.

We destroy a photon in mode b, a 2 omega 0. And now we create two photons at omega. We destroy a photon at 2 omega 0, create two photons at omega 0. And since the Hamiltonian has to be Hamiltonian, the opposite, the time-reverse process, has to be possible, too.

And that means we destroy two photons at omega 0, and create one photon at 2 omega 0. So now we forget about nonlinear crystals, about non-inversion symmetry in materials. We just take this Hamiltonian and play with it.

By simply looking at the Hamiltonian, what is the time evolution of a photon field under this Hamiltonian. We figure out what happens when you send light through a crystal, and what is the output.

And I want to show you now that the output of that is squeezed light, which is exactly what I promised you with these quasi-probabilities. We have a coherent state, which is a nice circle. We time-evolve the coherent state, our nice round circle, with this Hamiltonian. And what we get is an ellipse.

And if you want intuition, look at the classical example we did before, which really tells you in a more intuitive way what is happening.

OK. We want to make one simplifying assumption, here. And this is that we pump the crystal at 2 omega 0 with a strong laser beam. So we assume that the mode, b, is a powerful laser beam. Or in other words, a strong coherent state.

We assume that the mode, b, is in a coherent state. Coherent states are always labeled with a complex parameter, which I call beta, now. Well, it's mode b, therefore I call it, beta. For mode a, I've called it, alpha.

The coherent state has an amplitude, which I call, r over 2. And it has a phase.

We know, of course, that the operator, b, acting on beta, gives us beta times beta, because a coherent state is an eigenstate of the annihilation operator.

But when we look at the action of the operator b plus, the photon creation operator, the coherent state is not an eigenstate of the creation operator. It's only an eigenstate of the annihilation operator. But what sort of happens is, the coherent state is the sum over many, many number states with n. And the creation operator goes from n, to n plus 1, and has matrix elements which are square root n plus 1.

So in other words, if n is large, and if we don't care about the subtle difference between n, and n plus 1, in this limit the coherent state is also an eigenstate of the creation operator, with an eigenvalue, which is beta star. This means that we have a coherent state which is strong. Strong means, it has a large amplitude of the electric field. The photon states which are involved, n, are large. And we don't have whether it be n, or n plus 1.

This is actually, also, I should mention it here, explicitly-- this is sort of the step when we have a quantum description of light. And we replace the operators, p and p dega, by a c number, then we really go back to classical physics. Then we pretend that we have a classical electric field, which is described by the imaginary part of beta.

So when you have an Hamiltonian, where you write down an electric field, and the electric field is not changing-- you have an external electric field. This is really the limit of a quantum field, where you've eliminated the operator by a c number. This is essentially your electric field. And we do this approximation, here.

Because we are interested in the quantum features of mode a-- a is our quantum mode, with single photons, or with a vacuum state, and we want to squeeze it. b is just, they have parametric drive.

With this approximation, we have only the a operators. This is our operator.

Any question?

**AUDIENCE:** [INAUDIBLE] would give us a [INAUDIBLE], right?

**PROFESSOR:** Yes, thank you. That means, here should be a minus sign, yes.

OK. I've motivated our discussion with this nonlinear crystal, which generates pair of photons. This is the Hamiltonian which describes it. And if you want to have a time evolution by this Hamiltonian, you put this Hamiltonian into a time evolution operator.

In other words, you-- e to the minus iHt is the time evolution.

If you now evolve a quantum state of light for a fixed time, t, we apply the operator, e to the minus iHt, to the quantum state of light.

What I've just said is now the motivation for the definition of the squeezing operator.

The squeezing operator, S of r, is defined to be the exponent of minus r over 2, a squared minus a dega squared. This is related to the discussion above. You would say, hey, you want to do that time evolution, where is the i? Well, I've just made a choice of phi. If phi is chosen to be pi over 2, then the time evolution with the Hamiltonian, above, gives me the squeezing operator, below.

So with that motivation we are now studying, what is the squeezing operator doing to quantum states of light? Any questions about that? I know I spent a lot of time on it. I could have taught this class by just saying, here is an operator, the squeezing operator. Trust me, it does wonderful things. And then we can work out everything.

But I find his unsatisfying, so I wanted to show you what is really behind this operator. And I want you to have a feeling, where does this operator come from, and what is it doing?

In essence, what I've introduced into our description is now an operator, which is creating and destroying pairs of photons. And this will actually do wonderful things to our quantum states.

What are the properties of the squeezing operator? What is important is, it is unitary. It does a unitary time evolution. You may not see that immediately, so let me explain that.

You know from your basic quantum mechanics course, that e to the i operator A is unitary, when A is Hermitian. So the squeezing operator-- with the definition above-- can be written as, I factor out 2 i's over 2 a squared minus a dega squared. And you can immediately verify that this part, here, is Hermitian.

If you do the Hermitian conjugate, a squared turns into a dega squared. a dega squared turns into a squared. So we have a problem with a minus sign. But if you do the complex conjugate of i, this takes care of the minus sign. So this part is Hermitian. We multiply it with i, therefore this whole operator. Thus a unitary transformation in [INAUDIBLE].

Any questions?

OK. So after being familiar with this operator, we want to know, what is this operator doing?

I can describe, now, what this operator does, in a Schrodinger picture, or in a Heisenberg picture. I pick whatever is more convenient. And for now, this is the Heisenberg picture.

In the Heisenberg picture, what is changing are the operators. Therefore, in the Heisenberg picture, this unitary transformation transforms the operators. And we can study what happens when we transform the operator, x.

The unitary transformation is done by-- the operator, x, is transformed by multiplying from the left side with S, from the righthand side with S dega.

You are familiar with expressions like, this, and how to disentangle them. If you have an e to the i alpha, e to the minus alpha, if you could move the alpha past x. So if A and x commute, i A, minus i A would just give unity. So therefore, this expression is just x, unless you have non-Hermitian commutators between A and x.

I think you have solved, in your basic mechanics course, many such problems which involve identities of that form. Then there are higher order commutator, the commutator of A with the commutator of a x. Unless one of those commutator vanishes, you can get an infinite series.

Our operator, A, is nothing else than the annihilation operator, a squared minus the creation operator, a dega squared. So we can express everything in terms of a, and a dega.

The position operator in our harmonic oscillator can also be expressed by a, and a dega.

By doing elementary manipulations on the righthand side, and recouping terms, you find immediately that the unitary transformation of the Heisenberg operator, x, gives you an x operator back. But multiplied with an exponential, e to the r.

And if we would do the same to the momentum operator, which is a minus a dega over square root 2, we will find that the unitary transformation of the momentum operator is de-amplifying the momentum operator by an exponential factor.

If we would assume that we have a vacuum state in the harmonic oscillator, and while classically, it would be at x equals 0, p equals 0, quantum mechanically, we have single-point noise in x, and single-point noise in p.

Then you would find that the squeezing operator is amplifying the quantum noise in x. But it squeezes, or reduces, the noise in p.

If we apply this squeezing operator to the vacuum state, we obtain what is usually called, squeezed vacuum. And it means that, in this quasi-probability diagram, the action of the squeezing operator is turning the vacuum state into an ellipse.

What happens to energy, here? The vacuum state is the lowest-energy state. If you now act with a squeezing operator to it, we obtain a state which has-- the same energy? Is it energy-conserving, or very high energy?

**AUDIENCE:** Higher [INAUDIBLE].

**PROFESSOR:** Yes. Why?

**AUDIENCE:** It's no longer the [INAUDIBLE].

**PROFESSOR:** Sure, yeah. It's a vacuum state. We act on the vacuum state, but we get a state which is no longer the vacuum state.

The reason why we have extra energy-- the squeezed vacuum is very, very energetic. Because the squeezing operator had a dega squared, a squared. Well a squared, the annihilation operator acting on the vacuum, gives 0.

But what we are creating now, we are acting on the vacuum, and we are creating pairs of photons. So we are adding, literally, energy to the system. And the energy, of course, comes from the drive laser, from the laser 2 omega 0, which delivers the energy in forms of photons which are split into half, and they go into our quantum field.

In the limit of infinite squeezing-- I will show it to you, mathematically, but it's nice to discuss it already here. In the limit of infinite squeezing, what is the state we are getting?

**AUDIENCE:** Eigenstate of momentum.

**PROFESSOR:** Eigenstate momentum. We get the p equals 0 eigenstate. What is the energy of the p equals 0 eigenstate?

**AUDIENCE:** Infinite. It has to contain all number states.

**PROFESSOR:** It contains all number states? OK, you think immediately into number states, which is great. But in a more pedestrian way, the p equals 0 state has no kinetic energy. But if a state is localized in momentum, p equals 0, it has to be infinitely smeared out on the x-axis.

And don't forget, we have an harmonic oscillator potential. If you have a particle which is completely delocalized in x, it has infinite potential energy at the wings.

So therefore in the limit of extreme squeezing, we involve an extreme number of number states. Actually, I want to be more specific-- of photon pairs. We have states with 2n, and n can be infinitely large. But we'll see in the classical picture, what we get here when we squeeze it is, we get the p equals 0 eigenstate, which has infinite energy, due to the harmonic oscillator potential.

If we would allow with the system now, after we have squeezed it, to evolve for a quarter period in the harmonic oscillator, then the ellipse would turn into an vertical ellipse.

So this is now an eigenstate of x. It's the x equals 0 eigenstate. But the x equals 0 eigenstate has also infinite energy, because due to Heisenberg's uncertainty relation, it involves momentum states of infinite momentum.

Questions?

**AUDIENCE:** [INAUDIBLE] a is the photon field, right? So p is roughly the electrical field, right?

**PROFESSOR:** Yes.

**AUDIENCE:** So it's kind of that the electric field counts 0, and x is kind of the a, the-- and it-- because of [INAUDIBLE]. The electrical field is squeezed?

**PROFESSOR:** Yes.

**AUDIENCE:** It means we have no electrical field?

**PROFESSOR:** We'll come to that in a moment. I want to do a little bit more math, to show you. I wanted to derive for you an expression of the squeeze state, in number basis, and such.

Your question mentioned something which is absolutely correct. By squeezing that, we have now the p-axis is the electric field axis. So now we have, actually, in the limit of infinite squeezing, we have an electric field which has no uncertainty anymore. By squeezing the coherent state into a momentum eigenstate, we have created a sharp value for the electric field. We have created an electric field eigenstate.

Well you would say, it's pretty boring, because the only electric field state we have created is electric field e equals 0. But in the next half-hour, we want to discuss the displacement operator, and I will tell you what it is. That we can now move the ellipses, and move the circles, anywhere where we want. So once we have an electric field state which has a sharp value of the electric field at e equals 0, we can just translate it.

But before you get too excited about having an eigenstate of the electric field, I want you to think about what happened after one quarter-period it of the harmonic oscillator frequency. It turns upside down, and your electric field has an infinite variance.

That's what quantum mechanics tells us. We can create electric fields which are very precise, but only for a short moment. So in other words, this electric field state which we have created would have a sharp value. A moment later, it would be very smeared out, then it has a sharp value again, and then it's smeared out again. I mean, that's what squeezed states are.

Other questions?

**AUDIENCE:** That's why [INAUDIBLE].

**PROFESSOR:** That's why we need homodyne detection. Yes, exactly. If we have squeezed something, which is sort of narrow, that's great for measurement. Now we can do a measurement of, maybe, a LIGO measurement for gravitational waves with higher precision, because we have a more precise value in our quantum state.

But we have to look at it at the right time. We have to look at it synchronized with the harmonic motion. Homodyne detection means we look only at the sine component, or at the cosine component. Or if I want to simplify it, what you want to do is, if you have a state like this, you want to measure the electric field, so to speak, stroboscopically. You want to look at your system always when the ellipse is like this.

The stroboscopic measurement is, as I will show you, in essence, a lock-in measurement, which is phase-sensitive. And this will be homodyne detection.

So we can only take advantage of the squeezing, of having less uncertainty in one quadrature component, if you do phase-sensitive detection, which is homodyne detection. But now I'm already an hour ahead of the course.

OK. Back to basics.

We want to explicitly calculate, now, how does a squeezed vacuum look like. We actually want to do it twice, because it's useful. We have to see it in two different basis. One is, I want to write down the squeezed vacuum for you in a number representation. And then in a coherent state representation.

The squeezing operator is an exponential function involving a squared, and a dega squared. And of course, we're now using the Taylor expansion of that.

We are acting on the vacuum state. I will not do the calculation. It's again, elementary. You have n factorial, you have terms with a dega acts on c, well, you pay 2 photons. If it acts again, it adds 2 more photons, and the matrix element of a dega acting on n is square root n plus 1. You just sort of rearrange the terms. And what you find is, what I will write you down in the next line.

The important thing you should immediately realize is, the squeeze state is something very special. It is the superposition of number states, but all number states are even because our squeezing operator creates pairs of photons. This is what the parametric down-conversion does. We inject photons into the vacuum, but always exactly in pairs. And therefore, it's not a random state. It's a highly correlated state with very special properties.

OK. If you do the calculation and recoup the terms, you get factorials, you get 2 to the n, you get another factorial. You get hyperbolic tangent-- sorry, to the power n. And the normalization is done by the square root of the cos function.

And the more we squeeze, the larger are the amplitudes at higher and higher n. But this is also obvious from the graphic representation I've shown you.

Let me add the coherent state representation. The coherence states are related to the number states in that way.

If we transform now from number states to coherent states, the straightforward calculation gives, now, superposition over coherent states. Coherent states require an integral. e to the minus e to the r over 2, divided by--

Anyway, all this expressions, they may not be in its general form, too illuminating. But those things can be done analytically.

I just want to mention the interesting limiting case of infinite squeezing.

**AUDIENCE:** When you do the integral over alpha, is this over like, a magnitude of alpha, or a real part, or [INAUDIBLE]?

**PROFESSOR:** I remember, but I'm not 100% sure that alpha is real, here. I mean, it sort of makes sense, because we start with the vacuum state. And if we squeeze it, we are not really going into the imaginary direction. So I think what is involved here are only real alpha.

**AUDIENCE:** For negative r, we should get [INAUDIBLE].

**PROFESSOR:** For negative r, we need imaginary state.

**AUDIENCE:** So we should [INAUDIBLE].

**PROFESSOR:** Let me double-check. I don't remember that. You know that, sometimes, I admit it, the issue-- if you research material, prepare a course some years ago, you forget certain things. If I prepared the lecture, and everything worked out yesterday, I would know that. But certain things you don't remember. As far as I know, it's the real axis. But I have to double-check.

The limiting case is interesting. If r goes to infinity, you can show that this is simply the integral, d alpha over coherent states.

We have discussed, graphically, the situation where we had-- so these are quasi-probabilities.

In that case of infinite squeezing, we have the momentum eigenstate, p equals 0. This is the limit of the infinitely squeezed vacuum, and in a coherent state representation, it is the integral over coherent state alpha.

I'm pretty sure alpha is real here, seeing that now.

There is a second limit, which happens simply-- you can say, by rotation, or by time evolution-- which is the x equals 0 eigenstate. And this is proportional to the integral over alpha when we take the coherent state i alpha, and we integrate from minus to plus infinity.

OK. So we have connected our squeezed states, the squeezed vacuum, with number states, with coherent states. Now we need one more thing. So far we've only squeezed the vacuum, and we have defined the squeezing operator that it takes a vacuum state and elongates it. In order to generate more general states, we want to get away from the origin. And this is done by the displacement operator.

The definition of the displacement operator is given here. The displacement by a complex number, alpha, is done by putting alpha, and alpha star, into an exponential function. In many quantum mechanic courses, you show very easily the elementary properties. If the displacement operator is used to transform the annihilation operator, it just does that.

If you take the complex conjugate of it-- so in other words, what that means is, it's called the displacement operator, I just take that as the definition. But you immediately see why it's called the displacement operator when we do the unitary transformation of the annihilation operator, we get the annihilation operator displaced by a complex number.

So the action, the transformation of the annihilation operator is the annihilator operator itself, minus a c number. So therefore, we say, the annihilation operator data has been displaced.

So this is the action of the displacement operator on an operator-- on the annihilation operator. The question is now, how does the displacement operator act on quantum states?

And the simplest quantum state we want to test out is the vacuum state. And well, not surprisingly, the displacement operator, displacing the vector state by alpha, is creating the coherent state, alpha. This can be proven in one line.

We take our displaced vacuum, and we act on it with the annihilation operator. If we act with the annihilation operator on something, and we get the same thing back times an eigenvalue, we know it's a coherent state. Because this was the definition of coherent states.

So therefore, in order to show that this is a coherent state, we want to show that it's an eigenstate of the annihilation operator. So this is what we want to do. The proof is very easy. By multiplying this expression with unity, which is DD dega, we have this. And now we can use the result for the transformation of operators. Namely, that this is simply the annihilation operator, plus alpha.

If the annihilation operator acts on the vacuum state, we get 0. If alpha acts on the vacuum state, we get alpha times 0. So therefore, what we obtain is that. When the annihilation operator acts on this state, we get alpha times the state, and therefore the state is a coherent state with eigenvalue alpha.

In a graphical way, if you have a vacuum state the displacement operator, D alpha, takes a vacuum state and creates a coherent state alpha.

If you want to have squeezed states with a finite value, well, we just discussed the electric field. Related to the harmonic oscillator, we want squeeze states, which are not centered at the origin, which have a finite value of x or p. We can now create them by first squeezing the vacuum, and then displacing the state.

**AUDIENCE:** What's the physical realization of the displacement operator?

**PROFESSOR:** What is the physical realization of the displacement operator? Just one second. The physical representation of the displacement operator-- we'll do that on Monday-- is the following. If you pass an arbitrary state through a beam splitter-- but it's a beam splitter which has very, very high transmission-- and then, from the-- I'll just show that.

If you have a state-- this is a beam splitter-- which has a very high transmission, T is approximately 1, then the state passes through. But then from the other side of the beam splitter, you come with a very strong coherent state. You have a coherent state which is characterized by a large complex number, beta. And then there is a reflection coefficient, r, which is very small. It sort of reflects the coherent state with an amplitude r beta.

If you mix together the transmitted state and r beta-- I will show that to you explicitly, by doing a quantum treatment of the beam splitter-- what you get is, the initial state is pretty much transmitted without attenuation. But the reflected part of the strong coherent state-- you compensate for the small r by a large beta-- does actually an exact displacement of r beta.

It's actually great. The beam splitter is a wonderful device. You think you have a displacement operator formulated with a's and a dega's, it looks like something extract. But you can go to the lab, simply get one beam splitter, take a strong laser beam, and whatever you send through the beam splitter gets displaced, gets acted upon by the beam splitter.

Yes.

**AUDIENCE:** You showed the displacement operator, when you acted on the vacuum state, will displace the vacuum state to a state alpha. Does it still hold if you acted on, like another coherent state. Or in this case, a squeeze state like that?

**PROFESSOR:** Yes. I haven't shown it, but it's really-- it displaces-- When we use this representation with quasi-probabilities, it simply does a displacement in the plane. But no. To be honest, when I say it does a displacement on the plane, it reminds me that we have three different ways of defining quasi-probabilities. The w, the p, and the q representation.

I know we use it all the time, that we displace things in the plane. But I'm wondering if the displacement operator does an exact displacement of all representations, or only of the q representation. That's something I don't know for sure.

**AUDIENCE:** I was thinking it could also, like-- I mean, are you going to be able to displace all types of light, like thermal light, or any representation of light that you could put in, is the same displacement operator going to work? Or is its domain just the vacuum and coherent states?

**PROFESSOR:** The fact is, the coherent states-- I've shown you that it's a vacuum state. I know that's the next thing to show, the displacement operator if you have a displacement by alpha followed by displacement by beta, it is equal to displacement by alpha plus beta. So displacement operator forms a group, and if you do two displacements, they equal into one area of displacement, which is the sum of two complex numbers.

What I'm just saying, if you do the first displacement you can get an arbitrarily coherent state. So therefore, the displacement operator is exactly displacing a coherent state by the argument of the displacement operator. And now if you take an arbitrary quantum state and expand it into coherent states-- coherent states are not only complete, they are over-complete. All you have done is, you've done a displacement.

Now the over-completeness, of course, means you have to think about it, because you can represent states in more than one way by coherent states. But if you have your representation, you just displace it, and this is the result of the displacement operator.

So since the q representation is simply, you take the statistical operator and look for the elements in alpha, and if you displace alpha, the q representation has been moved around. So I'm sure that for the q representation, for the q quasi-probabilities, the displacement operator shifted around in this place. For the w and p representation, I'm not sure. Maybe there's an expert in the audience who knows more about it than I do.

OK. We have just five minutes left.

I want to discuss now the electric field of squeezed states. And for that, let me load a picture. Insert picture. Classroom files.

Let us discuss, now, the electric field of squeezed states. Just as a reminder, we can discuss the electric field by using the quasi-probability representation. And the electric field is the projection of the quasi-probabilities on the vertical axis.

And then the time evolution is, that everything rotates with omega in this complex plane. We discussed it already. For coherent state, we have a circle which rotates. Therefore, the projected fuzziness of the electric field is always the same. And as time goes by, we have a sinusoidal-bearing electric field.

Let me just make one comment. If you look into the literature, some people actually say, the electric field is the projection on the horizontal axis. So there are people who say, the electric field is given by the x-coordinate of the harmonic oscillator, whereas I'm telling you, it's the p-coordinate.

Well, if you think one person is wrong, I would suggest you just wait a quarter-period of the harmonic oscillator, and then the other person is right. It's really just a phase convention. What do you assume to be t equals 0-- it's really arbitrary.

But here in this course, I will use the projections on the vertical axis.

OK. If you project the number state, we get always, 0 electric field, with a large uncertainty. So that's just a reminder.

But now we have a squeezed state. It's a displaced squeezed state. If you project it onto the y-axis, we have first some large uncertainty. I think this plot assumes that we rotate with negative time, so I apologize for that. You can just invert time, if you want.

So after a quarter-period, the ellipse is now horizontal, and that means the electric field is very sharp. As time goes by, you see that the uncertainty of the electric field is large, small, large, small-- it modulates. It can become very extreme, when you do extreme squeezing, so you have an extremely precise value of the electric field, here, but you've a large uncertainty, there.

Sometimes you want to accurately measure the 0 crossing of the electric field. This may be something which interests you, for an experiment. In that case, you actually want to have an ellipse which is horizontally squeezed. Now, whenever the electric field is 0, there is very little noise. But after a quarter-period, when the electric field reaches its maximum, you have a lot of noise. So it's sort of your choice which way you squeeze. Whether you want the electric field to be precise, have little fluctuations when it goes through 0, or when it goes through the maximum.

So what we have done here is, we have first created the squeezed vacuum, and then we have acted on it with a displacement operator.

OK. I think that's a good moment to stop.

Let me just say what I wanted to take from this picture. The fact that the electric field is precise only at certain moments means that we can only take advantage of it when we do a phase-sensitive detection. We only want to sort of, measure, the electric field when it's sharp.

Or-- this is equivalent-- we should regard light is always composed of two quadrature components. You can say, the cosine, the sine oscillation, the x, and the p. And the squeezing is squeezed in one quadrature, by it is elongated in the other quadrature.

Therefore, we want to be phase-sensitive. We want to pick out either the cosine omega T, or the sine omega T oscillation. This is sort of, homodyne detection. We'll discuss it on Monday.

Any question? OK. Good.